Cálculo Diferencial e Integral II - Tagus Park **1o. Semestre 2009/2010**
3o. Teste **19/Dezembro/2009**
Justifique as suas respostas

1. **(3 val.)** Calcule caso exista

$$\int\int_D \frac{dxdy}{\sqrt{xy}} \qquad \text{onde} \qquad D = [0,1]\times[0,1]$$

Seja $0 < \epsilon < 1$ e $D_\epsilon = [\epsilon, 1]\times[\epsilon, 1]$.

$$\int\int_{D_\epsilon} \frac{dxdy}{\sqrt{xy}} = \int_\epsilon^1 dx x^{-1/2} \int_\epsilon^1 dy y^{-1/2} = \frac{1}{1/2}\Big[\sqrt{x}\Big]_{x=\epsilon}^{x=1} \cdot \frac{1}{1/2}\Big[\sqrt{y}\Big]_{y=\epsilon}^{y=1} =$$

$$= 2(1-\sqrt{\epsilon})\cdot 2(1-\sqrt{\epsilon}) \underset{\epsilon\mapsto 0}{\longrightarrow} 4$$

2. **(3 val.)** Calcule o integral de caminho $\int_{\mathbf{c}} f\, ds$ onde $f(x,y,z) = e^{\sqrt{z}}$ com $\mathbf{c}(t) = (1, 2, t^2)$, $t \in [0,1]$

$$||\mathbf{c}'(t)|| = \sqrt{0^2 + 0^2 + (2t)^2} = 2t \qquad\qquad f(\mathbf{c}(t)) = e^{\sqrt{t^2}} = e^t$$

$$\int_{\mathbf{c}} f\, ds = \int_0^1 e^t 2t dt = \cdots$$

$$\mathbf{P}te^t = te^t - \mathbf{P}e^t = te^t - e^t =$$

$$\cdots = 2\Big[te^t - e^t\Big]_0^1 = 2(1\cdot e^1 - e^1 - 0e^0 + e^0) = 2$$

3. **(3 val.)** Considere o campo da força gravitacional (com $G = M = m = 1$) dado por

$$\mathbf{F}(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \qquad (x,y,z) \neq (0,0,0)$$

Calcule o trabalho realizado por este campo quando uma partícula se desloca de (x_1, y_1, z_1) para (x_2, y_2, z_2).

$$f(x,y,z) = -\int dx\, \frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{1}{2}\int dx\,(2x)(x^2+y^2+z^2)^{-3/2} =$$
$$= -\frac{1}{2}\frac{1}{-\frac{3}{2}+1}(x^2+y^2+z^2)^{-3/2+1} = \frac{1}{\sqrt{x^2+y^2+z^2}} + f_1(y,z)$$

analogamente

$$f(x,y,z) = -\int dy\, \frac{y}{(x^2+y^2+z^2)^{3/2}} = \cdots = \frac{1}{\sqrt{x^2+y^2+z^2}} + f_2(x,z)$$

e

$$f(x,y,z) = -\int dz\, \frac{z}{(x^2+y^2+z^2)^{3/2}} = \cdots = \frac{1}{\sqrt{x^2+y^2+z^2}} + f_3(x,y)$$

donde a menos de uma constante aditiva numérica:

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

e portanto

$$W_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} = f(x_2,y_2,z_2) - f(x_1,y_1,z_1) = \frac{1}{\sqrt{x_2^2+y_2^2+z_2^2}} - \frac{1}{\sqrt{x_1^2+y_1^2+z_1^2}}$$

4. **(3 val.)** Calcule a área da superfície de uma esfera de raio $R > 0$.

$$\begin{aligned} x &= R\cos\theta\sin\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\phi \end{aligned}$$

$$\begin{aligned} T_\theta &= \frac{\partial x}{\partial \theta}\mathbf{i} + \frac{\partial y}{\partial \theta}\mathbf{j} + \frac{\partial z}{\partial \theta}\mathbf{k} = -R\sin\theta\sin\phi\,\mathbf{i} + R\cos\theta\sin\phi\,\mathbf{j} \\ T_\phi &= \frac{\partial x}{\partial \phi}\mathbf{i} + \frac{\partial y}{\partial \phi}\mathbf{j} + \frac{\partial z}{\partial \phi}\mathbf{k} = R\cos\theta\cos\phi\,\mathbf{i} + R\sin\theta\cos\phi\,\mathbf{j} - R\sin\phi\,\mathbf{k} \end{aligned}$$

