

# The Crystalline Algorithm for Computing Motion by Curvature

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**Abstract.** Motion by (weighted) mean curvature is a geometric evolution law for surfaces, representing steepest descent with respect to (anisotropic) surface energy. The crystalline algorithm is a numerical method for computing this motion. The main idea of this method is to solve the analogous evolution law for a crystalline surface energy which approximates the underlying smooth one. We have recently explored the nature of this method, demonstrating its convergence in some simple special cases. This paper summarizes our results.

## 1. Introduction

This is an expository discussion of the crystalline algorithm for computing motion by curvature. Our goal is to explain the nature of this numerical scheme, what is known about it, and what remains open. We focus mainly on our recent work [18] and [19]. However, we also discuss related work by other authors.

## 2. Motion by weighted mean curvature

We are interested in the motion of phase boundaries represented by curves in the plane, or surfaces in three-space.

The definition of *weighted mean curvature* is by now standard. Curvature is defined by the first variation of surface area, and weighted mean curvature is the analogous object for an anisotropic surface energy. To make this more explicit, we begin with a surface energy density  $f$  defined on unit vectors. It determines the surface energy functional

$$E = \int_C f(n) ds ,$$

defined for any sufficiently regular phase boundary  $C$ . Here  $n$  is the inward unit normal;  $ds$  denotes arc length for curves in the plane and Hausdorff measure for surfaces in three-space. Now consider a one-parameter family of interfaces  $C_\lambda$ , obtained by moving each point  $x \in C$  distance  $\lambda h(x)$  in the normal direction. The first variation of surface energy has the form

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} E(C_\lambda) = - \int_C \omega h ds$$

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for some  $\omega : C \rightarrow \mathbb{R}$ . This function  $\omega$  is the weighted mean curvature of  $C$ . When the surface energy is isotropic ( $f \equiv 1$ ) it becomes the mean curvature in the usual geometric sense. (Under our sign conventions, the curvature is  $1/r$  when  $C$  is a circle of radius  $r$ .)

Motion by weighted mean curvature is a dissipative geometrical evolution, amounting to steepest descent for surface energy. Its equation is

$$(1) \quad V = \omega ,$$

where  $V$  is the normal velocity of the interface.

Our interest arises from applications to materials science, see e.g. [20, 37]. In truth these applications rarely involve something so simple as (1). A more typical law is

$$V = M(n)[\omega + g(x, n)] ,$$

where  $M(n)$  represents the mobility of the interface and  $g(x, n)$  is the driving force. In many situations diffusion is crucial; then the driving force  $g$  depends on a temperature or concentration variable, which in turn satisfies a diffusion equation. We shall nevertheless concentrate on (1), viewing it as a special but central example and a laboratory for the development of new techniques.

We have not yet imposed any structural hypotheses upon the surface energy density  $f$ . We are mainly interested in “nice” surface energies – smooth functions  $f$  for which the associated evolution equation (1) is parabolic and well-posed. There are several equivalent characterizations of the acceptable  $f$ 's. One is that the Frank diagram (the polar plot of  $1/f$ ) should be strictly convex. Another is that the function  $\xi \rightarrow |\xi|f(\xi/|\xi|)$  should be convex, and strictly so except in the radial direction. For the motion of curves in the plane we can write  $f = f(\theta)$  by taking  $n = e^{i\theta}$ ; the equivalent condition is then  $f(\theta) + f''(\theta) > 0$ . See for example [6], [35] or the appendix of [19] for this basic material.

There are by now many numerical schemes for solving (1) and its generalizations. A comprehensive review is beyond the scope of this paper. We nevertheless mention several approaches, with selected references and apologies in advance to those whose work is omitted.

**Front-tracking** is the most classical and accurate method, especially for curves in the plane, but it has difficulty accommodating changes of topological type. The main idea is to represent the moving interface as a time-dependent, parameterized curve  $s \rightarrow \phi(s, t)$ . This leads to a nonlinear parabolic equation for  $\phi$ . Current ideas on efficient solution techniques can be found in [21].

