Existence and nonexistence of least energy solutions of the Neumann problem for a semilinear elliptic equation with critical Sobolev exponent and a critical lower-order perturbation

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Let Ω be a smooth bounded domain in $\mathcal{R}^N$, with $N \geq 5$, $a > 0$, $\alpha \geq 0$ and $2^* = \frac{2N}{N-2}$. We show that the the exponent $q = \frac{2(N-1)}{N-2}$ plays a critical role regarding the existence of least energy (or ground state) solutions of the Neumann problem

$$\begin{cases} -\Delta u + au = u^{2^*-1} - \alpha u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Namely, we prove that when $q = \frac{2(N-1)}{N-2}$ there exists an $\alpha_0 > 0$ such that the problem has a least energy solution if $\alpha < \alpha_0$ and has no least energy solution if $\alpha > \alpha_0$.

Key Words: Neumann problem, critical Sobolev exponent, least energy solutions, inequalities

1. INTRODUCTION

Let Ω be a smooth bounded domain in $\mathcal{R}^N$, with $N \geq 5$, $a > 0$ and $\alpha \geq 0$. Let $2^* = \frac{2N}{N-2}$ be the critical exponent for the Sobolev embedding

*Partially supported by FCT (Portugal).

$H^1(\Omega) \subset L^q(\Omega)$ and $2^\# = \frac{2(N-1)}{N-2}$. We consider the problem

$$\begin{cases} -\Delta u + au = u^{2^*-1} - \alpha u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathcal{P}_{\alpha,q})$$

We regard a as fixed and α as a parameter. From Theorem 3.2 of [18], due to X.J. Wang, we know that if $2 < q < 2^\#$, then problem $(\mathcal{P}_{\alpha,q})$ has a *least energy* solution for all values of $\alpha \geq 0$. (Wang's result actually holds for $N \geq 3$.) A question that naturally arises is the following: what happens for $q = 2^\#$?

It is well known that the solutions of $(\mathcal{P}_{\alpha,q})$ correspond to critical points of the functional $\Phi_\alpha : H^1(\Omega) \to \mathcal{R}$, defined by

$$\Phi_\alpha(u) := \frac{1}{2}|\nabla u|_2^2 + \frac{a}{2}|u|_2^2 + \frac{\alpha}{q}|u|_q^q - \frac{1}{2^*}|u|_{2^*}^{2^*},$$

where $|u|_p$ denotes the L^p norm of u in Ω. We recall that a *least energy* solution is a function $u \in H^1(\Omega)$ such that

$$\Phi_\alpha(u) = \inf_{\mathcal{N}} \Phi_\alpha.$$

The set $\mathcal{N}$ is the Nehari manifold, $\mathcal{N} := \{u \in H^1(\Omega) : \Phi'_\alpha(u)u = 0,\, u \neq 0\}$. It is interesting to note that when $q = 2^\#$ it is possible to determine explicitly the function $\Phi_\alpha|_{\mathcal{N}}$ by solving a quadratic equation. We take full advantage of this fact.

We recall that the infimum

$$S := \inf \left\{ \frac{\int_{\mathcal{R}^N} |\nabla u|^2}{\left(\int_{\mathcal{R}^N} |u|^{2^*}\right)^{2/2^*}} \,\middle|\, u \in L^{2^*}(\mathcal{R}^N), \nabla u \in L^2(\mathcal{R}^N), u \neq 0 \right\}$$

is achieved by the Talenti instanton $U(x) := \left(\frac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}}$. For $\varepsilon > 0$ and $y \in \mathcal{R}^N$, we define $U_{\varepsilon,y} := \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon}\right)$.

Heuristically, we can summarize the main idea behind the analysis of problem $(\mathcal{P}_{\alpha,q})$, when $q = 2^\#$, as follows. There exists an $\alpha_0 \in]0, +\infty]$ such that $\inf_{\mathcal{N}} \Phi_\alpha < \frac{S^{\frac{N}{2}}}{2N}$ for $\alpha < \alpha_0$, and $\inf_{\mathcal{N}} \Phi_\alpha = \frac{S^{\frac{N}{2}}}{2N}$ for $\alpha \geq \alpha_0$. If $\alpha < \alpha_0$, then $(\mathcal{P}_{\alpha,2^\#})$ has a least energy solution whereas, if $\alpha > \alpha_0$, then $(\mathcal{P}_{\alpha,2^\#})$ does not have a least energy solution. Suppose $\alpha_0 = +\infty$, so that there exist least energy solutions for all $\alpha \geq 0$. We choose a sequence $\alpha_k \to +\infty$ as $k \to +\infty$ and denote by u_k a corresponding sequence of least energy solutions. Then there would exist a sequence of positive numbers ε_k

converging to zero, and a sequence of points $P_k \in \partial\Omega$, such that, modulo a subsequence, $P_k \to P$ and $|\nabla(u_k - U_{\varepsilon_k,P_k})|_2 \to 0$, as $k \to +\infty$. We can use $\Phi_{\alpha_k}(U_{\varepsilon_k,P_k})$ to estimate $\Phi_{\alpha_k}(u_k)$ from below with an error that is $o(\alpha_k\varepsilon_k)$. However, from Adimurthi and Mancini [1] and X.J. Wang [18], we have the estimate

$$\Phi_{\alpha_k}(U_{\varepsilon_k,P_k}) = \frac{S^{\frac{N}{2}}}{2N} - \frac{S^{\frac{N}{2}}}{2}H(P_k)A(N)\varepsilon_k + \frac{1}{2}B(N)\alpha_k\varepsilon_k + o(\alpha_k\varepsilon_k),$$

where $A(N)$ and $B(N)$ are positive constants that only depend on N, and $H(P_k)$ is the mean curvature of $\partial\Omega$ at P_k with respect to the unit outward normal. This lower bound is greater than $\frac{S^{\frac{N}{2}}}{2N}$, for large k. This contradicts the hypothesis that $\alpha_0 = +\infty$.

It is somewhat delicate to justify the use of $\Phi_{\alpha_k}(U_{\varepsilon_k,P_k})$ to estimate $\Phi_{\alpha_k}(u_k)$ from below. This was first done by Adimurthi, Pacella and Yadava in [2], who treated the case where $\alpha = 0$. The argument involves an expansion to second order of the energy at U_{ε_k,P_k} and a comparison of the eigenvalues of the linearized problem at U_{ε_k,P_k} with the eigenvalues of a limiting problem.

The present analysis builds on the work [2] of Adimurthi, Pacella and Yadava, which we will frequently refer to as [APY]. Of course, the works of Talenti [17], Brézis and Nirenberg [9], P.L. Lions [15], Adimurthi and Mancini [1], and X.J. Wang [18] are also of major importance for our study.

Our main result is the following

Theorem. *Let Ω be a smooth bounded domain in $\mathcal{R}^N$, with $N \geq 5$, $a > 0$ and $\alpha \geq 0$. There exists a positive real number $\alpha_0 = \alpha_0(a,\Omega)$ such that*

(i) if $\alpha < \alpha_0$, then problem $(\mathcal{P}_{\alpha,2^\#})$ has a least energy solution;
(ii) if $\alpha > \alpha_0$, then problem $(\mathcal{P}_{\alpha,2^\#})$ has no least energy solution.

We remark that this result contrasts with Theorem 3.2 of [18], referred to above. Also, from this theorem we deduce an inequality (see (15)) which implies Aubin's inequality (16) (see [6] and Cherrier [11]).

We should mention that for any pair (a,α) (with $a > 0$ and $\alpha \geq 0$) problem $(\mathcal{P}_{\alpha,2^\#})$ has the constant solution

$$u = \kappa := \left(\tfrac{\alpha+\sqrt{\alpha^2+4a}}{2}\right)^{\frac{N-2}{2}}.$$

The energy of this solution is

$$\Phi_\alpha(\kappa) = \tfrac{|\Omega|}{2^\# N}\left[\left(\tfrac{\alpha+\sqrt{\alpha^2+4a}}{2}\right)^N + \tfrac{2^*}{2}a\left(\tfrac{\alpha+\sqrt{\alpha^2+4a}}{2}\right)^{N-2}\right],$$

where $|\Omega|$ denotes the N-dimensional Lebesgue measure of Ω. It follows that for $a > 0$ and $\alpha \geq 0$ sufficiently small, namely for $a \leq S/(2|\Omega|)^{\frac{2}{N}}$ and α such that $\Phi_\alpha(\kappa) \leq S^{\frac{N}{2}}/(2N)$, then the least energy solutions might be constant.

When the domain Ω is a ball and a is small, Adimurthi and Yadava [3] proved that $(\mathcal{P}_{0,2^\#})$ has more than one solution for $N = 4$, 5 and 6. However, when $N = 3$ a uniqueness result was proved by M. Zhu in [21] for convex domains, $\alpha = 0$ and small a.

Other works in the spirit of ours are those of Brézis and Lieb [8], Adimurthi and Yadava [4], M. Zhu [20], Z.Q. Wang [19] and Chabrowski and Willem [10].

The organization of this work is as follows. In Section 2 we give the setup of our work and the statement of the main result. In Section 3 we prove existence of least energy solutions. We then assume that the value α_0 is infinite and analyze the asymptotic behavior of the least energy solutions as $\alpha \to +\infty$. In Section 4 we prove nonexistence of least energy solutions. In Section 5 we give a lower bound for α_0 and, using the ideas of Chabrowski and Willem [10], give partial results concerning existence of least energy solutions for $\alpha = \alpha_0$. In Appendix A we check that the Nehari set $\mathcal{N}$ is a manifold and a natural constraint for Φ_α, we derive expressions for $\Phi_\alpha|_{\mathcal{N}}$, and we derive upper and lower bounds for $\Phi_\alpha|_{\mathcal{N}}$. Finally, in Appendix B we prove a technical estimate, used in our study, similar to those in Adimurthi and Mancini [1].

Motivated by this work, in [13] the second author has proved an inequality which improves inequality (15). In [14] he proves a family of inequalities which contains, as special cases, an inequality in Zhu's work [20] and the inequality in [13].

2. THE SETUP AND STATEMENT OF THE MAIN RESULT

Let Ω be a smooth bounded domain in $\mathcal{R}^N$, with $N \geq 5$. Let $2^* = \frac{2N}{N-2}$ be the critical exponent for the Sobolev embedding $H^1(\Omega) \subset L^q(\Omega)$ and $2^\# = \frac{2(N-1)}{N-2}$. Finally, let $a > 0$ and $\alpha \geq 0$. We are concerned with the problem of existence of a *least energy* solution of

$$\begin{cases} -\Delta u + au = u^{2^*-1} - \alpha u^{2^\#-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1_α}$$

Solutions of (1_α) correspond to critical points of the functional $\Phi_\alpha : H^1(\Omega) \to \mathcal{R}$ defined by

$$\Phi_\alpha(u) := \frac{1}{2}||u||^2 + \frac{\alpha}{2^\#}|u|_{2^\#}^{2^\#} - \frac{1}{2^*}|u|_{2^*}^{2^*}. \tag{2}$$

We use the notations

$$|u|_p := \left(\int |u|^p\right)^{\frac{1}{p}} \qquad \text{and} \qquad ||u|| := \left(|\nabla u|_2^2 + a|u|_2^2\right)^{\frac{1}{2}}.$$

Unless otherwise indicated, integrals are over Ω.

We recall that the Nehari manifold is

$$\mathcal{N} := \left\{u \in H^1(\Omega) : \Phi'_\alpha(u)u = 0, u \neq 0\right\}.$$

For any $u \in H^1(\Omega)\setminus\{0\}$, there exists a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}$; the value of $t(u)$ is given in expression (A.1) of Appendix A. We define $\Psi_\alpha : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ by

$$\Psi_\alpha(u) := \Phi_\alpha(t(u)u).$$

As can be checked in Appendix B,

$$\Psi_\alpha := \frac{1}{N}\frac{1}{2^\#}\frac{1}{2^N}\left[\left(\gamma + \sqrt{\gamma^2 + 4\beta}\right)^N + 2\cdot 2^*\beta\left(\gamma + \sqrt{\gamma^2 + 4\beta}\right)^{N-2}\right], \tag{3}$$

where β, $\gamma : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ are defined by

$$\beta(u) := \frac{||u||^2}{|u|_{2^*}^2} \tag{4}$$

and

$$\gamma(u) = \gamma_\alpha(u) := \alpha\frac{|u|_{2^\#}^{2^\#}}{|u|_{2^*}^{2^\#}}. \tag{5}$$

Equivalently,

$$\Psi_\alpha = \frac{1}{N}\frac{\beta^{\frac{N}{2}}}{2^\#}\left[\left(\delta + \sqrt{\delta^2 + 1}\right)^N + \frac{2^*}{2}\left(\delta + \sqrt{\delta^2 + 1}\right)^{N-2}\right], \tag{6}$$

with β as above and $\delta : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ defined by

$$\delta(u) = \delta_\alpha(u) := \frac{\gamma(u)}{2\sqrt{\beta(u)}} = \frac{1}{2}\frac{\alpha|u|_{2^\#}^{2^\#}}{||u||\cdot|u|_{2^*}^{\frac{2^*}{2}}}, \tag{7}$$

Obviously, every nonzero critical point of Φ_α is a critical point of Ψ_α. Since the Nehari manifold is a natural constraint for Φ_α, if u is a critical point of Ψ_α, then $t(u)u$ is a critical point of Φ_α.

