Instituto Superior Técnico - Álgebra Linear - 1^o Semestre 2011/2012

LEAN - LEMat - MEAmbi - MEBiol - MEQ

Resolução da 1^a Ficha de exercícios

1. (ii) $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^{1000} = \left(\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}^2 \right)^{500} = I^{500} = I$

(iv) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{222} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 + I \right) =$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{220} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(xi) se $ad - bc \neq 0$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I$$

2. (i) $\begin{bmatrix} -\frac{1}{3} & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -\pi \end{bmatrix} = [-2\pi - 1]$ **(ii)** Não é possível.

(iii) $\begin{bmatrix} \frac{3}{2} & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 2 & 4 \\ -3 & \sqrt{5} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{2} & 3 - 2\sqrt{5} & 5 \\ \frac{8}{3} & 2 - \sqrt{5} & \frac{7}{2} \end{bmatrix}$

(iv) $2 \begin{bmatrix} 1 & 0 \\ 3 & -\frac{1}{2} \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 0 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ \frac{20}{3} & -2 \end{bmatrix}$ **(v)** Não é possível. **(vi)** Não é possível.

(vii) $\begin{bmatrix} \sqrt{2} \\ -3 \end{bmatrix} \begin{bmatrix} 4 & -\frac{1}{2} & 2 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & -\frac{1}{2}\sqrt{2} & 2\sqrt{2} \\ -12 & \frac{3}{2} & -6 \end{bmatrix}$

viii) $\left(\begin{bmatrix} 2 \\ -\frac{1}{4} \\ 1 \end{bmatrix} 2 \begin{bmatrix} \sqrt{2} & -3 \end{bmatrix} \right)^T = \begin{bmatrix} 4\sqrt{2} & -\frac{1}{2}\sqrt{2} & 2\sqrt{2} \\ -12 & \frac{3}{2} & -6 \end{bmatrix}$

(ix) $\left(2 \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{2} & -1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{2} & -1 \end{bmatrix} - \begin{bmatrix} \frac{8}{9} & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{5}{3} & 1 & \frac{5}{2} \end{bmatrix} \right)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\textbf{(x)} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 \\ 0 & -1 & -\frac{1}{4} & 0 \\ 6 & 2 & 5 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ -\frac{1}{3} & -3 \end{bmatrix} = \begin{bmatrix} -1 & -18 \\ \frac{5}{6} & -10 \\ -\frac{7}{6} & -16 \\ -\frac{7}{3} & -3 \end{bmatrix}$$

$$\textbf{(xi)} \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ -\frac{1}{3} & -3 \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 \\ 0 & -1 & -\frac{1}{4} & 0 \\ 6 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{5}{6} & -\frac{7}{6} & -\frac{7}{3} \\ -18 & -10 & -16 & -3 \end{bmatrix}$$

3. Sendo x o n^o de livros e y o n^o de caixas, tem-se $\begin{cases} x - 7y = 1 \\ x - 8y = -7 \end{cases} \Leftrightarrow \begin{cases} x = 57 \\ y = 8. \end{cases}$
A solução geral do sistema é $\{(57, 8)\}$.

4.

$$\frac{212 - 32}{100 - 0} = \frac{9}{5}, \qquad \frac{F - 32}{C - 0} = \frac{9}{5} \Leftrightarrow F = \frac{9}{5}C + 32.$$

$$F = C, \qquad F = \frac{9}{5}F + 32 \Leftrightarrow F = -40.$$

5. a) $\begin{bmatrix} 1 & -4 & 9 & -16 \\ -1 & 4 & -9 & 16 \\ 1 & -4 & 9 & -16 \\ -1 & 4 & -9 & 16 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & 0 & \frac{1}{5} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}$

c) $\begin{bmatrix} 1 & -2 & -2 & -3 \\ 1 & 2 & -3 & -2 \\ 2 & 1 & 3 & -4 \\ 3 & 2 & 1 & 4 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 2 & 3 & 4 \\ -2 & 0 & 3 & 4 \\ -3 & -3 & 0 & 4 \\ -4 & -4 & -4 & 0 \end{bmatrix}$ $(a_{ii} = -a_{ii} \Leftrightarrow a_{ii} = 0)$

6. Seja $(a_{ij}) \in \mathcal{M}_{2\times 2}(\mathbb{R})$ tal que

$$a_{ij} = 3i + 2j$$

Como

$$a_{12} = 3 \times 1 + 2 \times 2 = 7 \neq 8 = 3 \times 2 + 2 \times 1 = a_{21}$$

então A não é simétrica.

7. (i) Seja $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. car $A = 0$, nul $A = 2$. Não existem pivots.

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \underset{-L_1+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, tem-se $\operatorname{car} A = 2$ e $\operatorname{nul} A = 1$. Pivots: 1 e 1.

(iii) $\begin{bmatrix} 2 & 1 \\ 2 & 4 \\ -1 & -2 \end{bmatrix} \underset{L_1\leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} -1 & -2 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \underset{\substack{2L_1+L_2\to L_2 \\ 2L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} -1 & -2 \\ 0 & 0 \\ 0 & -3 \end{bmatrix} \underset{L_1\leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} -1 & -2 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}$, $\operatorname{car} A = 2$ e $\operatorname{nul} A = 0$. Pivots: -1 e -3.

(iv) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \underset{\substack{-5L_1+L_2\to L_2 \\ -9L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix} \underset{-2L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\operatorname{car} A = 2$ e $\operatorname{nul} A = 2$. Pivots: 1 e -4.

(v) $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix} \underset{L_1\leftrightarrow L_2}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix} \underset{-L_1+L_3\to L_3}{\longrightarrow}$

$\underset{-L_1+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & -1 \end{bmatrix} \underset{-2L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix}$, tem-se $\operatorname{car} A = 3$ e $\operatorname{nul} A = 1$. Pivots: $1, 1$ e 3.

(vi) $\begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ -1 & 1 & 3 & -2 & -1 \\ 2 & 7 & -1 & 9 & 8 \\ 3 & 3 & -2 & 4 & -6 \end{bmatrix} \underset{\substack{L_1+L_2\to L_2 \\ -2L_1+L_3\to L_3 \\ -3L_1+L_4\to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 3 & 1 & 3 & 4 \\ 0 & -3 & 1 & -5 & -12 \end{bmatrix} \underset{\substack{-L_2+L_3\to L_3 \\ L_2+L_4\to L_4}}{\longrightarrow}$

$$\xrightarrow[L_2+L_4\to L_4]{-L_2+L_3\to L_3} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 3 & -4 & -11 \end{bmatrix} \xrightarrow[3L_3+L_4\to L_4]{} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 3 & 2 & 1 & 1 \\ 0 & 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}.$$

Assim, sendo $A = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ -1 & 1 & 3 & -2 & -1 \\ 2 & 7 & -1 & 9 & 8 \\ 3 & 3 & -2 & 4 & -6 \end{bmatrix}$, tem-se $\operatorname{car} A = 4$ e $\operatorname{nul} A = 1$. Pivots: $1, 3, -1$ e 2.

(vii) $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix} \xrightarrow[-4L_1+L_4\to L_4]{-2L_1+L_3\to L_3} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix} \xrightarrow[L_2+L_4\to L_4]{L_2+L_3\to L_3} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$, tem-se $\operatorname{car} A = 2$ e $\operatorname{nul} A = 2$. Pivots: 1 e 11.

(viii) Sendo $A = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, tem-se $\operatorname{car} A = 2$ e $\operatorname{nul} A = 1$. Pivots: 5 e 2.

(ix) $\begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[L_1\leftrightarrow L_3]{} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[-3L_1+L_3\to L_3]{-2L_1+L_2\to L_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, tem-se $\operatorname{car} A = 1$ e $\operatorname{nul} A = 2$. Pivot: 1.

(x) $\begin{bmatrix} 2 & 10 & -6 & 8 & -4 \\ -1 & -5 & 3 & -4 & 2 \\ -2 & -10 & 6 & -8 & 4 \end{bmatrix} \xrightarrow[L_1+L_3\to L_3]{\frac{1}{2}L_1+L_2\to L_2} \begin{bmatrix} 2 & 10 & -6 & 8 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Assim, sendo $A = \begin{bmatrix} 2 & 10 & -6 & 8 & -4 \\ -1 & -5 & 3 & -4 & 2 \\ -2 & -10 & 6 & -8 & 4 \end{bmatrix}$, $\operatorname{car} A = 1$ e $\operatorname{nul} A = 4$. Pivot: 2.

8. As equações das alíneas **(a)** e **(b)** são lineares.

9. O ponto $(1,-1)$ é a solução desse sistema de equações lineares.

10. Os pontos: $(1,-1,1,0)\,,(1,-1,1,2)\,,\left(3,-9,7,\dfrac{\sqrt[3]{\pi}}{2}\right)$ são soluções desse sistema de equações lineares.

11. **(i)** Para que o gráfico da função polinomial $p(x)=ax^3+bx^2+cx+d$ passe pelos pontos $P_1=(0,10), P_2=(1,7), P_3=(3,-11)$ e $P_4=(4,-14)$, é necessário que

$$\begin{cases} p(0)=10 \\ p(1)=7 \\ p(3)=-11 \\ p(4)=-14. \end{cases}$$

O que é equivalente a existir solução para o seguinte sistema de equações lineares nas variáveis a,b,c e d:

$$\begin{cases} d=10 \\ a+b+c+d=7 \\ 27a+9b+3c+d=-11 \\ 64a+16b+4c+d=-14. \end{cases}$$

Ou seja:

$$\begin{cases} d=10 \\ a+b+c=-3 \\ 27a+9b+3c=-21 \\ 16a+4b+c=-6. \end{cases}$$

Atendendo a que:

$$\begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 27 & 9 & 3 & | & -21 \\ 16 & 4 & 1 & | & -6 \end{bmatrix} \underset{\substack{-27L_1+L_2\to L_2 \\ -16L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 0 & -18 & -24 & | & 60 \\ 0 & -12 & -15 & | & 42 \end{bmatrix} \underset{\frac{1}{6}L_2\to L_2}{\longrightarrow}$$

$$\underset{\frac{1}{6}L_2\to L_2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 0 & -3 & -4 & | & 10 \\ 0 & -12 & -15 & | & 42 \end{bmatrix} \underset{-4L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & | & -3 \\ 0 & -3 & -4 & | & 10 \\ 0 & 0 & 1 & | & 2 \end{bmatrix},$$

tem-se

$$\begin{cases} a=1 \\ b=-6 \\ c=2 \\ d=10. \end{cases}$$

(ii) Para que os pontos $P_1=(-2,7), P_2=(-4,5)$ e $P_3=(4,-3)$ pertençam à circunferência de equação $x^2+y^2+ax+by+c=0$, é necessário que

$$\begin{cases} (-2)^2+7^2+a\,(-2)+7b+c=0 \\ (-4)^2+5^2+a\,(-4)+5b+c=0 \\ 4^2+(-3)^2+4a+b\,(-3)+c=0. \end{cases}$$