$$\begin{aligned} ||T_\theta \times T_\phi|| &= \left|\left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \end{vmatrix} \right|\right| = \\ &= ||(-\mathbf{i}R^2\cos\theta\sin^2\phi - \mathbf{j}R^2\sin\theta\sin^2\phi - \mathbf{k}R^2\sin\phi\cos\phi)|| = \\ &= \sqrt{R^4\cos^2\theta\sin^4\phi + R^4\sin^2\theta\sin^4\phi + R^4\sin^2\phi\cos^2\phi} = \\ &= R^2\sin\phi\sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi} = \cdots = R^2\sin\phi \end{aligned}$$

A área da esfera é então

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\phi R^2 \sin\phi = R^2 2\pi \Big[-\cos\phi \Big]_0^{\pi} = 4\pi R^2$$

5. **(3 val.)** Calcule

$$\int\int_S z\, dS$$

onde S é o triângulo com vértices $(1,0,0), (0,2,0), (0,1,1)$.

$$\mathbf{u} = (0,2,0) - (1,0,0) = (-1,2,0) \qquad \mathbf{v} = (0,1,1) - (0,2,0) = (0,-1,1)$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$0 = (2,1,1)\cdot(x-1, y-0, z-0) = 2(x-1) + y + z \qquad \text{isto é,} \qquad z = -2x - y + 2$$

$$T_x = \frac{\partial x}{\partial x}\mathbf{i} + \frac{\partial y}{\partial x}\mathbf{j} + \frac{\partial z}{\partial x}\mathbf{k} = \mathbf{i} - 2\mathbf{k}$$
$$T_y = \frac{\partial x}{\partial y}\mathbf{i} + \frac{\partial y}{\partial y}\mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} - \mathbf{k}$$

$$||T_x \times T_y|| = || \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} || = ||2\mathbf{i} + \mathbf{j} + \mathbf{k}|| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

O integral é então:

$$\int_0^1 dx \int_0^{-2x+2} dy \sqrt{6}(-2x+2-y) = \int_0^1 dx \sqrt{6}\Big[(-2x+2)y - \frac{1}{2}y^2\Big]_0^{-2x+2} =$$
$$= \int_0^1 dx \sqrt{6}\Big[(-2x+2)(-2x+2) - \frac{1}{2}(-2x+2)^2\Big] = \int_0^1 dx \sqrt{6}\frac{1}{2}(-2x+2)^2 =$$
$$= \sqrt{6}\frac{1}{2}\frac{1}{-2}\frac{1}{3}\Big[(-2x+2)^3\Big]_0^1 = -\sqrt{6}\frac{1}{12}\Big(-2^3\Big) = \frac{2}{3}\sqrt{6}$$

6. **(3 val.)** Calcule a área da figura plana delimitada por uma elipse de semi-eixos a e b.

Pelo teorema de Green a área em causa é, com $x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \leq \theta 2\pi$:

$$\frac{1}{2}\int_{\partial E}(xdy - ydx) = \frac{1}{2}\int_0^{2\pi} d\theta(a\cos\theta b\sin\theta - b\sin\theta(-a\cos\theta)) =$$
$$= \frac{1}{2}\int_0^{2\pi} d\theta(ab\cos^2\theta + ab\sin^2\theta) = \frac{ab}{2}\int_0^{2\pi} d\theta = \frac{ab}{2}2\pi = \pi ab$$

7. **(2 val.)** Verifique o teorema de Stokes para o hemisfério dado pela equação $z = \sqrt{1 - x^2 - y^2}$ e o campo vectorial $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Pelo Teorema de Stokes

$$\int\int_S rot\mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

$$rot\mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \cdots = \mathbf{0}$$

portanto

$$\int\int_S \mathbf{0} \cdot d\mathbf{S} = 0$$

Por outro lado, sobre ∂S, com $x = \cos\theta, y = \sin\theta, z = 0, \quad 0 \leq \theta \leq 2\pi$

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} d\theta(\cos\theta, \sin\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) =$$
$$= \int_0^{2\pi} d\theta(-\cos\theta\sin\theta + \sin\theta\cos\theta + 0) = \int_0^{2\pi} d\theta 0 = 0$$

Já que ambos os integrais valem 0, verifica-se o Teorema de Stokes para este campo $\mathbf{F}$ e para esta superfície S.