As is usual, we say that $u \neq 0$ is a *ground state* critical point of Φ_α, or a *least energy* solution of (1_α), if

$$\Phi_\alpha(u) = \inf_{\mathcal{N}} \Phi_\alpha = \inf_{H^1(\Omega)\setminus\{0\}} \Psi_\alpha.$$

Our aim is to establish existence and nonexistence of least energy solutions of (1_α). We will consider the minimization problem corresponding to

$$S_\alpha := \inf\left\{I_\alpha(u) | u \in H^1(\Omega) \setminus \{0\}\right\},$$

where $I_\alpha : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ is defined by

$$I_\alpha := (N\Psi_\alpha)^{\frac{2}{N}}. \tag{8}$$

From (3) and (6) we obtain

$$I_\alpha = \frac{1}{4(2^{\#})^{\frac{2}{N}}}\left[\left(\gamma + \sqrt{\gamma^2 + 4\beta}\right)^N + 2\cdot 2^*\beta\left(\gamma + \sqrt{\gamma^2+4\beta}\right)^{N-2}\right]^{\frac{2}{N}} \tag{9}$$

and

$$I_\alpha = \frac{\beta}{(2^{\#})^{\frac{2}{N}}}\left[\left(\delta + \sqrt{\delta^2+1}\right)^N + \frac{2^*}{2}\left(\delta + \sqrt{\delta^2+1}\right)^{N-2}\right]^{\frac{2}{N}}. \tag{10}$$

We observe that

$$I_\alpha \geq \beta, \tag{11}$$

since

$$\frac{1}{2^{\#}}\left(1 + \frac{2^*}{2}\right) = 1.$$

Before stating our main result, we recall that the infimum

$$S := \inf\left\{\left.\frac{\int_{\mathcal{R}^N} |\nabla u|^2}{\left(\int_{\mathcal{R}^N} |u|^{2^*}\right)^{2/2^*}}\right| u \in L^{2^*}(\mathcal{R}^N), \nabla u \in L^2(\mathcal{R}^N), u \neq 0\right\},$$

which depends on N, is achieved by the Talenti instanton

$$U(x) := \left(\frac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}}.$$

This instanton U satisfies

$$-\Delta U = U^{2^*-1}, \tag{12}$$

so that

$$\int_{\mathcal{R}^N} |\nabla U|^2 = \int_{\mathcal{R}^N} U^{2^*} = S^{\frac{N}{2}}. \tag{13}$$

Let $\varepsilon > 0$ and $y \in \mathcal{R}^N$. For later use, we define the rescaled instanton

$$U_{\varepsilon,y} := \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon}\right), \tag{14}$$

which also satisfies (12) and (13).

Our main result is

THEOREM 2.1. *Let Ω be a smooth bounded domain in $\mathcal{R}^N$, with $N \geq 5$, $a > 0$ and $\alpha \geq 0$. There exists a positive real number $\alpha_0 = \alpha_0(a, \Omega)$ such that*

(i) if $\alpha < \alpha_0$, then (1_α) has a least energy solution;
(ii) if $\alpha > \alpha_0$, then (1_α) does not have a least energy solution and

$$\frac{S}{2^{\frac{2}{N}}} \leq \frac{\beta}{(2^\#)^{\frac{2}{N}}} \left[\left(\delta + \sqrt{\delta^2+1}\right)^N + \frac{2^*}{2}\left(\delta + \sqrt{\delta^2+1}\right)^{N-2}\right]^{\frac{2}{N}} \tag{15}$$

in $H^1(\Omega)\setminus\{0\}$, where β and $\delta = \delta_\alpha$ are defined in (4) and (7), respectively. The constant on the left hand side of (15) is sharp.

COROLLARY 2.1 (Aubin's inequality). *Let Ω be a smooth bounded domain in $\mathcal{R}^N$, with $N \geq 5$. For every $\varsigma > 0$, there exists a $C(\varsigma, \Omega) > 0$ such that*

$$\frac{S}{2^{\frac{2}{N}}} - \varsigma \leq \frac{|\nabla u|_2^2 + C(\varsigma, \Omega)|u|_2^2}{|u|_{2^*}^2}, \tag{16}$$

for all $u \in H^1(\Omega) \setminus \{0\}$.

Proof. From Lemma 5.1, there exists a constant $\bar{c} > 0$ such that the right hand side of (15) is less than or equal to

$$\beta\left(1 + \frac{4}{2^\#}\delta + \bar{c}\delta^2\right)$$

and from Hölder's inequality $|u|_{2^\#}^{2^\#} \le |u|_2 |u|_{2^*}^{2^*/2}$. Hence $\delta(u) \le \frac{\alpha}{2}\frac{|u|_2}{||u||}$. Let $\epsilon > 0$. For all $u \in H^1(\Omega)$,

$$\begin{aligned}
\frac{S}{2^{\frac{2}{N}}}|u|_{2^*}^2 &\le ||u||^2\left(1+\frac{2}{2^\#}\alpha_0\frac{|u|_2}{||u||}+\bar{c}\frac{\alpha_0^2}{4}\frac{|u|_2^2}{||u||^2}\right)\\
&= ||u||^2+\frac{2}{2^\#}\alpha_0||u||\,|u|_2+\bar{c}\frac{\alpha_0^2}{4}|u|_2^2\\
&\le (1+\epsilon)|\nabla u|_2^2+\left(\frac{\alpha_0^2}{(2^\#)^2\epsilon}+a\epsilon+\bar{c}\frac{\alpha_0^2}{4}\right)|u|_2^2.
\end{aligned}$$

∎

Remark 2.1. Let $\kappa > 0$. By scaling, we easily check that

$$\alpha_0\left(\kappa^2 a, \frac{\Omega}{\kappa}\right) = \kappa\,\alpha_0(a,\Omega).$$

3. EXISTENCE OF LEAST ENERGY SOLUTIONS AND THEIR ASYMPTOTIC BEHAVIOR

In this section we start by proving the basic properties of the map $\alpha \mapsto S_\alpha$ and assertion (i) of Theorem 2.1. We then assume that the value α_0 in Theorem 2.1 is infinite and analyze the asymptotic behavior of the least energy solutions as $\alpha \to +\infty$.

As explained in the previous section, we consider the minimization problem corresponding to

$$S_\alpha := \inf\{I_\alpha(u) | u \in H^1(\Omega), u \neq 0\}.$$

From Adimurthi and Mancini [1] and X.J. Wang [18], we know that

$$0 < S_0 < \frac{S}{2^{\frac{2}{N}}} \tag{17}$$

(see (42) and (46) ahead). Obviously, S_α is nondecreasing as α increases. Choose any point $P \in \partial\Omega$. By testing I_α with $U_{\varepsilon,P}$ and letting $\varepsilon \to 0$, we conclude that $S_\alpha \le \frac{S}{2^{\frac{2}{N}}}$ for all $\alpha \ge 0$.

Lemma 3.1. *If $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$, then S_α is achieved.*

Proof. Let u_k be a minimizing sequence with $|u_k|_{2^*} = 1$. Since $\beta \le I_\alpha$, from (11), (u_k) is bounded in $H^1(\Omega)$. We can assume that $u_k \rightharpoonup u$ in

$H^1(\Omega)$, $u_k \to u$ a.e. on Ω, and $|\nabla(u_k-u)|^2 \rightharpoonup \mu$ and $|u_k-u|^{2^*} \rightharpoonup \nu$ in the sense of measures on $\bar\Omega$. Modulo a subsequence, the concentration-compactness lemma implies that

$$\lim_{k\to\infty} |\nabla u_k|_2^2 = |\nabla u|_2^2 + ||\mu||$$

and

$$\lim_{k\to\infty} |u_k|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + ||\nu|| = 1,$$

where

$$\frac{S}{2^{\frac{2}{N}}}||\nu||^{\frac{2}{2^*}} \le ||\mu||.$$

This last inequality is an immediate consequence of inequality (16). For $S_\alpha = \lim_{k\to\infty} I_\alpha(u_k)$, we obtain that S_α equals

$$\frac{1}{4(2^\#)^{\frac{2}{N}}}\left[\left(\gamma_\infty + \sqrt{\gamma_\infty^2 + 4\beta_\infty}\right)^N + 2\cdot 2^*\beta_\infty\left(\gamma_\infty + \sqrt{\gamma_\infty^2 + 4\beta_\infty}\right)^{N-2}\right]^{\frac{2}{N}},$$

with

$$\beta_\infty = ||u||^2 + ||\mu|| = \frac{||u||^2 + ||\mu||}{(|u|_{2^*}^{2^*} + ||\nu||)^{\frac{2}{2^*}}}$$

and

$$\gamma_\infty = \alpha|u|_{2^\#}^{2^\#} = \alpha\frac{|u|_{2^\#}^{2^\#}}{(|u|_{2^*}^{2^*} + ||\nu||)^{\frac{2^\#}{2^*}}}.$$

If $u = 0$, then

$$\beta_\infty = \frac{||\mu||}{||\nu||^{\frac{2}{2^*}}} \ge \frac{S}{2^{\frac{2}{N}}},$$

a contradiction. So $u \neq 0$.

We claim that $||\mu|| = 0$. We argue by contradiction and suppose that $||\mu|| \neq 0$. If $||\nu|| = 0$, then $S_\alpha > I_\alpha(u)$, which is impossible. So $||\nu|| \neq 0$.

Let $x_0 := |u|_{2^*}^{2^*}$, so that $1 - x_0 = ||\nu||$. We define f, g and $h : [0,1] \to \mathcal{R}$ by

$$f(x) := \underline{\gamma} x^{\frac{2^\#}{2^*}} + \sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4\underline{\beta} x^{\frac{2}{2^*}} + 4\tfrac{||\mu||}{||\nu||^{2/2^*}}(1-x)^{\frac{2}{2^*}}},$$

$$g(x) := \underline{\beta} x^{\frac{2}{2^*}} + \tfrac{||\mu||}{||\nu||^{2/2^*}}(1-x)^{\frac{2}{2^*}}$$

and

$$h := f^N + 2\cdot 2^* f^{N-2} g,$$

for $\underline{\beta} = \beta(u)$ and $\underline{\gamma} = \gamma(u)$. The value S_α is

$$S_\alpha = \frac{1}{4(2^\#)^{\frac{2}{N}}} \left[h(x_0)\right]^{\frac{2}{N}}.$$

We wish to prove that the minimum of h occurs at 0 or 1. The former case corresponds to $u = 0$ and the latter to $||\nu|| = 0$. In either case we are led to a contradiction. This will prove that $||\mu|| = 0$, thereby establishing the claim.

The derivative of h is

$$h' = f^{N-3}[N(f^2 + 4g)f' + 2 \cdot 2^* f g'].$$

Since

$$f^2 + 4g = 2f\sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4g},$$

we can write

$$h' = 2f^{N-2}\left[N\sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4g}\, f' + 2^* g'\right].$$

The expression for h' can be further simplified by computing f':

$$\begin{aligned}\sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4g}\, f' &= \frac{1}{2^*}\left[2^\# \underline{\gamma} x^{\frac{2^\#}{2^*}-1}\sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4g} + 2^\# \underline{\gamma}^2 x^{2\frac{2^\#}{2^*}-1} + 4g'\right] \\ &= \frac{2^\#}{2^*}\left[\underline{\gamma} x^{\frac{2^\#}{2^*}-1} f + 2\frac{2^*}{2^\#} g'\right].\end{aligned}$$

This yields

$$\begin{aligned} h' &= 2(N-1)f^{N-2}\left[\underline{\gamma} x^{\frac{2^\#}{2^*}-1} f + 2^* g'\right] \\ &= 2(N-1)f^{N-2} x^{\frac{2^\#}{2^*}-1}\left[\underline{\gamma} f + 2^* x^{1-\frac{2^\#}{2^*}} g'\right].\end{aligned}$$

We notice that $h'(0) = +\infty$ and $h'(1) = -\infty$; at a zero of h', $g' < 0$.

