Fractional dimensions

Nuno Luzia

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$\rightsquigarrow$ exemplos fractais em varias dimensoes: 0-1 Cantor, 1-2 Tapetes Sierpinski, Triangulo Sierpinski, Curva Weierstrass, 2-3 Esponjas Sierpinski

$\rightsquigarrow$ Hausdorff dimension $\rightarrow$ list of fractals by dimension

'Hausdorff dimension is a number that distinguishes different fractals'

Lecture 1: Self-affine Cantor sets: 1-D

Lecture 2: Self-affine Sierpinski carpets: 2-D

Lecture 3: Self-affine Sierpinski sponges: 3-D

Lecture 4: Non-linear fractal sets: conformal and non-conformal

Notes: www.math.ist.utl.pt/∼nluzia

Definition of Hausdorff dimension of a set F

This is done using **measures**.

Example: F bounded subset of $\mathbb{R}^2$, int$F \neq \emptyset$. Then dim$F = 2$ and we use Leb^2 to measure F:

$$\text{Leb}^2{}_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

$$\text{Leb}^2(F) = \lim_{\delta \to 0} \text{Leb}^2{}_\delta(F)$$

What if Leb$^2(F) = 0$, e.g. F is a segment of line. Then 2 is too big.

What if Leb$^2(F) = \infty$, e.g. F is a solid cube in $\mathbb{R}^3$. Then 2 is not enough.

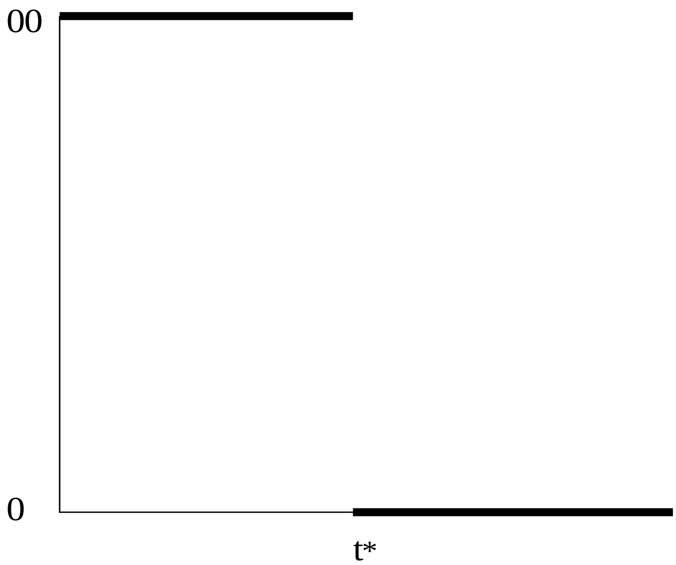
Let $t \in [0, 3]$, the candidate to the dimension. We define t-Hausdorff measure:

$$\mathcal{H}^t_\delta(F) = \text{inf} \left\{ \sum_{i=1}^{\infty} |U_i|^t : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$$

$$\mathcal{H}^t(F) = \lim_{\delta \to 0} \mathcal{H}^t_\delta(F)$$

What is the appropriate t for measuring F?

$t \longmapsto \mathcal{H}^t(F)$



$$\dim_{\mathsf{H}} F = t^*$$

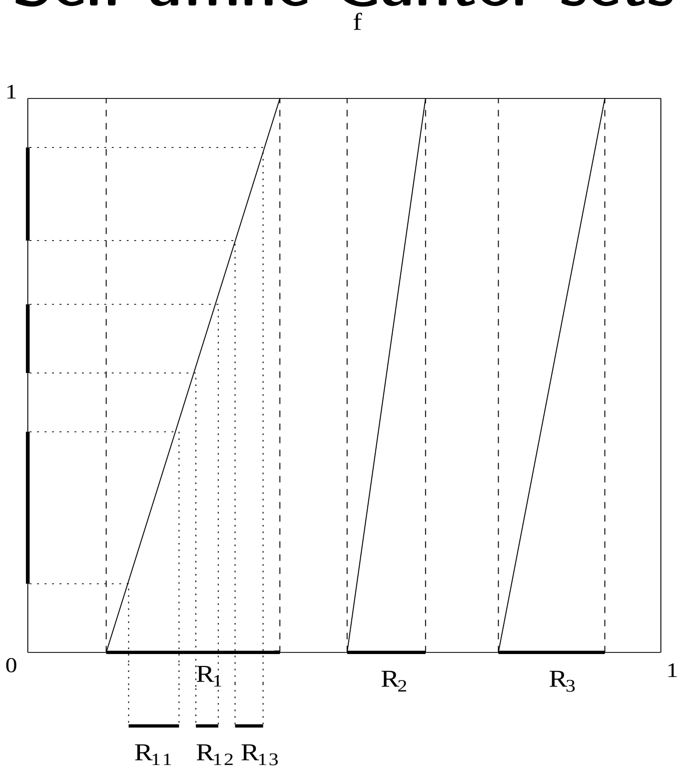
Remark 1. In the definition of Hausdorff measure/dimension instead of using any δ-cover we can use covers by balls, or squares, etc. $\rightsquigarrow$ What we cannot use is covers formed by sets which are 'too distorted'. **conformality**

In this lecture we will see that the Hausdorff dimension of the middle-third Cantor set is

$$\frac{\log 2}{\log 3}$$

How to compute Hausdorff dimension

Self-affine Cantor sets



$$R_{i_1 \ldots i_n} = R_{i_1} \cap f^{-1}(R_{i_2}) \cap \cdots \cap f^{-n+1}(R_{i_n})$$

basic sets of order n

$$K = \bigcap_{n=0}^{\infty} \bigcup_{i_1,\ldots,i_n} R_{i_1\ldots i_n}$$

Note: We are not interested in estimates but equality.