O que é equivalente a existir solução para o seguinte sistema de equações lineares nas variáveis a, b e c:

$$\begin{cases} -2a + 7b + c = -53 \\ -4a + 5b + c = -41 \\ 4a - 3b + c = -25. \end{cases}$$

Atendendo a que:

$$\left[\begin{array}{ccc|c} -2 & 7 & 1 & -53 \\ -4 & 5 & 1 & -41 \\ 4 & -3 & 1 & -25 \end{array}\right] \underset{\substack{-2L_1+L_2\to L_2 \\ 2L_1+L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} -2 & 7 & 1 & -53 \\ 0 & -9 & -1 & 65 \\ 0 & 11 & 3 & -131 \end{array}\right] \underset{\frac{11}{9}L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{\frac{11}{9}L_2+L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} -2 & 7 & 1 & -53 \\ 0 & -9 & -1 & 65 \\ 0 & 0 & 16/9 & -464/9 \end{array}\right],$$

tem-se

$$\begin{cases} a = -2 \\ b = -4 \\ c = -29. \end{cases}$$

12. **(i)** $$\begin{bmatrix} 1 & 0 & 1 \\ -1 & \alpha & \alpha \\ 0 & \alpha & 1 \end{bmatrix} \underset{L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \alpha & \alpha+1 \\ 0 & \alpha & 1 \end{bmatrix} \underset{-L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \alpha & \alpha+1 \\ 0 & 0 & \alpha \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ -1 & \alpha & \alpha \\ 0 & \alpha & 1 \end{bmatrix}$. Se $\alpha \neq 0$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 0$.

Se $\alpha = 0$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 1$.

Assim, A_α é invertível se e só se $\alpha \neq 0$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(ii) $$\begin{bmatrix} 1 & \alpha & \alpha \\ -2 & 1 & 2 \\ 3 & -2-\alpha & -1 \end{bmatrix} \underset{\substack{2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & \alpha & \alpha \\ 0 & 1+2\alpha & 2+2\alpha \\ 0 & -2-4\alpha & -1-3\alpha \end{bmatrix} \underset{2L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{2L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & \alpha & \alpha \\ 0 & 1+2\alpha & 2(1+\alpha) \\ 0 & 0 & 3+\alpha \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} 1 & \alpha & \alpha \\ -2 & 1 & 2 \\ 3 & -2-\alpha & -1 \end{bmatrix}$. Se $\alpha \neq -3$ e $\alpha \neq -\dfrac{1}{2}$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 0$.

Se $\alpha = -3$ ou $\alpha = -\dfrac{1}{2}$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 1$.

Assim, A_α é invertível se e só se $\alpha \neq -3$ e $\alpha \neq -\dfrac{1}{2}$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(iii) $$\begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 2 & 1 & 1 \\ 0 & \alpha^2-1 & \alpha+1 \end{bmatrix} \underset{-L_1+L_2 \to L_2}{\longrightarrow} \begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 0 & 1-\alpha^2 & 1+\alpha \\ 0 & \alpha^2-1 & \alpha+1 \end{bmatrix} \underset{L_2+L_3 \to L_3}{\longrightarrow}$$

$$\underset{L_2+L_3 \to L_3}{\longrightarrow} \begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 0 & (1-\alpha)(1+\alpha) & 1+\alpha \\ 0 & 0 & 2(\alpha+1) \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} 2 & \alpha^2 & -\alpha \\ 2 & 1 & 1 \\ 0 & \alpha^2-1 & \alpha+1 \end{bmatrix}$. Se $\alpha = -1$ então $\operatorname{car} A_\alpha = 1$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha \neq -1$ e $\alpha \neq 1$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq -1$ e $\alpha \neq 1$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(iv) $$\begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 3 & 0 & \alpha & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix} \underset{\substack{3L_1+L_3 \to L_3 \\ -L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha+3 & 3\alpha \\ 0 & -1 & 0 & 2-\alpha \end{bmatrix} \underset{L_2+L_4 \to L_4}{\longrightarrow}$$

$$\underset{L_2+L_4 \to L_4}{\longrightarrow} \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha+3 & 3\alpha \\ 0 & 0 & 0 & 2-\alpha \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & 0 & 0 \\ 3 & 0 & \alpha & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix}$. Se $\alpha = 2$ ou $\alpha = -3$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha \neq 2$ e $\alpha \neq -3$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq 2$ e $\alpha \neq -3$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(v) $$\begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix} \underset{\substack{L_1+L_3 \to L_3 \\ 2L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 0 & 0 & (1-\alpha)(1+\alpha) & \alpha-1 \\ 0 & 0 & 0 & 2(\alpha-1) \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix}$. Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha = -1$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha \neq 1$ e $\alpha \neq -1$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq 1$ e $\alpha \neq -1$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

(vi) $\begin{bmatrix} 1 & -1 & \alpha & 0 \\ 1 & \alpha & -1 & 0 \\ 1 & -1 & \alpha^3 & 0 \\ -1 & 1 & -\alpha & \alpha^2 - 1 \end{bmatrix} \underset{\substack{-L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3 \\ L_1+L_4 \to L_4}}{\longrightarrow}$

$$\underset{\substack{-L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3 \\ L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & -1 & \alpha & 0 \\ 0 & \alpha+1 & -1-\alpha & 0 \\ 0 & 0 & \alpha(\alpha-1)(\alpha+1) & 0 \\ 0 & 0 & 0 & (\alpha-1)(\alpha+1) \end{bmatrix}.$$

Seja $A_\alpha = \begin{bmatrix} -1 & 0 & 1 & \alpha \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -\alpha^2 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix}$. Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha = 0$ então $\operatorname{car} A_\alpha = 3$ e $\operatorname{nul} A_\alpha = 1$.

Se $\alpha = -1$ então $\operatorname{car} A_\alpha = 1$ e $\operatorname{nul} A_\alpha = 3$.

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = 2$ e $\operatorname{nul} A_\alpha = 2$.

Se $\alpha \neq 0$ e $\alpha \neq 1$ e $\alpha \neq -1$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_α é invertível se e só se $\alpha \neq 1$ e $\alpha \neq -1$, uma vez que é só neste caso que $\operatorname{car} A_\alpha = \text{n}^o$ de colunas de A_α.

13. (a) Tem-se $\begin{cases} 3x - z = 0 \\ 2y - 2z - w = 0 \\ 8x - 2w = 0 \end{cases}$ e assim,

$$\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 8 & 0 & 0 & -2 & 0 \end{array}\right] \underset{-\frac{8}{3}L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & \frac{8}{3} & -2 & 0 \end{array}\right].$$

Logo, $\begin{cases} 3x - z = 0 \\ 2y - 2z - w = 0 \\ \frac{8}{3}z - 2w = 0. \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{4}w \\ \\ y = \frac{5}{4}w \\ \\ z = \frac{3}{4}w. \end{cases}$ A solução geral do sistema é:

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \frac{1}{4}s \\ \\ \frac{5}{4}s \\ \\ \frac{3}{4}s \\ \\ s \end{bmatrix},$$

para qualquer $s \in \mathbb{R}$, isto é, o conjunto solução é dado por:

$$S = \left\{ \left(\frac{1}{4}s, \frac{5}{4}s, \frac{3}{4}s, s \right) : s \in \mathbb{R} \right\}.$$

Para $s = 4$, tem-se a seguinte solução para a equação química: $x = 1, y = 5, z = 3, w = 4$.

(b) Tem-se $\begin{cases} x - 6z = 0 \\ 2x + y - 6z - 2w = 0 \\ 2y - 12z = 0 \end{cases}$ e assim,

$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array} \right] \underset{-2L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array} \right] \underset{-2L_2+L_3 \to L_3}{\longrightarrow}$$

$$\underset{-2L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & 6 & -2 & 0 \\ 0 & 0 & -24 & 4 & 0 \end{array} \right].$$

Logo, $\begin{cases} x - 6z = 0 \\ y + 6z - 2w = 0 \\ -24z + 4w = 0. \end{cases} \Leftrightarrow \begin{cases} x = w \\ y = w \\ z = \frac{1}{6}w. \end{cases}$ A solução geral do sistema é

$$S = \left\{ \left(s, s, \frac{1}{6}s, s \right) : s \in \mathbb{R} \right\}.$$

Para $s = 6$, tem-se a seguinte solução para a equação química: $x = 6, y = 6, z = 1, w = 6$.

14. (a) $\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 5 & 7 & 3 \end{array} \right] \underset{-\frac{5}{2}L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{array} \right].$

Logo, $\begin{cases} 2x+3y=1 \\ -\frac{1}{2}y=\frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x=2 \\ y=-1. \end{cases}$

A solução geral do sistema é $S=\{(2,-1)\}$.

(b) $\begin{bmatrix} 2 & 4 & | & 10 \\ 3 & 6 & | & 15 \end{bmatrix} \underset{-\frac{3}{2}L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 2 & 4 & | & 10 \\ 0 & 0 & | & 0 \end{bmatrix}$. Logo, $2x+4y=10 \Leftrightarrow x=5-2y$.

A solução geral do sistema é $S=\{(5-2s,s): s\in\mathbb{R}\}$.

(c) $\begin{bmatrix} 4 & -2 & | & 5 \\ -6 & 3 & | & 1 \end{bmatrix} \underset{\frac{3}{2}L_1+L_2\to L_2}{\longrightarrow} \begin{bmatrix} 4 & -2 & | & 5 \\ 0 & 0 & | & \frac{17}{2} \end{bmatrix}$. Logo, o sistema não tem solução (é impossível). $S=\varnothing$.

(d) $\begin{bmatrix} 2 & 1 & -3 & | & 5 \\ 3 & -2 & 2 & | & 5 \\ 5 & -3 & -1 & | & 16 \end{bmatrix} \underset{\substack{-\frac{3}{2}L_1+L_2\to L_2 \\ -\frac{5}{2}L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 2 & 1 & -3 & | & 5 \\ 0 & -7/2 & 13/2 & | & -5/2 \\ 0 & -11/2 & 13/2 & | & 7/2 \end{bmatrix} \underset{-\frac{11}{7}L_2+L_3\to L_3}{\longrightarrow}$

$\underset{-\frac{11}{7}L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 2 & 1 & -3 & | & 5 \\ 0 & -7/2 & 13/2 & | & -5/2 \\ 0 & 0 & -26/7 & | & 52/7 \end{bmatrix}$.

