Instituto Superior Técnico - Álgebra Linear - 1^o Semestre 2011/2012

LEAN - LEMat - MEAmbi - MEBiol - MEQ

Resolução da 7^a Ficha de exercícios

1. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\right) = -(\lambda - 3)(\lambda - 1)^2.$$

Tem-se

$$\dim \mathcal{N}(A - 3I) = m_g(3) \underset{A \text{ é simétrica}}{=} m_a(3) = 1$$

$$\dim \mathcal{N}(A - 1I) = m_g(1) \underset{A \text{ é simétrica}}{=} m_a(1) = 2.$$

2. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & 2 \\ 3 & 0 & 1 \end{bmatrix}$. Como A não é simétrica, não é possível encontrar uma base ortogonal para $\mathbb{R}^3$ formada só por vectores próprios de A.

3. $A = \begin{bmatrix} -3 & -1 & -2 \\ -1 & -3 & 2 \\ -2 & 2 & 0 \end{bmatrix}$. Como A é simétrica então é ortogonalmente diagonalizável, isto é, existem uma matriz ortogonal P^T ($P^TP = PP^T = I$, isto é, $P^T = P^{-1}$) e uma matriz diagonal D tais que

$$D = PAP^T.$$

Como

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -1 & -2 \\ -1 & -3-\lambda & 2 \\ -2 & 2 & -\lambda \end{vmatrix} = -(\lambda - 2)(\lambda + 4)^2,$$

os valores próprios de A são 2 e -4 e tem-se

$$\mathcal{N}(A - 2I) = L\left(\left\{\left(-\frac{1}{2}, \frac{1}{2}, 1\right)\right\}\right)$$

$$\mathcal{N}(A + 4I) = L\left(\left\{(1, 1, 0), (2, 0, 1)\right\}\right).$$

Note-se que os vectores de $\mathcal{N}(A - 2I)$ são ortogonais aos vectores de $\mathcal{N}(A + 4I)$. Para determinar uma base ortonormada de $\mathbb{R}^3$ formada só com vectores próprios de A basta aplicar o método de Gram-Schmidt ao conjunto

$$\left\{\left(-\frac{1}{2}, \frac{1}{2}, 1\right), (2, 0, 1), (1, 1, 0)\right\}.$$

Como, relativamente ao produto interno usual em $\mathbb{R}^3$,

$$\left\langle \left(-\frac{1}{2},\frac{1}{2},1\right),(2,0,1)\right\rangle = 0$$

basta substituir $(1,1,0)$ pelo vector: $(1,1,0)-\text{proj}_{\left(-\frac{1}{2},\frac{1}{2},1\right)}(1,1,0)-\text{proj}_{(2,0,1)}(1,1,0)=$

$$(1,1,0)-\frac{\left\langle(1,1,0),\left(-\frac{1}{2},\frac{1}{2},1\right)\right\rangle}{\left\langle\left(-\frac{1}{2},\frac{1}{2},1\right),\left(-\frac{1}{2},\frac{1}{2},1\right)\right\rangle}\left(-\frac{1}{2},\frac{1}{2},1\right)-\frac{\langle(1,1,0),(2,0,1)\rangle}{\langle(2,0,1),(2,0,1)\rangle}(2,0,1)=$$

$$=(1,1,0)-\frac{0}{\frac{3}{2}}\left(-\frac{1}{2},\frac{1}{2},1\right)-\frac{2}{5}(2,0,1)=\left(\frac{1}{5},1,-\frac{2}{5}\right).$$

Base ortonormada de $\mathbb{R}^3$ formada só com vectores próprios de A:

$$\left\{\frac{1}{\left\|\left(-\frac{1}{2},\frac{1}{2},1\right)\right\|}\left(-\frac{1}{2},\frac{1}{2},1\right),\frac{1}{\|(2,0,1)\|}(2,0,1),\frac{1}{\left\|\left(\frac{1}{5},1,-\frac{2}{5}\right)\right\|}\left(\frac{1}{5},1,-\frac{2}{5}\right)\right\}=$$

$$=\left\{\left(-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3}\right),\left(\frac{2\sqrt{5}}{5},0,\frac{\sqrt{5}}{5}\right),\left(\frac{\sqrt{30}}{30},\frac{\sqrt{30}}{6},-\frac{\sqrt{30}}{15}\right)\right\}.$$