At a point of minimum of h in the interior of $[0, 1]$, $h' = 0$ and

$$\begin{aligned}\sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4g}\, f' &= \frac{2^\#}{2^*}\left[\underline{\gamma} x^{\frac{2^\#}{2^*}-1} f + 2^* g' - 2^*\left(1 - \frac{2}{2^\#}\right) g'\right] \\ &= -(2^\# - 2) g';\end{aligned}$$

we notice that at a zero of h', $f' > 0$.

We consider

$$\kappa := -2^* x^{1-\frac{2^\#}{2^*}} g',$$

whose derivative is

$$\begin{aligned}
\kappa' &= -(2^* - 2^\#)x^{-\frac{2^\#}{2^*}} g' - 2^* x^{\frac{2^*-2^\#}{2^*}} g'' \\
&> -(2^* - 2^\#)x^{-\frac{2^\#}{2^*}} g' \\
&= x^{-\frac{2^\#}{2^*}} \sqrt{\underline{\gamma}^2 x^{2\frac{2^\#}{2^*}} + 4g}\, f' \quad \text{for } h' = 0 \\
&> \underline{\gamma} f'.
\end{aligned}$$

The zeros of h' occur when $\underline{\gamma} f = \kappa$. We just proved that $\kappa' > \underline{\gamma} f'$ at the zeros of h'. This implies that the graphs of $\underline{\gamma} f$ and κ can cross at most once, and that h' has at most one zero. If the function h were to have a minimum in the interior of $[0, 1]$, then h' would have at least three zeros because $h'(0) = +\infty$ and $h'(1) = -\infty$. We conclude that h has no minimum inside $[0, 1]$. (The conditions on the derivative of h at the end points of the interval, or the fact that the graphs of $\underline{\gamma} f$ and κ cross, imply that h' does vanish inside $[0, 1]$, at a point of maximum of h.) Therefore the minimum of h occurs either at 0 or 1 and we have proved our claim.

Since $||\mu|| = 0$, the function u is a minimizer for I_α. ▮

Lemma 3.2. *The map $\alpha \mapsto S_\alpha$ is continuous on $[0, +\infty[$.*

Proof. Let $\bar\alpha \in [0, +\infty[$. First we prove that $\alpha \mapsto S_\alpha$ is continuous from the right at $\bar\alpha$. If $S_{\bar\alpha} = \frac{S}{2^{\frac{2}{N}}}$, then continuity from the right at $\bar\alpha$ is obvious. If $S_{\bar\alpha} < \frac{S}{2^{\frac{2}{N}}}$, let $u_{\bar\alpha}$ be a minimizer of $I_{\bar\alpha}$, which exists by the previous lemma. If $\alpha > \bar\alpha$, then $S_{\bar\alpha} \le S_\alpha \le I_\alpha(u_{\bar\alpha}) \to S_{\bar\alpha}$ as $\alpha \searrow \bar\alpha$. This proves continuity from the right at $\bar\alpha$.

To prove continuity from the left we show that $\lim_{\alpha\nearrow\bar\alpha} S_\alpha = S_{\bar\alpha}$. If the value of the limit on the left hand side is $\frac{S}{2^{\frac{2}{N}}}$, then this equality is obvious. So suppose $\lim_{\alpha\nearrow\bar\alpha} S_\alpha < \frac{S}{2^{\frac{2}{N}}}$. Choose a sequence $\alpha_k \nearrow \bar\alpha$ and $u_k \in H^1(\Omega)$, with $|u_k|_{2^*} = 1$, such that $I_{\alpha_k}(u_k) = S_{\alpha_k}$. By (11), the sequence (u_k) is bounded in $H^1(\Omega)$ and we can assume that $u_k \rightharpoonup u$ in $H^1(\Omega)$. An application of the concentration-compactness principle, as in the previous lemma, shows that $u \ne 0$ and

$$\lim_{k\to\infty} I_{\alpha_k}(u_k) \ge I_{\bar\alpha}(u).$$

So,

$$S_{\bar\alpha} \le I_{\bar\alpha}(u) \le \lim_{k\to\infty} I_{\alpha_k}(u_k) = \lim_{\alpha_k \nearrow \bar\alpha} S_\alpha$$

and $S_{\bar\alpha} = \lim_{\alpha_k\nearrow\bar\alpha} S_\alpha$. ▮

By the previous lemma, the value

$$\alpha_0 := \begin{cases} +\infty, \text{ if } S_\alpha < \frac{S}{2^{\frac{2}{N}}} \text{ for all } \alpha \in [0,+\infty[, \\ \min\left\{\alpha \in [0,+\infty[\;\middle|\; S_\alpha = \frac{S}{2^{\frac{2}{N}}}\right\}, \text{ otherwise.} \end{cases} \tag{18}$$

is well defined. By (17) it is not zero. Lemma 3.1 implies the following two corollaries:

Corollary 3.1. *The map $\alpha \mapsto S_\alpha$ is strictly increasing on $[0,\alpha_0]$.*

Corollary 3.2. *If $\alpha \in [0,\alpha_0[$, then (1_α) has a least energy solution u_α. If $\alpha \in]\alpha_0,+\infty[$, then (1_α) does not have a least energy solution.*

This proves (i) of Theorem 2.1. Assertion (ii) of Theorem 2.1 will also follow once we establish that α_0 is finite.

Lemma 3.3. *If $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$ for all $\alpha \ge 0$, then*

$$\lim_{\alpha\to+\infty} S_\alpha = \frac{S}{2^{\frac{2}{N}}}. \tag{19}$$

Suppose $\alpha_k \to +\infty$ as $k \to +\infty$ and u_k is a minimizer for I_{α_k} satisfying (1_{α_k}). Then $u_k \rightharpoonup 0$ in $H^1(\Omega)$ and

$$M_k := \max_\Omega u_k$$

converges to $+\infty$, as $k \to \infty$.

Proof. Suppose $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$ for all $\alpha \ge 0$ and choose any sequence $\alpha_k \to +\infty$ as $k \to +\infty$. Let u_k be a minimizer for I_{α_k} satisfying (1_{α_k}), which necessarily exists by Lemma 3.1 and rescaling. We claim that u_k is bounded in $H^1(\Omega)$. Indeed, by (9),

$$\frac{1}{(2^\#)^{\frac{2}{N}}}\,\gamma^2(u_k) \le I_\alpha(u_k) \le \frac{S}{2^{\frac{2}{N}}}.$$

So,

$$\alpha_k|u_k|_{2^\#}^{2^\#} \leq \left(\frac{2^\#}{2}\right)^{\frac{1}{N}} S^{\frac{1}{2}}|u_k|_{2^*}^{2^\#}.$$

By (1_{α_k}),

$$|u_k|_{2^*}^{2^*} = ||u_k||^2 + \alpha_k|u_k|_{2^\#}^{2^\#}. \tag{20}$$

Together,

$$\begin{aligned} |u_k|_{2^*}^{2^*-2} &\leq \beta(u_k) + \left(\frac{2^\#}{2}\right)^{\frac{1}{N}} S^{\frac{1}{2}}|u_k|_{2^*}^{2^\#-2} \\ &\leq \frac{S}{2^{\frac{2}{N}}} + \left(\frac{2^\#}{2}\right)^{\frac{1}{N}} S^{\frac{1}{2}}|u_k|_{2^*}^{2^\#-2}, \end{aligned}$$

since, by (11), $\beta(u_k) \leq I_{\alpha_k}(u_k) \leq \frac{S}{2^{\frac{2}{N}}}$. So $|u_k|_{2^*}$ is bounded. Recalling that $\beta(u_k) = \frac{||u_k||^2}{|u_k|_{2^*}^2}$, we conclude that u_k is bounded in $H^1(\Omega)$.

From (20), we conclude that $u_k \rightharpoonup 0$ in $H^1(\Omega)$. We can assume that $u_k \to 0$ a.e. on Ω, and $|\nabla u_k|^2 \rightharpoonup \mu$ and $|u_k|^{2^*} \rightharpoonup \nu$ in the sense of measures on $\bar{\Omega}$. Then

$$\lim_{k\to\infty} |\nabla u_k|_2^2 = ||\mu|| \tag{21}$$

and

$$\lim_{k\to\infty} |u_k|_{2^*}^{2^*} = ||\nu||, \tag{22}$$

where

$$\frac{S}{2^{\frac{2}{N}}}||\nu||^{\frac{2}{2^*}} \leq ||\mu||. \tag{23}$$

Thus

$$\frac{S}{2^{\frac{2}{N}}} \geq \lim_{k\to\infty} S_{\alpha_k} = \lim_{k\to\infty} I_{\alpha_k}(u_k) \geq \frac{||\mu||}{||\nu||^{\frac{2}{2^*}}} \geq \frac{S}{2^{\frac{2}{N}}}, \tag{24}$$

and the inequalities in (24) are equalities. This proves (19).

From (1_{α_k}), the values M_k satisfy

$$a + \alpha_k M_k^{2^\#-2} \leq M_k^{2^*-2} \tag{25}$$

and consequently $M_k \to +\infty$ as $k \to +\infty$. ▮

Lemma 3.4. *Let* $S_{\alpha_k} < \frac{S}{2^{\frac{2}{N}}}$ *and* $S_{\alpha_k} \to \frac{S}{2^{\frac{2}{N}}}$ *as* $\alpha_k \to \alpha_0 \in]0,+\infty]$. *Denote by* $u_k \in H^1(\Omega)$ *a minimizer for* I_{α_k} *satisfying* (1_{α_k}). *In case* $\alpha_0 < +\infty$ *suppose that* $u_k \rightharpoonup 0$. *Then*

$$\lim_{k\to\infty} |\nabla u_k|_2^2 = \lim_{k\to\infty} |u_k|_{2^*}^{2^*} = \frac{S^{\frac{N}{2}}}{2}. \tag{26}$$

Moreover, if $\alpha_0 = +\infty$, *or if* $\alpha_0 < +\infty$ *and we further assume that* $\lim_{\alpha_k\to\alpha_0} M_k = +\infty$, *then we also have*

$$\lim_{k\to\infty} \alpha_k \delta_k = 0, \tag{27}$$

$$\lim_{k\to\infty} |\nabla u_k - \nabla U_{\delta_k,P_k}|_2 = 0 \tag{28}$$

and $P_k \in \partial\Omega$, *for large* k. *Here, we are denoting*

$$\delta_k := M_k^{-\frac{2}{N-2}},$$

and P_k *is such that* $M_k = u_k(P_k)$.

Note. If $\alpha_0 = +\infty$, Lemma 3.3 guarantees the conditions $S_{\alpha_k} \to \frac{S}{2^{\frac{2}{N}}}$, $u_k \rightharpoonup 0$ and $M_k \to +\infty$ are satisfied.

Proof. By (1_{α_k}), u_k satisfies (20). Since $u_k \rightharpoonup 0$ in $H^1(\Omega)$, u_k is bounded in $H^1(\Omega)$. Therefore, (21), (22), (23) and (24) hold, with equalities in (24). Hence, $\beta(u_k) \to \frac{S}{2^{\frac{2}{N}}}$. From (10), $\delta(u_k) \to 0$ and

$$\lim_{k\to\infty} \alpha_k |u_k|_{2^\#}^{2^\#} = 0,$$

as u_k is bounded in $H^1(\Omega)$. Taking limits in (20) as $k \to \infty$,

$$||\nu|| = ||\mu||. \tag{29}$$

Combining (24) and (29), equalities (26) follow.

