On the exterior Neumann problem involving the critical Sobolev exponent

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1 Introduction

We are concerned with obtaining solutions of the exterior Neumann problem

$$\text{(P)} \quad \begin{cases} -\Delta u + au = Q|u|^{2^*-2}u & \text{in } \Omega^C, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega^C = \mathbb{R}^N \setminus \Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$, the value 2^* is the critical Sobolev exponent, Q is positive, bounded and locally Hölder continuous on $\overline{\Omega^C}$, and $a \in L^{\frac{N}{2}}(\Omega^C) \cap L^\infty(\Omega^C) \cap C^1(\Omega^C)$ is such that $\{x \in \Omega^C;\ a(x) < 0\} \neq \emptyset$.

Related to our work are [3], by Chen and Li, who considered the exterior Dirichlet problem in the case where Q is constant, and [4], by Chabrowski and Ruf, who considered the case where the function a is identically equal to a positive constant.

Our argument relies on the analysis of an eigenvalue problem and on a linking argument, similar to the ones in [3]. We give conditions that guarantee existence of solutions to (P). These depend on the relative values of $2^{2/(N-2)}Q_m = 2^{2/(N-2)} \max_{\partial\Omega} Q$, $Q_M = \sup_{\Omega^C} Q$, and $Q(\infty) = \lim_{|x|\to\infty} Q(x)$, where the last limit is always assumed to exist. The conditions are related to the ones in [4], where the proof relies entirely on constrained minimization.

The importance of the shape of Q on the existence of solutions to Neumann semilinear elliptic equations with critical Sobolev exponents was analyzed in [5] by Chabrowski and Willem.

We should also mention that the first work, to our knowledge, to investigate semilinear Neumann problems with critical Sobolev exponents in exterior domains was [9], by Pan and Wang. Existence results in the subcritical case were proved in [11], by Z.Q. Wang. We refer the reader to [6] for other relevant work.

Of course, all the works mentioned above build upon [1] and [10], by Adimurthi and Mancini and X.P. Wang, respectively, who studied the critical Neumann problem on a bounded domain, in the spirit of the work [2], by Brezis and Nirenberg.

The organization of this work is as follows. In Section 2 we give the setup of problem (P), and the related eigenvalue problem. When this problem has its first eigenvalue greater than one, solutions to (P) can be obtained by constrained minimization. This is done in Section 3. Otherwise, solutions are obtained by a linking argument in Section 4.

2 The setup

Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be a bounded domain with a smooth boundary and $\Omega^C = \mathbb{R}^N \setminus \Omega$. We consider the homogeneous Neumann problem

$$\begin{cases} -\Delta u + au = Q|u|^{2^*-2}u & \text{in } \Omega^C, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. The function Q is positive, bounded and locally Hölder continuous on $\overline{\Omega^C}$. The function a satisfies

$$a \in L^{\frac{N}{2}}(\Omega^C) \cap L^\infty(\Omega^C) \cap C^1(\Omega^C). \tag{2}$$

Furthermore the set $\{x \in \Omega^C;\ a(x) < 0\} \neq \emptyset$.

We denote by $\mathcal{D}^{1,2}(\Omega^C) = \{u \in L^{2^*}(\Omega^C);\ Du \in L^2(\Omega^C)\}$ equipped with the norm $\|\nabla u\| := \|\nabla u\|_{L^2(\Omega^C)}$. Proposition 2.1 of [9] implies that this is indeed a norm on $\mathcal{D}^{1,2}(\Omega^C)$.

We also denote by $a^+ = \max\{a, 0\}$ and by $a^- = \max\{-a, 0\}$, so that $a = a^+ - a^-$. We fix a nonnegative function $g \in L^{\frac{N}{2}}(\Omega^C) \cap L^\infty(\Omega^C) \cap C^1(\Omega^C)$ such that $a^- + g > 0$ in Ω^C. Hölder's inequality implies that the norm $\|u\|_{a^+,g} := (\|\nabla u\|^2_{L^2(\Omega^C)} + \int_{\Omega^C}(a^+ + g)u^2)^{\frac{1}{2}}$ is equivalent to the norm of $\mathcal{D}^{1,2}(\Omega^C)$.