Estimate $\leq$: If, for every $\delta > 0$, we find *some* δ-cover $\{U_i\}$ of K such that

$$\sum_i |U_i|^t \leq C < \infty,$$

where C is a constant independent of δ, then $\mathcal{H}^t(K) < \infty$ which implies

$$\dim_{\mathsf{H}} K \leq t.$$

Since our set is dynamically defined we have natural covers of arbitrarily small diameter: covers by *basic sets of order* n: $R_{i_1\ldots i_n}$. Note that, since we are in the real line, these sets are balls and they are good to compute Hausdorff dimension.

Let $a_i = |R_i|$, $i = 1, ..., m$, the contraction ratios. Then $|R_{i_1...i_n}| = a_{i_1} a_{i_2} ... a_{i_n}$ and

$$\sum_{i_1, i_2, ..., i_n} |R_{i_1...i_n}|^t = \left(\sum_{i=1}^{m} a_i^t \right)^n$$

The least t for which the above is finite for every n is the unique solution $t = t_0$ of

$$\textbf{Moran formula}: \quad \sum_{i=1}^{m} a_i^{t_0} = 1$$

$$\dim_{\mathsf{H}} K \leq t_0$$

Estimate $\geq$: For *any* δ-cover $\{U_i\}$ of K

$$\sum_i |U_i|^t \geq C.........$$

Mass Distibution Principle

μ measure $\mu(K) = 1$ (fractal measure). Suppose that

$$\mu(U) \leq C|U|^t$$

for any set U.

If $\{U_i\}$ is *any* δ-cover of K

$$\begin{aligned}\sum_i |U_i|^t &\geq C^{-1} \sum_i \mu(U_i) \\ &\geq C^{-1} \mu(\sum_i U_i) \\ &\geq C^{-1} \mu(K) = C^{-1}\end{aligned}$$

$\Rightarrow \quad \mathcal{H}^t(F) \geq C^{-1} > 0$

$\Rightarrow \quad \dim_{\mathsf{H}} F \geq t$

Important class of fractal measures: *Bernoulli measures*

Assign weights p_i to the sets R_i, $i = 1, ..., m$ $(p_1, ..., p_m)$ ia a probability vector

$$\sum_{i=1}^{m} p_i = 1$$

By Kolmogorov, μ is well defined by

$$\mu(R_{i_1 \dots i_n}) = p_{i_1} \dots . p_{i_n}$$

In our case let

$$p_i = a_i^{t_0}$$

where t_0 is given by Moran formula (it is a probability measure!). Then

$$\mu(R_{i_1 \dots i_n}) = (a_{i_1} \dots . a_{i_n})^{t_0} = |R_{i_1 \dots i_n}|^{t_0}$$

$\Rightarrow \dim_{\mathsf{H}} K \geq t_0$

Conclusion:

$$\dim_{\mathsf{H}} K = t_0 \qquad : \qquad \sum_{i=1}^{m} a_i^{t_0} = 1$$

Example: Middle-third Cantor set

$$2\left(\frac{1}{3}\right)^t = 1$$

$$\dim_{\mathsf{H}} K = \frac{\log 2}{\log 3}$$

Note that this is a formula of the type

$$\text{dimension} = \frac{\text{entropy}}{\text{Lyapunov exponent}}$$

entropy $= \log(\sharp\, pieces)$

Lyapunov exponent $= \log(expansion)$

This method also applies to **conformal** (self-similar) fractal sets

Sierpinski carpet

contraction: 3^{-1}
pieces: 8

$$\dim_{\mathsf{H}} K = \frac{\log 8}{\log 3}$$

Sierpinski gasket

contraction: 2^{-1}
pieces: 3

$$\dim_{\mathsf{H}} K = \frac{\log 3}{\log 2}$$

Sierpinski sponge

contraction: 3^{-1}
pieces: 20

$$\dim_{\mathsf{H}} K = \frac{\log 20}{\log 3}$$

Example Middle-α Cantor set, $\alpha \in (0, 1)$

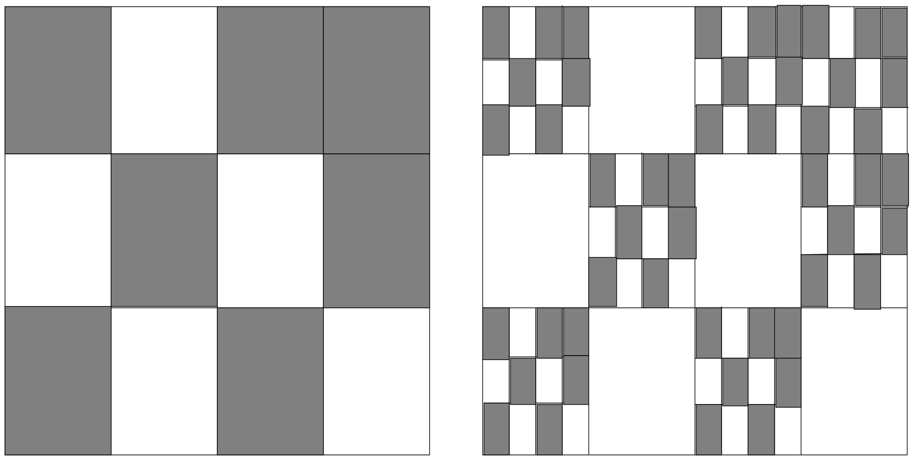
This gives a family of Cantor sets whose dimensions runs from 0 to 1 continuously

$$\dim_{\mathsf{H}} K_\alpha = \frac{\log 2}{\log(2/(1-\alpha))}$$

Non-conformal fractals

Example: General Sierpinski carpets (Bedford, McMullen 1984)