The **level-set method** is especially convenient for problems involving complex initial data or changes of topological type. The main idea is to represent the interface as the level set of a function  $\{x : v(x, t) = 0\}$ . The condition that the interface move by curvature is a degenerate parabolic equation  $v_t/|\nabla v| = \operatorname{div}(\nabla v/|\nabla v|)$ . First introduced as a numerical tool [26, 31], this approach has also become an essential analytical technique [12, 15]. Some convergence results are given in [39].

The **Allen-Cahn equation** is of interest partly because it makes contact with the phase-field approach to solidification. The interface is again a level set  $\{x : u(x, t) = 0\}$ , but the function  $u(x, t)$  solves a different equation  $u_t - \Delta u + \epsilon^{-2}(u^3 - u) = 0$ . Motion by curvature is achieved in the limit  $\epsilon \rightarrow 0$ , not only for classical solutions but also for singular ones [9, 14, 32]. The Allen-Cahn approach appears awkward numerically, because it requires the solution of a very stiff system. Phase field models have, however, been implemented numerically with considerable success, see for example [10, 22, 38, 40, 41]. A different but related “double obstacle” approach to motion by curvature is discussed in [24], and convergence results are given in [25].

The **time-implicit method** solves a geometric variational method at each time step. The theory of flat curvature flows developed in [2] makes use of such an approximation. Numerically, a scheme of this type was used to simulate solidification in [3].

A **time-splitting method** was recently introduced in [23]. The main goal of that work was to deal with multiphase problems, however the method is also interesting in the two-phase context. The basic iteration has two parts: first one blurs the interface by diffusion, then one sharpens it again. Convergence results are given in [8, 13].

Our list omits the **crystalline algorithm**, since it is the focus of the rest of this paper.

### 3. Crystalline surface energies

We said we were mainly interested in “nice” surface energies, i.e. smooth  $f$ ’s with strictly convex Frank diagrams. But that is not the whole truth. We are also interested in “crystalline” surface energies, by which we mean those whose Frank diagrams are *convex polygons*<sup>1</sup>. (Remember, the Frank diagram is the polar plot of  $1/f$ , in other words the curve defined in polar coordinates by  $r = 1/f(e^{i\theta})$ .)

A crystalline energy determines a finite family of distinguished normal vectors — the vertices of the Frank diagram. It is natural to consider the restricted class of piecewise linear surfaces with just these normals. Such surfaces have no curvature in the conventional geometric sense (more precisely, the curvature is zero on each face and infinite at each vertex). But one can still define their weighted mean curvature *variationally*: taking the first variation of surface energy within this *restricted* class of surfaces associates to each face a well-defined (constant) weighted mean curvature. Angenent and Gurtin [6] and Taylor [33, 36] used this observation to define motion by weighted mean curvature in the crystalline case. If no new faces

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<sup>1</sup>Our use of the term “crystalline” is not quite the standard one. Many authors would call a nonconvex surface energy crystalline if the *convexification* of its Frank diagram is a convex polygon. In this paper, we never consider surface energies with nonconvex Frank diagrams.

are created, then the equation  $V = \omega$  amounts to a system of ODE's for the normal displacements of the initial faces.

This idea works better in the plane than in three-space: for surfaces in  $\mathbb{R}^3$  the resulting system of ODE's may cease to have a solution in finite time [36]. This shouldn't be too puzzling – it just means that the restricted class of surfaces was too small. According to Taylor, the motion must be continued as a varifold.

Even in the plane the situation is more subtle than it seems. We assumed above that no new faces are created. But for problems of the type  $V = \omega + g$  this hypothesis seems to be wrong. The shattering process discussed in [28, 29, 30] is specifically designed to detect the creation of new faces.