Similarly to Lemma 1 of [3], we can show that the problem

$$\begin{cases} -\Delta u + (a^+ + g)u = \lambda(a^- + g)u & \text{in } \Omega^C, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

has an increasing sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, with $\lambda_n \to +\infty$ as $n \to \infty$. The fact that $\lambda_1 > 0$ follows from $\int_{\Omega^C}(a^- + g)u^2 \leq C\|u\|^2$, with $C > 0$. We denote by $\{e_i\}_{i=1,2,\ldots}$ the corresponding orthonormal sequence of eigenfunctions in the inner product given by $\|\cdot\|_{a^+,g}$. Consider a fixed positive integer i. By Theorem 8.17 of [7], for any R such that the ball $B_{2R}(y)$, of radius $2R$ with center in y, is contained in Ω^C,

$$\sup_{B_R(y)} |e_i| \le C R^{-\frac{N-2}{2}} \|e_i\|_{L^{2^*}(B_{2R}(y))},$$

where the constant C depends only on N and R. Therefore, $\lim_{|x|\to+\infty} e_i(x) = 0$.

3 Constrained Minimization

In this section we consider the situation where $\lambda_1 > 1$. In this case, $\|u\|_a := \left(|\nabla u|_{L^2(\Omega^C)} + \int_{\Omega^C} au^2\right)^{\frac{1}{2}}$ is a norm in $\mathcal{D}^{1,2}(\Omega^C)$, equivalent to $\|\ \cdot\ \|$. Therefore, we can show existence of least energy solutions to (1) using constrained minimization, as in [4]. Indeed, let

$$S_a = \inf\{\textstyle\int_{\Omega^C}(|\nabla u|^2 + au^2);\ u \in \mathcal{D}^{1,2}(\Omega^C),\ \int_{\Omega^C} Q|u|^{2^*} = 1\}, \tag{4}$$

$$Q_m = \max_{\partial\Omega} Q, \quad Q_M = \sup_{\Omega^C} Q, \quad Q(\infty) = \lim_{|x|\to\infty} Q(x).$$

Throughout we assume that the last limit exists.

We denote by U the Talenti instanton

$$U(x) := \left(\tfrac{N(N-2)}{N(N-2)+|x|^2}\right)^{\frac{N-2}{2}},$$

by $U_{\varepsilon,P}(\ \cdot\ ) := \varepsilon^{-\frac{N-2}{2}} U\left(\frac{\cdot\ - P}{\varepsilon}\right)$, and by $S := \frac{\int_{\mathbb{R}^N} |\nabla U|^2}{\left(\int_{\mathbb{R}^N} U^{2^*}\right)^{2/2^*}}$.

Lemma 3.1. *If*

$$S_a < \hat{S} := \min\left\{\tfrac{S}{2^{2/N} Q_m^{(N-2)/N}}, \tfrac{S}{Q_M^{(N-2)/N}}, \tfrac{S}{Q_\infty^{(N-2)/N}}\right\} \tag{5}$$

then (4) *has a minimizer, i.e.* (1) *has a least energy solution.*

Proof. Suppose that $\{u_m\} \in \mathcal{D}^{1,2}(\Omega^C)$ is such that $\int_{\Omega^C} Q|u_m|^{2^*} = 1$ and $\int_{\Omega^C}(|\nabla u_m|^2 + au_m^2) \to S_a$. Then, up to a subsequence, $u_m \rightharpoonup u$ on $\mathcal{D}^{1,2}(\Omega^C)$, $u_m \to u$ a.e. on Ω^C, $|\nabla(u_m - u)|^2 \rightharpoonup \mu$ and $|u_m - u|^{2^*} \rightharpoonup \nu$ in $M(\overline{\Omega^C})$, the space of finite measures on $\overline{\Omega^C}$. Let

$$\mu_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\Omega^C\setminus B_R(0)} |\nabla u_m|^2,$$

$$\nu_\infty = \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\Omega^C\setminus B_R(0)} |u_m|^{2^*}.$$