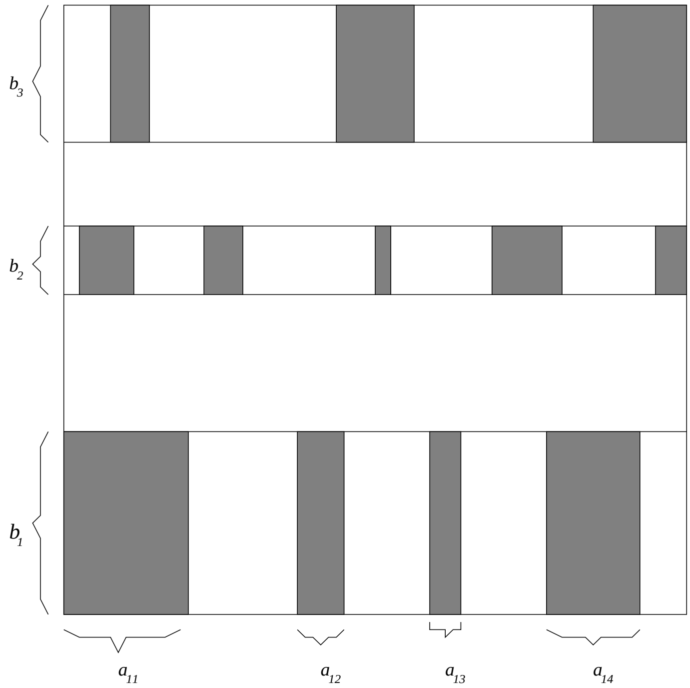
$$f(x,y) = (lx, my) \qquad 1 < m < l$$



$$\dim_{\mathsf{H}} \Lambda = \frac{\log(\sum_{i=1}^{m} n_i^{\alpha})}{\log m}$$

$$\alpha = \frac{\log m}{\log l}$$

Self-affine Sierpinski carpets



$$\mathbf{p} = (p_1, p_2, ..., p_m) \quad \text{probability vector}$$

$$\dim_{\mathrm{H}} \Lambda = \sup_{\mathbf{p}} \left\{ \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i} + t(\mathbf{p}) \right\}$$

$$t(\mathbf{p}): \qquad \sum_{i=1}^{m} p_i \log \left(\sum_{j=1}^{n_i} a_{ij}^{t(\mathbf{p})} \right) = 0$$

random 1-dimensional self-affine Cantor set

$G_{\mathbf{p}}$ set of vertical points y that distribute according to $\mathbf{p}$

$$y \in G_{\mathbf{p}} \quad \Rightarrow \quad \dim_{\mathsf{H}} \Lambda_y = t(\mathbf{p})$$

$$\dim_{\mathsf{H}} G_{\mathbf{p}} = \frac{\text{entropy}(\mathbf{p})}{\text{Lyapunov exponent}(\mathbf{p})}$$

$$= \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i}$$

Points that distribute according to some distribution $\mathbf{p}$ carry all Hausdorff dimension $\to$ Variational Principle for Hausdorff dimension

Exceptional points do too!

To prove an estimate

$$\dim_{\mathsf{H}} \Lambda \geq t$$

we used probability measures on Λ and the Mass Distribution Principle:

$$\mu(U) \leq C|U|^t$$

'Non-uniform version': **Volume Lemma**

Suppose

$$d_\mu(z) = \lim_{r \to 0} \frac{\log \mu(B(z,r))}{\log r}$$

exists $|\underline{\overline{\mu - a.e.\, z}}|$ and is constant:

$$\dim_{\mathsf{H}} \mu$$

Then

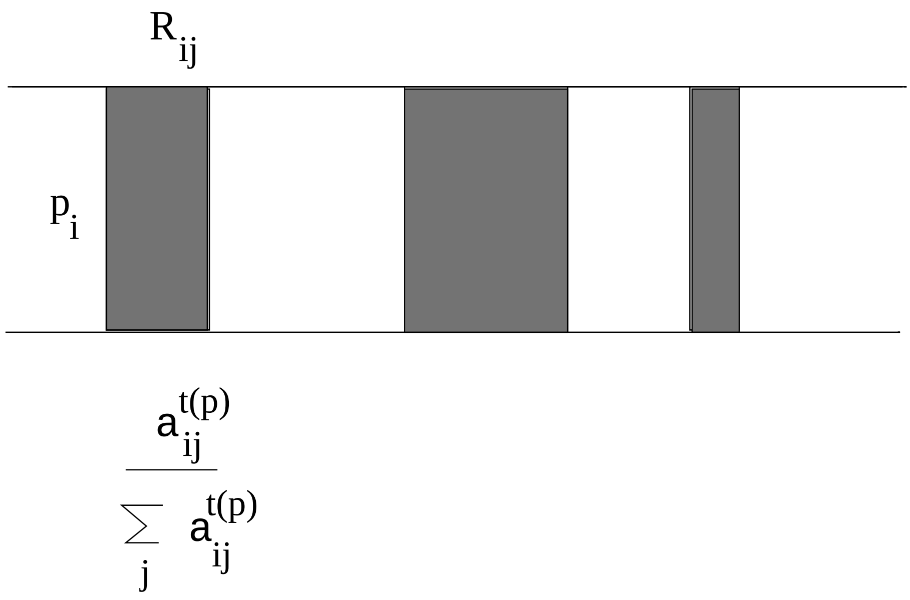
$$\dim_{\mathsf{H}} \Lambda \geq \dim_{\mathsf{H}} \mu$$

Given $\mathbf{p} = (p_1, p_2, ..., p_m)$
construct Bernoulli measures $\mu_{\mathbf{p}}$ on Λ

$$\dim_{\mathrm{H}} \mu_{\mathbf{p}} = \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i} + t(\mathbf{p})$$

$$\Rightarrow \quad \dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \mu_{\mathbf{p}}$$