These difficulties don't arise, however, when solving  $V = \omega$  for curves in the plane with crystalline initial data. This assertion is far from self-evident; it follows from the analysis summarized herein and also from the work of Taylor [36]. As a result, the use of a crystalline rather than smooth surface energy radically simplifies the problem of motion-by-curvature for plane curves, by replacing a PDE with a system of ODE's.

#### 4. Crystalline approximation as a numerical method

Thus crystalline approximation presents itself as a physically natural numerical method for computing motion by curvature [27]. Suppose our goal is to compute the evolution of a smooth curve  $C$ , under the motion by weighted curvature associated with a convex surface energy  $f$ . The crystalline algorithm (i) approximates  $f$  with a crystalline surface energy  $\tilde{f}$ , then (ii) approximates  $C$  with a polygonal curve  $\tilde{C}$  whose normals lie in the directions preferred by  $\tilde{f}$ , then finally (iii) solves the ODE's describing motion by weighted curvature for this crystalline problem.

What kind of method is this? Basically, a sort of *nonlinear Galerkin scheme*. A traditional Galerkin method for solving a nonlinear evolution equation starts with a basis  $\{\phi_k(x)\}_{k=1}^{\infty}$ . One seeks an approximate solution of the form  $u(x, t) = \sum_{k=1}^N a_k(t)\phi_k(x)$  by solving a suitable ODE for the coefficients  $a_k(t)$ . In crystalline approximation the spatial discretization is different and more nonlinear: rather than restrict  $u(\cdot, t)$  to a finite dimensional linear space, we restrict the normals of the moving curve instead.

Curiously, for closed convex curves in the plane crystalline approximation can also be viewed quite differently. When expressed in appropriate variables it becomes a sort of *finite difference scheme*, as we shall explain below.

#### 5. Graphs in the plane

To examine this method in detail, we now consider the motion of a graph in the plane [19]. The unknown is a scalar-valued function  $u(x, t)$  defined for  $x \in [0, 1]$  and  $t > 0$ . Its (anisotropic) surface energy can be expressed as

$$\int_0^1 W(u_x) dx$$

with  $W$  convex, and motion by weighted curvature is described by the quasilinear parabolic PDE

$$(2) \quad \frac{u_t}{(1+u_x^2)^{1/2}} = W''(u_x)u_{xx}.$$

We shall suppose that the ends are pinned, so the boundary condition is  $u(0, t) = u(1, t) = 0$ .

The crystalline algorithm considers only functions with special tangents. This means we permit only special values of  $u_x$ . The list of permitted  $u_x$  determines a piecewise linear approximation to  $W$ , whose graph has corners at the associated points  $(u_x, W(u_x))$ . This is our crystalline energy.

We now specify the discrete equation of motion. Let

$$(3) \quad m = \text{the maximum distance between permitted values of } u_x;$$

this is the ‘‘mesh size’’ of the crystalline approximation. The numerical solution  $u^m(x, t)$  should be piecewise linear in  $x$  at each time  $t$ , with vertices at  $0 = x_0 < x_1(t) < \dots < x_{N-1}(t) < x_N = 1$ . Its slopes may take only permitted values, with adjacent segments taking adjacent slopes. We shall refer to the segment between  $x_{i-1}$  and  $x_i$  as the  $i$ th face. Let  $(u_x^m)_i$ ,  $(u_t^m)_i$  be the (constant-in-space) values of  $u_x^m$  and  $u_t^m$  on the  $i$ th face. A calculation using only the continuity of  $u^m$  gives

$$\dot{x}_i = -\frac{(u_t^m)_{i+1} - (u_t^m)_i}{(u_x^m)_{i+1} - (u_x^m)_i},$$

so the motion of the curve is fully determined by specifying the values of  $(u_t^m)_i$ . They are determined, of course, by the crystalline analogue of (2). The first variation of crystalline energy turns out to be  $-\Delta_i/L_i$  on the  $i$ th face, where  $L_i = x_i - x_{i-1}$  is the length of the  $i$ th face,