There exists at most a countable set J such that

$$\mu \geq \sum_{j\in J} \mu_j \delta_{x_j},$$

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j},$$

with $\mu_j \geq \frac{S}{2^{2/N}} \nu_j^{2/2^*}$ if $x_j \in \partial\Omega$, $\mu_j \geq S\nu_j^{2/2^*}$ if $x_j \in \Omega^C$, and $\mu_\infty \geq S\nu_\infty^{2/2^*}$. Furthermore,

$$\limsup_{m\to\infty} \int_{\Omega^C} |\nabla u_m|^2 = \int_{\Omega^C} |\nabla u|^2 + \|\mu\| + \mu_\infty,$$

$$1 = \int_{\Omega^C} Q|u|^{2^*} + \sum_{x_j\in\partial\Omega} Q(x_j)\nu_j + \sum_{x_j\in\Omega^C} Q(x_j)\nu_j + Q_\infty \nu_\infty.$$

Note also (see Lemma 2.13 of [12]) that $\lim_{m\to\infty} \int_{\Omega^C} au_m^2 = \int_{\Omega^C} au^2$, because $a \in L^{N/2}(\Omega^C)$. Hence,

$$\begin{aligned} S_a \;&\geq\; \int_{\Omega^C} (|\nabla u|^2 + au^2) + \sum_{x_j\in\partial\Omega} \frac{S}{2^{2/N}} \nu_j^{2/2^*} + \sum_{x_j\in\Omega^C} S\nu_j^{2/2^*} + S\nu_\infty^{2/2^*} \\ &\geq\; S_a \left(\int_{\Omega^C} Q|u|^{2^*} \right)^{2/2^*} + \sum_{x_j\in\partial\Omega} \frac{S}{2^{2/N} Q_m^{2/2^*}} (Q(x_j)\nu_j)^{2/2^*} \\ &\quad + \sum_{x_j\in\Omega^C} \frac{S}{Q_M^{2/2^*}} (Q(x_j)\nu_j)^{2/2^*} + \frac{S}{Q_\infty^{2/2^*}} (Q_\infty \nu_\infty)^{2/2^*}. \end{aligned}$$

Under the hypothesis all the ν's must be zero. It follows that $\int_{\Omega^C} Q|u|^{2^*} = 1$ and then $\int_{\Omega^C} (|\nabla u|^2 + au^2) = S_a$. □

We now turn to conditions that insure (5). We denote by $H(x)$ the mean curvature of $\partial\Omega$ at x, with respect to the outward unit normal to Ω.

Case 1. Assume $N \geq 5$, $\hat{S} = \frac{S}{2^{2/N} Q_m^{(N-2)/N}}$, $Q_m = Q(x_0)$ with $H(x_0) < 0$, and $|Q(x) - Q(x_0)| = o(|x - x_0|)$ as $x \to x_0$. Then (5) is satisfied. For the proof we refer to [5]. Hence, there exists a least energy solution.

Case 2. Assume $N \geq 7$, $\hat{S} = \frac{S}{Q_M^{(N-2)/N}}$, $Q_M = Q(x_0)$ with $a(x_0) < 0$, and $|Q(x) - Q(x_0)| = o(|x - x_0|^2)$ as $x \to x_0$. Then condition (5) is satisfied. Indeed,

$$\frac{\int_{\Omega^C} (|\nabla U_{\varepsilon,x_0}|^2 + aU_{\varepsilon,x_0}^2)}{(\int_{\Omega^C} QU_{\varepsilon,x_0}^{2^*})^{2/2^*}} = \hat{S} + Ca(x_0)\varepsilon^2 + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