Logo, $\begin{cases} 2x+y-3z=5 \\ -\frac{7}{2}y+\frac{13}{2}z=-\frac{5}{2} \\ -\frac{26}{7}z=\frac{52}{7} \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=-3 \\ z=-2. \end{cases}$ A solução geral do sistema é $S=\{(1,-3,-2)\}$.

(e) $\begin{bmatrix} 2 & 3 & -2 & | & 5 \\ 1 & -2 & 3 & | & 2 \\ 4 & -1 & 4 & | & 1 \end{bmatrix} \underset{\substack{-\frac{1}{2}L_1+L_2\to L_2 \\ -2L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 2 & 3 & -2 & | & 5 \\ 0 & -7/2 & 4 & | & -1/2 \\ 0 & -7 & 8 & | & -9 \end{bmatrix} \underset{-2L_2+L_3\to L_3}{\longrightarrow}$

$\underset{-2L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 2 & 3 & -2 & | & 5 \\ 0 & -7/2 & 4 & | & -1/2 \\ 0 & 0 & 0 & | & -8 \end{bmatrix}$.

Logo, o sistema não tem solução (é impossível). $S=\varnothing$.

(f) $\begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 2 & 3 & 8 & | & 4 \\ 3 & 2 & 17 & | & 1 \end{bmatrix} \underset{\substack{-2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & -1 & 2 & | & -2 \\ 0 & -4 & 8 & | & -8 \end{bmatrix} \underset{-4L_2+L_3\to L_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 3 & | & 3 \\ 0 & -1 & 2 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.

Logo, $\left\{\begin{array}{l} x+2y+3z=3 \\ -y+2z=-2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x=-7z-1 \\ y=2z+2. \end{array}\right.$

A solução geral do sistema é $S=\{(-7s-1, 2s+2, s) : s \in \mathbb{R}\}$.

(g) $$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & -2 & 5 \\ 3 & 2 & 7 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 2 & 3 & 3 \\ 3 & 2 & 7 \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -3L_1+L_3 \to L_3}}{\longrightarrow}$$

$$\underset{\substack{-2L_1+L_2 \to L_2 \\ -3L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 7 & -7 \\ 0 & 8 & -8 \end{array}\right] \underset{-\frac{8}{7}L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{array}\right].$$

Logo, $\left\{\begin{array}{l} x-2y=5 \\ 7y=-7 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x=3 \\ y=-1. \end{array}\right.$ A solução geral do sistema é $S=\{(3,-1)\}$.

(h) $$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 2 & 4 & 4 & 3 & 9 \\ 3 & 6 & -1 & 8 & 10 \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -3L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 0 & 6 & -3 & 3 \\ 0 & 0 & 2 & -1 & 1 \end{array}\right] \underset{-\frac{1}{3}L_2+L_3 \to L_3}{\longrightarrow}$$

$$\underset{-\frac{1}{3}L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 0 & 6 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Logo, $\left\{\begin{array}{l} x+2y-z+3w=3 \\ 6z-3w=3 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x=-2y-\frac{5}{2}w+\frac{7}{2} \\ z=\frac{1}{2}w+\frac{1}{2}. \end{array}\right.$

A solução geral do sistema é $S=\left\{\left(-2s-\frac{5}{2}t+\frac{7}{2}, s, \frac{1}{2}t+\frac{1}{2}, t\right) : s,t \in \mathbb{R}\right\}$.

(i) $$\left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 3 & -1 & 2 & 5 & 2 \\ 2 & 2 & 3 & -4 & 1 \end{array}\right] \underset{\substack{-3L_1+L_2 \to L_2 \\ -2L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & -8 & -5 & 22 & -5 \end{array}\right] \underset{-\frac{1}{2}L_2+L_3 \to L_3}{\longrightarrow}$$

$$\underset{-\frac{1}{2}L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right].$$

Logo, o sistema não tem solução (é impossível). $S=\varnothing$.

(j) $$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 3 & 4 \\ 2 & 0 & -6 & 9 & 7 \\ 2 & 2 & -5 & 2 & 4 \\ 0 & 100 & 150 & -200 & 50 \end{array}\right] \underset{\substack{L_1 \leftrightarrow L_3 \\ \frac{1}{50}L_4 \to L_4}}{\longrightarrow} \left[\begin{array}{cccc|c} 2 & 2 & -5 & 2 & 4 \\ 2 & 0 & -6 & 9 & 7 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \end{array}\right] \underset{-L_1+L_2 \to L_2}{\longrightarrow}$$

$$\xrightarrow[-L_1+L_2\to L_2]{} \begin{bmatrix} 2 & 2 & -5 & 2 & | & 4 \\ 0 & -2 & -1 & 7 & | & 3 \\ 0 & 0 & 2 & 3 & | & 4 \\ 0 & 2 & 3 & -4 & | & 1 \end{bmatrix} \xrightarrow[L_2+L_4\to L_4]{} \begin{bmatrix} 2 & 2 & -5 & 2 & | & 4 \\ 0 & -2 & -1 & 7 & | & 3 \\ 0 & 0 & 2 & 3 & | & 4 \\ 0 & 0 & 2 & 3 & | & 4 \end{bmatrix} \xrightarrow[-L_3+L_4\to L_4]{}$$

$$\xrightarrow[-L_3+L_4\to L_4]{} \begin{bmatrix} 2 & 2 & -5 & 2 & | & 4 \\ 0 & -2 & -1 & 7 & | & 3 \\ 0 & 0 & 2 & 3 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Logo, $\begin{cases} 2x_1 + 2x_2 - 5x_3 + 2x_4 = 4 \\ -2x_2 - x_3 + 7x_4 = 3 \\ 2x_3 + 3x_4 = 4 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{19}{2} - 9x_4 \\ x_2 = \frac{17}{4}x_4 - \frac{5}{2} \\ x_3 = -\frac{3}{2}x_4 + 2 \end{cases}$

A solução geral do sistema é dada por

$$S = \left\{ \left(\frac{19}{2} - 9s, \frac{17}{4}s - \frac{5}{2}, -\frac{3}{2}s + 2, s \right) : s \in \mathbb{R} \right\}.$$

(k) $\begin{bmatrix} 1 & -2 & 3 & -1 & | & 1 \\ 3 & -1 & 2 & 5 & | & 2 \\ -3 & 6 & -9 & 3 & | & -6 \end{bmatrix} \xrightarrow[\substack{-3L_1+L_2\to L_2 \\ 3L_1+L_3\to L_3}]{} \begin{bmatrix} 1 & -2 & 3 & -1 & | & 1 \\ 0 & 5 & -7 & 8 & | & -1 \\ 0 & 0 & 0 & 0 & | & -3 \end{bmatrix}.$

Logo, o sistema não tem solução (é impossível). $S = \varnothing$.

15. (a) Sejam $A_\alpha = \begin{bmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{bmatrix}$ e $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$[A_\alpha \mid B] = \begin{bmatrix} \alpha & 1 & 1 & | & 1 \\ 1 & \alpha & 1 & | & 1 \\ 1 & 1 & \alpha & | & 1 \end{bmatrix} \xrightarrow[L_1\leftrightarrow L_3]{} \begin{bmatrix} 1 & 1 & \alpha & | & 1 \\ 1 & \alpha & 1 & | & 1 \\ \alpha & 1 & 1 & | & 1 \end{bmatrix} \xrightarrow[\substack{-L_1+L_2\to L_2 \\ -\alpha L_1+L_3\to L_3}]{}$$

$$\xrightarrow[\substack{-L_1+L_2\to L_2 \\ -\alpha L_1+L_3\to L_3}]{} \begin{bmatrix} 1 & 1 & \alpha & | & 1 \\ 0 & \alpha-1 & 1-\alpha & | & 0 \\ 0 & 1-\alpha & 1-\alpha^2 & | & 1-\alpha \end{bmatrix} \xrightarrow[L_2+L_3\to L_3]{} \begin{bmatrix} 1 & 1 & \alpha & | & 1 \\ 0 & \alpha-1 & 1-\alpha & | & 0 \\ 0 & 0 & (1-\alpha)(\alpha+2) & | & 1-\alpha \end{bmatrix}.$$

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B] = 1 < 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se $x + y + z = 1$. A solução geral deste sistema é então dada por

$$S_1 = \{(1 - s - t, s, t) : s, t \in \mathbb{R}\}.$$

Se $\alpha = -2$ então $\underbrace{\text{car}\, A_\alpha}_{=2} < \underbrace{\text{car}\, [A_\alpha \mid B]}_{=3}$. Logo, o sistema não tem solução (é impossível). $S_{-2} = \varnothing$.

Se $\alpha \neq 1$ e $\alpha \neq -2$ então $\text{car}\, A_\alpha = \text{car}\, [A_\alpha \mid B] = 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x + y + \alpha z = 1 \\ (\alpha - 1)\, y + (1 - \alpha)\, z = 0 \\ (1 - \alpha)(\alpha + 2)\, z = 1 - \alpha \end{cases} \Leftrightarrow \begin{cases} x = 1/(\alpha + 2) \\ y = 1/(\alpha + 2) \\ z = 1/(\alpha + 2). \end{cases}$$

A solução geral do sistema é então dada por $S_\alpha = \left\{ \left(\dfrac{1}{\alpha+2}, \dfrac{1}{\alpha+2}, \dfrac{1}{\alpha+2} \right) \right\}$.

(b) Sejam $A_\alpha = \begin{bmatrix} 1 & 2 & \alpha \\ 2 & \alpha & 8 \end{bmatrix}$ e $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$[A_\alpha \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & \alpha & 1 \\ 2 & \alpha & 8 & 3 \end{array}\right] \underset{-2L_1+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & \alpha & 1 \\ 0 & \alpha - 4 & 8 - 2\alpha & 1 \end{array}\right].$$

Se $\alpha \neq 4$ então $\text{car}\, A_\alpha = \text{car}\, [A_\alpha \mid B] = 2 < 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x + 2y + \alpha z = 1 \\ (\alpha - 4)\, y + (8 - 2\alpha)\, z = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 - \dfrac{2}{\alpha - 4} - (\alpha + 4)\, z \\[2ex] y = \dfrac{1}{\alpha - 4} + 2z. \end{cases}$$

A solução geral deste sistema é então dada por

$$S_\alpha = \left\{ \left(1 - \frac{2}{\alpha - 4} - (\alpha + 4)\, s, \frac{1}{\alpha - 4} + 2s, s \right) : s \in \mathbb{R} \right\}.$$

Se $\alpha = 4$ então $\underbrace{\text{car}\, A_\alpha}_{=1} < \underbrace{\text{car}\, [A_\alpha \mid B]}_{=2}$. Logo, o sistema não tem solução (é impossível). $S_4 = \varnothing$.

