Partial Differential Equations

First Exam - June 17, 2010

Duration: 3 hours

Show all computations

1. Using the method of characteristics, determine all solutions of the first order nonlinear equation (3)

$$u + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 = 1$$

which satisfy $u(x, 0) = 0$.

2. Using the Hopf-Lax formula, determine a solution of the first order equation (2)

$$u_t + \frac{1}{2}u_x^2 = 0$$

which satisfies $u(x, 0) = x^2$. Sketch the projection of the characteristics on the (x, t) plane.

3.

a) Suppose $g \in C([0, 1])$. Consider the second order ordinary differential equation (3)

$$-(r^2 v'(r))' = g(r),$$

for $r \in]0, 1[$, with boundary conditions $v'(0) = 0$, $v(1) = 0$. Integrating this equation twice, determine the continuous function $G : [0, 1]^2 \setminus \{(0, 0)\} \to \mathbb{R}$ such that

$$v(r) = \int_0^1 G(r, s)g(s)\,ds.$$

What equations does G satisfy?

b) Let $f \in C([0, 1])$. Consider Poisson's equation in $B_1(0)$, the unit ball in $\mathbb{R}^3$ centered at the origin, (2)

$$-\Delta u\,(x) = f(|x|),$$

with $u = 0$ on $\partial B_1(0)$. Use your answer to the previous question to determine a radial solution of this equation. Might this equation have a nonradial solution?

c) What is the relation between the function $G(r, s)$ and Green's function $\mathcal{G}(\xi, x)$ for the Dirichlet Laplacian on $B_1(0)$? (1)

4. Let $h \in C_c(\mathbb{R}^n \times \mathbb{R}_0^+)$. Consider the non-homogeneous heat equation in $n \geq 1$ spatial dimensions (3)

$$(u_t - \Delta u)(x, t) = h(x, t)$$

with initial condition $u(x, 0) = 0$. Solve using the Fourier transform and recover Duhamel's formula. Note: you can use the fact that the Fourier transform of the convolution is the product of the Fourier transforms.

5. Let $c > 0$ and $g \in C^2([0, +\infty[)$. Consider the wave equation in 3 spatial dimensions (3)

$$u_{tt} = c^2 \Delta u$$

with initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = g(|x|)$. Compute $u(0, t)$.

6. Let $d\mu$ be the surface measure of the unit sphere in $\mathbb{R}^3$.

a) Using the Fourier transform, prove $d\mu \in H^{-1}(\mathbb{R}^3)$. (2)
b) Confirm using the Trace Theorem. (1)