We now use the Gidas and Spruck blow up technique [12]. Let $v_k(x) := \delta_k^{\frac{N-2}{2}} u_k(\delta_k x + P_k)$ for $x \in \Omega_k := (\Omega - P_k)/\delta_k$, so that

$$\begin{cases} -\Delta v_k + a\delta_k^2 v_k + \alpha_k \delta_k v_k^{2^\# - 1} = v_k^{2^*-1} & \text{in } \Omega_k, \\ 0 < v_k \le v_k(0) = 1 & \text{in } \Omega_k, \\ \frac{\partial v_k}{\partial \nu} = 0 & \text{on } \partial\Omega_k. \end{cases}$$

Rewriting (25) in terms of the δ_k,

$$a\delta_k^2 + \alpha_k \delta_k \le 1.$$

So, we can assume that $P_k \to P_0$,

$$\operatorname{dist}(P_k, \partial\Omega)/\delta_k \to L \in [0, +\infty],$$

$$\Omega_k \to \Omega_\infty := \{(\tilde{x}, x_N) \in \mathcal{R}^{N-1} \times \mathcal{R} : x_N > -L\}$$

and $\alpha_k \delta_k \to \bar{\alpha}$. By the elliptic estimates in [5],

$$v_k \to v \text{ in } C^2_{\mathrm{loc}}(\Omega_\infty) \tag{30}$$

where v satisfies

$$\begin{cases} -\Delta v + \bar{\alpha} v^{2^{\#}-1} = v^{2^*-1} & \text{in } \Omega_\infty, \\ 0 < v \le v(0) = 1 & \text{in } \Omega_\infty, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega_\infty \end{cases}$$

as $a\delta_k^2 \to 0$. By lower semicontinuity of the norm, $v \in L^{2^*}(\Omega_\infty)$ and $\nabla v \in L^2(\Omega_\infty)$. So, we can apply Pohozaev's identity and get $\bar{\alpha} = 0$, and thus $v = U$.

If $L = +\infty$, then $\Omega_\infty = \mathcal{R}^N$. From (26),

$$S^{\frac{N}{2}} = \int_{\mathcal{R}^N} |\nabla U|^2 \le \lim_{k\to\infty} |\nabla u_k|_2^2 = \frac{S^{\frac{N}{2}}}{2},$$

which is impossible.

So L is finite. This implies that $P_0 \in \partial\Omega$. In fact, L has to be zero since $v \le v(0)$. Using a diffeomorphism to straighten a boundary portion of Ω, the argument in Lemma 2.2 of [APY] shows that $P_k \in \partial\Omega$ for large k. Finally, from (26), (30) and

$$\int_{\mathcal{R}^N_+} |\nabla U|^2 = \frac{S^{\frac{N}{2}}}{2},$$

we deduce (28). ▮

As in [2] and [7], let

$$\mathcal{M} := \{CU_{\varepsilon,y}, C \in \mathcal{R}, \varepsilon > 0, y \in \partial\Omega\}$$

and $d(u,\mathcal{M}) := \inf\{|\nabla(u-V)|_2, V \in \mathcal{M}\}$. The set $\mathcal{M}\setminus\{0\}$ is a manifold of dimension $N+1$. The tangent space $T_{C_l,\varepsilon_l,y_l}(\mathcal{M})$ at $C_lU_{\varepsilon_l,y_l}$ is given by

$$T_{C_l,\varepsilon_l,y_l}(\mathcal{M}) = \mathrm{span}\left\{U_{\varepsilon,y}, C\frac{\partial}{\partial\varepsilon}U_{\varepsilon,y}, C\frac{\partial}{\partial\tau_i}U_{\varepsilon,y}, 1\le i\le N-1\right\}_{(C_l,\varepsilon_l,y_l)}$$

where $T_x(\partial\Omega) = \mathrm{span}\{\tau_1,\ldots,\tau_{N-1}\}$.

For large k, the infimum $d(u_k,\mathcal{M})$ is achieved:

$$d(u_k,\mathcal{M}) = |\nabla(u_k - C_kU_{\varepsilon_k,y_k})|_2 \text{ for } C_kU_{\varepsilon_k,y_k}\in\mathcal{M}. \tag{31}$$

Furthermore,

$$C_k = 1 + o(1) \tag{32}$$

$y_k \to P_0$ and $\varepsilon_k/\delta_k \to 1$ (see Lemma 1 of [7] and Lemma 2.3 of [2]). From (27),

$$\alpha_k\varepsilon_k \to 0. \tag{33}$$

We define

$$w_k := u_k - C_kU_{\varepsilon_k,y_k},$$

so that

$$\int \nabla U_{\varepsilon_k,y_k}\cdot\nabla w_k = 0. \tag{34}$$

Now, on the one hand, from (28),

$$\lim_{k\to\infty}|\nabla(u_k - C_kU_{\varepsilon_k,y_k})|_2 = 0.$$

On the other hand, from Poincaré's inequality, and the fact that both the average of u_k and the average of $C_kU_{\varepsilon_k,y_k}$, in Ω, converge to zero,

$$\lim_{k\to\infty}|u_k - C_kU_{\varepsilon_k,y_k}|_{2^*} = 0.$$

Together,

$$\lim_{k\to\infty}||w_k|| = 0. \tag{35}$$

Our next objective is the upper bound in Lemma 3.8 for $\int U_{\varepsilon_k,y_k}^{2^*-2}\,w_k^2$ in terms of $|\nabla w_k|_2^2 + (2^\# - 1)\alpha_k\int U_{\varepsilon_k,y_k}^{2^\#-2}\,w_k^2$. This will be crucial in the lower bound estimates for the energy in Section 4.

The eigenvalue problems arising from the linearization of (1_{α_k}) at U_{ε_k,y_k} are related to the eigenvalue problem in

LEMMA 3.5 (Bianchi and Egnell [7], Rey [16]). *The eigenvalue problem*

$$\begin{cases} -\Delta\varphi = \mu U^{2^*-2}\varphi & \text{in } \mathcal{R}^N_+, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\mathcal{R}^N_+, \\ \int_{\mathcal{R}^N_+} U^{2^*-2}\varphi^2 < \infty \end{cases}$$

admits a discrete spectrum $\mu_1 < \mu_2 \le \mu_3 \le \dots$ such that $\mu_1 = 1$, $\mu_2 = \mu_3 = \dots = \mu_N = 2^* - 1$ and $\mu_{N+1} > 2^* - 1$. The eigenspaces V_1 and $V_{(2^*-1)}$, corresponding to 1 and $(2^* - 1)$, are given by

$$\begin{aligned} V_1 &= \text{span } U, \\ V_{(2^*-1)} &= \text{span}\left\{ \left.\frac{\partial U_{1,y}}{\partial y_i}\right|_{y=0}, \text{ for } 1 \le i \le N-1 \right\}. \end{aligned}$$

We will consider the eigenvalue problems arising from the linearization of (1_{α_k}) at U_{ε_k,y_k}. Let $\varepsilon > 0$, $\nu_\varepsilon > 0$, and $y_\varepsilon \in \partial\Omega$ with $\lim_{\varepsilon\to 0} y_\varepsilon = y_0$. Let $\{\varphi_{i,\varepsilon}\}_{i=1}^\infty$ be a complete set of orthogonal eigenfunctions with eigenvalues $\mu_{1,\varepsilon} < \mu_{2,\varepsilon} \le \mu_{3,\varepsilon} \le \dots$ for the weighted eigenvalue problem

$$\begin{cases} -\Delta\varphi + \nu_\varepsilon U_{\varepsilon,y_\varepsilon}^{2^\#-2}\varphi = \mu U_{\varepsilon,y_\varepsilon}^{2^*-2}\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\varphi_{1,\varepsilon} > 0$ and

$$\int_\Omega U^{2^*-2}\varphi_{i,\varepsilon}\varphi_{j,\varepsilon} = \delta_{i,j}.$$

Let

$$\Omega_\varepsilon := (\Omega - y_\varepsilon)/\varepsilon.$$

The sets Ω_ε converge to a half space as $\varepsilon \to 0$. For a function v on Ω, we define $\tilde{v}$ on Ω_ε by

$$\tilde{v}(x) := \varepsilon^{\frac{N-2}{2}} v(\varepsilon x + y_\varepsilon).$$

The relation between these eigenvalue problems and the one considered in Lemma 3.5 is given in

LEMMA 3.6. *Suppose $y_\varepsilon \in \partial\Omega$, $\lim_{\varepsilon\to 0} y_\varepsilon = y_0$, $\lim_{\varepsilon\to 0}(\varepsilon\nu_\varepsilon) = 0$ and the sets Ω_ε converge to $\mathcal{R}^N_+$. Then, up to a subsequence,*

$$\lim_{\varepsilon\to 0} \mu_{i,\varepsilon} = \mu_i$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} U^{2^*-2}(\tilde\varphi_{i,\varepsilon}-\tilde\varphi_i)^2=0.$$

The μ_i *and* $\tilde\varphi_i$ *satisfy*

$$\begin{cases} -\Delta\tilde\varphi_i=\mu_i U^{2^*-2}\tilde\varphi_i & \text{in } \mathcal{R}^N_+,\\ \frac{\partial\tilde\varphi_i}{\partial\nu}=0 & \text{on } \partial\mathcal{R}^N_+,\\ \int_{\mathcal{R}^N_+} U^{2^*-2}\tilde\varphi_i^2=1, & \end{cases}$$

and the functions $\tilde\varphi_i$ *are supposed extended to* $\mathcal{R}^N$ *by reflection. In particular, from the previous lemma,* $\mu_1=1$, $\tilde\varphi_1=CU$ *for some constant* $C>0$, $\mu_i=2^*-1$ *for* $2\le i\le N$ *and* $\mu_{N+1}>2^*-1$. *Also,* $\{\tilde\varphi_i\}_{i=2}^N$ *is in the span of* $\{\partial U_{1,y}/\partial y_i|_{y=0}\,,$ *for* $1\le i\le N-1\}$.

The proof of Lemma 3.6 is a consequence of the arguments in the proof of Lemma 3.3 of [APY], of Lemma 3.7 and of Remark 3.1. For the details we refer to the proof of Lemma 5.5 of [14] for parameter s there equal to one.

Lemma 3.7. *Suppose* $y_\varepsilon\in\bar\Omega$, $\varphi_\varepsilon\in H^1(\Omega)$,

$$\begin{cases} \displaystyle\int U_{\varepsilon,y_\varepsilon}^{2^\#-2}\varphi_\varepsilon^2\to 0,\\ \displaystyle\int |\nabla\varphi_\varepsilon|^2\to 0, \end{cases}$$

as $\varepsilon\to 0$. *Then*

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\varphi_\varepsilon^2\to 0,$$

as $\varepsilon\to 0$.

Proof. We denote the average of φ_ε in Ω by $\bar\varphi_\varepsilon$. By Poincaré's inequality,

$$|\varphi_\varepsilon-\bar\varphi_\varepsilon|_{2^*}\to 0.$$

The limits in this proof are taken as $\varepsilon\to 0$. So we can write $\varphi_\varepsilon=\bar\varphi_\varepsilon+\eta_\varepsilon$, with $\eta_\varepsilon\to 0$ in L^{2^*}. We know that

$$\int U_{\varepsilon,y_\varepsilon}^{2^\#-2}(\bar\varphi_\varepsilon^2+2\bar\varphi_\varepsilon\eta_\varepsilon+\eta_\varepsilon^2)=o(1).$$

We have the following estimates for the three terms on the left hand side:

$$\int U_{\varepsilon,y_\varepsilon}^{2^\#-2}\bar\varphi_\varepsilon^2 \geq b\bar\varphi_\varepsilon^2\varepsilon,$$

for some $b > 0$, and

$$\begin{aligned}\left|\int U_{\varepsilon,y_\varepsilon}^{2^\#-2}\eta_\varepsilon\bar\varphi_\varepsilon\right| &\leq |\eta_\varepsilon|_{2^*}|\bar\varphi_\varepsilon|\left(\int U_{\varepsilon,y_\varepsilon}^{\frac{N}{N-2}\frac{4}{N+2}}\right)^{\frac{N+2}{2N}}\\ &\leq C|\eta_\varepsilon|_{2^*}|\bar\varphi_\varepsilon|\varepsilon,\end{aligned}$$

by (39); and

$$\begin{aligned}\int U_{\varepsilon,y_\varepsilon}^{2^\#-2}\eta_\varepsilon^2 &\leq |\eta_\varepsilon|_{2^*}^2\left(\int U_{\varepsilon,y_\varepsilon}^{\frac{N}{N-2}}\right)^{\frac{2}{N}}\\ &\leq C|\eta_\varepsilon|_{2^*}^2\varepsilon|\log\varepsilon|^{\frac{2}{N}},\end{aligned}\tag{36}$$

by (40). (Inequalities (39), (40) and (41) are in the beginning of the next section.) Thus

$$b\bar\varphi_\varepsilon^2\varepsilon \leq C|\bar\varphi_\varepsilon|\varepsilon + o(1).$$