Logo

$$P^T=S_{\mathcal{B}_{vp}\to\mathcal{B}_c}=\begin{bmatrix} -\frac{\sqrt{6}}{6} & \frac{2\sqrt{5}}{5} & \frac{\sqrt{30}}{30} \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{5}}{5} & -\frac{\sqrt{30}}{15} \end{bmatrix}$$

e

$$D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}=\underbrace{\begin{bmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} \\ \frac{2}{5}\sqrt{5} & 0 & \frac{1}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} & \frac{1}{6}\sqrt{30} & -\frac{1}{15}\sqrt{30} \end{bmatrix}}_{=P}\begin{bmatrix} -3 & -1 & -2 \\ -1 & -3 & 2 \\ -2 & 2 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} -\frac{\sqrt{6}}{6} & \frac{2\sqrt{5}}{5} & \frac{\sqrt{30}}{30} \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{5}}{5} & -\frac{\sqrt{30}}{15} \end{bmatrix}}_{=P^T}$$

4. $A=\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. Como

$$\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix}=-(\lambda-1)(\lambda-3)(\lambda-4),$$

os valores próprios de A são 1, 3 e 4. Como A é simétrica ($A=A^T$) e definida positiva uma vez que os valores próprios de A são todos positivos, então existe uma única raíz quadrada definida positiva B, isto é, existe uma única matriz simétrica B definida positiva tal que $A=B^2$. De facto, como A é ortogonalmente diagonalizável (por ser simétrica) tem-se

$$A=P^TDP=P^T(D')^2P=\left(P^TD'P\right)\left(P^TD'P\right)=BB=B^2$$

com

$$D=\begin{bmatrix}1&0&0\\0&3&0\\0&0&4\end{bmatrix}$$

Note-se que sendo os 3 valores próprios distintos, os correspondentes espaços próprios são ortogonais entre si. Base ordenada ortonormada de $\mathbb{R}^3$ formada só com vectores próprios de A:

$$\left\{\frac{1}{\|(1,2,1)\|}(1,2,1), \frac{1}{\|(-1,0,1)\|}(-1,0,1), \frac{1}{\|(1,-1,1)\|}(1,-1,1)\right\}=$$

$$=\left\{\left(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right),\left(-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right),\left(\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)\right\}.$$

Logo

$$P^T=S_{\mathcal{B}_{vp}\to\mathcal{B}_c}=\begin{bmatrix}\frac{\sqrt{6}}{6}&-\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{3}&0&-\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{6}&\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\end{bmatrix}.$$

Assim, com

$$D'=\begin{bmatrix}1&0&0\\0&\sqrt{3}&0\\0&0&2\end{bmatrix}$$

tem-se

$$B=P^TD'P=\begin{bmatrix}\frac{1}{2}\sqrt{3}+\frac{5}{6}&-\frac{1}{3}&\frac{5}{6}-\frac{1}{2}\sqrt{3}\\-\frac{1}{3}&\frac{4}{3}&-\frac{1}{3}\\\frac{5}{6}-\frac{1}{2}\sqrt{3}&-\frac{1}{3}&\frac{1}{2}\sqrt{3}+\frac{5}{6}\end{bmatrix}$$

definida positiva e

$$BB=A.$$

No entanto, se fizermos ou

$$D'=\begin{bmatrix}-1&0&0\\0&\sqrt{3}&0\\0&0&2\end{bmatrix}\quad\text{ou}\quad D'=\begin{bmatrix}1&0&0\\0&-\sqrt{3}&0\\0&0&2\end{bmatrix}$$

tem-se as 3 raízes quadradas seguintes, isto 3 matrizes B tais que $B^2=A$:

$$B=\begin{bmatrix}\frac{\sqrt{6}}{6}&-\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{3}&0&-\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{6}&\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&\sqrt{3}&0\\0&0&2\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{6}&-\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{3}&0&-\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{6}&\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\end{bmatrix}^T=\begin{bmatrix}\frac{1}{2}\sqrt{3}+\frac{5}{6}&-\frac{1}{3}&\frac{5}{6}-\frac{1}{2}\sqrt{3}\\-\frac{1}{3}&\frac{4}{3}&-\frac{1}{3}\\\frac{5}{6}-\frac{1}{2}\sqrt{3}&-\frac{1}{3}&\frac{1}{2}\sqrt{3}+\frac{5}{6}\end{bmatrix}$$