(c) Sejam $A_\alpha = \begin{bmatrix} 1 & 1 & \alpha \\ 3 & 4 & 2 \\ 2 & 3 & -1 \end{bmatrix}$ e $B_\alpha = \begin{bmatrix} 2 \\ \alpha \\ 1 \end{bmatrix}$. $[A_\alpha \mid B_\alpha] = \left[\begin{array}{ccc|c} 1 & 1 & \alpha & 2 \\ 3 & 4 & 2 & \alpha \\ 2 & 3 & -1 & 1 \end{array}\right] \underset{\substack{-3L_1+L_2 \to L_2 \\ -2L_1+L_3 \to L_3}}{\longrightarrow}$

$$\underset{\substack{-3L_1+L_2\to L_2\\-2L_1+L_3\to L_3}}{\longrightarrow}\left[\begin{array}{ccc|c}1 & 1 & \alpha & 2\\0 & 1 & 2-3\alpha & \alpha-6\\0 & 1 & -1-2\alpha & -3\end{array}\right]\underset{-L_2+L_3\to L_3}{\longrightarrow}\left[\begin{array}{ccc|c}1 & 1 & \alpha & 2\\0 & 1 & 2-3\alpha & \alpha-6\\0 & 0 & -3+\alpha & 3-\alpha\end{array}\right].$$

Se $\alpha = 3$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\alpha] = 2 < 3 = \mathrm{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x+y+3z=2\\ y-7z=-3\end{cases} \Leftrightarrow \begin{cases} x=5-10z\\ \\ y=-3+7z.\end{cases}$$

A solução geral deste sistema é então dada por

$$S_3 = \{(8-\alpha+(2-4\alpha)\,s, \alpha-6+(3\alpha-2)\,s, s) : s \in \mathbb{R}\}.$$

Se $\alpha \neq 3$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\alpha] = 3 = \mathrm{n}^o$ de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x+y+\alpha z=2\\ y+(2-3\alpha)\,z=\alpha-6\\ (-3+\alpha)\,z=3-\alpha\end{cases} \Leftrightarrow \begin{cases} x=6+3\alpha\\ y=-4-2\alpha\\ z=-1.\end{cases}$$

A solução geral do sistema é então dada por

$$S_\alpha = \{(6+3\alpha, -4-2\alpha, -1)\}.$$

(d) Sejam $A_\alpha = \begin{bmatrix}\alpha & 1 & 1\\1 & \alpha & 1\\1 & 1 & \alpha\end{bmatrix}$ e $B_\alpha = \begin{bmatrix}1\\\alpha\\\alpha^2\end{bmatrix}$. $[A_\alpha \mid B_\alpha] = \left[\begin{array}{ccc|c}\alpha & 1 & 1 & 1\\1 & \alpha & 1 & \alpha\\1 & 1 & \alpha & \alpha^2\end{array}\right]\underset{L_1\leftrightarrow L_3}{\longrightarrow}$

$$\underset{L_1\leftrightarrow L_3}{\longrightarrow}\left[\begin{array}{ccc|c}1 & 1 & \alpha & \alpha^2\\1 & \alpha & 1 & \alpha\\\alpha & 1 & 1 & 1\end{array}\right]\underset{\substack{-L_1+L_2\to L_2\\-\alpha L_1+L_3\to L_3}}{\longrightarrow}\left[\begin{array}{ccc|c}1 & 1 & \alpha & \alpha^2\\0 & \alpha-1 & 1-\alpha & \alpha-\alpha^2\\0 & 1-\alpha & 1-\alpha^2 & 1-\alpha^3\end{array}\right]\underset{L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{L_2+L_3\to L_3}{\longrightarrow}\left[\begin{array}{ccc|c}1 & 1 & \alpha & \alpha^2\\0 & \alpha-1 & 1-\alpha & \alpha\,(1-\alpha)\\0 & 0 & (1-\alpha)\,(\alpha+2) & (1+\alpha)\,(1-\alpha^2)\end{array}\right].$$

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\alpha] = 1 < 3 = \mathrm{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se $x+y+z=1$. A solução geral deste sistema é então dada por $S_1 = \{(1-s-t, s, t) : s, t \in \mathbb{R}\}$.

Se $\alpha = -2$ então $\underbrace{\operatorname{car} A_\alpha}_{=2} < \underbrace{\operatorname{car}[A_\alpha \mid B_\alpha]}_{=3}$. O sistema não tem solução (é impossível). $S_{-2} = \varnothing$.

Se $\alpha \neq 1$ e $\alpha \neq -2$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\alpha] = 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x + y + \alpha z = \alpha^2 \\ (\alpha - 1) y + (1 - \alpha) z = \alpha (1 - \alpha) \\ (1 - \alpha)(\alpha + 2) z = (1 + \alpha)(1 - \alpha^2) \end{cases} \Leftrightarrow \begin{cases} x = -(\alpha + 1)/(\alpha + 2) \\ y = 1/(\alpha + 2) \\ z = (1 + \alpha)^2/(\alpha + 2). \end{cases}$$

A solução geral do sistema é então dada por

$$S_\alpha = \left\{ \left(-\frac{\alpha + 1}{\alpha + 2}, \frac{1}{\alpha + 2}, \frac{(1 + \alpha)^2}{\alpha + 2} \right) \right\}.$$

(e) $\begin{cases} -x + y + \alpha z = 1 \\ 2x + \alpha y - 2\alpha z = \alpha \\ -\alpha x + \alpha y + z = -1 + 2\alpha \end{cases}$ Sejam $A_\alpha = \begin{bmatrix} -1 & 1 & \alpha \\ 2 & \alpha & -2\alpha \\ -\alpha & \alpha & 1 \end{bmatrix}$ e $B = \begin{bmatrix} 1 \\ \alpha \\ -1 + 2\alpha \end{bmatrix}$.

$$[A_\alpha \mid B] = \left[\begin{array}{ccc|c} -1 & 1 & \alpha & 1 \\ 2 & \alpha & -2\alpha & \alpha \\ -\alpha & \alpha & 1 & -1 + 2\alpha \end{array}\right] \underset{\substack{2L_1 + L_2 \to L_2 \\ -\alpha L_1 + L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} -1 & 1 & \alpha & 1 \\ 0 & \alpha + 2 & 0 & \alpha + 2 \\ 0 & 0 & (1 - \alpha)(1 + \alpha) & -1 + \alpha \end{array}\right].$$

Se $\alpha = 1$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B] = 2 < 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} -x + y + z = 1 \\ 3y = 3. \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = 1. \end{cases}$$

A solução geral deste sistema é então dada por

$$S_1 = \{(s, 1, s) : s \in \mathbb{R}\}.$$

Se $\alpha = -2$ então então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B] = 2 < 3 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} -x + y - 2z = 1 \\ -3z = -3. \end{cases} \Leftrightarrow \begin{cases} x = y - 3 \\ z = 1. \end{cases}$$

A solução geral deste sistema é então dada por

$$S_{-2} = \{(s - 3, s, 1) : s \in \mathbb{R}\}.$$

Se $\alpha = -1$ então $\underbrace{\operatorname{car} A_\alpha}_{=2} < \underbrace{\operatorname{car}[A_\alpha \mid B]}_{=3}$. Logo, o sistema não tem solução (é impossível).

$$S_{-1} = \varnothing.$$

Se $\alpha \neq 1$ e $\alpha \neq -1$ e $\alpha \neq -2$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B] = 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} -x + y + \alpha z = 1 \\ (\alpha + 2)\, y = \alpha + 2 \\ (1-\alpha)(1+\alpha)\, z = -1 + \alpha \end{cases} \Leftrightarrow \begin{cases} x = -\alpha / (\alpha + 1) \\ y = 1 \\ z = -1/(\alpha + 1). \end{cases}$$

A solução geral do sistema é então dada por

$$S_\alpha = \left\{ \left(-\frac{\alpha}{\alpha + 1}, 1, -\frac{1}{\alpha + 1} \right) \right\}.$$

16. **(a)** Sejam $A_\alpha = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & -2 \\ 1 & 5 & \alpha \end{bmatrix}$ e $B_\beta = \begin{bmatrix} 10 \\ 10 \\ \beta \end{bmatrix}$.

$$[A_\alpha \mid B_\beta] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & 10 \\ 2 & 7 & -2 & 10 \\ 1 & 5 & \alpha & \beta \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3}}{\longrightarrow}$$

$$\underset{\substack{-2L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 10 \\ 0 & -1 & -8 & -10 \\ 0 & 1 & \alpha - 3 & \beta - 10 \end{array}\right] \underset{L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 10 \\ 0 & -1 & -8 & -10 \\ 0 & 0 & \alpha - 11 & \beta - 20 \end{array}\right].$$

Se $\alpha = 11$ e $\beta = 20$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 2 < 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x + 4y + 3z = 10 \\ -y - 8z = -10 \end{cases} \Leftrightarrow \begin{cases} x = -30 + 29z \\ y = 10 - 8z. \end{cases}$$

A solução geral deste sistema é então dada por $S_{\alpha,\beta} = \{(-30 + 29s, 10 - 8s, s) : s \in \mathbb{R}\}$.

Se $\alpha = 11$ e $\beta \neq -20$ então $\underbrace{\operatorname{car} A_\alpha}_{=2} < \underbrace{\operatorname{car}[A_\alpha \mid B_\beta]}_{=3}$. Logo, o sistema não tem solução (é impossível). $S_{\alpha,\beta} = \varnothing$.

