Partial Differential Equations

Second Exam - July 7, 2010

Duration: 3 hours

Show all computations

1. Consider the second order pde (2)

$$u_{tt} = u_x + u_t$$

with initial conditions $u(x,0) = x(3+x)$ and $u_t(x,0) = -1 - 2x$. Compute the solution by forming its Taylor series about the origin, as in the proof of the Cauchy-Kovalevski Theorem.

2. Consider the first order linear pde

$$u_x + \frac{y}{2} u_y = u.$$

a) Using the method of characteristics, determine the solution u that satisfies $u(x, e^x) = -1$. (2)

b) Sketch the projections of the characteristics in the (x, y) plane. (1)

c) Check that (1)

$$\phi(x, y, a, b) = ae^x + bye^{\frac{x}{2}}$$

is a complete integral.

d) Let $x_0 \in \mathbb{R}$. Determine the pair (a, b) such that the graph of $(x, y) \mapsto \phi(x, y, a, b)$ is tangent to the curve parametrized by $x \mapsto (x, e^x, -1)$ at $(x_0, e^{x_0}, -1)$. Determine the corresponding solution. (2)

e) Use your answer to **d)** to recover the answer to **a)**. (1)

f) Use the complete integral to obtain, once again, the characteristics. (1)

3. Let $m > 0$. Consider the Klein-Gordon equation in one spatial dimension

$$v_{tt} = v_{rr} - m^2 v.$$

Introduce the function $w(r, s, t) = \cos(ms)v(r, t)$.

a) Write down a pde satisfied by w and the initial conditions in terms of $F(r) = v(r, 0)$ and $G(r) = v_t(r, 0)$. (1)

b) Assume $F = 0$ and G is of class C^2. Determine a formula for w. Simplify it and determine a formula for v. (3)

Consider now the Klein-Gordon equation

$$u_{tt} = \Delta u - m^2 u$$

in 3 spatial dimensions. Let $M_u(x, r, t)$ be the spherical mean of $u(\,\cdot\,, t)$ on the sphere of radius r centered at x.

c) Write down a pde satisfied by M_u and prove it. You can use the formulas deduced in class. Write the initial conditions of the pde supposing $u(x, 0) = 0$ and $u_t(x, 0) = g(x)$, with g of class C^2. (2)

d) Without actually carrying out the computations, explain how one could obtain a formula for u. (1)

4. Let $d\mu$ be the surface measure of the unit sphere in $\mathbb{R}^3$.

a) Determine a radial solution $u \in H^1(\mathbb{R}^3)$ of (2)

$$-\Delta u + u = d\mu.$$

b) Does the differential equation have other solutions in $H^1(\mathbb{R}^3)$? Justify your answer. (1)