R_{ij} basic sets of order 1



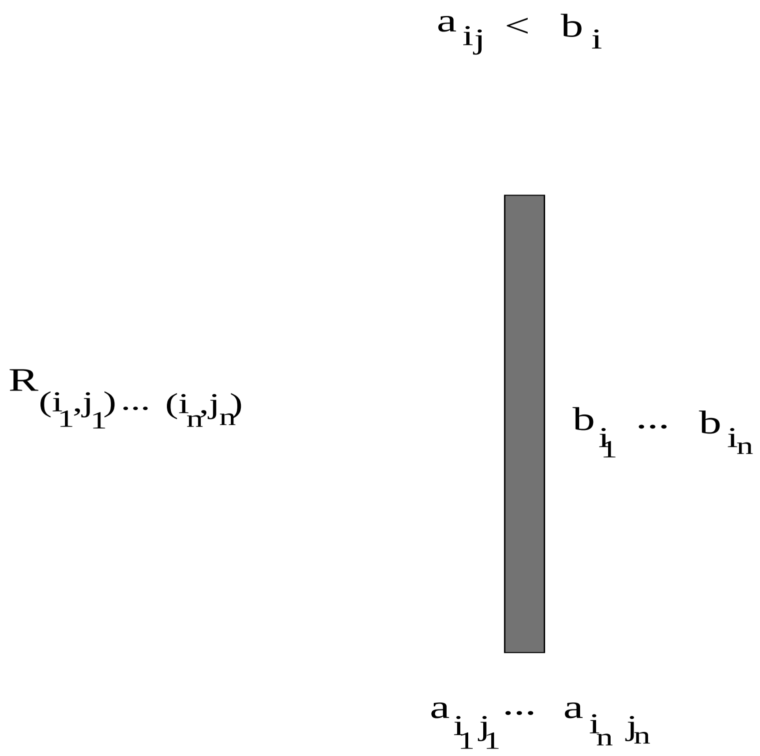
$$\mu_{\mathbf{p}}(R_{ij}) = p_i \cdot \frac{a_{ij}^{t(\mathbf{p})}}{\sum_j a_{ij}^{t(\mathbf{p})}} = p_{ij}$$

$R_{(i_1 j_1)\dots(i_n j_n)}$ basic sets of order n

$$\mu_{\mathbf{p}}(R_{(i_1 j_1)\dots(i_n j_n)}) = \prod_{l=1}^{n} p_{i_l j_l}$$

$R_{(i_1 j_1)\dots(i_n j_n)}$ 'too distorted'

no good for calculating dimension

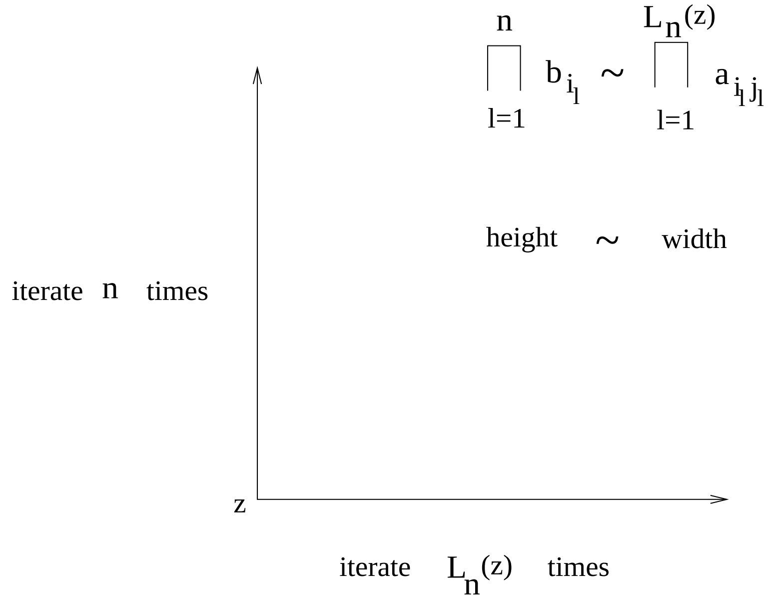


Solution: Make union of basic sets of order n to make it an 'approximate square'

$B_n(z)$

$$z = \bigcap_{n=1}^{\infty} R_{(i_1 j_1)\ldots(i_n j_n)}.$$

$$z \equiv ((i_1 j_1)(i_2 j_2)\ldots(i_n j_n)\ldots.)$$



approximate square of order n centered at z

$$B_n(z) = \{\, z' \in \Lambda : i'_l = i_l,\ l = 1, \ldots, n \\ j'_l = j_l,\ l = 1, \ldots, L_n(z) \,\}$$

$$\mu_{\mathbf{p}}(B_n(z)) = \prod_{l=1}^{n} p_{i_l} \cdot \prod_{l=1}^{L_n(z)} \frac{a_{i_l j_l}^{t(\mathbf{p})}}{\sum_j a_{i_l j}^{t(\mathbf{p})}}$$

dimension of $\mu_{\mathbf{p}}$ using Volume Lemma

$$\frac{\log \mu_{\mathbf{p}}(B_n(z))}{\sum_{l=1}^{n} \log b_{i_l}} = v + h_1 + h_2$$

By KSLLN, $|\overline{\underline{\mu_{\mathbf{p}} - a.e.\, z}}|$

$$v = \frac{\frac{1}{n}\sum_{l=1}^{n} \log p_{i_l}}{\frac{1}{n}\sum_{l=1}^{n} \log b_{i_l}} \quad \underset{n\to\infty}{\longrightarrow} \quad \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i}$$

$$h_1 = t(\mathbf{p}) \cdot \frac{\sum_{l=1}^{L_n(z)} \log a_{i_l j_l}}{\sum_{l=1}^{n} \log b_{i_l}} \quad \xrightarrow[n\to\infty]{} \quad t(\mathbf{p})$$

$$h_2 = \frac{\frac{1}{L_n(z)} \sum_{l=1}^{L_n(z)} \log\Big(\sum_{j=1}^{n_{i_l}} a_{i_l j}^{t(\mathbf{p})}\Big)}{\frac{1}{L_n(z)} \sum_{l=1}^{n} \log b_{i_l}} \quad \xrightarrow[n\to\infty]{} \quad 0$$