This shows that $\bar\varphi_\varepsilon\sqrt{\varepsilon}$ is bounded. But if $\bar\varphi_\varepsilon\sqrt{\varepsilon}$ is bounded this shows that

$$\bar\varphi_\varepsilon\sqrt{\varepsilon} \to 0.\tag{37}$$

We want to prove that

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\bar\varphi_\varepsilon^2 + 2\bar\varphi_\varepsilon\eta_\varepsilon + \eta_\varepsilon^2) = o(1).$$

For the first term on the left hand side we have, by (39) and then (37),

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\bar\varphi_\varepsilon^2 \leq C\bar\varphi_\varepsilon^2\varepsilon^2 \to 0.$$

For the third term we have

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\eta_\varepsilon^2 \leq C|\eta_\varepsilon|_{2^*}^2 \to 0.$$

We claim that the remaining term also converges to zero. This will prove the lemma. For the second term we have the estimate

$$\zeta_\varepsilon := \left|\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\bar\varphi_\varepsilon\eta_\varepsilon\right| \leq |\eta_\varepsilon|_{2^*}|\bar\varphi_\varepsilon|\left(\int U_{\varepsilon,y_\varepsilon}^{\frac{N}{N-2}\frac{8}{N+2}}\right)^{\frac{N+2}{2N}}.$$

If $N = 5$, by (41),

$$\zeta_\varepsilon \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^{N\left(1-\frac{4}{N+2}\right)\frac{N+2}{2N}} \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^{\frac{N-2}{2}} = C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^{\frac{3}{2}}.$$

If $N = 6$, by (40),

$$\zeta_\varepsilon \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^2|\log\varepsilon|^{\frac{2}{3}}.$$

Finally, if $N \geq 7$, by (39),

$$\zeta_\varepsilon \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^2.$$

In all three cases, (37) implies that $\zeta_\varepsilon \to 0$. ▌

Remark 3.1. If in the previous lemma, instead of assuming $\int|\nabla\varphi_\varepsilon|^2 \to 0$, we assume that $\int|\nabla\varphi_\varepsilon|^2$ is bounded, then we can still conclude $\bar{\varphi}_\varepsilon\sqrt{\varepsilon} \to 0$ and $\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\left(\bar{\varphi}_\varepsilon^2 + 2\bar{\varphi}_\varepsilon\eta_\varepsilon\right) = \int_{\Omega_\varepsilon} U^{2^*-2}\left(\tilde{\bar{\varphi}}_\varepsilon^2 + 2\tilde{\bar{\varphi}}_\varepsilon\tilde{\eta}_\varepsilon\right) \to 0$, as $\varepsilon \to 0$.

Using Lemma 3.6 and the arguments in the proof of Lemma 3.4 of [APY], we deduce

LEMMA 3.8. *Suppose $y_\varepsilon \in \partial\Omega$, $\lim_{\varepsilon\to 0} y_\varepsilon = y_0$ and $\lim_{\varepsilon\to 0}(\varepsilon\nu_\varepsilon) = 0$. There exists a constant $\gamma_1 > 0$ such that, for sufficiently small ε,*

$$|\nabla w|_2^2 + \nu_\varepsilon \int U_{\varepsilon,y_\varepsilon}^{2^\#-2} w^2 \geq (2^* - 1 + \gamma_1)\int U_{\varepsilon,y_\varepsilon}^{2^*-2} w^2 + O(\varepsilon^2||w||^2)$$

for w orthogonal to $T_{1,\varepsilon,y_\varepsilon}(\mathcal{M})$.

4. NONEXISTENCE OF LEAST ENERGY SOLUTIONS

In this section we prove (ii) of Theorem 2.1. The idea of the proof is to obtain a lower bound for I_α and show that if α_0, defined in (18), is infinite, then the least energy solutions u_k of (1_{α_k}) have energy $I_{\alpha_k}(u_k) > \frac{S}{2^{\frac{2}{N}}}$, for large α_k. This is impossible. Therefore α_0 is finite. By Corollary 3.2, (ii) of Theorem 2.1 follows.

Assume

$$u_k = C_k U_{\varepsilon_k,y_k} + w_k,$$

(26), (31), (32), (33) and (35). From (10), I_α has the lower bound

$$I_\alpha \geq \beta\left(1 + \frac{4}{2^\#}\delta\right) \tag{38}$$

(this is also checked in (A.2) of Appendix A). We will expand β and δ to second order around U_{ε_k,y_k}. We start by deriving estimates for the terms that appear in this expansion.

We recall, from Brézis and Nirenberg [9], that, for $y \in \bar{\Omega}$, there exist positive constants c_1 and c_2 such that:
if $1 \leq q < \frac{N}{N-2}$, then

$$c_1 \varepsilon^{q\left(\frac{N-2}{2}\right)} \leq |U_{\varepsilon,y}|_q^q \leq c_2 \varepsilon^{q\left(\frac{N-2}{2}\right)}; \tag{39}$$

if $q = \frac{N}{N-2}$, then

$$c_1 \varepsilon^{\frac{N}{2}} |\log \varepsilon| \leq |U_{\varepsilon,y}|_q^q \leq c_2 \varepsilon^{\frac{N}{2}} |\log \varepsilon|; \tag{40}$$

and if $\frac{N}{N-2} < q \leq \frac{2N}{N-2}$, then

$$c_1 \varepsilon^{N\left(1-\frac{q}{2^*}\right)} \leq |U_{\varepsilon,y}|_q^q \leq c_2 \varepsilon^{N\left(1-\frac{q}{2^*}\right)}. \tag{41}$$

For brevity, we shall write

$$U_k := U_{\varepsilon_k,y_k}.$$

Estimate for $|U_k|_2^2$*:* For $N \geq 5$, $\frac{N}{N-2} < 2$. From (41),

$$|U_k|_2^2 = O({\varepsilon_k}^2). \tag{42}$$

Estimate for $|U_k|_{2^\#}^{2^\#}$*:* Since $y_k \in \partial\Omega$ and we are supposing that the domain is smooth,

$$|U_k|_{2^\#}^{2^\#} = \frac{2^\# B(N) \varepsilon_k}{2} + o(\varepsilon_k), \tag{43}$$

with

$$\begin{aligned} B(N) &= \frac{1}{2^\#} \int_{\mathcal{R}^N} U^{2^\#} \\ &= \omega_N \frac{1}{2^N} \sqrt{\pi} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)} [N(N-2)]^{\frac{N}{2}}, \end{aligned}$$

as proved in Appendix B. Here ω_N is the volume of the $N-1$ dimensional unit sphere.

Estimate for $|\nabla U_k|_2^2$ *and for* $|U_k|_{2^*}^{2^*}$*:* From Adimurthi and Mancini [1], since $N \geq 5$,

$$|\nabla U_k|_2^2 = \frac{S^{\frac{N}{2}}}{2} - \bar{C}_1 \varepsilon_k + O({\varepsilon_k}^2) \tag{44}$$

and

$$|U_k|_{2^*}^{2^*} = \frac{S^{\frac{N}{2}}}{2} - \bar{C}_2\varepsilon_k + O({\varepsilon_k}^2), \tag{45}$$

where

$$\bar{C}_1 = H(y_k)\frac{\omega_{N-1}(N-2)^2}{4}\frac{\Gamma\left(\frac{N+3}{2}\right)\Gamma\left(\frac{N-3}{2}\right)}{\Gamma(N)}[N(N-2)]^{\frac{N-2}{2}}$$

and

$$\bar{C}_2 = H(y_k)\frac{\omega_{N-1}}{4}\frac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma(N)}[N(N-2)]^{\frac{N}{2}}.$$

Here $H(y_k)$ denotes the mean curvature of $\partial\Omega$ at y_k with respect to the unit outward normal and, as above, ω_N is the volume of the $N-1$ dimensional unit sphere. This yields

$$\frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} = \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}}SH(y_k)A(N)\varepsilon_k + O(\varepsilon_k^2) \tag{46}$$

with

$$A(N) = \frac{2}{N}\frac{\omega_{N-1}}{\omega_N}\frac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N-3}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-2}{2}\right)} = \tfrac{N-1}{N}\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{N-3}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}. \tag{47}$$

To justify the last equality we recall that if $\omega_N(r)$ is the volume of the $N-1$ dimensional sphere with radius r, then

$$\omega_N(r) = \int_0^\pi \omega_{N-1}(r\sin\varphi)\,rd\varphi = r^{N-1}\omega_{N-1}(1)\int_0^\pi \sin^{N-2}\varphi\,d\varphi,$$

which yields

$$\frac{\omega_{N-1}}{\omega_N} = \frac{\omega_{N-1}(1)}{\omega_N(1)} = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}.$$

We mention that the Talenti instanton we use does not coincide with the one in [1]. Denoting the Talenti instanton in Adimurthi and Mancini by V, $V(\,\cdot\,) = U((N(N-2))^{1/2}\,\cdot\,)$.

Estimate for $\int U_k w_k$:

Lemma 4.1.

$$\left|\int U_k w_k\right| \le \begin{cases} O\left(\varepsilon_k^{\frac{3}{2}}||w_k||\right) & \textit{if } N=5,\\ O\left(\varepsilon_k^2|\log\varepsilon_k|^{\frac{2}{3}}||w_k||\right) & \textit{if } N=6,\\ O\left(\varepsilon_k^2||w_k||\right) & \textit{if } N\ge 7.\end{cases} \tag{48}$$

Proof.

$$\left|\int U_k w_k\right| \le |w_k|_{2^*} \left(\int U_k^{\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}}.$$

If $N = 5$, then $\frac{2N}{N+2} < \frac{N}{N-2}$. By (39),

$$\left|\int U_k w_k\right| \le C||w_k||\varepsilon_k^{\frac{N-2}{2}} = O\left(\varepsilon_k^{\frac{3}{2}}||w_k||\right).$$

If $N = 6$, then $\frac{2N}{N+2} = \frac{N}{N-2}$. By (40),

$$\left|\int U_k w_k\right| \le C||w_k|| \left(\varepsilon_k^{\frac{N}{2}}|\log\varepsilon_k|\right)^{\frac{N+2}{2N}} = O\left(\varepsilon_k^2|\log\varepsilon_k|^{\frac{2}{3}}||w_k||\right).$$

If $N \ge 7$, then $\frac{N}{N-2} < \frac{2N}{N+2}$. By (41),

$$\left|\int U_k w_k\right| \le C||w_k|| \left(\varepsilon_k^{N\left(1-\frac{2N}{N+2}\frac{N-2}{2N}\right)}\right)^{\frac{N+2}{2N}} = O\left(\varepsilon_k^2||w_k||\right).$$

∎

Estimate for $\int U_k^{2^*-1}w_k$*:* From [APY], Equations (3.15), for $N \ge 5$,

$$\int U_k^{2^*-1}w_k = O(\varepsilon_k||w_k||). \tag{49}$$

Estimate for $\int U_k^{2^\#-1}|w_k|$*:* Since $\frac{2N}{N+2} > 1$,

$$\begin{aligned}\int U_k^{2^\#-1}|w_k| &\le |w_k|_{2^*}\left(\int U_k^{\frac{N}{N-2}\frac{2N}{N+2}}\right)^{\frac{N+2}{2N}} \\ &\le C|w_k|_{2^*}\varepsilon_k^{\left(1-\frac{N}{N+2}\right)\frac{N+2}{2}} \\ &= O(\varepsilon_k||w_k||). \end{aligned} \tag{50}$$

Estimate for $\int U_k^{2^*-2}w_k^2$*:*

$$\int U_k^{2^*-2}w_k^2 = O(||w_k||^2). \tag{51}$$

Now we will obtain a lower bound for $I_{\alpha_k}(u_k)$. Let $v_k = u_k/C_k = U_k + \tilde{w}_k = U_k + w_k/C_k$. Because of (32), the sequence (v_k) satisfies (26) and the sequence $\tilde{w}_k$ satisfies (35). Of course, $d(v_k, M)$ is achieved by U_k. Because I is homogeneous of degree zero, $I_{\alpha_k}(u_k) = I_{\alpha_k}(v_k)$. We will compute $I_{\alpha_k}(v_k)$ but we will still call v_k by u_k, and $\tilde{w}_k$ by w_k.

