Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

18 de Outubro de 2004
Duração: 45 min.
Teste A

Resolução do 1º TESTE DE ÁLGEBRA LINEAR

CURSOS: Lic. Eng. Geológica e Mineira, Lic. Eng. de Materiais e Lic. Eng. Mecânica

1) a)

$$A_\alpha = \begin{bmatrix} 1 & \alpha & \alpha \\ -2 & 1 & 2 \\ 3 & -2-\alpha & -1 \end{bmatrix} \underset{\substack{2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \begin{bmatrix} 1 & \alpha & \alpha \\ 0 & 1+2\alpha & 2+2\alpha \\ 0 & -2-4\alpha & -1-3\alpha \end{bmatrix} \underset{2L_2+L_3\to L_3}{\longrightarrow}$$

$$\longrightarrow \begin{bmatrix} 1 & \alpha & \alpha \\ 0 & 1+2\alpha & 2+2\alpha \\ 0 & 0 & 3+\alpha \end{bmatrix}.$$

Logo, car $A_\alpha = 3$ se e só se $\alpha \neq -\frac{1}{2}$ e $\alpha \neq -3$.

A matriz $A_\alpha = 3$ é invertível se e só se $\alpha \neq -\frac{1}{2}$ e $\alpha \neq -3$.

car $A_\alpha = 2$ se e só se $\alpha = -\frac{1}{2}$ ou $\alpha = -3$.

b) Para $\alpha = 0$,

$$[A_{-1} \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 1 & 0 \\ 3 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{2L_1+L_2\to L_2 \\ -3L_1+L_3\to L_3}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right] \underset{2L_2+L_3\to L_3}{\longrightarrow}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & 1 \end{array}\right] \underset{-\frac{2}{3}L_3+L_2\to L_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4/3 & -1/3 & -2/3 \\ 0 & 0 & 3 & 1 & 2 & 1 \end{array}\right] \underset{\substack{-L_2\to L_2 \\ -\frac{1}{3}L_3\to L_3}}{\longrightarrow}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4/3 & -1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 2/3 & 1/3 \end{array}\right].$$

Logo,

$$A_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & -1/3 & -2/3 \\ 1/3 & 2/3 & 1/3 \end{bmatrix}.$$

c) $A_0 X = B \Longleftrightarrow X = A_0^{-1}B \Longleftrightarrow X = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & -1/3 & -2/3 \\ 1/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$

d) Com $E_{12}(2) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{13}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ e $E_{23}(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ tem-se

$$E_{23}(2)E_{13}(-3)E_{12}(2)A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Logo,

$$A_0 = LDU, \quad \text{com} \quad L = (E_{12}(2))^{-1}(E_{13}(-3))^{-1}(E_{23}(2))^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

e)

$$\mathcal{N}(A_{-3}) = \left\{u \in \mathbb{R}^3 : A_{-3}u = \mathbf{0}\right\}.$$

$$A_{-3}u = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 1 & -3 & -3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \text{ (Método de eliminação de Gauss).}$$

$$\begin{cases} x - 3y - 3z = 0 \\ \\ -5y - 4z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{3}{4}y \\ \\ z = -\frac{5}{4}y. \end{cases}$$

Logo,

$$\mathcal{N}(A_{-3}) = \left\{\left(-\frac{3}{4}y, y, -\frac{5}{4}y\right) : y \in \mathbb{R}\right\} = L\left(\left\{\left(-\frac{3}{4}, 1, -\frac{5}{4}\right)\right\}\right).$$

f) Solução geral de $A_{-3}X = C$ é igual a:

$$\text{(Solução particular de } A_{-3}X = C) + \text{(Solução geral de } A_{-3}X = \mathbf{0}).$$