ou

$$B=\begin{bmatrix}\frac{\sqrt{6}}{6}&-\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{3}&0&-\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{6}&\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\end{bmatrix}\begin{bmatrix}-1&0&0\\0&\sqrt{3}&0\\0&0&2\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{6}&-\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{3}&0&-\frac{\sqrt{3}}{3}\\\frac{\sqrt{6}}{6}&\frac{\sqrt{2}}{2}&\frac{\sqrt{3}}{3}\end{bmatrix}^T=\begin{bmatrix}\frac{1}{2}\sqrt{3}+\frac{1}{2}&-1&\frac{1}{2}-\frac{1}{2}\sqrt{3}\\-1&0&-1\\\frac{1}{2}-\frac{1}{2}\sqrt{3}&-1&\frac{1}{2}\sqrt{3}+\frac{1}{2}\end{bmatrix}$$

ou

$$B=\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -\sqrt{3} & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}^T=\begin{bmatrix}\frac{5}{6}-\frac{1}{2}\sqrt{3} & -\frac{1}{3} & \frac{1}{2}\sqrt{3}+\frac{5}{6}\\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3}\\ \frac{1}{2}\sqrt{3}+\frac{5}{6} & -\frac{1}{3} & \frac{5}{6}-\frac{1}{2}\sqrt{3}\end{bmatrix}.$$

De facto:

$$\begin{bmatrix}\frac{1}{2}\sqrt{3}+\frac{5}{6} & -\frac{1}{3} & \frac{5}{6}-\frac{1}{2}\sqrt{3}\\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3}\\ \frac{5}{6}-\frac{1}{2}\sqrt{3} & -\frac{1}{3} & \frac{1}{2}\sqrt{3}+\frac{5}{6}\end{bmatrix}^2=\begin{bmatrix}\frac{1}{2}\sqrt{3}+\frac{1}{2} & -1 & \frac{1}{2}-\frac{1}{2}\sqrt{3}\\ -1 & 0 & -1\\ \frac{1}{2}-\frac{1}{2}\sqrt{3} & -1 & \frac{1}{2}\sqrt{3}+\frac{1}{2}\end{bmatrix}^2=\begin{bmatrix}\frac{5}{6}-\frac{1}{2}\sqrt{3} & -\frac{1}{3} & \frac{1}{2}\sqrt{3}+\frac{5}{6}\\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3}\\ \frac{1}{2}\sqrt{3}+\frac{5}{6} & -\frac{1}{3} & \frac{5}{6}-\frac{1}{2}\sqrt{3}\end{bmatrix}^2=$$

$$=\begin{bmatrix}3 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix}.$$

5.

$$Q:\mathbb{R}^3\to\mathbb{R}$$

$$Q(x_1,x_2,x_3)=3x_1^2-2x_1x_2+2x_2^2-2x_2x_3+3x_3^2=$$

$$=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}3 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=$$

$$\underset{3)}{=}\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}^T\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=$$

$$=\left(\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}^T\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\right)^T\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}^T\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=$$

$$=\begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

logo a forma quadrática é definida positiva pois os valores próprios de A são todos positivos. Fazendo a mudança de variável

$$\begin{bmatrix}\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{3} & 0 & -\frac{\sqrt{3}}{3}\\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3}\end{bmatrix}^T\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}.$$

Tem-se então a forma quadrática diagonal:

$$Q'(y_1,y_2,y_3)=y_1^2+3y_2^2+4y_3^2.$$

6. $A=\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$.

(i)

$$AA^H=\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}^H=\begin{bmatrix} 6 & 5+5i \\ 5-5i & 11 \end{bmatrix}$$

$$A^HA=\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}^H\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}=\begin{bmatrix} 6 & 5+5i \\ 5-5i & 11 \end{bmatrix}.$$

Logo A é normal.

(ii)

$$A=\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}=\begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}^H=A^H$$

logo A é hermitiana e em particular é normal:

$$AA^H=AA=A^HA.$$

(iii) Como A é hermitiana então é unitariamente diagonalizável (embora o recíproco não seja verdadeiro), isto é, existem uma matriz unitária U^H ($U^HU=UU^H=I$, isto é, $U^H=U^{-1}$) e uma matriz diagonal D tais que

$$D=UAU^H.$$

Note-se que: A normal $\Leftrightarrow$ A unitariamente diagonalizável.