$$\Delta_i = \frac{W[(u_x^m)_{i+1}] - W[(u_x^m)_i]}{(u_x^m)_{i+1} - (u_x^m)_i} - \frac{W[(u_x^m)_i] - W[(u_x^m)_{i-1}]}{(u_x^m)_i - (u_x^m)_{i-1}}$$

for  $2 \leq i \leq N-1$ , and  $\Delta_1 = \Delta_N = 0$ . So the discrete evolution is

$$(4) \quad (u_t^m)_i = \sqrt{1 + (u_x^m)_i^2} \cdot \frac{\Delta_i}{L_i}.$$

We must also discretize the initial data, i.e.  $u^m(x, 0)$  must be a crystalline approximation of  $u(x, 0)$ . One convenient method starts by drawing those tangents to the graph of  $u(x, 0)$  which lie in permitted directions. They can be pieced together (modifying them a bit at the endpoints) to obtain  $u^m(x, 0)$ .

Viewed this way, there is nothing intrinsically geometric about the crystalline approximation scheme. It is a nonlinear Galerkin scheme, suitable for solving any parabolic equation of the form  $u_t = g(u_x)W''(u_x)u_{xx}$ . It can also be used to solve the *linear* heat equation,  $u_t = u_{xx}$ : then (4) simplifies to

$$(u_t^m)_i = \frac{(u_x^m)_{i+1} - (u_x^m)_{i-1}}{2L_i}.$$

In deriving the crystalline motion, we assumed that neighboring faces of  $u^m$  have neighboring slopes  $u_x^m$ . One can show that this property is preserved by the flow, even as faces disappear [19, 36]. Therefore the crystalline motion is well-defined. (One must of course re-index at the times when faces disappear.)

Our paper [19] demonstrates the convergence of this scheme. The analysis is simplest and the result strongest for the linear heat equation. In that case

$$(5) \quad \frac{d}{dt} \int_0^1 |u_x - u_x^m|^2 dx \leq -2 \int_0^1 |u_t - u_t^m|^2 dx + m \|u_{xxx}\|_\infty(t),$$

where  $m$  is defined by (3). The proof resembles the standard one for a linear Galerkin method: it uses only the evolution equations, the boundary conditions, and Taylor expansion of  $u$ . We may suppose that the initial data satisfies

$$\max_x |u_x - u_x^m| \leq m \quad \text{at } t = 0,$$

for example by choosing it as indicated above. Then (5) leads immediately to the  $H^1$  convergence result:

$$\int_0^1 |u_x - u_x^m|^2 dx \leq m^2 + m \int_0^t \|u_{xxx}\|_\infty(\tau) d\tau \quad \text{for each } t \geq 0.$$

The situation is similar though more complicated for the nonlinear equation (4). One must use the Taylor expansion of  $W$  as well as that of  $u$ , and the resulting differential inequality is

$$\frac{d}{dt} \int_0^1 |u_x - u_x^m|^2 dx \leq A(t) \int_0^1 |u_x - u_x^m|^2 dx + B(t)m$$

rather than (5). Here  $A(t)$  and  $B(t)$  depend on the (smooth) solution  $u(x, t)$  but they are independent of  $m$ . An application of Gronwall's inequality gives once again a convergence result of the form

$$\max_{0 \leq t \leq T} \|u - u^m\|_{H^1([0,1])} \leq C_T m^{1/2}.$$

Our discussion has focussed on the homogeneous Dirichlet boundary condition  $u(0, t) = u(1, t) = 0$ , but the situation is similar for the homogeneous Neumann boundary condition  $u_x(0, t) = u_x(1, t) = 0$ .

## 6. Closed convex curves in the plane

We have shown that the crystalline approximation scheme is a sort of nonlinear Galerkin method. Curiously, however, when applied to closed convex curves and viewed in the proper variables it looks more like a finite difference scheme [18].