Se $\alpha \neq 11$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 3 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e determinado, tendo-se

$$\begin{cases} x + 4y + 3z = 10 \\ -y - 8z = -10 \\ (\alpha - 11)\, z = \beta - 20 \end{cases} \Leftrightarrow \begin{cases} x = -(30\alpha - 29\beta + 250)/(\alpha - 11) \\ y = (10\alpha - 8\beta + 50)/(\alpha - 11) \\ z = (\beta - 20)/(\alpha - 11). \end{cases}$$

A solução geral do sistema é então dada por

$$S_{\alpha,\beta} = \left\{ \left(-\frac{30\alpha - 29\beta + 250}{\alpha - 11}, \frac{10\alpha - 8\beta + 50}{\alpha - 11}, \frac{\beta - 20}{\alpha - 11} \right) \right\}.$$

(b) Sejam $A_\alpha = \begin{bmatrix} 0 & 0 & 2 & \alpha \\ 1 & 1 & 1 & 3 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 3 & 14 \end{bmatrix}$ e $B_\beta = \begin{bmatrix} \beta \\ 1 \\ 2 \\ 4 \end{bmatrix}$. $[A_\alpha \mid B_\beta] = \left[\begin{array}{cccc|c} 0 & 0 & 2 & \alpha & \beta \\ 1 & 1 & 1 & 3 & 1 \\ 2 & 2 & 1 & 1 & 2 \\ 1 & 1 & 3 & 14 & 4 \end{array}\right] \underset{L_1 \leftrightarrow L_3}{\longrightarrow}$

$$\underset{L_1 \leftrightarrow L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 2 & 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & 2 & \alpha & \beta \\ 1 & 1 & 3 & 14 & 4 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 2 & \alpha & \beta \\ 1 & 1 & 3 & 14 & 4 \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -L_1+L_4 \to L_4}}{\longrightarrow}$$

$$\underset{\substack{-2L_1+L_2 \to L_2 \\ -L_1+L_4 \to L_4}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 2 & \alpha & \beta \\ 0 & 0 & 2 & 11 & 3 \end{array}\right] \underset{\substack{2L_2+L_3 \to L_3 \\ 2L_2+L_4 \to L_4}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & \alpha - 10 & \beta \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\longrightarrow}$$

$$\underset{L_1 \leftrightarrow L_2}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & \alpha - 10 & \beta \end{array}\right] \underset{-(\alpha-10)L_3+L_4 \to L_4}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 1 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -3(\alpha - 10) + \beta \end{array}\right].$$

Se $\beta = 3(\alpha - 10)$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 3 < 4 =$ n^o de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\begin{cases} x + y + z + 3w = 1 \\ -z - 5w = 0 \\ w = 3 \end{cases} \Leftrightarrow \begin{cases} x = 7 - y \\ z = -15 \\ w = 3. \end{cases}.$$

A solução geral deste sistema é então dada por $S_{\alpha,\beta} = \{(7 - s, s, -15, 3) : s \in \mathbb{R}\}$.

Se $\beta \neq 3(\alpha - 10)$ então $\underbrace{\operatorname{car} A_\alpha}_{=3} < \underbrace{\operatorname{car}[A_\alpha \mid B_\beta]}_{=4}$. Logo, o sistema não tem solução (é impossível). $S_{\alpha,\beta} = \varnothing$.

(c) Sejam $A_\alpha = \begin{bmatrix} \alpha & 1 & -1 & \alpha \\ 1 & -2 & 2 & 1 \\ 1 & -1 & 1 & \alpha + 1 \end{bmatrix}$ e $B_\beta = \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix}$.

$$[A_\alpha \mid B_\beta] = \left[\begin{array}{cccc|c} \alpha & 1 & -1 & \alpha & 0 \\ 1 & -2 & 2 & 1 & 1 \\ 1 & -1 & 1 & \alpha + 1 & \beta \end{array}\right] \underset{L_1 \leftrightarrow L_3}{\longrightarrow}$$

$$\underset{L_1 \leftrightarrow L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & -1 & 1 & \alpha+1 & \beta \\ 1 & -2 & 2 & 1 & 1 \\ \alpha & 1 & -1 & \alpha & 0 \end{array}\right] \underset{\substack{-L_1+L_2 \to L_2 \\ -\alpha L_1+L_3 \to L_3}}{\longrightarrow}$$

$$\underset{\substack{-L_1+L_2 \to L_2 \\ -\alpha L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & -1 & 1 & \alpha+1 & \beta \\ 0 & -1 & 1 & -\alpha & 1-\beta \\ 0 & \alpha+1 & -\alpha-1 & -\alpha^2 & -\alpha\beta \end{array}\right] \underset{(\alpha+1)L_2+L_3 \to L_3}{\longrightarrow}$$

$$\underset{(\alpha+1)L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & -1 & 1 & \alpha+1 & \beta \\ 0 & -1 & 1 & -\alpha & 1-\beta \\ 0 & 0 & 0 & (-2\alpha-1)\,\alpha & \alpha-2\alpha\beta+1-\beta \end{array}\right].$$

Se $\alpha \neq 0$ e $\alpha \neq -\dfrac{1}{2}$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 3 < 4 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\left\{\begin{array}{l} x-y+z+(\alpha+1)\,w=\beta \\ -y+z-\alpha w = 1-\beta \\ (-2\alpha-1)\,\alpha w = \alpha+1+(-2\alpha-1)\,\beta \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x = -\frac{\alpha+1}{2\alpha+1}-1-\frac{(\alpha+1)^2}{(-2\alpha-1)\alpha}-\frac{\beta}{\alpha} \\ y = z-\frac{\alpha+1}{2\alpha+1}-1 \\ w = \frac{\alpha+1}{(-2\alpha-1)\alpha}+\frac{\beta}{\alpha}. \end{array}\right.$$

A solução geral do sistema é então dada por

$$S_{\alpha,\beta} = \left\{\left(-\frac{\alpha+1}{2\alpha+1}-1-\frac{(\alpha+1)^2}{(-2\alpha-1)\,\alpha}-\frac{\beta}{\alpha}, s-\frac{\alpha+1}{2\alpha+1}-1, s, \frac{\alpha+1}{(-2\alpha-1)\,\alpha}+\frac{\beta}{\alpha}\right)\right\}.$$

Se $\alpha = 0$ e $\beta = 1$ então $\operatorname{car} A_\alpha = \operatorname{car}[A_\alpha \mid B_\beta] = 2 < 4 = \text{n}^o$ de incógnitas do sistema. Logo o sistema é possível e indeterminado, tendo-se

$$\left\{\begin{array}{l} x-y+z+w=1 \\ -y+z=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x = 1-w \\ y = z. \end{array}\right.$$

A solução geral deste sistema é então dada por $S_{\alpha,\beta} = \{(1-s,t,t,s) : s,t \in \mathbb{R}\}$.

Se ($\alpha = 0$ e $\beta \neq 1$) ou $\alpha = -\dfrac{1}{2}$ então $\underbrace{\operatorname{car} A_\alpha}_{=2} < \underbrace{\operatorname{car}[A_\alpha \mid B_\beta]}_{=3}$. Logo, o sistema não tem solução (é impossível). $S_{\alpha,\beta} = \varnothing$.

17. (a) Sejam $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 1 & -5 & 8 \end{bmatrix}$ e $B_{a,b,c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$[A \mid B_{a,b,c}] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 3 & -1 & 2 & b \\ 1 & -5 & 8 & c \end{array}\right] \underset{\substack{-3L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3}}{\longrightarrow}$$

$$\underset{\substack{-3L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & -7 & 11 & c-a \end{array}\right] \underset{-L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & -7 & 11 & b-3a \\ 0 & 0 & 0 & c-b+2a \end{array}\right].$$

Para que haja solução é necessário que car $A = \text{car}\,[A \mid B_{a,b,c}]$, isto é, é necessário que

$$c - b + 2a = 0.$$

(b) Sejam $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 3 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ e $B_{a,b,c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$[A \mid B_{a,b,c}] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & a \\ 2 & 3 & -1 & b \\ 3 & 1 & 2 & c \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -3L_1+L_3 \to L_3}}{\longrightarrow}$$

$$\underset{\substack{-2L_1+L_2 \to L_2 \\ -3L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 4 & a \\ 0 & 7 & -9 & b-2a \\ 0 & 7 & -10 & c-3a \end{array}\right] \underset{-L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -2 & 4 & a \\ 0 & 7 & -9 & b-2a \\ 0 & 0 & -1 & c-b-a \end{array}\right].$$

Como car $A = \text{car}\,[A \mid B_{a,b,c}]$, este sistema tem solução para quaisquer valores de a, b, c.

18. (a) Sejam $x = 1 + t$ e $y = 1 - t$. Logo

$$x + y = 2.$$

(b) Sejam $x = t$, $y = 1 - 2t$ e $z = 1$. Tem-se então o seguinte sistema:

$$\begin{cases} 2x + y = 1 \\ z = 1. \end{cases}$$

(c) Sejam $x = 3t$, $y = 2t$ e $z = t$. Tem-se então o seguinte sistema:

$$\begin{cases} x - 3z = 0 \\ y - 2z = 0. \end{cases}$$

(d) Sejam $x = 3t - s$, $y = t + 2s - 1$ e $z = s - 2t + 1$. Logo $s = 3t - x$ e assim

$$y = t + 2(3t - x) - 1 = 7t - 2x - 1 \Leftrightarrow t = \frac{y + 2x + 1}{7}.$$

Deste modo:

$$s = 3\frac{y + 2x + 1}{7} - x = \frac{3y - x + 3}{7}$$

Com

$$s = \frac{3y - x + 3}{7} \quad \text{e} \quad t = \frac{y + 2x + 1}{7}$$

Tem-se então a seguinte equação linear:

$$z = s - 2t + 1 = \frac{3y - x + 3}{7} - 2\frac{y + 2x + 1}{7} + 1.$$

Isto é:

$$5x - y + 7z = 8.$$

(e) Sejam $x = 2t - 3s$, $y = t + s - 1$, $z = 2s + 1$ e $w = t - 1$. Logo $t = w + 1$ e $s = \frac{z - 1}{2}$.
Assim:

$$\begin{cases} x = 2(w + 1) - 3\frac{z - 1}{2} \\ \\ y = w + 1 + \frac{z - 1}{2} - 1. \end{cases}$$

Deste modo, obtém-se o sistema de equações lineares:

$$\begin{cases} 2x + 3z - 4w = 7 \\ \\ 2y - z - 2w = -1. \end{cases}$$

(f) Seja $S = \{(1 - s, s - t, 2s, t - 1) : s, t \in \mathbb{R}\}$.

Sejam $x = 1 - s$, $y = s - t$, $z = 2s$, $w = t - 1$. Uma vez que $s = 1 - x$ e $t = w + 1$, tem-se então o seguinte sistema linear não homogéneo

$$\begin{cases} y = 1 - x - (w + 1) \\ z = 2(1 - x) \end{cases} \Leftrightarrow \begin{cases} x + y + w = 0 \\ 2x + z = 2 \end{cases}$$

(g) Por exemplo:

$$\begin{cases} x + y = 1 \\ \\ x + y = 0. \end{cases}$$

19. Pretende-se determinar $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ tal que

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Tem-se então

$$\begin{cases} a + 2c = a + 3b \\ b + 2d = 2a + 4b \\ 3a + 4c = c + 3d \\ 3b + 4d = 2c + 4d \end{cases} \Leftrightarrow \begin{cases} c = \frac{3}{2}b \\ d = a + \frac{3}{2}b \end{cases}$$

As matrizes reais que comutam com $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ são da forma: $\begin{bmatrix} a & b \\ \frac{3}{2}b & a + \frac{3}{2}b \end{bmatrix}$, com $a, b \in \mathbb{R}$.