Thus

$$\dim_{\mathsf{H}} \mu_{\mathbf{p}} = \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i} + t(\mathbf{p})$$

and

$$\dim_{\mathsf{H}} \Lambda \geq \dim_{\mathsf{H}} \mu_{\mathbf{p}}$$

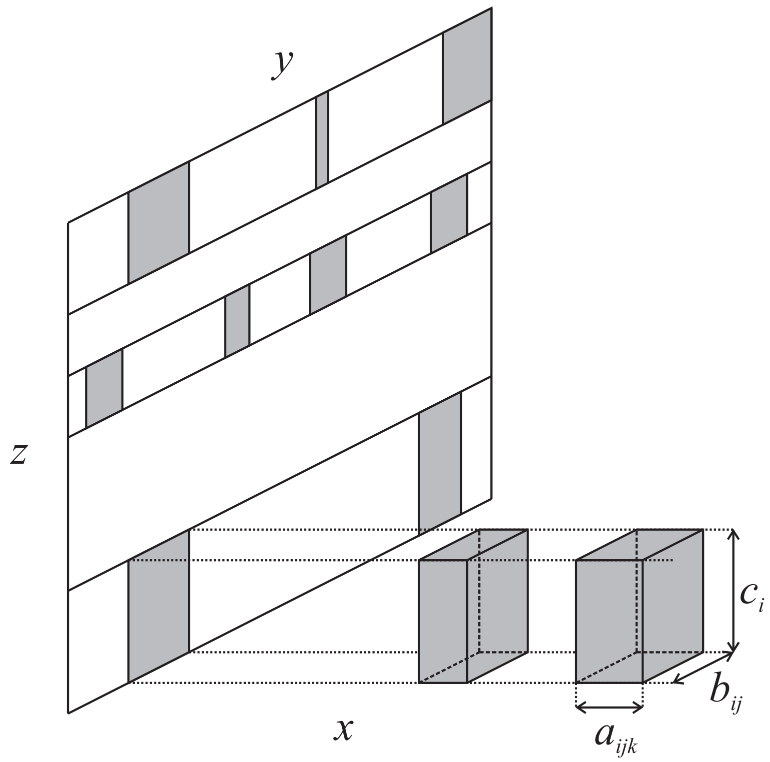
Exceptional points

The estimate $\leq$ is more difficult because we cannot disregard the set of *exceptional points* i.e. the set of points that do not follow any distribution $\mathbf{p}$: they have full Hausdorff dimension!

$\rightsquigarrow$ mostrar problema convergencia em KSLLN na dimensao medida, sublimites diferentes

We have to be more careful with the choice of appropriate measures that will take into account the 'oscillation phenomenon'. We will construct a special family of Bernoulli measures in the more general 3-D context

Self-affine Sierpinski sponges



$\mathbf{p} = (p_{ij})_{ij}$ 'vertical' probability vector

$$p_i = \sum_j p_{ij}$$

$$\dim_{\mathsf{H}} \Lambda = \sup_{\mathbf{p}} \{\lambda_1(\mathbf{p}) + \lambda_2(\mathbf{p}) + t(\mathbf{p})\}$$

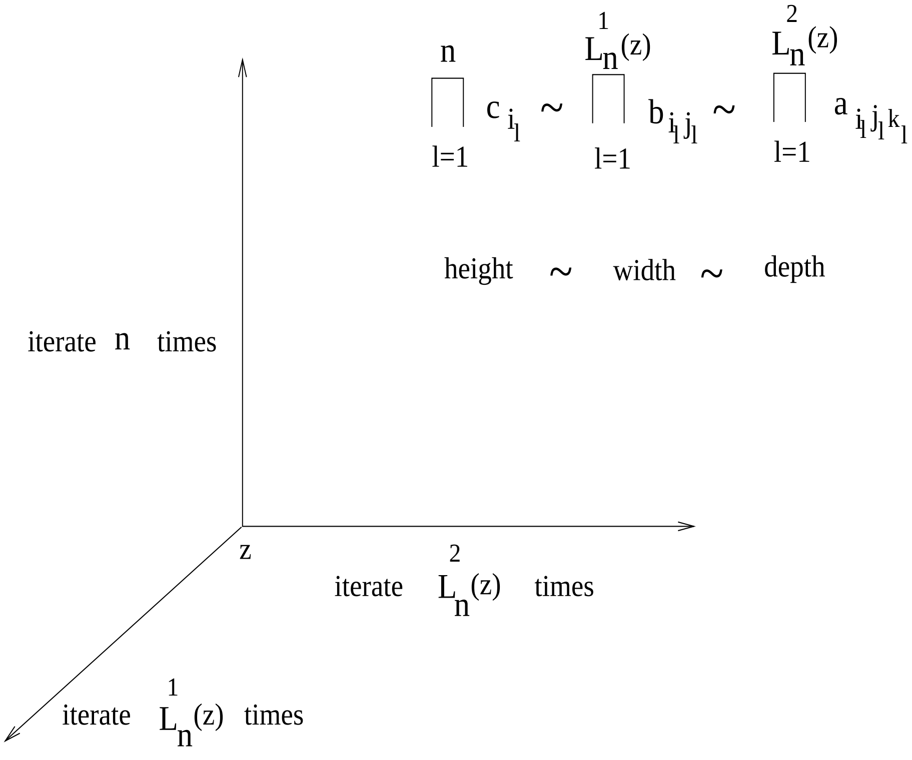
$$\lambda_1(\mathbf{p}) = \frac{\sum_{i=1}^m p_i \log p_i}{\sum_{i=1}^m p_i \log c_i}$$

$$\lambda_2(\mathbf{p}) = \frac{\sum_{i,j} p_{ij} \log p_{ij} - \sum_i p_i \log p_i}{\sum_{i,j} p_{ij} \log b_{ij}}$$

$$t(\mathbf{p}) : \qquad \sum_{i,j} p_{ij} \log \left(\sum_k a_{ijk}^{t(\mathbf{p})} \right) = 0$$