where C is a positive constant. This follows from

$$\begin{array}{rcl}\int_{\Omega^C}|a(x_0)-a|U_{\varepsilon,x_0}^2 & \le & \int_{\Omega^C\cap B_\rho(x_0)} C|\cdot -x_0|U_{\varepsilon,x_0}^2 \\ & & +\int_{\Omega^C\setminus B_\rho(x_0)} CU_{\varepsilon,x_0}^2 \\ & \le & C\varepsilon^3\int_{B_{\frac{\rho}{\varepsilon}}(0)} zU^2(z)\,dz + C\varepsilon^2\int_{B^C_{\frac{\rho}{\varepsilon}}(0)} U^2 \\ & \le & C\varepsilon^3 + C\varepsilon^{N-2} \\ & \le & C\varepsilon^3\end{array}$$

and

$$\int_{\Omega^C} QU_{\varepsilon,x_0}^{2^*}(x) = \int_{\Omega^C} Q(x_0)U_{\varepsilon,x_0}^{2^*} + o(\varepsilon^2) = Q(x_0)\int_{\mathbb{R}^N} U_{\varepsilon,x_0}^{2^*} + o(\varepsilon^2). \quad (6)$$

In fact, let $\delta > 0$. Fixing ρ such that $|Q(x)-Q(x_0)| < \delta|x-x_0|^2$ for $|x-x_0| < \rho$,

$$\begin{array}{rcl}\int_{\Omega^C}|Q-Q(x_0)|U_{\varepsilon,x_0}^{2^*} & \le & \int_{\Omega^C\cap B_\rho(x_0)} \delta|\cdot -x_0|^2U_{\varepsilon,x_0}^{2^*} \\ & & +\int_{\Omega^C\setminus B_\rho(x_0)} |Q-Q(x_0)|U_{\varepsilon,x_0}^{2^*} \\ & \le & \delta\varepsilon^2\int_{B_{\frac{\rho}{\varepsilon}}(0)} z^2U^{2^*}(z)\,dz + O(\varepsilon^N) \\ & \le & C\delta\varepsilon^2 + O(\varepsilon^N) \\ & \le & C\delta\varepsilon^2,\end{array}$$

if ε is sufficiently small.

Case 3. Finally, assume $N \ge 7$ and $\hat{S} = \frac{S}{Q_\infty^{(N-2)/N}}$. Suppose also, in addition to (2), there exist constants $\alpha > 2$, $C_1 > 0$ and $\rho_1 > 0$ such that

$$a(x) \le -\frac{C_1}{|x|^\alpha} \text{ for } |x| > \rho_1,$$

and there exist constants $C_2 > 0$ and $\rho_2 > 0$ such that

$$0 < Q_\infty - Q(x) < \frac{C_2}{|x|^p} \text{ for } |x| > \rho_2, \quad (7)$$

where

$$p > \frac{N-2}{N-6}\alpha.$$

Then condition (5) is satisfied. The proof is similar to that of Theorem 4.2 ahead.

4 Linking

In this section we consider the situation where $0 < \lambda_1 \leq \ldots \leq \lambda_{n-1} \leq 1 < \lambda_n \leq \ldots$, for some $n \geq 2$. To prove existence of solutions to (1) we use the linking theorem, instead of constrained minimization.

Consider $\varphi : \mathcal{D}^{1,2}(\Omega^C) \to \mathbb{R}$, defined by

$$\varphi(u) := \int_{\Omega^C} \left(\frac{|\nabla u|^2}{2} + \frac{a|u|^2}{2} - \frac{Q|u|^{2^*}}{2^*} \right).$$

Lemma 4.1. *If $\{u_m\}$ is a sequence in $\mathcal{D}^{1,2}(\Omega^C)$ such that $\varphi(u_m) \to c < \frac{\hat{S}^{N/2}}{N}$ and $\varphi'(u_m) \to 0$ in $\mathcal{D}^{-1,2}(\Omega^C)$, then $\{u_m\}$ is relatively compact in $\mathcal{D}^{1,2}(\Omega^C)$.*