O vector $(1, 0, 0)$ é uma solução particular de $A_{-3}X = C$. Seja S o conjunto solução (ou solução geral) de $A_{-3}X = C$, isto é,

$$S = \{X : A_{-3}X = C\}$$

Tem-se

$$S = \{(1,0,0)\} + \mathcal{N}(A_{-3}) = \{(1,0,0)\} + \left\{\left(-\frac{3}{4}\lambda, \lambda, -\frac{5}{4}\lambda\right) : \lambda \in \mathbb{R}\right\} = \left\{\left(1 - \frac{3}{4}\lambda, \lambda, -\frac{5}{4}\lambda\right) : \lambda \in \mathbb{R}\right\}.$$

2)

$$U = \left\{a_0 + a_1t + a_2t^2 \in P_2 : a_0 - 2a_1 + 3a_2 = 0\right\} = \left\{(2a_1 - 3a_2) + a_1t + a_2t^2 : a_1, a_2 \in \mathbb{R}\right\} =$$

$$= \left\{a_1(2 + t) + a_2\left(-3 + t^2\right) : a_1, a_2 \in \mathbb{R}\right\} = L\left(\left\{2 + t, -3 + t^2\right\}\right).$$

3) Em $\mathbb{R}^3$, considere os subespaços:

$$U = L(\{(-1,0,1), (2,2,2)\}) \quad \text{e} \quad V = L(\{(-2,1,2), (3,1,1)\}).$$

Seja $v \in U$, então

$$v = \alpha(-1,0,1) + \beta(2,2,2) = (-\alpha + 2\beta, 2\beta, \alpha + 2\beta),$$

com $\alpha, \beta \in \mathbb{R}$. Para que v esteja também em V é preciso que:

$$\begin{aligned}(-\alpha + 2\beta, 2\beta, \alpha + 2\beta) &= \lambda(-2,1,2) + \mu(3,1,1) = \\ &= (-2\lambda + 3\mu, \lambda + \mu, 2\lambda + \mu)\,,\end{aligned}$$

com $\lambda, \mu \in \mathbb{R}$. Deste modo,

$$\begin{cases} -\alpha + 2\beta = -2\lambda + 3\mu \\ \\ 2\beta = \lambda + \mu \\ \\ \alpha + 2\beta = 2\lambda + \mu. \end{cases}$$

Considerando a matriz aumentada tem-se

$$\left[\begin{array}{cccc} -1 & 2 & | & -2\lambda + 3\mu \\ 0 & 2 & | & \lambda + \mu \\ 1 & 2 & | & 2\lambda + \mu \end{array}\right] \underset{L_1 + L_3 \to L_3}{\longrightarrow} \left[\begin{array}{cccc} -1 & 2 & | & -2\lambda + 3\mu \\ 0 & 2 & | & \lambda + \mu \\ 0 & 4 & | & 4\mu \end{array}\right] \underset{-2L_2 + L_3 \to L_3}{\longrightarrow}.$$

$$\longrightarrow \left[\begin{array}{cccc} -1 & 2 & | & -2\lambda + 3\mu \\ 0 & 2 & | & \lambda + \mu \\ 0 & 0 & | & -2\lambda + 2\mu \end{array}\right].$$

Logo,

$$\begin{cases} -\alpha + 2\beta = -2\lambda + 3\mu \\ \\ 2\beta = \lambda + \mu \\ \\ 0 = -2\lambda + 2\mu. \end{cases} \iff \begin{cases} \alpha = \mu \\ \\ \beta = \mu \\ \\ \lambda = \mu. \end{cases}$$

Assim,

$$\alpha(-1,0,1) + \beta(2,2,2) = \mu(-2,1,2) + \mu(3,1,1) = (\mu, 2\mu, 3\mu) = \mu(1,2,3).$$

Logo,

$$U \cap V = \{(\mu, 2\mu, 3\mu) : \mu \in \mathbb{R}\} = \{\mu(1,2,3) : \mu \in \mathbb{R}\} = L\left(\{(1,2,3)\}\right).$$

O vector $(-1,0,1) \notin U \cap V$, pois não existe $\mu \in \mathbb{R}$ tal que

$$(-1,0,1) = \mu(1,2,3).$$