The estimate $\geq$ is silimar to the 2-D case: we use Bernoulli measures and the Volume Lemma

$$\mu_{\mathbf{p}}(R_{ijk}) = p_{ij} \cdot \frac{a_{ijk}^{t(\mathbf{p})}}{\sum_k a_{ijk}^{t(\mathbf{p})}}$$



approximate cube of order n *centered at* z

$$\begin{aligned} B_n(z) = \{\, z' \in \Lambda : & i'_l = i_l,\, l = 1, ..., n \\ & j'_l = j_l,\, l = 1, ..., L^1_n(z)\,\} \\ & k'_l = k_l,\, l = 1, ..., L^2_n(z)\,\} \end{aligned}$$

$$\dim_{\mathsf{H}} \mu_{\mathbf{p}} = \lambda_1(\mathbf{p}) + \lambda_2(\mathbf{p}) + t(\mathbf{p})$$

Estimate $\leq$

Lemma 1. *Let* $\mu(\Lambda) = 1$. *If*

$$\underline{d}_\mu(z) \leq d \quad \textit{for } \textbf{\textit{all}} \quad z \in \Lambda$$

then

$$\dim_{\mathrm{H}} \Lambda \leq d$$

1 measure $\longrightarrow$ finite number measures $\longrightarrow$ continuous family measures

We use a 2-parameter family of Bernoulli measures:

t horizontal dimensional parameter

ρ non-conformal parameter

Lemma 2. $\forall\, t \in (\underline{t}, \overline{t}),\ \rho \in (0,1]$

$\exists\, \mathbf{p} = \mathbf{p}(t,\rho)$ *continuous :*

$t(\mathbf{p}) = t$

$$p_{ij} = c_i^{\lambda_1(\mathbf{p})}\, b_{ij}^{\lambda_2(\mathbf{p})} \Big(\sum_k a_{ijk}^t\Big)^{\alpha} \Big(\sum_j b_{ij}^{\lambda_2(\mathbf{p})} \Big(\sum_k a_{ijk}^t\Big)^{\alpha}\Big)^{\rho-1}$$

$\alpha = \alpha(t,\rho)$ *is* C^1

$\alpha(\underline{t},\rho) = -\infty,\ \alpha(\overline{t},\rho) = \infty$

pointwise dimension of the measures $\mu_{\mathbf{p}(t,\rho)}$

$$\frac{\log \mu_{\mathbf{p}(t,\rho)}(B_n(z))}{\sum_{l=1}^{n} \log c_{i_l}} =$$

$$\lambda_1(\mathbf{p}(t,\rho)) + \beta_n \lambda_2(\mathbf{p}(t,\rho)) + \gamma_n t + \textbf{Remain}$$

where

$$\beta_n \to 1, \quad \gamma_n \to 1$$

and

$$\textbf{Remain} =$$

$$\frac{1}{n} \frac{\rho \sum_{l=1}^{n} \log \gamma_{i_l}(t,\rho) - \sum_{l=1}^{L_n^1(z)} \log \gamma_{i_l}(t,\rho)}{\frac{1}{n}\sum_{l=1}^{n} \log c_{i_l}}$$

$$+\frac{1}{n} \frac{\alpha(t,\rho) \sum_{l=1}^{L_n^1(z)} \log\Big(\sum_k a_{i_l j_l k}^t\Big) - \sum_{l=1}^{L_n^2(z)} \log\Big(\sum_k a_{i_l j_l k}^t\Big)}{\frac{1}{n}\sum_{l=1}^{n} \log c_{i_l}}$$

where

$$\gamma_i(t,\rho) = \sum_j b_{ij}^{\lambda_2(\mathbf{p}(t,\rho))} \Big(\sum_k a_{ijk}^t \Big)^{\alpha(t,\rho)}$$

If z is a generic point for some distribution $\mathbf{p}$ then choosing (t,ρ) such that

$$\rho = \lim_{n\to\infty} \frac{L_n^1(z)}{n}$$

$$\alpha(t,\rho) = \lim_{n\to\infty} \frac{L_n^2(z)}{L_n^1(z)}$$

then

Remain $\to 0$

If z is not generic then **Remain** $\nrightarrow 0$

But we will see that

$$\liminf_{n\to\infty} \textbf{Remain} \leq 0$$

This is equivalent to

$$\limsup_{n\to\infty} \widetilde{\textbf{Remain}} \geq 0$$

where

$$\widetilde{\textbf{Remain}} = \frac{1}{n}\left\{\rho\, f_1(n) - f_1(\,L_n^1(z)\,)\right\} +$$

$$\frac{1}{n}\left\{\alpha(t,\rho)\, f_2(\,L_n^1(z)\,) - f_2(\,L_n^2(z)\,)\right\}$$

Choose (t_n, ρ_n) such that

$$\rho_n = \frac{L_n^1(z)}{n}$$

$$\alpha(t_n, \rho_n) = \frac{L_n^2(z)}{L_n^1(z)}$$

$\rightsquigarrow$ se f_1, f_2 lineares o.k.

This follows from the following abstract result:

Lemma 3. *Assume*

- $f_k \colon (0,\infty) \to \mathbb{R}$ *Lipschitz* $k = 1, 2, ..., r$

- $\alpha_k \colon (0,\infty) \to \mathbb{R}$ C^1

- $\delta \le \alpha_k(u) \le \delta^{-1}$

- $|\alpha_k'(u)| \le \dfrac{C}{u}$

Then

$$\limsup_{u\to\infty} \frac{1}{u} \sum_{k=1}^{r} \Big(\alpha_k(u)\, f_k(u) - f_k(\alpha_k(u)u)\Big) \ge 0$$

Conclusion

$$\dim_{\mathrm{H}} \Lambda = \sup_{\mathbf{p}} \{\lambda_1(\mathbf{p}) + \lambda_2(\mathbf{p}) + t(\mathbf{p})\}$$