Proof. The first step in the proof is to show that $\{u_m\}$ is bounded in $\mathcal{D}^{1,2}(\Omega^C)$. We argue by contradiction and suppose that $\|u_m\| \to +\infty$. We set $v_m := \frac{u_m}{\|u_m\|}$ and may assume that $v_m \rightharpoonup v$ in $\mathcal{D}^{1,2}(\Omega^C)$. For every $\varphi \in \mathcal{D}^{1,2}(\Omega^C)$,

$$\frac{1}{\|u_m\|^{2^*-2}} \int_{\Omega^C} (\nabla v_m \cdot \nabla \varphi + a v_m \varphi) = \int_{\Omega^C} Q|v_m|^{2^*-2} v_m \varphi + o(1),$$

and so $\int_{\Omega^C} Q|v|^{2^*-2} v\varphi = 0$. Hence $v = 0$ a.e. and $v_m \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\Omega^C)$, $\int_{\Omega^C} a v_m^2 \to 0$. It follows that

$$\frac{1}{2} \int_{\Omega^C} |\nabla v_m|^2 + o(1) = \frac{\|u_m\|^{2^*-2}}{2^*} \int_{\Omega^C} Q|v_m|^{2^*}$$

and

$$\int_{\Omega^C} |\nabla v_m|^2 + o(1) = \|u_m\|^{2^*-2} \int_{\Omega^C} Q|v_m|^{2^*}.$$

Together, $\|v_m\| \to 0$, which is impossible. We conclude that $\|u_m\|$ is bounded.

The second and last step of the proof is to show that $\{u_m\}$ converges in $\mathcal{D}^{1,2}(\Omega^C)$. This step is similar to the argument in the proof of Lemma 3.1. The key point is the following. With similar notations as above, $\mu_j = Q(x_j)\nu_j$ and $\mu_\infty = Q_\infty \nu_\infty$. This implies $\nu_j \geq \frac{S^{N/2}}{2Q(x_j)^{N/2}}$ if $x_j \in \partial\Omega$, $\nu_j \geq \frac{S^{N/2}}{Q(x_j)^{N/2}}$ if $x_j \in \Omega^C$ and $\nu_\infty \geq \frac{S^{N/2}}{Q_\infty^{N/2}}$. Also,

$$c = \lim_{n\to\infty} \left[\varphi(u_m) - \frac{1}{2}\varphi'(u_m)u_m \right] = \lim_{n\to\infty} \frac{1}{N} \int_{\Omega^C} Q|u_m|^{2^*}.$$

$\square$

We set $Y = \text{span}\{e_1, \dots, e_{n-1}\}$ and $Z = (Y)^\perp$ in the inner product given by $\| \cdot \|_{a^+,g}$. Let $R > r > 0$ and $z \in Z$ with $\|z\| = r$, to be chosen below. Consider

$$\begin{array}{lcl} M & := & \{u = y + \lambda z;\ \|u\| \leq R,\ \lambda \geq 0,\ y \in Y\}, \\ M_0 & := & \{u = y + \lambda z;\ y \in Y,\ \|u\| = R \text{ and } \lambda \geq 0,\ \text{or } \|u\| \leq R \text{ and } \lambda = 0\}, \\ N & := & \{u \in Z;\ \|u\| = r\}, \end{array}$$

$$\Gamma := \{\gamma \in \mathcal{C}(M, \mathcal{D}^{1,2}(\Omega^C));\ \gamma|_{M_0} = \text{id}\},$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)).$$

We will choose r and R such that

$$\max_{M_0} J < \inf_N J, \tag{8}$$

and z such that

$$c < \frac{1}{N} \hat{S}^{N/2}. \tag{9}$$

By the previous lemma, the functional φ satisfies the Palais-Smale condition at level c. By the linking theorem, c is a critical value of φ.

