

# A sharp inequality for Sobolev functions

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**Abstract.** Let  $N \geq 5$ ,  $a > 0$ ,  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$ ,  $2^* = \frac{2N}{N-2}$ ,  $2^\# = \frac{2(N-1)}{N-2}$  and  $\|u\|^2 = |\nabla u|_2^2 + a|u|_2^2$ . We prove there exists an  $\alpha_0 > 0$  such that, for all  $u \in H^1(\Omega) \setminus \{0\}$ ,

$$\frac{S}{2^{\frac{2}{N}}} \leq \frac{\|u\|^2}{|u|_{2^*}^2} \left( 1 + \alpha_0 \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}} \right).$$

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## Une inégalité dans un espace de Sobolev

**Résumé.** Nous considérons  $N \geq 5$ ,  $a > 0$ ,  $\Omega$  un ouvert borné régulier de  $\mathbb{R}^N$ ,  $2^* = \frac{2N}{N-2}$ ,  $2^\# = \frac{2(N-1)}{N-2}$  et  $\|u\|^2 = |\nabla u|_2^2 + a|u|_2^2$ . Nous prouvons qu'il existe  $\alpha_0 > 0$  tel que, pour toute fonction  $u \in H^1(\Omega) \setminus \{0\}$ ,

$$\frac{S}{2^{\frac{2}{N}}} \leq \frac{\|u\|^2}{|u|_{2^*}^2} \left( 1 + \alpha_0 \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}} \right).$$

Cette inégalité implique l'inégalité de Cherrier. Pour citer cet article : P.M. Girão, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 105-108. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Let  $N \geq 5$ ,  $a > 0$ ,  $\alpha \geq 0$ ,  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^N$ ,  $2^* = \frac{2N}{N-2}$  and  $2^\# = \frac{2(N-1)}{N-2}$ . We regard  $a$  as fixed and  $\alpha$  as a parameter. Denote the  $L^p$  and  $H^1$  norms of  $u$  in  $\Omega$  by

$$|u|_p := \left( \int |u|^p \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\| := \left( |\nabla u|_2^2 + a|u|_2^2 \right)^{\frac{1}{2}},$$

respectively. All our integrals are over  $\Omega$ . We define the functional  $\delta : H^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ , homogeneous of degree zero, by

$$\delta(u) := \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*}^{2^*/2}}$$

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and consider the system

$$\begin{cases} (1 + \frac{1}{2}\alpha\delta(u))(-\Delta u + au) + \frac{2^\#}{2}\alpha u^{2^\#-1} = \left(1 + \frac{2^\#+1}{2}\alpha\delta(u)\right)u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)_\alpha$$

We claim that the solutions of  $(1)_\alpha$  correspond to critical points of the functional  $\Phi_\alpha : H^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ , defined by

$$\Phi_\alpha(u) := \left(\frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}\right) \left(1 + \alpha \frac{|u|_{2^\#}^{2^\#}}{\|u\| \cdot |u|_{2^*/2}^{2^*/2}}\right)^{\frac{N}{2}} = \Phi_0(u)(1 + \alpha\delta(u))^{\frac{N}{2}}. \quad (2)$$

In fact, since

$$\Phi'_\alpha = (1 + \alpha\delta)^{\frac{N}{2}-1} [\Phi'_0(1 + \alpha\delta) + \frac{N}{2}\Phi_0\alpha\delta']$$

and, for  $\varphi \in H^1(\Omega)$ ,

$$\delta'(u)(\varphi) = -\frac{\delta(u)}{\|u\|^2} \int (\nabla u \cdot \nabla \varphi + au\varphi) + 2^\# \frac{\delta(u)}{|u|_{2^\#}^{2^\#}} \int (|u|^{2^\#-2}u\varphi) - \frac{2^*}{2} \frac{\delta(u)}{|u|_{2^*}^{2^*}} \int (|u|^{2^*-2}u\varphi),$$

the critical points of  $\Phi_\alpha$  satisfy

$$\begin{aligned} (-\Delta u + au) \left(1 + \frac{4-N}{4}\alpha\delta(u) + \frac{N-2}{4} \frac{|u|_{2^*}^{2^*}}{\|u\|^2} \alpha\delta(u)\right) + \frac{2^\#N}{2}\alpha|u|^{2^\#-2}u \left(\frac{1}{2} \frac{\|u\|}{|u|_{2^*/2}^{2^*/2}} - \frac{1}{2^*} \frac{|u|_{2^*}^{2^*/2}}{\|u\|}\right) \\ - |u|^{2^*-2}u \left(1 + \frac{4-N}{4}\alpha\delta(u) + \frac{2^*N}{8} \frac{\|u\|^2}{|u|_{2^*}^{2^*}} \alpha\delta(u)\right) = 0 \end{aligned}$$

in  $\Omega$ , ( $\frac{\partial u}{\partial \nu} = 0$  on  $\partial\Omega$ ). However, multiplying this equation by  $u$  and integrating over  $\Omega$  (i.e. differentiating (2) along the radial direction) we get  $\|u\|^2 = |u|_{2^*}^{2^*}$ . Conversely, the solutions of  $(1)_\alpha$  are solutions of the previous equation: multiplying  $(1)_\alpha$  by  $u$  and integrating over  $\Omega$  we get a quadratic equation in  $\|u\|/|u|_{2^*}^{2^*/2}$ , whose solution is  $\|u\|/|u|_{2^*}^{2^*/2} = 1$ . This proves our claim.