The motion of a convex curve in the plane has a convenient curvature-angle representation, in which its (weighted) curvature  $\omega$  is viewed as a function of time and the angle  $\theta$  between its normal and some fixed axis. This is very well known

for isotropic motion by curvature; the more general, anisotropic case is treated for example in [6]. If  $f$  is the surface energy density represented as a function of  $\theta$ , then the weighted curvature  $\omega$  can be expressed in terms of the ordinary curvature  $\kappa$  as

$$\omega = (f + f'')\kappa ,$$

and motion by weighted curvature is expressed by the PDE

$$(6) \quad \omega_t = (f + f'')^{-1}(\omega^2\omega_{\theta\theta} + \omega^3) .$$

The crystalline algorithm considers only curves with special tangents, i.e. convex polygons. The faces are restricted to certain directions, corresponding to a list of permitted angles  $\{\theta_i\}_{i=1}^N$ . In the following analysis we take these angles to be equally spaced:

$$\theta_i = i\Delta\theta , \quad \Delta\theta = 2\pi/N .$$

There is an associated crystalline energy. It agrees with  $f$  at the special angles  $\theta_i$ , and its Frank diagram is the  $N$ -sided polygon determined by these vertices.

The discretized evolution is motion by weighted mean curvature for this crystalline energy. It says

$$(7) \quad V_i = \omega_i ,$$

where  $V_i$  and  $\omega_i$  are respectively the normal velocity and weighted curvature of the  $i$ th face (the face with angle  $i\Delta\theta$ ). We must also discretize the initial data, replacing a smooth convex curve by a polygon with permitted faces. The method suggested earlier for graphs works well here too: it forms the initial polygon as a union of line segments tangent to the initial curve at the points with permitted normals.

With some work, (7) can be put in a form analogous to (6). We outline the main steps, referring to [18] for the details. Starting from the definition of  $\omega$  as the first variation of surface energy one finds that

$$\omega_i = \left[ f_i + \frac{f_{i+1} - 2f_i + f_{i-1}}{2(1 - \cos \Delta\theta)} \right] \kappa_i$$

where

$$\kappa_i = 2 \tan \left( \frac{\Delta\theta}{2} \right) \frac{1}{L_i}$$

is the curvature of the  $i$ th face (defined by the first variation of isotropic surface energy),  $f_i = f(i\Delta\theta)$ , and  $L_i$  is the length of the  $i$ th face. Also, from kinematic considerations, a solution of (7) has

$$\dot{L}_i = 2\omega_i \cot \Delta\theta - \omega_{i-1} \csc \Delta\theta - \omega_{i+1} \csc \Delta\theta .$$

Using these formulas, one arrives with some calculation at the desired analogue of (6):

$$(8) \quad \dot{\omega}_i = \left[ f_i + \frac{f_{i+1} - 2f_i + f_{i-1}}{2(1 - \cos \Delta\theta)} \right]^{-1} \left[ \omega_i^2 \frac{\omega_{i+1} - 2\omega_i + \omega_{i-1}}{2(1 - \cos \Delta\theta)} + \omega_i^3 \right].$$

This is very nearly a standard finite-difference approximation to (6)! The only difference is the use of  $2(1 - \cos \Delta\theta)$  rather than  $(\Delta\theta)^2$  in the denominator. These two expressions are the same to principal order as  $\Delta\theta \rightarrow 0$ , so the convergence of this scheme is hardly in doubt.