We start by choosing r and R. First we note that on Z

$$\varphi(u) \geq \frac{1}{2}(1 - \lambda_n^{-1}) \int_{\Omega^C} (|\nabla u|^2 + (a^+ + g)u^2) - \frac{1}{2^*} \int_{\Omega^C} Q|u|^{2^*}.$$

Using the Sobolev inequality and the fact that Q is bounded, we take $r > 0$ sufficiently small so that

$$\inf_{\substack{\|u\|=r \\ u \in Z}} \varphi(u) > 0.$$

On Y, we have

$$\varphi(u) \leq \frac{1}{2}(1 - \lambda_{n-1}^{-1}) \int_{\Omega^C} (|\nabla u|^2 + (a^+ + g)u^2) - \frac{1}{2^*} \int_{\Omega^C} Q|u|^{2^*} \leq 0, \tag{10}$$

and on the finite dimensional space $Y \oplus \mathbb{R}z$,

$$\varphi(u) \leq C\|u\|^2 - \frac{1}{2^*} \int_{\Omega^C} Q|u|^{2^*} \to -\infty \text{ as } \|u\| \to +\infty.$$

We take R sufficiently big so that

$$\max_{M_0} \varphi = 0.$$

With these choices of r and R, inequality (8) is satisfied.

To insure (9) it is enough to choose z so that

$$\max_{Y\oplus\mathbb{R}z}\varphi < \frac{1}{N}\hat{S}^{N/2}.$$

For $u\in\mathcal{D}^{1,2}(\Omega^C)$ such that $\int_{\Omega^C}(|\nabla u|^2+au^2)>0$,

$$\max_{t\geq 0}\varphi(tu)=\frac{1}{N}\left(\frac{\int_{\Omega^C}(|\nabla u|^2+au^2)}{\left(\int_{\Omega^C}Q|u|^{2^*}\right)^{2/2^*}}\right)^{N/2}$$

(for other $u\in\mathcal{D}^{1,2}(\Omega^C)$ this maximum is zero). Hence, it is enough to prove there exist a small positive ε and a point $P_\varepsilon\in\overline{\Omega^C}$ for which

$$\max_{u\in Y\oplus\mathbb{R}U_{\varepsilon,P_\varepsilon}}\frac{\int_{\Omega^C}(|\nabla u|^2+au^2)}{\left(\int_{\Omega^C}Q|u|^{2^*}\right)^{2/2^*}}<\hat{S}. \tag{11}$$

Notice that $U_{\varepsilon,P_\varepsilon}\notin Y$ for small ε, independently of the choice of P_ε. So suppose $u=y+tU_{\varepsilon,P_\varepsilon}$ with $y\in Y$ and $t\geq 0$, chosen so that

$$\int_{\Omega^C}Q|u|^{2^*}=1. \tag{12}$$

Let us prove that t and $\int_{\Omega^C}Q|y|^{2^*}$ (like any norm of y) are uniformly bounded for small ε. Since there exists a constant $\eta>0$ such that

$$|a+b|^{2^*}\geq|a|^{2^*}+|b|^{2^*}-\eta(|a|^{2^*-1}|b|+|a||b|^{2^*-1}),$$

for all a, $b\in\mathbb{R}$, it follows that

$$\begin{aligned}
1 &\geq \int_{\Omega^C}Q|y|^{2^*}+\int_{\Omega^C}Q|tU_{\varepsilon,P_\varepsilon}|^{2^*}\\
&\quad -\eta\left(\int_{\Omega^C}Q|y|^{2^*-1}|tU_{\varepsilon,P_\varepsilon}|+\int_{\Omega^C}Q|y||tU_{\varepsilon,P_\varepsilon}|^{2^*-1}\right)\\
&\geq \int_{\Omega^C}Q|y|^{2^*}+\int_{\Omega^C}Q|tU_{\varepsilon,P_\varepsilon}|^{2^*}\\
&\quad -\eta\left(|Q|_\infty|y|_\infty^{2^*-1}|t|C\varepsilon^{(N-2)/2}+|Q|_\infty|y|_\infty|t|^{2^*-1}C\varepsilon^{(N-2)/2}\right)\\
&\geq \int_{\Omega^C}Q|y|^{2^*}+\int_{\Omega^C}Q|tU_{\varepsilon,P_\varepsilon}|^{2^*}-\eta|Q|_\infty(|y|^{2^*}+|t|^{2^*})C\varepsilon^{(N-2)/2}\\
&\geq \left(\int_{\Omega^C}Q|y|^{2^*}+C|t|^{2^*}\right)(1-C\varepsilon^{(N-2)/2})
\end{aligned}$$

This shows that $\int_{\Omega^C}Q|y|^{2^*}$ and t are uniformly bounded for small ε.