Como

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 3-\lambda \end{vmatrix}=(\lambda-1)(\lambda-4),$$

os valores próprios de A são 1 e 4 e tem-se

$$\mathcal{N}(A-1I)=L(\{(-1-i,1)\})$$

$$\mathcal{N}(A-4I)=L\left(\left\{\left(\frac{1}{2}+\frac{1}{2}i,1\right)\right\}\right).$$

Note-se que os vectores de $\mathcal{N}(A-1I)$ são ortogonais aos vectores de $\mathcal{N}(A-4I)$. Logo, uma base ortonormada de $\mathbb{C}^2$ formada só com vectores próprios de A pode ser:

$$\left\{\frac{1}{\|(-1-i,1)\|}(-1-i,1),\frac{1}{\left\|\left(\frac{1}{2}+\frac{1}{2}i,1\right)\right\|}\left(\frac{1}{2}+\frac{1}{2}i,1\right)\right\}=$$

$$=\left\{\left(-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}i,\frac{\sqrt{3}}{3}\right),\left(\frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}i,\frac{\sqrt{6}}{3}\right)\right\}.$$

Logo

$$U^H=S_{\mathcal{B}_{vp}\to\mathcal{B}_c}=\begin{bmatrix} -\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}i & \frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}i \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix}$$

e

$$D=\begin{bmatrix}1&0\\0&4\end{bmatrix}=\underbrace{\begin{bmatrix}-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}i & \frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}i\\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3}\end{bmatrix}^H}_{=U}\begin{bmatrix}2&1+i\\1-i&3\end{bmatrix}\underbrace{\begin{bmatrix}-\frac{\sqrt{3}}{3}-\frac{\sqrt{3}}{3}i & \frac{\sqrt{6}}{6}+\frac{\sqrt{6}}{6}i\\ \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3}\end{bmatrix}}_{=U^H}.$$

7. Determine a solução de mínimos quadrados de $Au=b$, com

$$A=\begin{bmatrix}1&2\\2&4\\1&-2\end{bmatrix}\quad\text{e}\quad b=\begin{bmatrix}3\\2\\1\end{bmatrix}$$

calculando o correspondente vector erro de mínimos quadrados.

Tem-se $\operatorname{car}\left(A^TA\right)=\operatorname{car}A=2$ e como tal a solução de mínimos quadrados é única e dada por:

$$u=\left(A^TA\right)^{-1}A^Tb=$$

$$=\left(\begin{bmatrix}1&2&1\\2&4&-2\end{bmatrix}\begin{bmatrix}1&2\\2&4\\1&-2\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2&1\\2&4&-2\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6/5\\1/10\end{bmatrix}.$$

O vector erro de mínimos quadrados $b-Au$ é dado por:

$$b-Au=\begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}1&2\\2&4\\1&-2\end{bmatrix}\begin{bmatrix}6/5\\1/10\end{bmatrix}=\begin{bmatrix}\frac{8}{5}\\-\frac{4}{5}\\0\end{bmatrix}$$

sendo o erro de mínimos quadrados dado por:

$$\|b-Au\|=\left\|\left(\frac{8}{5},-\frac{4}{5},0\right)\right\|=\frac{4}{5}\sqrt{5}.$$

8. (i)

$$\begin{cases}u_1'=-2u_3\\u_2'=u_1+2u_2+u_3\\u_3'=u_1+3u_3\end{cases}$$

$$\begin{cases}u_1'=-2u_3\\u_2'=u_1+2u_2+u_3\\u_3'=u_1+3u_3\end{cases}\Leftrightarrow\begin{bmatrix}u_1'(t)\\u_2'(t)\\u_3'(t)\end{bmatrix}=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}\begin{bmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{bmatrix}.\ (*)$$

1 e 2 são os valores próprios da matriz $A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}$, sendo os espaços próprios dados por:

$$\mathcal{N}(A-1I)=L\left(\{(-2,1,1)\}\right)$$

$$\mathcal{N}\left(A-2I\right)=L\left(\left\{\left(0,1,0\right),\left(-1,0,1\right)\right\}\right).$$