20. Existem 16 matrizes 2×2 só com 0 e 1 nas respectivas entradas. 6 são invertíveis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

21. Como

$$A^2 + 2A + 2I = \mathbf{0} \Leftrightarrow A\left(-\frac{1}{2}A - I\right) = \left(-\frac{1}{2}A - I\right)A = I$$

então A é invertível e

$$A^{-1} = -\frac{1}{2}A - I.$$

22. Sejam $A, B, X \in \mathcal{M}_{n\times n}(\mathbb{R})$ matrizes invertíveis tais que

$$(AB)^2 = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}.$$

(i)

$$AXB + AB = \mathbf{0} \Leftrightarrow AXB = -AB \Leftrightarrow A^{-1}(AXB)B^{-1} = A^{-1}(-AB)B^{-1} \Leftrightarrow$$

$$\Leftrightarrow \left(A^{-1}A\right)X\left(BB^{-1}\right) = -\left(A^{-1}A\right)\left(BB^{-1}\right) \Leftrightarrow$$

$$\Leftrightarrow IXI = -II \Leftrightarrow X = -I.$$

(ii)

$$BXA - A^{-1}B^{-1} = \mathbf{0} \Leftrightarrow B^{-1}(BXA)A^{-1} = B^{-1}\left(A^{-1}B^{-1}\right)A^{-1} \Leftrightarrow$$

$$\Leftrightarrow \left(B^{-1}B\right)X\left(AA^{-1}\right) = \left(B^{-1}A^{-1}\right)\left(B^{-1}A^{-1}\right) \Leftrightarrow$$

$$\Leftrightarrow IXI = \left[(AB)^2\right]^{-1} \Leftrightarrow$$

$$\Leftrightarrow X = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}.$$

23. **(i)** $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$,

$$\left(2I - \left(3A^{-1}\right)^T\right)^{-1} = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix} \Leftrightarrow 2I - \left(3A^{-1}\right)^T = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}^{-1} \Leftrightarrow$$

$$\Leftrightarrow 2I - \left(3A^{-1}\right)^T = \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix} \Leftrightarrow \left(3A^{-1}\right)^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -5 & 3 \\ 7 & -4 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow A = \left(\frac{1}{3}\begin{bmatrix} 7 & -3 \\ -7 & 6 \end{bmatrix}^T\right)^{-1} = \begin{bmatrix} \frac{7}{3} & -\frac{7}{3} \\ -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{6}{7} & 1 \\ \frac{3}{7} & 1 \end{bmatrix}.$$

(ii)

$$\left(A^T + 4I\right)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \Leftrightarrow A^T + 4I = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1} \Leftrightarrow$$

$$\Leftrightarrow A = \left(\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)^T = \begin{bmatrix} -9 & 2 \\ 3 & -5 \end{bmatrix}.$$

24. **(i)** $\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_1 \leftrightarrow L_2]{} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$. Logo $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(iii)** $[1]^{-1} = [1]$

(iv) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow[-3L_1 + L_2 \to L_2]{} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow[L_2 + L_1 \to L_1]{}$

$\xrightarrow[L_2 + L_1 \to L_1]{} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}L_2 \to L_2]{} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$.

Logo $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

$$\textbf{(v)}\ \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-4L_1+L_2\to L_2 \\ -7L_1+L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \underset{-2L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{-2L_2+L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right].$$

Logo, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ é singular e como tal não é invertível.

Nas alíneas (vi) e (vii) só se apresentam as soluções:

$$\textbf{(vi)}\ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(vii)}\ \left(\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}\right)^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\textbf{(viii)}\ \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \underset{-4L_1+L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array}\right] \underset{\frac{2}{3}L_3+L_1\to L_1}{\longrightarrow}$$

$$\underset{\frac{2}{3}L_3+L_1\to L_1}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array}\right] \underset{\substack{\frac{1}{3}L_2\to L_2 \\ -\frac{1}{3}L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & -\frac{1}{3} \end{array}\right].$$

Logo $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{4}{3} & 0 & -\frac{1}{3} \end{bmatrix}$.

$$\textbf{(ix)}\ \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 0 & 6 & 0 & 1 & 0 \\ 1 & 8 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-4L_1+L_2\to L_2 \\ -L_1+L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & 2 & -4 & 1 & 0 \\ 0 & 6 & 0 & -1 & 0 & 1 \end{array}\right] \underset{\frac{3}{4}L_2+L_3\to L_3}{\longrightarrow}$$

$$\underset{\frac{3}{4}L_2+L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & 2 & -4 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & -4 & \frac{3}{4} & 1 \end{array}\right] \underset{\frac{2}{3}L_3\to L_3}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -8 & 2 & -4 & 1 & 0 \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3} \end{array}\right] \underset{\substack{-2L_3+L_2\to L_2 \\ -L_3+L_1\to L_1}}{\longrightarrow}$$

$$\underset{\substack{-2L_3+L_2\to L_2 \\ -L_3+L_1\to L_1}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{11}{3} & -\frac{1}{2} & -\frac{2}{3} \\ 0 & -8 & 0 & \frac{4}{3} & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3} \end{array}\right] \underset{-\frac{1}{8}L_2\to L_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{11}{3} & -\frac{1}{2} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3} \end{array}\right] \underset{-2L_2+L_1\to L_1}{\longrightarrow}$$

$$\xrightarrow[-2L_2+L_1\to L_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{2} & \frac{2}{3} \end{array}\right].$$

Logo $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 6 \\ 1 & 8 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -\frac{1}{2} & -1 \\ -\frac{1}{6} & 0 & \frac{1}{6} \\ -\frac{8}{3} & \frac{1}{2} & \frac{2}{3} \end{bmatrix}.$

(x) Para $\alpha \neq k\frac{\pi}{2}$, $(k \in \mathbb{Z})$

$$\left[\begin{array}{cc|cc} \cos\alpha & -\operatorname{sen}\alpha & 1 & 0 \\ \operatorname{sen}\alpha & \cos\alpha & 0 & 1 \end{array}\right] \xrightarrow[\substack{(\cos\alpha)L_1\to L_1 \\ (\operatorname{sen}\alpha)L_2\to L_2}]{} \left[\begin{array}{cc|cc} \cos^2\alpha & -\cos\alpha\operatorname{sen}\alpha & \cos\alpha & 0 \\ \operatorname{sen}^2\alpha & \operatorname{sen}\alpha\cos\alpha & 0 & \operatorname{sen}\alpha \end{array}\right] \xrightarrow[L_2+L_1\to L_1]{}$$

$$\xrightarrow[L_2+L_1\to L_1]{} \left[\begin{array}{cc|cc} 1 & 0 & \cos\alpha & \operatorname{sen}\alpha \\ \operatorname{sen}^2\alpha & \operatorname{sen}\alpha\cos\alpha & 0 & \operatorname{sen}\alpha \end{array}\right] \xrightarrow[(-\operatorname{sen}^2\alpha)L_1+L_2\to L_2]{}$$

$$\xrightarrow[(-\operatorname{sen}^2\alpha)L_1+L_2\to L_2]{} \left[\begin{array}{cc|cc} 1 & 0 & \cos\alpha & \operatorname{sen}\alpha \\ 0 & \operatorname{sen}\alpha\cos\alpha & -\operatorname{sen}^2\alpha\cos\alpha & \operatorname{sen}\alpha\,(1-\operatorname{sen}^2\alpha) \end{array}\right] \xrightarrow[\frac{1}{\operatorname{sen}\alpha\cos\alpha}L_2\to L_2]{}$$

$\xrightarrow[\frac{1}{\operatorname{sen}\alpha\cos\alpha}L_2\to L_2]{} \left[\begin{array}{cc|cc} 1 & 0 & \cos\alpha & \operatorname{sen}\alpha \\ 0 & 1 & -\operatorname{sen}\alpha & \cos\alpha \end{array}\right]$. Note que $\operatorname{sen}\alpha\cos\alpha \neq 0$ para todo o $\alpha \neq k\frac{\pi}{2}$, $(k \in \mathbb{Z})$.

Logo $\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}$, para todo o $\alpha \neq k\frac{\pi}{2}$, $(k \in \mathbb{Z})$

Se $\alpha = \frac{\pi}{2} + 2k\pi$, $(k \in \mathbb{Z})$,

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

Se $\alpha = 2k\pi$, $(k \in \mathbb{Z})$,

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

Se $\alpha = \pi + 2k\pi$, $(k \in \mathbb{Z})$,

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

Se $\alpha = \dfrac{3\pi}{2} + 2k\pi$, $(k \in \mathbb{Z})$,

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

Logo, para todo o $\alpha \in \mathbb{R}$

$$\begin{bmatrix} \cos\alpha & -\operatorname{sen}\alpha \\ \operatorname{sen}\alpha & \cos\alpha \end{bmatrix}^{-1} = \begin{bmatrix} \cos\alpha & \operatorname{sen}\alpha \\ -\operatorname{sen}\alpha & \cos\alpha \end{bmatrix}.$$

(xi) Seja $k \neq 0$.

$$\left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & k & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & k & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & k & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-\frac{1}{k}L_1+L_2\to L_2 \\ \frac{1}{k}L_3\to L_3 \\ \frac{1}{k}L_4\to L_4}}{\longrightarrow} \left[\begin{array}{cccc|cccc} k & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & -\frac{1}{k} & 1 & 0 & 0 \\ 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right] \underset{\substack{-\frac{1}{k^2}L_2+L_3\to L_3 \\ \frac{1}{k}L_1\to L_1}}{\longrightarrow}$$

$$\underset{\substack{-\frac{1}{k^2}L_2+L_3\to L_3 \\ \frac{1}{k}L_1\to L_1}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & k & 0 & 0 & -\frac{1}{k} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} & 1 & 0 & 0 & 0 & \frac{1}{k} \end{array}\right] \underset{\substack{-\frac{1}{k}L_3+L_4\to L_4 \\ \frac{1}{k}L_2\to L_2}}{\longrightarrow}$$

$$\underset{\substack{-\frac{1}{k}L_3+L_4\to L_4 \\ \frac{1}{k}L_2\to L_2}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{array}\right].$$

Logo $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 \\ -\frac{1}{k^2} & \frac{1}{k} & 0 & 0 \\ \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} & 0 \\ -\frac{1}{k^4} & \frac{1}{k^3} & -\frac{1}{k^2} & \frac{1}{k} \end{bmatrix}.$

(xii) Sejam $k_1, k_2, k_3, k_4 \neq 0$.