For $u=y+tU_{\varepsilon,P_\varepsilon}$ satisfying (12) we have the following estimates.

$$\begin{aligned}
\int_{\Omega^C}(|\nabla u|^2+au^2) &= \int_{\Omega^C}(|\nabla y|^2+ay^2)+\int_{\Omega^C}[|\nabla(tU_{\varepsilon,P_\varepsilon})|^2+a(tU_{\varepsilon,P_\varepsilon})^2]\\
&\quad +2\int_{\Omega^C}[\nabla y\cdot\nabla(tU_{\varepsilon,P_\varepsilon})+ay(tU_{\varepsilon,P_\varepsilon})].
\end{aligned}$$

Writing y as $y = \sum_{i=1}^{n-1} c_{\varepsilon,i} e_i$,

$$\begin{aligned} \int_{\Omega^C} [\nabla u \cdot \nabla (tU_{\varepsilon,P_\varepsilon}) + ay(tU_{\varepsilon,P_\varepsilon})] &\leq \sum_{i=1}^{n-1} \int_{\Omega^C} t(1-\lambda_i)(a^- + g)|c_{\varepsilon,i}||e_i|U_{\varepsilon,P_\varepsilon} \\ &\leq C\varepsilon^{(N-2)/2}. \end{aligned}$$

From the calculus inequality

$$|1+x|^{2^*} \geq 1 + 2^* x, \text{ for } x \in \mathbb{R},$$

$$\begin{aligned} 1 = \int_{\Omega^C} Q|u|^{2^*} &\geq \int_{\Omega^C} Q|tU_{\varepsilon,P_\varepsilon}|^{2^*} + 2^* \int_{\Omega^C} Qy(tU_{\varepsilon,P_\varepsilon})^{2^*-1} \\ &\geq \int_{\Omega^C} Q|tU_{\varepsilon,P_\varepsilon}|^{2^*} - C\varepsilon^{(N-2)/2}. \end{aligned}$$

As observed above in (10),

$$\int_{\Omega^C} (|\nabla y|^2 + ay^2) \leq 0.$$

Combining the previous estimates,

$$\begin{aligned} \int_{\Omega^C} (|\nabla u|^2 + au^2) &\leq \int_{\Omega^C} [|\nabla (tU_{\varepsilon,P_\varepsilon})|^2 + a(tU_{\varepsilon,P_\varepsilon})^2] + C\varepsilon^{(N-2)/2} \\ &\leq \left(\frac{1+C\varepsilon^{(N-2)/2}}{\int_{\Omega^C} Q|tU_{\varepsilon,P_\varepsilon}|^{2^*}}\right)^{2/2^*} \int_{\Omega^C} [|\nabla (tU_{\varepsilon,P_\varepsilon})|^2 + a(tU_{\varepsilon,P_\varepsilon})^2] \\ &\quad + C\varepsilon^{(N-2)/2} \\ &\leq \frac{\int_{\Omega^C} [|\nabla U_{\varepsilon,P_\varepsilon}|^2 + a(U_{\varepsilon,P_\varepsilon})^2]}{\left(\int_{\Omega^C} Q|U_{\varepsilon,P_\varepsilon}|^{2^*}\right)^{2/2^*}} + C\varepsilon^{(N-2)/2}. \end{aligned} \tag{13}$$

We are now in position to prove

Theorem 4.2. *Under the assumptions of* Cases 1, 2 *or* 3 *above, problem* (1) *has a solution.*

Proof. As observed above, it is enough to establish (11) to guarantee (9). In Case 1 we use estimate (13), $N \geq 5$ and refer to [5]. In Case 2 we use estimates (6), (13), and $N \geq 7$.