There is, however, some work to do. In deriving the equation of motion we assumed that each  $\omega_i$  was well defined, i.e. that the evolving polygon makes use of each permitted angle. This can and must be proved. In fact, as the polygon shrinks its area  $A_{\Delta\theta}(t)$  decreases at a constant rate

$$\dot{A}_{\Delta\theta} = - \sum_i \frac{2(1 - \cos \Delta\theta)}{\sin \Delta\theta} \left[ f_i + \frac{f_{i+1} - 2f_i + f_{i-1}}{2(1 - \cos \Delta\theta)} \right]$$

until the extinction time  $T_{\Delta\theta}$  when  $A_{\Delta\theta} = 0$ , and no face disappears prior to  $T_{\Delta\theta}$ . This is the crystalline analogue of the theorem of Gage and Hamilton [17], that a smooth convex curve moving by curvature in the plane remains smooth and convex until its extinction time  $T = A(0) / \int_0^{2\pi} (f + f'') d\theta$ . Interestingly, the proof that no face disappears before the extinction time uses the presence of  $2(1 - \cos \Delta\theta)$  rather than  $(\Delta\theta)^2$  in (8).

We turn to the matter of convergence. Let  $\omega(\theta, t)$  describe the smooth curve as it moves by weighted curvature, and let  $\{\omega_i(t)\}$  be the crystalline solution described above with  $\Delta\theta$  sufficiently small. Our method of choosing the initial polygon turns out to satisfy

$$|\omega(i\Delta\theta, 0) - \omega_i(0)| \leq C(\Delta\theta)^2.$$

Since the difference approximation (8) is second-order accurate, it is not hard to deduce

$$|\omega(i\Delta\theta, t) - \omega_i(t)| \leq C_\tau(\Delta\theta)^2 \quad \text{for } t \leq \tau,$$

for any  $\tau$  less than the extinction time. A similar argument also shows

$$\left| \omega_\theta(i\Delta\theta, t) - \frac{\omega_{i+1}(t) - \omega_i(t)}{\sin \Delta\theta} \right| \leq C_\tau \Delta\theta \quad \text{for } t \leq \tau.$$

This gives convergence of the crystalline scheme in curvature-angle coordinates. However the natural question is the convergence of the curves themselves. This requires some extra work; the result proved in [18] is

$$D(C(t), P_{\Delta\theta}(t)) \leq C_\tau(\Delta\theta)^2 \quad \text{for } t \leq \tau,$$

where  $C(t)$  is the evolving smooth curve,  $P_{\Delta\theta}(t)$  is the crystalline approximation, and  $D$  is the Hausdorff metric.

## 7. Alternative viewpoints

We have carefully avoided discussing crystalline motion by curvature for non-crystalline initial data, i.e. for initial curves whose initial normals are not restricted to the vertices of the Frank diagram. We have no need to consider such curves, since a numerical scheme is entitled to use approximate initial data. In our treatment of graphs, the error due to approximating the initial data is small compared to the error due to approximating the PDE; for simple closed curves the two errors are of the same order.

Other authors have taken a different but equally valid viewpoint. Angenent and Gurtin [6] and Taylor [36] consider it important to discuss crystalline motion by curvature for non-crystalline initial data, because some real materials are described by crystalline surface energies. The treatments just cited are based on approximating noncrystalline data by oscillatory crystalline data, then passing to a suitable limit.

The crystalline evolution of graphs in the plane has also been considered by Fukui and Giga [16]. We summarize briefly their approach. Remembering (2), one is tempted to say that the crystalline evolution solves

$$u_t = \sqrt{1 + u_x^2} W''(u_x) u_{xx}$$

with  $W$  piecewise linear. This equation doesn't make sense as written, but it is formally equivalent to

$$(9) \quad u_t = (\bar{W}'(u_x))_x$$

with  $\bar{W}'(t) = \sqrt{1 + t^2} W'(t) - \int_0^t (\sqrt{1 + s^2})' W'(s) ds$ . There is an extensive theory concerning divergence-form heat equations such as (9), including theorems of existence and well-posedness even when  $\bar{W}$  is merely convex and piecewise  $C^1$ . This gives an alternative definition of crystalline evolution. Fukui and Giga show that for crystalline initial data, the solution obtained this way is identical to the one obtained using (4).