Como existe uma base de $\mathbb{R}^3$ formada só por vectores próprios: $\left\{\left(0,1,0\right),\left(-1,0,1\right),\left(-2,1,1\right)\right\}$ então a matriz $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ é diagonalizável. Assim, fazendo

$$P^{-1}=\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{tem-se} \quad P=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

e

$$\Leftrightarrow D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}=P\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}P^{-1}\Leftrightarrow$$

$$\Leftrightarrow\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}=P^{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}P.$$

o sistema (*) é equivalente a

$$\begin{bmatrix} u_1'\left(t\right) \\ u_2'\left(t\right) \\ u_3'\left(t\right) \end{bmatrix}=\left(P^{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}P\right)\begin{bmatrix} u_1\left(t\right) \\ u_2\left(t\right) \\ u_3\left(t\right) \end{bmatrix}\Leftrightarrow$$

$$\Leftrightarrow P\begin{bmatrix} u_1'\left(t\right) \\ u_2'\left(t\right) \\ u_3'\left(t\right) \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(P\begin{bmatrix} u_1\left(t\right) \\ u_2\left(t\right) \\ u_3\left(t\right) \end{bmatrix}\right)$$

Assim, considerando a mudança de variável

$$\begin{bmatrix} v_1\left(t\right) \\ v_2\left(t\right) \\ v_3\left(t\right) \end{bmatrix}=P\begin{bmatrix} u_1\left(t\right) \\ u_2\left(t\right) \\ u_3\left(t\right) \end{bmatrix}$$

tem-se

$$\begin{bmatrix} v_1'\left(t\right) \\ v_2'\left(t\right) \\ v_3'\left(t\right) \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(P\begin{bmatrix} v_1\left(t\right) \\ v_2\left(t\right) \\ v_3\left(t\right) \end{bmatrix}\right)\Leftrightarrow$$

$$\Leftrightarrow\left\{\begin{array}{l} v_1'\left(t\right)=2v_1\left(t\right) \\ v_2'\left(t\right)=2v_2\left(t\right) \\ v_3'\left(t\right)=v_3\left(t\right) \end{array}\right. \underset{\substack{\text{Se } v_1\neq\mathbf{0}, \\ v_2\neq\mathbf{0} \text{ e } v_3\neq\mathbf{0}}}{\Leftrightarrow}\left\{\begin{array}{l} \dfrac{v_1'\left(t\right)}{v_1\left(t\right)}=2 \\ \dfrac{v_2'\left(t\right)}{v_2\left(t\right)}=2 \\ \dfrac{v_3'\left(t\right)}{v_3\left(t\right)}=1 \end{array}\right.\Leftrightarrow$$

$$\Leftrightarrow\left\{\begin{array}{l} \log\left|v_1\left(t\right)\right|=2t+k_1 \\ \log\left|v_2\left(t\right)\right|=2t+k_2 \\ \log\left|v_3\left(t\right)\right|=t+k_2 \end{array}\right.\Leftrightarrow\left\{\begin{array}{l} v_1\left(t\right)=c_1e^{2t} \\ v_2\left(t\right)=c_2e^{2t} \\ v_3\left(t\right)=c_3e^{t} \end{array}\right.$$

com $c_1, c_2, c_3 \in \mathbb{R}$. De facto, se $u(t)$ fôr solução de $u'(t) = \alpha u(t)$ então $u(t)e^{-\alpha t} = c$ (constante) uma vez que $\left(u(t)e^{-\alpha t}\right)' = 0$. Logo $u(t) = ce^{\alpha t}$.

Assim, a solução geral do sistema de equações diferenciais lineares é dada por:

$$\begin{aligned}\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} &= P^{-1}\begin{bmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{t} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{t} \end{bmatrix} = \begin{bmatrix} -c_2e^{2t} - 2c_3e^{t} \\ c_3e^{t} + c_1e^{2t} \\ c_2e^{2t} + c_3e^{t} \end{bmatrix} = \\ &= c_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}e^{t}.\end{aligned}$$

(ii) Com $\begin{cases} u_1(0) = 2 \\ u_2(0) = -1 \\ u_3(0) = 0 \end{cases}$ tem-se

$$\begin{bmatrix} -c_2 - 2c_3 \\ c_1 + c_3 \\ c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

então a única solução geral do sistema de equações diferenciais lineares que satisfaz $\begin{cases} u_1(0) = 2 \\ u_2(0) = -1 \\ u_3(0) = 0 \end{cases}$ é dada por

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} -2e^{t} + 4e^{t} \\ -2e^{t} + e^{2t} \\ 2e^{t} - 2e^{t} \end{bmatrix} = u_1(0)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{2t} + u_2(0)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{2t} + u_3(0)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}e^{t}.$$