The functional  $\Phi_\alpha$  restricted to the Nehari manifold,

$$\mathcal{N} := \{u \in H^1(\Omega) \setminus \{0\} : \Phi'_\alpha(u)u = 0\} = \left\{u \in H^1(\Omega) \setminus \{0\} : \|u\|^2 = |u|_{2^*}^{2^*}\right\},$$

is  $\frac{1}{N}[\beta(1 + \alpha\delta)]^{\frac{N}{2}}$ , where  $\beta : H^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$  is defined by

$$\beta(u) := \frac{\|u\|^2}{|u|_{2^*}^{2^*}}.$$

So, we consider the functional  $\Psi_\alpha : H^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ , defined by

$$\Psi_\alpha := \beta(1 + \alpha\delta).$$

A *least energy* solution of  $(1)_\alpha$  is a function  $u \in H^1(\Omega) \setminus \{0\}$ , such that

$$\Phi_\alpha(u) = \inf_{\mathcal{N}} \Phi_\alpha = \inf_{H^1(\Omega) \setminus \{0\}} \frac{1}{N}(\Psi_\alpha)^{\frac{N}{2}}.$$

We are interested in proving existence and nonexistence of least energy solutions of  $(1)_\alpha$ . We note that every critical point of  $\Phi_\alpha$  is a critical point of  $\Psi_\alpha$ . It is easy to check that the Nehari manifold

is a natural constraint for  $\Phi_\alpha$ . So conversely, if  $u$  is a critical point of  $\Psi_\alpha$ , then there exists a unique  $t(u) > 0$ , such that  $t(u)u$  is a critical point of  $\Phi_\alpha$  ( $t(u) = (\|u\|^2/|u|_{2^*}^{2^*})^{\frac{N-2}{4}}$ ).

We consider the minimization problem corresponding to

$$S_\alpha := \inf \{ \Psi_\alpha(u) | u \in H^1(\Omega) \setminus \{0\} \}.$$

We recall that  $S := \inf \left\{ \frac{|\nabla u|_{L^2(\mathbb{R}^N)}^2}{|u|_{L^{2^*}(\mathbb{R}^N)}^2} \mid u \in L^{2^*}(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N), u \neq 0 \right\}$  is achieved by the instanton  $U(x) := \left( \frac{N(N-2)}{N(N-2)+|x|^2} \right)^{\frac{N-2}{2}}$ . Our main result is

**THEOREM 1.** – *There exists a positive real number  $\alpha_0 = \alpha_0(a, \Omega) = \min \{ \alpha \mid S_\alpha = S/2^{\frac{2}{N}} \}$  such that*

- (i) *if  $\alpha < \alpha_0$ , then  $(1)_\alpha$  has a least energy solution  $u_\alpha$ ;*
- (ii) *if  $\alpha > \alpha_0$ , then  $(1)_\alpha$  does not have a least energy solution and  $\Psi_\alpha \geq S/2^{\frac{2}{N}}$ . The constant  $S/2^{\frac{2}{N}}$  is sharp.*

*Remark 1.* – *Obviously,  $\alpha_0$  is a nonincreasing function of  $a$ . By testing  $\Psi_\alpha$  with constant functions and instantons we can prove that  $\alpha_0 \geq \max \{ [S/(2|\Omega|)^{\frac{2}{N}} - a]/\sqrt{a}, C(N) \max_{\partial\Omega} H \}$  where  $|\Omega|$  is the Lebesgue measure of  $\Omega$ ,  $H$  is the mean curvature of  $\partial\Omega$  and  $C(N)$  is a constant that only depends on  $N$ . The least energy solutions might be constant ( $a^{\frac{N-2}{4}}$ ) for  $a \leq S/(2|\Omega|)^{\frac{2}{N}}$  if  $\alpha \leq [S/(2|\Omega|)^{\frac{2}{N}} - a]/\sqrt{a}$ .*

**COROLLARY 2.** – *For all  $u \in H^1(\Omega) \setminus \{0\}$ ,*

$$\frac{S}{2^{\frac{2}{N}}} \leq \frac{\|u\|^2}{|u|_{2^*}^2} \left( 1 + \alpha_0 \frac{|u|_{2^{\#}}^{2^{\#}}}{\|u\| \cdot |u|_{2^*}^{2^*/2}} \right).$$

**COROLLARY 3.** – *For all  $u \in H^1(\Omega)$ ,*

$$\frac{S}{2^{\frac{2}{N}}} |u|_{2^*}^2 \leq \|u\|^2 + \alpha_0 \|u\| \cdot |u|_2 \quad \text{and} \quad \frac{S}{2^{\frac{2}{N}}} |u|_{2^*}^2 \leq (|\nabla u|_2 + c_{a,\alpha_0} |u|_2)^2,$$

with  $c_{a,\alpha_0} = \max \left\{ \alpha_0/2, \sqrt{a + \alpha_0 \sqrt{a}} \right\}$ .