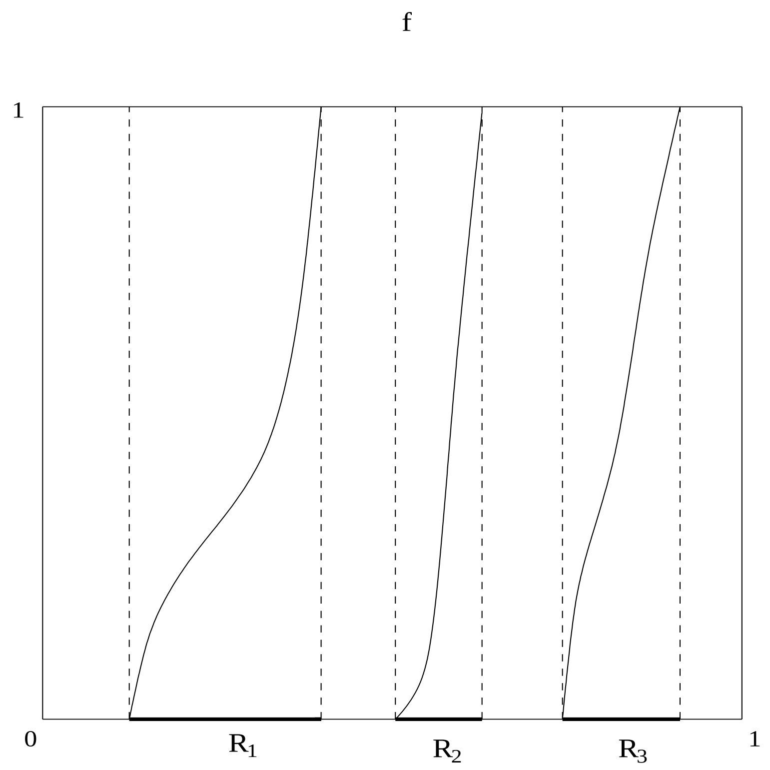
$$= \sup_{\mathbf{p}} \dim_{\mathrm{H}} \mu_{\mathbf{p}}$$

Corollary 1.

$$\exists\, \mathbf{p} \; : \qquad \dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{H}} \mu_{\mathbf{p}}$$

Question: Is there a *unique* measure of maximal dimension?

Non-linear Cantor sets



f is $C^{1+\delta}$, $\quad f' > 1$

Basic intervals of order n : $R_{i_1 i_2 \dots i_n}$

$|R_{i_1 i_2 \dots i_n}| = \Big((f^n)'(x)\Big)^{-1}$ for some $x \in R_{i_1 i_2 \dots i_n}$

f is $\mathrm{C}^{1+\delta} \Rightarrow$ Bounded Distortion

f^n acts as if it were self-affine at proper scales: on each basic interval of order n

Proposition 1. *(exercise)*

$$\exists\, C > 1 : \quad C^{-1} \leq \frac{(f^n)'(x)}{(f^n)'(y)} \leq C$$

$\forall\, n \in \mathbb{N},\, x, y \in R_{i_1 i_2 \dots i_n}$

Notation: $(f^n)'(x) \sim (f^n)'(y)$

$S_n \varphi(x) = \sum_{j=0}^{n-1} \varphi(f^j x)$

To get an estimate $\dim_{\mathsf{H}} \Lambda \leq t$: use covers

$$\bigcup_{i_1 \ldots i_n} R_{i_1 \ldots i_n}$$

$$\sum_{i_1 \ldots i_n} |R_{i_1 \ldots i_n}|^t \sim \sum_{i_1 \ldots i_n} \sup_{x \in R_{i_1 \ldots i_n}} e^{-t \, \log (f^n)'(x)}$$

$$\sim \sum_{i_1 \ldots i_n} \sup_{x \in R_{i_1 \ldots i_n}} e^{-t \, (S_n \log f)(x)}$$

$$\limsup_{n \to \infty} \frac{1}{n} \log \sum_{i_1, \ldots, i_n} \sup_{x \in R_{i_1 \ldots i_n}} e^{-t_0 (S_n \log f')(x)} = 0$$

$\Rightarrow \dim_{\mathsf{H}} \Lambda \leq t_0$

Pressure

φ Hölder

$$P_{f|\Lambda}(\varphi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\ldots,i_n} \sup_{x\in R_{i_1\cdots i_n}} e^{S_n\varphi(x)}.$$

Bowen's equation

$$P_{f|\Lambda}(-t_0 \log f') = 0$$

self-affine: Moran formula

non-linear: Bowen's equation

To get the estimate $\dim_{\mathrm{H}} \Lambda \geq t_0$: **measures** (and Volume Lemma)

self-affine: Bernoulli measures

non-linear: Ergodic measures

Bernoulli $\subset$ Ergodic

$\rightsquigarrow$ conjunto mais complicado: dim infinita, nao-fechado, denso

Entropy of ergodic measure μ

$$h_\mu(f) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(R_{i_1 \ldots i_n})$$

for μ-a.e. z.

Hausdorff dimension of ergodic measure μ

$$d_\mu(z) = \lim_{n\to\infty} \frac{\log \mu(R_{i_1\ldots i_n})}{\log |R_{i_1\ldots i_n}|}$$

$$= \lim_{n\to\infty} \frac{-\frac{1}{n}\log \mu(R_{i_1\ldots i_n})}{\frac{1}{n}(S_n \log f')(z)}$$

$$= \frac{h_\mu(f)}{\int \log f'\, d\mu} \qquad \mu - a.e.\ z$$

$$\Rightarrow \quad \dim_{\mathsf{H}} \mu = \frac{h_\mu(f)}{\int \log f'\, d\mu}$$

Variational Principle for the Pressure (S-R-B)

φ Hölder

$$P_{f|\Lambda}(\varphi) = \sup_{\mu \text{ ergodic}} \left\{ h_\mu(f) + \int \varphi \, d\mu \right\}.$$