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{L_1\leftrightarrow L_4 \\ L_2\leftrightarrow L_3}}{\longrightarrow} \left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right] \underset{\substack{\frac{1}{k_4}L_1\to L_1 \\ \frac{1}{k_3}L_2\to L_2 \\ \frac{1}{k_2}L_3\to L_3 \\ \frac{1}{k_1}L_4\to L_4}}{\longrightarrow}$$

$$\xrightarrow[\substack{\frac{1}{k_4}L_1\to L_1\\ \frac{1}{k_3}L_2\to L_2\\ \frac{1}{k_2}L_3\to L_3\\ \frac{1}{k_1}L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1}1 & 0 & 0 & 0 \end{array}\right]. \text{ Logo } \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1}1 & 0 & 0 & 0 \end{bmatrix}.$$

(xiii) $$\begin{bmatrix} \frac{5}{13} & \frac{2}{13} & \frac{2}{13} & -\frac{8}{13} \\ \frac{2}{13} & -\frac{7}{13} & \frac{6}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{6}{13} & -\frac{7}{13} & \frac{2}{13} \\ -\frac{8}{13} & \frac{2}{13} & \frac{2}{13} & \frac{5}{13} \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 5 & 2 & 2 & -8 \\ 2 & -7 & 6 & 2 \\ 2 & 6 & -7 & 2 \\ -8 & 2 & 2 & 5 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|cccc} 5 & 2 & 2 & -8 & 1 & 0 & 0 & 0 \\ 2 & -7 & 6 & 2 & 0 & 1 & 0 & 0 \\ 2 & 6 & -7 & 2 & 0 & 0 & 1 & 0 \\ -8 & 2 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_1\leftrightarrow L_3]{} \left[\begin{array}{cccc|cccc} 2 & 6 & -7 & 2 & 0 & 0 & 1 & 0 \\ 2 & -7 & 6 & 2 & 0 & 1 & 0 & 0 \\ 5 & 2 & 2 & -8 & 1 & 0 & 0 & 0 \\ -8 & 2 & 2 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-L_1+L_2\to L_2\\ -\frac{5}{2}L_1+L_3\to L_3\\ 4L_1+L_4\to L_4}]{}$$

$$\xrightarrow[\substack{-L_1+L_2\to L_2\\ -\frac{5}{2}L_1+L_3\to L_3\\ 4L_1+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 2 & 6 & -7 & 2 & 0 & 0 & 1 & 0 \\ 0 & -13 & 13 & 0 & 0 & 1 & -1 & 0 \\ 0 & -13 & \frac{39}{2} & -13 & 1 & 0 & -\frac{5}{2} & 0 \\ 0 & 26 & -26 & 13 & 0 & 0 & 4 & 1 \end{array}\right] \xrightarrow[\substack{-L_2+L_3\to L_3\\ 2L_2+L_4\to L_4}]{}$$

$$\xrightarrow[\substack{-L_2+L_3\to L_3\\ 2L_2+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 2 & 6 & -7 & 2 & 0 & 0 & 1 & 0 \\ 0 & -13 & 13 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & \frac{13}{2} & -13 & 1 & -1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 13 & 0 & 2 & 2 & 1 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}L_1\to L_1\\ -\frac{1}{13}L_2\to L_2\\ \frac{2}{13}L_3\to L_3\\ \frac{1}{13}L_4\to L_4}]{}$$

$$\xrightarrow[\substack{\frac{1}{2}L_1\to L_1\\ -\frac{1}{13}L_2\to L_2\\ \frac{2}{13}L_3\to L_3\\ \frac{1}{13}L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & 3 & -\frac{7}{2} & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{13} & \frac{1}{13} & 0 \\ 0 & 0 & 1 & -2 & \frac{2}{13} & -\frac{2}{13} & -\frac{3}{13} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} \end{array}\right] \xrightarrow[\substack{-L_4+L_1\to L_1\\ 2L_4+L_3\to L_3}]{}$$

$$\xrightarrow[\substack{-L_4+L_1\to L_1\\ 2L_4+L_3\to L_3}]{} \left[\begin{array}{cccc|cccc} 1 & 3 & -\frac{7}{2} & 0 & 0 & -\frac{2}{13} & \frac{9}{26} & -\frac{1}{13} \\ 0 & 1 & -1 & 0 & 0 & -\frac{1}{13} & \frac{1}{13} & 0 \\ 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} \end{array}\right] \xrightarrow[\substack{L_3+L_2\to L_2\\ \frac{7}{2}L_3+L_1\to L_1}]{}$$

$$\xrightarrow[\substack{L_3+L_2\to L_2\\ \frac{7}{2}L_3+L_1\to L_1}]{} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & \frac{7}{13} & \frac{5}{13} & \frac{8}{13} & \frac{6}{13} \\ 0 & 1 & 0 & 0 & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} & \frac{2}{13} \\ 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} \end{array}\right] \xrightarrow[-3L_2+L_1\to L_1]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{13} & \frac{2}{13} & \frac{2}{13} & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} & \frac{2}{13} \\ 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} \end{array}\right].$$

Logo $\begin{bmatrix} \frac{5}{13} & \frac{2}{13} & \frac{2}{13} & -\frac{8}{13} \\ \frac{2}{13} & -\frac{7}{13} & \frac{6}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{6}{13} & -\frac{7}{13} & \frac{2}{13} \\ -\frac{8}{13} & \frac{2}{13} & \frac{2}{13} & \frac{5}{13} \end{bmatrix}^{-1} = \left(\frac{1}{13}\begin{bmatrix} 5 & 2 & 2 & -8 \\ 2 & -7 & 6 & 2 \\ 2 & 6 & -7 & 2 \\ -8 & 2 & 2 & 5 \end{bmatrix}\right)^{-1} =$

$$= 13\begin{bmatrix} \frac{1}{13} & \frac{2}{13} & \frac{2}{13} & 0 \\ \frac{2}{13} & \frac{1}{13} & \frac{2}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{2}{13} & \frac{1}{13} & \frac{2}{13} \\ 0 & \frac{2}{13} & \frac{2}{13} & \frac{1}{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

(xiv) $\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$

$$\left[\begin{array}{cccc|cccc} 2 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_1\leftrightarrow L_4]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{L_1+L_2\to L_2\\ L_1+L_3\to L_3\\ -2L_1+L_4\to L_4}]{}$$

$$\xrightarrow[\substack{L_1+L_2\to L_2\\ L_1+L_3\to L_3\\ -2L_1+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 & 0 & 0 & -2 \end{array}\right] \xrightarrow[\substack{L_2+L_3\to L_3\\ -L_2+L_4\to L_4}]{}$$

$$\xrightarrow[\substack{L_2+L_3\to L_3\\ -L_2+L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & -4 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow[L_3\leftrightarrow L_4]{}$$

$$\xrightarrow[L_3\leftrightarrow L_4]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -4 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\substack{2L_4+L_3\to L_3\\ -\frac{1}{2}L_4+L_2\to L_2\\ -L_4+L_1\to L_1}]{}$$

$$\xrightarrow[\substack{2L_4+L_3\to L_3\\ -\frac{1}{2}L_4+L_2\to L_2\\ -L_4+L_1\to L_1}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\substack{\frac{1}{2}L_3\to L_3\\ \frac{1}{2}L_4\to L_4}]{}$$

$$\xrightarrow[\substack{\frac{1}{2}L_3\to L_3\\ \frac{1}{2}L_4\to L_4}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[\substack{L_3+L_2\to L_2\\ L_3+L_1\to L_1}]{}$$

$$\xrightarrow[\substack{L_3+L_2\to L_2\\ L_3+L_1\to L_1}]{} \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[L_2+L_1\to L_1]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right].$$

$$\text{Logo} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \left(\frac{1}{2} \begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 2 \end{bmatrix} \right)^{-1} =$$

$$= 2 \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

25. **(i)** Seja $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ tal que

$$A^k = \mathbf{0}$$

para algum $k \in \mathbb{N}\setminus\{1\}$.

$(I - A)\left(I + A + ... + A^{k-1}\right) = I + A + ... + A^{k-1} - A - A^2 - ... - A^{k-1} - A^k = I - A^k = I$

ou seja, $I - A$ é invertível e

$$(I - A)^{-1} = I + A + ... + A^{k-1}.$$

(ii)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \left(I - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \underset{\text{(i)}}{=}$$

$$I + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

uma vez que $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

26. $A = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}$.

(i) $A^3 = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(ii) Por (i): $(I - A)(I + A + A^2) = I$

27. a) Usando o método de eliminação de Gauss, tem-se

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 2 & 7 & 2 & 20 \\ 1 & 5 & \alpha & 10 \end{array}\right] \underset{\substack{-2L_1+L_2 \to L_2 \\ -L_1+L_3 \to L_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & \alpha - 2 & 0 \end{array}\right] \underset{L_2+L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 10 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & \alpha - 4 & 0 \end{array}\right].$$

Se $\alpha \neq 4$ então o sistema é possível e determinado, existindo uma única solução. Se $\alpha = 4$ então o sistema é possível e indeterminado, existindo um n^o infinito de soluções.

b) Para $\alpha = 4$, tem-se o sistema de equações lineares

$$\begin{cases} x + 4y + 2z = 10 \\ -y - 2z = 0. \end{cases} \Leftrightarrow \begin{cases} x = 10 + 6z \\ y = -2z. \end{cases}$$

Colocando $z = s$, a solução geral do sistema é dada por: $S = \{(10 + 6s, -2s, s) : s \in \mathbb{R}\}$.

28. $A_{\lambda,\mu} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & \lambda & \lambda^2 + \mu & \lambda\mu \\ 0 & 1 & \lambda & \mu \\ 1 & \lambda & \lambda^2 + \mu & \lambda + \lambda\mu \end{bmatrix} \underset{L_2 \leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \lambda & \mu \\ 1 & \lambda & \lambda^2 + \mu & \lambda\mu \\ 1 & \lambda & \lambda^2 + \mu & \lambda + \lambda\mu \end{bmatrix} \underset{\substack{-L_1+L_3 \to L_3 \\ -L_1+L_4 \to L_4}}{\longrightarrow}$

$$\underset{\substack{-L_1+L_3 \to L_3 \\ -L_1+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \lambda & \mu \\ 0 & \lambda & \lambda^2 + \mu + 1 & \lambda\mu \\ 0 & \lambda & \lambda^2 + \mu + 1 & \lambda + \lambda\mu \end{bmatrix} \underset{\substack{-\lambda L_2+L_3 \to L_3 \\ -\lambda L_2+L_4 \to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \lambda & \mu \\ 0 & 0 & \mu + 1 & 0 \\ 0 & 0 & \mu + 1 & \lambda \end{bmatrix} \underset{-L_3+L_4 \to L_4}{\longrightarrow}$$

$$\underset{-L_3+L_4 \to L_4}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & \lambda & \mu \\ 0 & 0 & \mu + 1 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}.$$

Se $\mu = -1$ e $\lambda = 0$ então car $A = 2$ e nul $A = 2$.