The work of Fukui and Giga has two important consequences. First, it applies even to noncrystalline initial data, giving a definition of such motion which is clearly well-posed and independent of any specific method for approximating the initial data. Second, their work gives another proof of convergence of the crystalline approximation scheme for graphs, since the solution depends continuously on the surface energy as well as on the data. However this argument does not give a convergence rate.

We have thus far discussed only curves in the plane, using methods that are intrinsically one-dimensional. A much more general approach has been developed by Almgren, Taylor, and Wang [2]. They prove the existence of a “flat curvature flow” – essentially, a weak solution of motion by weighted curvature – for a broad class of surface energies, including crystalline ones. Their approach is something like an implicit numerical method: it starts with a discrete-in-time solution, defined

by solving an appropriate variational problem at each time step; then estimates are proved yielding the existence of a continuous-time limit.

This approach gives existence but not uniqueness. However, that is not the fault of the method: flow by weighted mean curvature is not necessarily unique. There are examples of smooth hypersurfaces, even in  $\mathbb{R}^3$ , whose flow by (isotropic) curvature develops a “fattening singularity” and ceases to be unique thereafter [5, 7]. For graphs in the plane and convex closed curves the flat curvature flow is unique, since it agrees with the classical solution so long as one exists.

When applied to a crystalline energy, the flat curvature flow provides an alternative definition of motion by weighted curvature. But as in the work of Fukui and Giga, it has to be shown that this is the same as the evolution described by (4) or (7) when the initial data is crystalline. This fundamental consistency result is proved for curves in the plane in [1], provided that in the crystalline solution two adjacent faces do not vanish simultaneously.

The paper [2] does not discuss continuity with respect to perturbations of the surface energy integrand. If, however, the flat curvature flow could be shown to depend continuously on the surface energy, this would give another convergence proof for crystalline approximation as a numerical scheme. As in [16], such an argument would provide no convergence rate.

## 8. Discussion

We have discussed the crystalline approximation as a numerical scheme for computing the motion of smooth curves by weighted mean curvature. For graphs in the plane we have shown it to be a nonlinear Galerkin scheme, and demonstrated convergence with an error of order  $m^{1/2}$  in the  $H^1$  norm. For convex closed curves we have shown it to be a finite difference scheme, and demonstrated convergence with an error of order  $(\Delta\theta)^2$  in the Hausdorff distance. These results show, in particular, that the definition of crystalline motion by weighted mean curvature is “correct.”

Our methods are, unfortunately, rather special. For example, our analysis of closed curves in the plane requires convexity because it uses the curvature-angle representation  $\omega(\theta, t)$ . The analysis of nonconvex closed curves seems to require a different method.

Our problems are rather special as well. In defining crystalline motion, we implicitly assumed that no new faces are ever created. Our convergence results justify this assumption. However such an assumption seems to be false for closely related problems such as  $V = \omega + g$ . To achieve an efficient numerical scheme one needs some guidance concerning when and how to create new faces. We refer to [29, 30] for discussion of a successful implementation, and [28] for some analysis. We mention in passing that crystalline approximation has begun to be used for other problems of phase interface motion, including Saffman-Taylor fingering [4] and surface diffusion [11].

Another issue arises for the motion of two-dimensional surfaces in  $\mathbb{R}^3$ . If the crystalline solution ceases to exist in the classical sense then our whole ap-

proximation scheme breaks down. This situation requires an efficient method of representing varifold solutions. It is not at all clear how this should be done in general; see [34] for a report on recent progress.

Returning to the 1D models considered here, we note an unpleasant feature of crystalline approximation: it is intrinsically piecewise linear. The smooth solution is always approximated by a polygonal one, a representation much more coarse than one could get using front-tracking with splines. We wonder whether there might be a related method which approximates the surface by smoother curves. Perhaps when expressed in curvature-angle coordinates it would become a higher-order difference scheme for (6).

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