The supremum is attained at a unique ergodic measure: μ_φ Gibbs state

Take

$$\varphi = -t_0 \log f'$$

$$\mu_\varphi = \mu_0$$

$$0 = P_{f|\Lambda}(-t_0 \log f') = h_{\mu_0}(f) - t_0 \int \log f' \, d\mu_0$$

$$t_0 = \frac{h_{\mu_0}(f)}{\int \log f' \, d\mu_0} = \dim_{\mathrm{H}} \mu_0.$$

$$\dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \mu_0 = t_0$$

Conclusion

$$\mathsf{dim}_\mathsf{H}\,\Lambda = t_0$$

Bowen's formula: $\quad P_{f|\Lambda}(-t_0 \log f') = 0$

Variational Principle for Dimension

$$\mathsf{dim}_\mathsf{H}\,\Lambda = \sup_{\mu \ \text{ergodic}} \mathsf{dim}_\mathsf{H}\,\mu$$

$$= \mathsf{dim}_\mathsf{H}\,\mu_0 = \frac{h_{\mu_0}(f)}{\int \log f'\, d\mu_0}$$

Non-linear Sierpinski carpets

f preserves horizontal lines:

$$f(x,y) = (a(x,y), b(y))$$

f expanding + domination condition:

$$\partial_x \, a(x,y) > b'(y) > 1 \qquad (a_{ij}^{-1} > b_i^{-1} > 1)$$

bounded distortion:

$$f \in \mathrm{C}^{1+\delta}$$

Λ is repeller for Markov partition f

SELF-AFFINE

$$\dim_{\mathrm{H}} \Lambda = \sup_{\mathbf{p}} \left\{ \frac{\sum_{i=1}^{m} p_i \log p_i}{\sum_{i=1}^{m} p_i \log b_i} + t(\mathbf{p}) \right\}$$

$$t(\mathbf{p}): \qquad \sum_{i=1}^{m} p_i \log \left(\sum_{j=1}^{n_i} a_{ij}^{t(\mathbf{p})} \right) = 0$$

NON-LINEAR

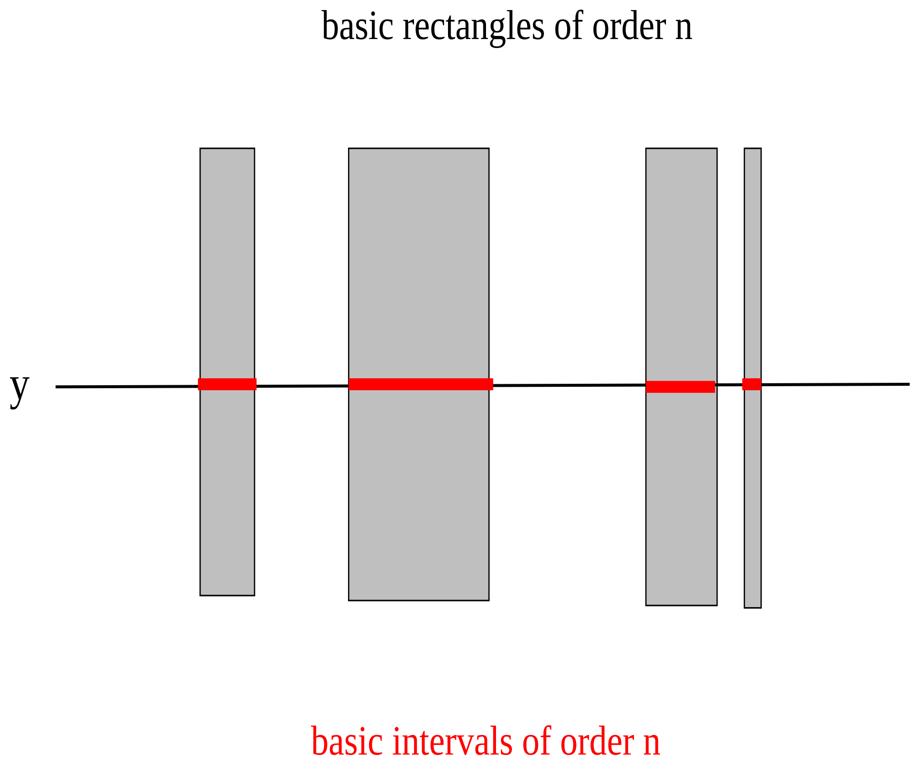
$$\dim_{\mathrm{H}} \Lambda = \sup_{\nu \text{ ergodic}} \left\{ \frac{h_\nu(b)}{\int \log b' \, d\nu} + t(\nu) \right\}$$

$t(\nu)$ *random Bowen's equation*

$\pi(x, y) = y$

For each y in the vertical we have a 1-dimensional Cantor set

$$\Lambda_y = \Lambda \cap \pi^{-1}(y)$$



Relative Pressure of f with respect to the fibre $\pi^{-1}(y)$

$\varphi \colon \Lambda \to \mathbb{R}$ Hölder

$y \equiv (i_1, i_2, \ldots)$

$P(f|\Lambda, \varphi, \pi^{-1}(y)) =$

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{j_1,\cdots,j_n} \sup_{x \in R_{(i_1 j_1)\ldots(i_n j_n)} \cap \pi^{-1}(y)} e^{S_n \varphi(x,y)}$$

$\varphi(x, y) = \log \partial_x \, a(x, y)$

Random Bowen's equation

$$\int_{\pi(\Lambda)} P(f|\Lambda,\, -t(\nu)\,\log \partial_x a,\, \pi^{-1}(y))\, d\nu(y) = 0$$

$$\dim_{\mathrm{H}} \Lambda = \sup_{\nu\,\text{ergodic}} \left\{ \frac{h_\nu(b)}{\int \log b'\, d\nu} + t(\nu) \right\}$$

$$= \sup_{\nu\,\text{ergodic}} \dim_{\mathrm{H}} \mu_\nu$$

$$\pi_* \mu_\nu = \nu$$

Problem: Is there an ergodic measure of maximal dimension?

$$\nu \text{ ergodic} \mapsto \frac{h_\nu(b)}{\int \log b' \, d\nu} + t(\nu)$$

is upper-semicontinuous but the set of ergodic measures is not closed (is dense).

Answer: Yes. Is it unique?