Se ($\mu = -1$ e $\lambda \neq 0$) ou ($\mu \neq -1$ e $\lambda = 0$) então car $A = 3$ e nul $A = 1$.

Se $\mu \neq -1$ e $\lambda \neq 0$ então car $A = 4$ e nul $A = 0$.

Assim, $A_{\lambda,\mu}$ é invertível se e só se $\mu \neq -1$ e $\lambda \neq 0$, uma vez que é só neste caso que car $A_{\lambda,\mu}$ = n^o de colunas de $A_{\lambda,\mu}$.

29. **(a)** Tem-se

$$A_\beta = \begin{bmatrix} 1 & 0 & \beta & 2 \\ 2 & \beta & \beta^2 & 4 \\ -4 & 0 & -\beta^3 & -8 \\ \beta & 0 & \beta^2 & \beta^2 \end{bmatrix} \underset{\substack{-2L_1+L_2\to L_2 \\ 4L_1+L_3\to L_3 \\ -\beta L_1+L_4\to L_4}}{\longrightarrow} \begin{bmatrix} 1 & 0 & \beta & 2 \\ 0 & \beta & \beta(\beta-2) & 0 \\ 0 & 0 & (2-\beta)(2+\beta)\beta & 0 \\ 0 & 0 & 0 & \beta(\beta-2) \end{bmatrix}.$$

Logo, como $\operatorname{car} A_\beta + \operatorname{nul} A_\beta = 4$,
se $\beta = 0$ então $\operatorname{car} A_\beta = 1$ e $\operatorname{nul} A_\beta = 3$;
se $\beta = 2$ então $\operatorname{car} A_\beta = 2$ e $\operatorname{nul} A_\beta = 2$;
se $\beta = -2$ então $\operatorname{car} A_\beta = 3$ e $\operatorname{nul} A_\beta = 1$;
se $\beta \neq 0$ e $\beta \neq 2$ e $\beta \neq -2$ então $\operatorname{car} A_\alpha = 4$ e $\operatorname{nul} A_\alpha = 0$.

Assim, A_β é invertível se e só se $\beta \in \mathbb{R}\backslash\{-2,0,2\}$, uma vez que é só nestes casos que $\operatorname{car} A_\beta =$ nº de colunas de A_β.

(b) $\left[\begin{array}{c|c} A_1 & I \end{array}\right] =$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 4 & 0 & 1 & 0 & 0 \\ -4 & 0 & -1 & -8 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-2L_1+L_2\to L_2 \\ 4L_1+L_3\to L_3 \\ -L_1+L_4\to L_4}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{2L_4+L_1\to L_1 \\ -\frac{1}{3}L_3+L_1\to L_1 \\ \frac{1}{3}L_3+L_2\to L_2}}{\longrightarrow}$$

$$\underset{\substack{2L_4+L_1\to L_1 \\ -\frac{1}{3}L_3+L_1\to L_1 \\ \frac{1}{3}L_3+L_2\to L_2}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{7}{3} & 0 & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 3 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-L_4\to L_4 \\ \frac{1}{3}L_3\to L_3}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{7}{3} & 0 & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right]$$

Logo

$$(A_1)^{-1} = \begin{bmatrix} -\frac{7}{3} & 0 & -\frac{1}{3} & 2 \\ -\frac{2}{3} & 1 & \frac{1}{3} & 0 \\ \frac{4}{3} & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

30. **(a)** $B_{a,b} = \begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & a & b \\ 3 & 0 & 6 & 0 \end{bmatrix} \underset{L_1\leftrightarrow L_2}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 0 & 0 & a & b \\ 3 & 0 & 6 & 0 \end{bmatrix} \underset{L_2\leftrightarrow L_4}{\longrightarrow}$

$$\underset{L_2\leftrightarrow L_4}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 3 & 0 & 6 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & a & 1 \end{bmatrix} \underset{\substack{-\frac{3}{2}L_1+L_2\to L_2 \\ -L_3+L_4\to L_4}}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 0 & -3 & 6 & -\frac{3}{2}a \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 1-b \end{bmatrix}.$$

Se $a = 0$ ou ($a \neq 0$ e $b = 1$) então $\operatorname{car} B_{a,b} = 3$ e $\operatorname{nul} B_{a,b} = 1$.

Se $a \neq 0$ e $b \neq 1$ então $\operatorname{car} B_{a,b} = 4$ e $\operatorname{nul} B_{a,b} = 0$.

(b) $[B_{1,0} \mid I] = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_4}{\longrightarrow} \left[\begin{array}{cccc|cccc} 3 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \underset{\substack{-\frac{2}{3}L_1+L_2 \to L_2 \\ -L_3+L_4 \to L_4}}{\longrightarrow}$

$$\underset{\substack{-\frac{2}{3}L_1+L_2 \to L_2 \\ -L_3+L_4 \to L_4}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 3 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & -4 & 1 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right] \underset{\substack{-L_4+L_2 \to L_2 \\ -6L_3+L_1 \to L_1}}{\longrightarrow}$$

$$\underset{\substack{-L_4+L_2 \to L_2 \\ -6L_3+L_1 \to L_1}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 0 & 0 & -6 & 1 \\ 0 & 2 & -4 & 0 & -1 & 1 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right] \underset{\substack{\frac{1}{2}L_2 \to L_2 \\ \frac{1}{3}L_1 \to L_1}}{\longrightarrow}$$

$$\underset{\substack{\frac{1}{2}L_2 \to L_2 \\ \frac{1}{3}L_1 \to L_1}}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{1}{3} \\ 0 & 1 & -2 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right] \underset{2L_3+L_2 \to L_2}{\longrightarrow} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right].$$

Logo $(B_{1,0})^{-1} = \begin{bmatrix} 0 & 0 & -2 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

(c) Como $B_{1,0}$ é invertível,

$$B_{1,0}X = C \Leftrightarrow X = (B_{1,0})^{-1}C \Leftrightarrow X = \begin{bmatrix} 0 & 0 & -2 & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{19}{3} \\ \frac{19}{3} \\ 3 \\ -2 \end{bmatrix}.$$

(d) Seja $X = (x_1, x_2, x_3, x_4)$.

$$B_{a,1}X = D \Leftrightarrow \begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 3 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ -a \\ -6 \end{bmatrix}.$$

A solução geral de $B_{a,1}X = D$ é dada por:

$$(\text{Solução particular de } B_{a,1}X = D) + (\text{Solução geral de } B_{a,1}X = \mathbf{0}).$$

O vector $(0,0,-1,0)$ é uma solução particular de $B_{a,1}X = D$. Determinemos a solução geral de $B_{a,1}X = \mathbf{0}$.

Tem-se $\begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 3 & 0 & 6 & 0 \end{bmatrix} \underset{\substack{-L_1+L_3\to L_3 \\ -\frac{3}{2}L_2+L_4\to L_4}}{\longrightarrow} \begin{bmatrix} 0 & 0 & a & 1 \\ 2 & 2 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 6 & -\frac{3}{2}a \end{bmatrix} \underset{\substack{L_1\leftrightarrow L_2 \\ L_3\leftrightarrow L_4}}{\longrightarrow}$

$$\underset{\substack{L_1\leftrightarrow L_2 \\ L_3\leftrightarrow L_4}}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 0 & 0 & a & 1 \\ 0 & -3 & 6 & -\frac{3}{2}a \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{L_2\leftrightarrow L_3}{\longrightarrow} \begin{bmatrix} 2 & 2 & 0 & a \\ 0 & -3 & 6 & -\frac{3}{2}a \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Logo, $\begin{cases} 2x_1 + 2x_2 + ax_4 = 0 \\ -3x_2 + 6x_3 - \frac{3}{2}ax_4 = 0 \\ ax_3 + x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -2x_3 \\ x_2 = \left(2 + \dfrac{a^2}{2}\right) x_3 \\ x_4 = -ax_3 \end{cases}$

Assim, a solução geral de $B_{a,1}X = \mathbf{0}$ é dada por:

$$\left\{(-2s, \left(2 + \frac{a^2}{2}\right) s, s, -as) : s \in \mathbb{R}\right\}$$

Logo, a solução geral do sistema linear $B_{a,1}X = D$ é dada por:

$$\{(0,0,-1,0)\} + \left\{\left(-2s, \left(2 + \frac{a^2}{2}\right) s, s, -as\right) : s \in \mathbb{R}\right\} = \left\{\left(-2s, \left(2 + \frac{a^2}{2}\right) s, s-1, -as\right) : s \in \mathbb{R}\right\}.$$

Resolução Alternativa.

$[B_{a,1} \mid D] = \left[\begin{array}{cccc|c} 0 & 0 & a & 1 & -a \\ 2 & 2 & 0 & a & 0 \\ 0 & 0 & a & 1 & -a \\ 3 & 0 & 6 & 0 & -6 \end{array}\right] \underset{\substack{-L_1+L_3\to L_3 \\ -\frac{3}{2}L_2+L_4\to L_4}}{\longrightarrow} \left[\begin{array}{cccc|c} 0 & 0 & a & 1 & -a \\ 2 & 2 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 6 & -\frac{3}{2}a & -6 \end{array}\right] \underset{\substack{L_1\leftrightarrow L_2 \\ L_3\leftrightarrow L_4}}{\longrightarrow}$

$$\underset{\substack{L_1\leftrightarrow L_2 \\ L_3\leftrightarrow L_4}}{\longrightarrow} \left[\begin{array}{cccc|c} 2 & 2 & 0 & a & 0 \\ 0 & 0 & a & 1 & -a \\ 0 & -3 & 6 & -\frac{3}{2}a & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{L_2\leftrightarrow L_3}{\longrightarrow} \left[\begin{array}{cccc|c} 2 & 2 & 0 & a & 0 \\ 0 & -3 & 6 & -\frac{3}{2}a & -6 \\ 0 & 0 & a & 1 & -a \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Tem-se então $\begin{cases} 2x + 2y \ + aw = 0 \\ -3y + 6z - \dfrac{3}{2}aw = -6 \\ az + w = -a \end{cases} \Leftrightarrow \begin{cases} x = -2z - 2 \\ y = \left(\frac{a^2}{2} + 2\right)(z+1) \\ w = -a - az \end{cases}$

Logo, a solução geral do sistema linear $B_{a,1}X = D$ é dada por:

$$\left\{\left(-2s - 2, \left(\frac{a^2}{2} + 2\right)(s+1), s, -a - as\right) : s \in \mathbb{R}\right\} = \left\{\left(-2s, \left(2 + \frac{a^2}{2}\right) s, s-1, -as\right) : s \in \mathbb{R}\right\}.$$