Suppose the hypotheses of Case 3 hold. Let $\omega \in \mathbb{R}^N$ with $|\omega| = 1$. We denote by A, B and C various positive constants and assume $\rho > 0$ is big enough that $\Omega \subset B_{\rho/2}(0)$. Then,

$$\begin{aligned}\int_{\Omega^C} |\nabla U_{\varepsilon,\rho\omega}|^2 &= \int_{\mathbb{R}^N} |\nabla U_{\varepsilon,\rho\omega}|^2 \\ &\quad -\varepsilon^{N-2}(N-2)^2 \int_\Omega \frac{[N(N-2)]^{N-2}|x-\rho\omega|^2}{(\varepsilon^2 N(N-2)+|x-\rho\omega|^2)^N}\,dx \\ &\leq K_1 - \frac{C\varepsilon^{N-2}}{\rho^{2N-2}},\end{aligned}$$

$$\begin{aligned}\int_{\Omega^C} aU^2_{\varepsilon,\rho\omega} &= \int_{\Omega^C\cap B_{\rho/2}(\rho\omega)} aU^2_{\varepsilon,\rho\omega} + \int_{\Omega^C\setminus B_{\rho/2}(\rho\omega)} aU^2_{\varepsilon,\rho\omega} \\ &\leq -\frac{C\varepsilon^2}{\rho^\alpha} + \frac{C\varepsilon^{N-2}}{\rho^{N-4}},\end{aligned}$$

$$\begin{aligned}\int_{\Omega^C} Q|U_{\varepsilon,\rho\omega}|^{2^*} &= \int_{\mathbb{R}^N} Q_\infty |U_{\varepsilon,\rho\omega}|^{2^*} - \int_\Omega Q_\infty |U_{\varepsilon,\rho\omega}|^{2^*} \\ &\quad + \int_{\Omega^C\setminus B_{\rho/2}(\rho\omega)} (Q-Q_\infty)|U_{\varepsilon,\rho\omega}|^{2^*} \\ &\quad + \int_{\Omega^C\cap B_{\rho/2}(\rho\omega)} (Q-Q_\infty)|U_{\varepsilon,\rho\omega}|^{2^*} \\ &\geq K_2 Q_\infty - Q_\infty \int_{\mathbb{R}^N\setminus B_{\rho/2}(\rho\omega)} |U_{\varepsilon,\rho\omega}|^{2^*} \\ &\quad -C\int_{\mathbb{R}^N\setminus B_{\rho/(2\varepsilon)}(0)} |U|^{2^*} - \frac{C_2}{(\rho/2)^p}\int_{\mathbb{R}^N} |U_{\varepsilon,\rho\omega}|^{2^*} \\ &\geq K_2 Q_\infty - \frac{C\varepsilon^N}{\rho^N} - \frac{CK_2}{\rho^p}.\end{aligned}$$

Combining (13) with the last three estimates,

$$\int_{\Omega^C} (|\nabla u|^2 + au^2) \leq \frac{S}{Q_\infty^{(N-2)/N}} - \frac{A\varepsilon^2}{\rho^\alpha} + B\varepsilon^{(N-2)/2} + \frac{C}{\rho^p}.$$

We choose ρ satisfying

$$-\tfrac{A\varepsilon^2}{2\rho^\alpha} + B\varepsilon^{(N-2)/2} = 0 \;\Leftrightarrow\; \tfrac{1}{\rho} = \left(\tfrac{2B}{A}\right)^{1/\alpha} \varepsilon^{(N-6)/(2\alpha)}.$$

It follows that

$$\int_{\Omega^C} (|\nabla u|^2 + au^2) \leq \frac{S}{Q_\infty^{(N-2)/N}} - B\varepsilon^{(N-2)/2} + C\left(\frac{2B}{A}\right)^{p/\alpha} \varepsilon^{(N-6)p/(2\alpha)}.$$

The assumption $p > \frac{N-2}{N-6}\alpha$ implies that (11) is satisfied for small ε. This finishes the proof. $\square$

Remark 4.3. *The proof above goes through if we just assume* (7) *in a cone.*

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