Going back to (38), $I_\alpha(u_k)$ is bounded below by the sum of $\beta(u_k)$ and $\frac{4}{2^\#}\beta(u_k)\delta(u_k)$. We start by obtaining lower bounds for $\beta(u_k)$ and $\frac{4}{2^\#}\beta(u_k)\delta(u_k)$ separately. The expression for $\beta(u_k)$ involves two terms: $||u_k||^2$ and $|u_k|_{2^*}^{2^*}$. The first one is obviously

$$\begin{aligned} ||u_k||^2 &= ||U_k||^2 + 2\big(\textstyle\int \nabla U_k \cdot \nabla w_k + a \int U_k w_k\big) + \; ||w_k||^2 \\ &= A_1 + A_2 + A_3. \end{aligned} \tag{52}$$

For the second term we use

Lemma 4.2 ([APY] Lemma 3.5). *Let $q > 1$ and L be a non negative integer with $L \le q$. Let V and ω be measurable functions on Ω with $V \ge 0$ and $V + \omega \ge 0$. Then*

$$\begin{aligned} \int (V+\omega)^q &= \sum_{i=0}^{L} \frac{q(q-1)\dots(q-i+1)}{i!} \int V^{q-i}\omega^i \\ &\quad + O\left(\int [V^{q-r}|\omega|^r + |\omega|^q]\right), \end{aligned}$$

where $r = \min\{L+1, q\}$.

Taking $L = 2$ and $q = 2^*$,

$$|u_k|_{2^*}^{2^*} = |U_k|_{2^*}^{2^*} + 2^* \int U_k^{2^*-1} w_k + \tfrac{2^*(2^*-1)}{2} \int U_k^{2^*-2} w_k^2 + O(||w_k||^r), \tag{53}$$

where $r = \min\{2^*, 3\}$, i.e., $r = 3$ if $N = 5$, and $r = 2^*$ if $N > 5$. The inequality

$$(1+z)^{-\eta} \ge 1 - \eta z, \tag{54}$$

for $\eta > 0$ and $z \ge -1$, implies

$$\begin{aligned} |u_k|_{2^*}^{-2} &\ge |U_k|_{2^*}^{-2} \Bigg(1 - \frac{2\int U_k^{2^*-1} w_k}{|U_k|_{2^*}^{2^*}} \\ &\qquad - \frac{(2^*-1)\int U_k^{2^*-2} w_k^2}{|U_k|_{2^*}^{2^*}} + O(||w_k||^r)\Bigg) \\ &= B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{55}$$

Let

$$l := \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^{2^*}}.$$

From (44) and (45),

$$l = 1 + O(\varepsilon_k). \tag{56}$$

Using (52) and (55), we can write,

$$\beta(u_k) \geq \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4,$$

where

$$\begin{aligned}
\bar{I}_1 &= \frac{||U_k||^2}{|U_k|_{2^*}^2} \\
&= A_1 B_1 \\
\bar{I}_2 &= \frac{2}{|U_k|_{2^*}^2}\left[\int \nabla U_k \cdot \nabla w_k + a\int U_k w_k - l\int\ U_k^{2^*-1} w_k\right] \\
&= A_2 B_1 + A_1 B_2 \\
\bar{I}_3 &= \frac{1}{|U_k|_{2^*}^2}\left[||w_k||^2 - l(2^*-1)\int\ U_k^{2^*-2} w_k^2\right] \\
&= A_3 B_1 + A_1 B_3
\end{aligned}$$

and

$$\begin{aligned}
\bar{I}_4 &= [(A_1 + A_3)B_4] + [A_2(B_2 + B_3 + B_4)] + A_3 B_2 + A_3 B_3 \\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

By (42) and (45),

$$\bar{I}_1 = \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + o(\varepsilon_k).$$

We recall (35), $w_k \to 0$ in $H^1(\Omega)$.

By (34), the first of the four terms in $\bar{I}_2$ is zero; by Lemma 4.1 and by (49) the second and the third ones are $o(\varepsilon_k)$:

$$\bar{I}_2 = o(\varepsilon_k).$$

By (45), (51) and (56),

$$\bar{I}_3 = 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}}\left[||w_k||^2 - (2^*-1)\int\ U_k^{2^*-2} w_k^2\right]$$

The term E_1 is $o(||w_k||^2)$ because B_4 is $o(||w_k||^2)$. The term E_2 is $o(\varepsilon_k)$ because, from Lemma 4.1, A_2 is $o(\varepsilon_k)$. The term E_3 is $o(\varepsilon_k)$ because, from (49), B_2 is $o(\varepsilon_k)$. Finally, the term E_4 is $o(||w_k||^2)$ because both A_3 and B_3 are $O(||w_k||^2)$. Therefore,

$$\bar{I}_4 = o(\varepsilon_k) + o(||w_k||^2).$$

Combining the expressions for $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ and $\bar{I}_4$,

$$\begin{aligned}
\beta(u_k) &= \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \left[||w_k||^2 - (2^*-1)\int U_k^{2^*-2} w_k^2 \right] \\
&\quad + o(\varepsilon_k) + o\left(||w_k||^2\right) \\
&\geq \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \left[\gamma_2 ||w_k||^2 - (2^*-1)\int U_k^{2^*-2} w_k^2 \right] + o(\varepsilon_k),
\end{aligned}$$

for any fixed number $\gamma_2 < 1$, because $a > 0$. This is our lower bound for $\beta(u_k)$.

Now we turn to the term $\frac{4}{2^\#}\beta(u_k)\delta(u_k)$ and write

$$\frac{4}{2^\#}\beta(u_k)\delta(u_k) = \frac{2}{2^\#} \frac{||u_k||}{|u_k|_{2^*}^{2+2^*/2}} \alpha_k |u_k|_{2^\#}^{2^\#} \tag{57}$$

We obtain a lower bound for $||u_k||$ from (52). Using (34), (42), (44) and Lemma 4.1,

$$||u_k|| \geq \left(\frac{S^{\frac{N}{2}}}{2} \right)^{\frac{1}{2}} + O(\varepsilon_k) + O(||w_k||^2).$$

We obtain a lower bound for $|u_k|_{2^*}^{-(2+2^*/2)}$ from (53). Using (45), (49), (51) and (54),

$$|u_k|_{2^*}^{-(2+2^*/2)} \geq \left(\frac{S^{\frac{N}{2}}}{2} \right)^{-\frac{1}{2}-\frac{2}{2^*}} + O(\varepsilon_k) + O(||w_k||^2).$$

For the product we obtain the lower bound

$$\begin{aligned}
\frac{||u_k||}{|u_k|_{2^*}^{2+2^*/2}} &\geq 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} + O(\varepsilon_k) + O(||w_k||^2) \\
&= D_1 + D_2 + D_3.
\end{aligned} \tag{58}$$

To estimate the term $\alpha_k|u_k|_{2^\#}^{2^\#}$ we do not use Lemma 4.2 because it would give rise to a term $O\left(\alpha_k||w_k||^{2^\#}\right)$, for which we do not have estimates. Instead we use this calculus

LEMMA 4.3. *Let $\eta > 2$. For any $z \geq -1$,*

$$\frac{\eta(\eta-1)}{2}z^2 - \tilde{C}|z| + 1 \leq (z+1)^\eta, \tag{59}$$

where $\tilde{C} = 1 + \eta(\eta-1)/2$.

Proof. The difference between the right hand side and the left hand side is zero for $z = -1$ and $z = 0$. It is increasing for $z > 0$ and concave for $-1 < z < 0$. ▌

((59) also hold for $\eta = 2$, with equality for negative values of z.)
As a consequence of Lemma 4.3,

$$|u_k|_{2^\#}^{2^\#} \geq |U_k|_{2^\#}^{2^\#} - 2^\#\hat{C}\int U_k^{2^\#-1}|w_k| + \tfrac{2^\#(2^\#-1)}{2}\int U_k^{2^\#-2}w_k^2,$$

with

$$\hat{C} := \frac{\tilde{C}}{2^\#} = \frac{1}{2^\#} + \frac{2^\#-1}{2}.$$

Using (43) and (49),

$$\begin{aligned} \frac{2}{2^\#}\alpha_k|u_k|_{2^\#}^{2^\#} &\geq B(N)\alpha_k\varepsilon_k + (2^\#-1)\alpha_k\int U_k^{2^\#-2}w_k^2 + o(\alpha_k\varepsilon_k) \\ &= F_1 + F_2 + F_3. \end{aligned} \tag{60}$$

We will now substitute (58) and (60) in (57). On the one hand,

$$(D_1 + D_2 + D_3)F_3 = o(\alpha_k\varepsilon_k)$$

and

$$(D_2 + D_3)F_1 = o(\alpha_k\varepsilon_k).$$

On the other hand, by (36),

$$D_2F_2 = O\left(\alpha_k\varepsilon_k^2|\log\varepsilon_k|^{\frac{2}{N}}||w_k||^2\right) = o(\alpha_k\varepsilon_k).$$

So,

$$\begin{aligned}
\frac{4}{2^{\#}}\beta(u_k)\delta(u_k) \;\geq\;& 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} B(N)\alpha_k\varepsilon_k \\
&+ 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} (2^{\#}-1)\alpha_k \int U_k^{2^{\#}-2} w_k^2 \\
&+ O\left(||w_k||^2\right)\alpha_k \int U_k^{2^{\#}-2} w_k^2 + o(\alpha_k\varepsilon_k) \\
\geq\;& 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \left[B(N)\alpha_k\varepsilon_k + \gamma_2(2^{\#}-1)\alpha_k \int U_k^{2^{\#}-2} w_k^2\right] \\
&+ o(\alpha_k\varepsilon_k),
\end{aligned}$$

for any fixed number $\gamma_2 < 1$. This is our lower bound for $\frac{4}{2^{\#}}\beta(u_k)\delta(u_k)$.

Combining the lower bounds for $\beta(u_k)$ and for $\frac{4}{2^{\#}}\beta(u_k)\delta(u_k)$,

$$\begin{aligned}
I_{\alpha_k}(u_k) \;\geq\;& \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} B(N)\alpha_k\varepsilon_k \\
&+ 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \left[\gamma_2||w_k||^2 + \gamma_2(2^{\#}-1)\alpha_k \int U_k^{2^{\#}-2} w_k^2 \right. \\
&\left. - (2^*-1)\int U_k^{2^*-2} w_k^2\right] + o(\alpha_k\varepsilon_k).
\end{aligned}$$

From Lemma 3.8, the term inside the square parenthesis is greater than

$$\left[\left(\gamma_2 - \frac{(2^*-1)}{(2^*-1)+\gamma_1}\right)\left(||w_k||^2 + (2^{\#}-1)\alpha_k \int U_k^{2^{\#}-2} w_k^2\right) + o(\varepsilon_k)\right].$$

Choosing $\gamma_2 \geq \frac{(2^*-1)}{(2^*-1)+\gamma_1}$, yields that this term is greater than $o(\varepsilon_k)$. Hence,

$$I_{\alpha_k}(u_k) \geq \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} B(N)\alpha_k\varepsilon_k + o(\alpha_k\varepsilon_k).$$

Substituting (46) into this expression, we obtain

$$\begin{aligned}
I_{\alpha_k}(u_k) \;\geq\;& \frac{S}{2^{\frac{2}{N}}} + 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} B(N)\alpha_k\varepsilon_k \left[1 - S^{\frac{N}{2}}\frac{A(N)}{B(N)} H(y_k)\frac{1}{\alpha_k} + o(1)\right] \\
>\;& \frac{S}{2^{\frac{2}{N}}},
\end{aligned}$$

for large k.

So assume α_0, in (18), is $+\infty$. Choose a sequence $\alpha_k \to +\infty$ as $k \to +\infty$ and denote by u_k a minimizer for I_{α_k} satisfying (1_{α_k}). From Lemmas 3.3

and 3.4, the conditions (26), (31), (32), (33) and (35) hold. Therefore $S_{\alpha_k} = I_{\alpha_k}(u_k) > \frac{S}{2^{\frac{2}{N}}}$ for large k, which is impossible. By Corollary 3.2, this establishes (ii) of Theorem 2.1.

Remark 4.1. Since $S^{\frac{N}{2}} = \int_{\mathcal{R}^N} U^{2^*} = \omega_N \frac{1}{2^N}\sqrt{\pi}\frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N+1}{2})}[N(N-2)]^{\frac{N}{2}}$, it follows that

$$B(N) = S^{\frac{N}{2}}.$$

Using

$$\omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)},$$

the common value is

$$B(N) = S^{\frac{N}{2}} = \frac{\pi^{\frac{N+1}{2}}}{2^{N-1}}\frac{1}{\Gamma\left(\frac{N+1}{2}\right)}[N(N-2)]^{\frac{N}{2}}.$$

5. LEAST ENERGY SOLUTIONS OF (1_{α_0})

In this section we give a lower bound for $\alpha_0 = \min\left\{\alpha \mid S_\alpha = S/2^{\frac{2}{N}}\right\}$, and give partial results concerning existence of least energy solutions of (1_{α_0}).

From (10) we obtain

LEMMA 5.1. *There exists a constant $\bar{c} > \frac{4}{(2^{\#})^{2/N}}$ such that*

$$I_\alpha \le \beta\left(1 + \frac{4}{2^{\#}}\delta + \bar{c}\delta^2\right). \tag{61}$$

Proof. Consider $\Lambda : [0, +\infty[\to \mathcal{R}$, defined by

$$\Lambda(\underline{\delta}) := \frac{1}{(2^{\#})^{\frac{2}{N}}}\left[\left(\underline{\delta} + \sqrt{\underline{\delta}^2+1}\right)^N + \frac{2^*}{2}\left(\underline{\delta} + \sqrt{\underline{\delta}^2+1}\right)^{N-2}\right]^{\frac{2}{N}}.$$

Since $\frac{\partial}{\partial\underline{\delta}}\sqrt{\underline{\delta}^2+1}\Big|_{\underline{\delta}=0} = 0$ and $\frac{\partial}{\partial\underline{\delta}}\frac{1}{\sqrt{\underline{\delta}^2+1}}\Big|_{\underline{\delta}=0} = 0$, the first two derivatives of Λ at zero are

$$\Lambda'(0) = \frac{1}{(2^{\#})^{\frac{2}{N}}}\frac{2}{N}(2^{\#})^{\frac{2}{N}-1}\left[N + \frac{2^*}{2}(N-2)\right] = \frac{4}{2^{\#}}$$

and

$$\Lambda''(0) = \tfrac{1}{(2^{\#})^{\frac{2}{N}}} \tfrac{2}{N} \left(\tfrac{2}{N} - 1\right) (2^{\#})^{\frac{2}{N}-2} (2N)^2 + \tfrac{2}{2^{\#}} [N + (N-2)] = \tfrac{4}{2^{\#}} \tfrac{2N-3}{N-1}.$$

Fix any number $c_1 > \frac{2}{2^{\#}} \frac{2N-3}{N-1}$. There exists an $\epsilon > 0$ such that (61) holds for $\bar{c} = c_1$ and $0 \leq \underline{\delta} < \epsilon$.

Fix any number $c_2 > \frac{4}{(2^{\#})^{2/N}}$. From (10), there exists an $L > 0$ such that (61) holds for $\bar{c} = c_2$ and $\underline{\delta} > L$.

The inequalities $\frac{2}{2^{\#}} \frac{2N-3}{N-1} < \frac{4}{2^{\#}} < \frac{4}{(2^{\#})^{2/N}}$ show that $\max\left\{\frac{2}{2^{\#}} \frac{2N-3}{N-1}, \frac{4}{(2^{\#})^{2/N}}\right\} = \frac{4}{(2^{\#})^{2/N}}$.

By taking $\bar{c} \geq \max\{c_1, c_2\}$, $\bar{c}$ sufficiently large, we can guarantee (61) for all $\underline{\delta} \in [\epsilon, L]$. ▮

LEMMA 5.2. *If $\alpha < A(N) \max_{\partial\Omega} H$, then $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$.*

Proof. Choose $P \in \partial\Omega$ such that $H(P) = \max_{\partial\Omega} H$. From (42) and (46),

$$\beta(U_{\varepsilon,P}) = \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}} S H(P) A(N) \varepsilon + o(\varepsilon),$$

whereas, from (7) and (42)-(45),

$$\delta(U_{\varepsilon,P}) = \frac{2}{S^{\frac{N}{2}}} \frac{2^{\#}}{4} B(N) \alpha \varepsilon + o(\varepsilon).$$

The previous lemma implies that

$$\begin{aligned} S_\alpha &\leq I_\alpha(U_{\varepsilon,P}) \\ &\leq \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} B(N) \alpha \varepsilon \left[S^{\frac{N}{2}} \frac{A(N)}{B(N)} H(P) \frac{1}{\alpha} - 1 + o(1) \right] \\ &= \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}} S \alpha \varepsilon \left[A(N) H(P) \frac{1}{\alpha} - 1 + o(1) \right] \end{aligned}$$

as $\varepsilon \to 0$. Since, by assumption, $\alpha < A(N) \max_{\partial\Omega} H = A(N) H(P)$, $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$. ▮

COROLLARY 5.1. *The value α_0 is greater than or equal to $A(N) \max_{\partial\Omega} H$.*

We let $|\Omega|$ denote the Lebesgue measure of Ω. By testing I_α with constant functions we obtain

LEMMA 5.3. *If* $a \leq \frac{S}{(2|\Omega|)^{\frac{2}{N}}}$, *then* $\alpha_0 \geq \max\left\{\alpha \in [0,+\infty[\,\middle|\, I_\alpha(1) \leq \frac{S}{2^{\frac{2}{N}}}\right\}$.

Note. The value of $I_\alpha(1)$ is

$$I_\alpha(1) = \frac{|\Omega|^{\frac{2}{N}}}{(2^{\#})^{\frac{2}{N}}}\left[\left(\frac{\alpha+\sqrt{\alpha^2+4a}}{2}\right)^N + \frac{2^*}{2}a\left(\frac{\alpha+\sqrt{\alpha^2+4a}}{2}\right)^{N-2}\right]^{\frac{2}{N}}.$$

We have not determined the exact value of α_0. However, using the ideas of Chabrowski and Willem [10], we have the following proposition concerning existence of least energy solutions for $\alpha = \alpha_0$:

PROPOSITION 5.1. *If* $\alpha_0 > A(N)\max_{\partial\Omega} H$ *then there exists a least energy solution of* (1_{α_0}).

Proof. Choose a sequence $\alpha_k \nearrow \alpha_0$. Let u_k be a minimizer of I_{α_k} satisfying (1_{α_k}). As in the proof of Lemma 3.3, we conclude that the sequence (u_k) is bounded in $H^1(\Omega)$. So we can assume $u_k \rightharpoonup u$.

We claim that $u \neq 0$. Suppose, by contradiction, that $u = 0$. If the norms $|u_k|_{L^\infty(\Omega)}$ are uniformly bounded, then, from (26), $|u|_{2^*}^{2^*} = \frac{S^{\frac{N}{2}}}{2}$, which contradicts $u = 0$. If $|u_k|_{L^\infty(\Omega)} \to +\infty$, then Lemma 3.4 implies that we can repeat the argument of the previous sections to conclude that $S_{\alpha_k} > \frac{S}{2^{\frac{2}{N}}}$, for large k. This is also a contradiction. So $u \neq 0$.

Since $u \neq 0$, the argument in the proof of Lemmas 3.1 and 3.2 yields that u is a least energy solution of (1_{α_0}). Indeed, with the notations in the proof of Lemma 3.1, $x_0 \neq 0$. If $[h(1)]^{\frac{2}{N}}/[4(2^{\#})^{\frac{2}{N}}] > \frac{S}{2^{\frac{2}{N}}}$, then $S_{\alpha_0} = [h(x_0)]^{\frac{2}{N}}/[4(2^{\#})^{\frac{2}{N}}] > \frac{S}{2^{\frac{2}{N}}}$. Hence $I_{\alpha_0}(u) = [h(1)]^{\frac{2}{N}}/[4(2^{\#})^{\frac{2}{N}}] = \frac{S}{2^{\frac{2}{N}}}$. ∎

Remark 5.1. If a is sufficiently small and Ω is a (unit) ball, then the lower bound for α_0 in Corollary 5.1 is smaller than the lower bound for α_0 in Lemma 5.3 so that the previous proposition applies.

Proof. The lower bound for α_0 in Corollary 5.1 is $A(N)$, given in (47). As $a \to 0$, the lower bound for α_0 in Lemma 5.3 tends to

$$\begin{aligned}\left(\frac{(2^{\#})^{\frac{2}{N}}}{|\Omega|^{\frac{2}{N}}}\frac{S}{2^{\frac{2}{N}}}\right)^{\frac{1}{2}} &= \left(\frac{2^{\#}}{2}\right)^{\frac{1}{N}} S^{\frac{1}{2}}\frac{1}{|\Omega|^{\frac{1}{N}}}\\ &= \left(\frac{N-1}{N-2}\right)^{\frac{1}{N}}\pi^{\frac{N+1}{2N}}\frac{1}{2^{\frac{N-1}{N}}\left[\Gamma\left(\frac{N+1}{2}\right)\right]^{\frac{1}{N}}}[N(N-2)]^{\frac{1}{2}}\frac{\left[\Gamma\left(\frac{N+2}{2}\right)\right]^{\frac{1}{N}}}{\pi^{\frac{1}{2}}}\end{aligned}$$

$$= \left[\frac{\pi^{\frac{1}{2}}}{2^{N-1}} \left(\tfrac{N-1}{N-2}\right) \frac{\Gamma\left(\frac{N+2}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)} \right]^{\frac{1}{N}} [N(N-2)]^{\frac{1}{2}}.$$

∎

Suppose now $\alpha_0 = A(N)\max_{\partial\Omega} H$. Once again, choose a sequence $\alpha_k \nearrow \alpha_0$ and let u_k be a minimizer of I_{α_k} satisfying (1_{α_k}). The argument in the proof of the previous proposition shows that, modulo a subsequence, either $u_k \rightharpoonup u \neq 0$, or $u_k \rightharpoonup 0$ and $|u_k|_{L^\infty(\Omega)} \to +\infty$. *We have not determined which of these alternatives holds.* In the first case u is a least energy solution of (1_{α_0}). In the second case let, as before, P_k be such that $u_k(P_k) = |u_k|_{L^\infty(\Omega)}$. Any limit point of (P_k) is contained in the set of points of maximum mean curvature of $\partial\Omega$. For if y_0 is a limit point of P_k, then

$$\begin{aligned} -2^{\frac{N-2}{N}} SH(y_k)A(N)\varepsilon_k &= \left[2^{\frac{N-2}{N}} SH(y_0)A(N)\varepsilon_k - 2^{\frac{N-2}{N}} SH(y_k)A(N)\varepsilon_k\right] \\ &\quad -2^{\frac{N-2}{N}} SH(y_0)A(N)\varepsilon_k \\ &= -2^{\frac{N-2}{N}} SH(y_0)A(N)\varepsilon_k + o(\varepsilon_k). \end{aligned}$$

If $H(y_0) < \max_{\partial\Omega} H$, then the argument in the previous section shows that $S_{\alpha_k} > \frac{S}{2^{\frac{2}{N}}}$, for large k.

We summarize these observations in

Proposition 5.2. *Suppose $\alpha_0 = A(N)\max_{\partial\Omega} H$. Then*

(i) either there exists a least energy solution of (1_{α_0}),

(ii) or any sequence, u_k, of least energy solutions of (1_{α_k}), for $\alpha_k < \alpha_0$, $\alpha_k \to \alpha_0$, has a subsequence, u_k, $u_k \rightharpoonup 0$, $|u_k|_{L^\infty(\Omega)} \to +\infty$; the limit points of any sequence of maximums of u_k are contained in the set of points of maximum mean curvature of the boundary of Ω.

APPENDIX A
The functional restricted to the Nehari manifold

In this Appendix we start by checking, using standard arguments, that the Nehari set $\mathcal{N}$ is a manifold and a natural constraint for Φ_α (defined in (2)). We then derive the expressions (3) and (6) for Φ_α restricted to $\mathcal{N}$, we derive an expression for I_α (defined in (8)) equivalent to (9) and to (10), and we derive upper and lower bounds for I_α.

Consider the set

$$\mathcal{N} := \left\{ u \in H^1(\Omega) : \Phi'_\alpha(u)u = 0, u \neq 0 \right\},$$

where Φ_α is the C^2 functional defined in (2), and define $J_\alpha : H^1(\Omega) \to \mathcal{R}$ by

$$J_\alpha(u) := \Phi'_\alpha(u)u = ||u||^2 + \alpha|u|_{2^\#}^{2^\#} - |u|_{2^*}^{2^*}.$$

The set $\mathcal{N} = \left\{u \in H^1(\Omega) : J_\alpha(u) = 0, u \neq 0\right\}$, is a manifold (called the Nehari manifold). Indeed, if $u \in \mathcal{N}$, then $J'_\alpha(u) \neq 0$, because if $J_\alpha(u) = 0$ and $J'_\alpha(u)u = 0$, then

$$0 = 2^* J_\alpha(u) - J'_\alpha(u)u = (2^* - 2)||u||^2 + (2^* - 2^\#)\alpha|u|_{2^\#}^{2^\#}.$$

This yields $u = 0$. Furthermore, the Nehari manifold is a natural constraint for Φ_α, by which we mean that any critical point of $\Phi_\alpha|_\mathcal{N}$ is a critical point of Φ_α. In fact, suppose that $u \in \mathcal{N}$ is a critical point of $\Phi_\alpha|_\mathcal{N}$. Then there exists a $\lambda \in \mathcal{R}$ such that $\Phi'_\alpha(u) = \lambda J'_\alpha(u)$. Applying both sides to u, $0 = J_\alpha(u) = \Phi'_\alpha(u)u = \lambda J'_\alpha(u)u$. However, we just saw that $J'_\alpha(u)u \neq 0$ if $J_\alpha(u) = 0$ (and $u \neq 0$). It follows that $\lambda = 0$ and u is a critical point of Φ_α.

For any $u \in H^1(\Omega) \setminus \{0\}$ there exists a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}$, i.e. $\Phi'_\alpha(t(u)u)t(u)u = 0$. The value of $t(u)$ is the solution of

$$||u||^2 + \alpha|u|_{2^\#}^{2^\#}[t(u)]^{2^\# - 2} - |u|_{2^*}^{2^*}[t(u)]^{2^* - 2} = 0.$$

Since $2^\# - 2 = \frac{2}{N-2}$ is half of $2^* - 2$, the equation

$$\mathrm{a} + bt^{2^\# - 2} - ct^{2^* - 2} = 0.$$

is quadratic in $t^{\frac{2}{N-2}}$. Define the functionals a, b and $c : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ by

$$\begin{aligned} \mathrm{a}(u) &:= ||u||^2, \\ b(u) &:= \alpha|u|_{2^\#}^{2^\#} = b_\alpha(u), \\ c(u) &:= |u|_{2^*}^{2^*}. \end{aligned}$$

(Note that $\mathrm{a} \neq a$.) The value of $t(u)$ is

$$t(u) = \left(\frac{b + \sqrt{b^2 + 4\mathrm{a}c}}{2c}\right)^{\frac{N-2}{2}}(u). \tag{A.1}$$

The functional $t : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ is obviously continuous and the map $u \mapsto t(u)u$ defines a homeomorphism of the unit sphere in $H^1(\Omega)$ with $\mathcal{N}$. Its inverse is the retraction $u \mapsto \frac{u}{||u||}$.

We define $\Psi_\alpha : H^1(\Omega) \setminus \{0\} \to \mathcal{R}$ by

$$\Psi_\alpha(u) := \Phi_\alpha(t(u)u).$$

In terms of a, b, c and t,

$$\Psi_\alpha = \frac{1}{2}at^2 + \frac{1}{2^\#}bt^{2^\#} - \frac{1}{2^*}ct^{2^*}.$$

Replacing (A.1) into this expression for Ψ_α, and simplifying, leads to

$$\Psi_\alpha = \frac{1}{N}\frac{1}{2^\#}\left[\left(\frac{b+\sqrt{b^2+4ac}}{2c}\right)^N c + \frac{2^*}{2}\left(\frac{b+\sqrt{b^2+4ac}}{2c}\right)^{N-2} a\right].$$

We now introduce the functionals β, $\gamma : H^1(\Omega)\setminus\{0\} \to \mathcal{R}$, defined by

$$\beta := \frac{a}{c^{\frac{N-2}{N}}}$$

and

$$\gamma = \gamma_\alpha := \frac{b}{c^{\frac{N-1}{N}}},$$

as in expressions (4) and (5), respectively. In terms of β and γ, the expression for Ψ_α is

$$\Psi_\alpha = \frac{1}{N}\frac{1}{2^\#}\frac{1}{2^N}\left[\left(\gamma+\sqrt{\gamma^2+4\beta}\right)^N + 2\cdot 2^*\beta\left(\gamma+\sqrt{\gamma^2+4\beta}\right)^{N-2}\right].$$

This is (3). If we introduce still another functional $\delta : H^1(\Omega)\setminus\{0\} \to \mathcal{R}$, defined by

$$\delta = \delta_\alpha := \frac{\gamma}{2\sqrt{\beta}},$$

as in expression (7), then we can write Ψ_α as

$$\Psi_\alpha = \frac{1}{N}\frac{\beta^{\frac{N}{2}}}{2^\#}\left[\left(\delta+\sqrt{\delta^2+1}\right)^N + \frac{2^*}{2}\left(\delta+\sqrt{\delta^2+1}\right)^{N-2}\right].$$

This is (6).

We give an expression for $I_\alpha = (N\Psi_\alpha)^{\frac{2}{N}}$, defined in (8), equivalent to (9) and to (10):

$$I_\alpha = \beta\left(\delta+\sqrt{\delta^2+1}\right)^{\frac{4}{2^*}}\left(\frac{2}{2^\#}\delta^2 + \frac{2}{2^\#}\delta\sqrt{\delta^2+1}+1\right)^{\frac{2}{N}}.$$

Since

$$\frac{4}{2^*} + \frac{2}{N}\frac{2}{2^\#} = \frac{4}{2^\#},$$

I_α has the lower bound

$$I_\alpha \geq \beta\left(1+\frac{4}{2^\#}\delta\right). \tag{A.2}$$

For an upper bound for I_α we refer to Lemma 5.1.

APPENDIX B
The estimate for $|U_k|_{2^\#}^{2^\#}$

In this Appendix we use the ideas of Adimurthi and Mancini [1] to prove (43).

We wish to estimate $|U_{\varepsilon,y}|_{2^\#}^{2^\#}$, where $U_{\varepsilon,y}$ is defined in (14) and $y \in \partial\Omega$. By a change of coordinates we can assume that $y = 0$,

$$B_R(0) \cap \Omega = \{(x', x_N) \in B_R(0) | x_N > \rho(x')\}$$

and

$$B_R(0) \cap \partial\Omega = \{(x', x_N) \in B_R(0) | x_N = \rho(x')\},$$

for some $R > 0$, where $x' = (x_1, \ldots, x_{N-1})$,

$$\rho(x') = \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3),$$

$\lambda_i \in \mathcal{R}$, $1 \leq i \leq N-1$.

We begin by supposing all the λ_i's are positive. Let $U_\varepsilon := U_{\varepsilon,0}$ and $\Sigma := \{(x', x_N) \in B_R(0) | 0 < x_N < \rho(x')\}$. Then

$$|U_\varepsilon|_{2^\#}^{2^\#} = \frac{1}{2}\int_{B_R(0)} U_\varepsilon^{2^\#} - \int_\Sigma U_\varepsilon^{2^\#} + \int_{B_R^C(0)\cap\Omega} U_\varepsilon^{2^\#}. \tag{B.1}$$

We will estimate each of the three terms on the right hand side of (B.1). For the third term we have

$$\begin{aligned}
\int_{B_R^C(0)\cap\Omega} U_\varepsilon^{2^\#} &\leq \int_{B_R^C(0)} U_\varepsilon^{2^\#} \\
&= O\left(\varepsilon \int_{R/\varepsilon}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{N-1}} dr\right) \\
&= O(\varepsilon \times \varepsilon^{N-2}) \\
&= O(\varepsilon^{N-1})
\end{aligned}$$

Using this estimate, for the first term on the right hand side of (B.1) we have

$$\begin{aligned}\frac{1}{2}\int_{B_R(0)} U_\varepsilon^{2^\#} &= \frac{1}{2}\int_{\mathcal{R}^N} U_\varepsilon^{2^\#} + O(\varepsilon^{N-1})\\ &= \frac{1}{2}\varepsilon\int_{\mathcal{R}^N} U^{2^\#} + O(\varepsilon^{N-1})\\ &= \frac{2^\#}{2}B(N)\varepsilon + O(\varepsilon^{N-1}),\end{aligned}$$

with

$$\begin{aligned}B(N) &:= \frac{1}{2^\#}\int_{\mathcal{R}^N} U^{2^\#}\\ &= \frac{1}{2^\#}\,\omega_N\int_0^{+\infty}\frac{r^{N-1}}{(1+r^2)^{N-1}}\,dr\times[N(N-2)]^{\frac{N}{2}}\\ &= \tfrac{N-2}{2(N-1)}\,\omega_N\times\frac{1}{2^{N-1}}\sqrt{\pi}\,\frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\times[N(N-2)]^{\frac{N}{2}}\\ &= \omega_N\frac{1}{2^N}\sqrt{\pi}\,\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)}[N(N-2)]^{\frac{N}{2}};\end{aligned}$$

here ω_N is the volume of the $N-1$ dimensional unit sphere.

So we are left with the estimate of the second term on the right hand side of (B.1). Let $\sigma > 0$ be such that

$$L_\sigma := \{x\in\mathcal{R}^N \,|\, |x_i| < \sigma, 1\le i\le N\}\subset B_{\frac{R}{4}}(0)$$

and define

$$\Delta_\sigma := \{x' \,|\, |x_i| < \sigma, 1\le i\le N-1\}.$$

For the second term on the right hand side of (B.1),

$$\begin{aligned}\int_\Sigma U_\varepsilon^{2^\#} &= \int_{\Sigma\cap L_\sigma} U_\varepsilon^{2^\#} + O(\varepsilon^{N-1})\\ &= \int_{\Delta_\sigma}\int_0^{\rho(x')} U_\varepsilon^{2^\#}\,dx_N\,dx' + O(\varepsilon^{N-1})\\ &= O\left(\int_{\Delta_\sigma}\int_0^{\rho(x')}\frac{\varepsilon^{N-1}}{(\varepsilon^2+|x|^2)^{N-1}}\,dx_N\,dx'\right) + O(\varepsilon^{N-1});\end{aligned}$$

using the change of variables $\sqrt{\varepsilon^2+|x'|^2}\,y_N = x_N$,

$$
\begin{aligned}
&= O\left(\int_{\Delta_\sigma} \frac{\varepsilon^{N-1}}{(\varepsilon^2+|x'|^2)^{N-\frac{3}{2}}} \int_0^{\frac{\rho(x')}{\sqrt{\varepsilon^2+|x'|^2}}} \frac{1}{(1+y_N^2)^{N-1}}\, dy_N\, dx'\right)\\
&\quad + O(\varepsilon^{N-1});
\end{aligned}
$$

since $\int_0^s \frac{1}{(1+t^2)^{N-1}}\, dt = s + O(s^3)$,

$$
\begin{aligned}
&= O\left(\varepsilon^{N-1}\int_{\Delta_\sigma} \frac{\sum \lambda_i x_i^2}{(\varepsilon^2+|x'|^2)^{N-1}}\, dx'\right)\\
&\quad + O\left(\varepsilon^{N-1}\int_{\Delta_\sigma} \frac{|x'|^3}{(\varepsilon^2+|x'|^2)^{N-1}}\, dx'\right)\\
&\quad + O(\varepsilon^{N-1})\\
&= O\left(\varepsilon^{2}\int_{\Delta_\sigma/\varepsilon} \frac{|y'|^2}{(1+|y'|^2)^{N-1}}\, dy'\right)\\
&\quad + O\left(\varepsilon^{3}\int_{\Delta_\sigma/\varepsilon} \frac{|y'|^3}{(1+|y'|^2)^{N-1}}\, dy'\right)\\
&\quad + O(\varepsilon^{N-1})\\
&= O(\varepsilon^2).
\end{aligned}
$$

Combining the estimates for the three terms on the right hand side of (B.1),

$$
|U_\varepsilon|_{2^\#}^{2^\#} = \frac{2^\#}{2} B(N)\varepsilon + O(\varepsilon^2), \tag{B.2}
$$

if all the λ_i's are positive. If all the λ_i's are negative, then the minus sign on the right hand side of (B.1) turns into a plus sign, and (B.2) follows. From these two cases we deduce that (B.2) holds no matter what the sign of the λ_i's is.

ACKNOWLEDGMENTS

This work was initiated during the visit of the first author to Instituto Superior Técnico, whose kind hospitality and support he gratefully acknowledges.

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