*Proof.* – From Hölder's inequality  $|u|_{2^{\#}}^{2^{\#}} \leq |u|_2 |u|_{2^*}^{2^*/2}$ , so  $\delta(u) \leq \frac{|u|_2}{\|u\|}$ . □

**COROLLARY 4.** – *(Cherrier's inequality). Let  $\varepsilon > 0$ . For all  $u \in H^1(\Omega)$ ,*

$$\frac{S}{2^{\frac{2}{N}}} |u|_{2^*}^2 \leq (1 + \varepsilon) \|u\|^2 + \frac{\alpha_0^2}{4\varepsilon} |u|_2^2 = (1 + \varepsilon) |\nabla u|_2^2 + \left( \frac{\alpha_0^2}{4\varepsilon} + a\varepsilon \right) |u|_2^2.$$

*Sketch of the proof of Theorem 1.* – By testing  $\Psi_\alpha$  with instantons,  $S_\alpha \leq \frac{S}{2^{\frac{2}{N}}}$ , for all  $\alpha \geq 0$ . We claim that if  $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$ , then  $S_\alpha$  is achieved. This is a consequence of the concentration-compactness principle. A minimizing sequence  $u_k$  with  $|u_k|_{2^*} = 1$  is bounded and we can assume  $u_k \rightharpoonup u$  in  $H^1(\Omega)$ ,  $\lim_{k \rightarrow \infty} |\nabla u_k|_2^2 = |\nabla u|_2^2 + \|\mu\|$  and  $\lim_{k \rightarrow \infty} |u_k|_{2^*}^{2^*} = |u|_{2^*}^{2^*} + \|\nu\| = 1$ , where  $\frac{S}{2^{\frac{2}{N}}} \|\nu\|^{2^*} \leq \|\mu\|$  (we remark that this inequality follows from Cherrier's inequality). We can write

$\beta(u)\delta(u) = \gamma(u)\sqrt{\beta(u)}$  for  $\gamma(u) = |u|_{2^*}^{2^*} / |u|_{2^*}^{2^*}$ . The key step in the proof of the claim is the following observation. Define  $f$  and  $g : [0, 1] \rightarrow \mathbb{R}$ , by

$$\begin{aligned} f(x) &:= \beta x^{\frac{2}{2^*}} + \frac{S}{2^{\frac{2}{N}}}(1-x)^{\frac{2}{2^*}} + \alpha \gamma x^{\frac{2^*}{2}} \sqrt{\beta x^{\frac{2}{2^*}} + \frac{S}{2^{\frac{2}{N}}}(1-x)^{\frac{2}{2^*}}} \\ &\geq \beta x + \frac{S}{2^{\frac{2}{N}}}(1-x) + \alpha \gamma x \sqrt{\beta x + \frac{S}{2^{\frac{2}{N}}}(1-x)} =: g(x). \end{aligned}$$

Suppose  $\min f < \frac{S}{2^{\frac{2}{N}}}$ . It follows that  $\beta < \frac{S}{2^{\frac{2}{N}}}$ , and this in turn implies that  $g$  is concave. Since  $f$  and  $g$  coincide at 0 and 1, the minimum of  $f$  occurs at 1. This proves the claim.

A similar argument shows  $\alpha \mapsto S_\alpha$  is continuous. In particular, the supremum  $\alpha_0 := \sup\{\alpha | S_\alpha < S/2^{\frac{2}{N}}\}$  is either  $+\infty$  or a maximum. The map  $\alpha \mapsto S_\alpha$  is strictly increasing on  $[0, \alpha_0]$ . If  $\alpha \in [0, \alpha_0]$ , then  $(1)_\alpha$  has a least energy solution. If  $\alpha \in ]\alpha_0, +\infty[$ , then  $(1)_\alpha$  does not have a least energy solution. It remains to prove that  $\alpha_0$  is finite.

Suppose  $\alpha_0 = +\infty$ . Choose  $\alpha_k \rightarrow +\infty$  as  $k \rightarrow +\infty$  and let  $u_k$  be minimizer for  $S_{\alpha_k}$  satisfying  $(1)_{\alpha_k}$ . It is easy to prove that  $\lim_{\alpha \rightarrow \infty} S_\alpha = S/2^{\frac{2}{N}}$ ,  $M_k := \max_{\bar{\Omega}} u_k = u_k(P_k) \rightarrow +\infty$ ,  $\lim_{k \rightarrow \infty} |\nabla u_k|_2^2 = \lim_{k \rightarrow \infty} |u_k|_{2^*}^{2^*} = S^{\frac{N}{2}}/2$  and  $\alpha_k \delta(u_k) \rightarrow 0$ . We can apply the Gidas-Spruck blow up technique to  $(1)_{\alpha_k}$  because  $\alpha_k \delta(u_k) \rightarrow 0$ . Define  $\varepsilon_k := M_k^{-2/(N-2)}$  and  $U_{\varepsilon, y} := \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon}\right)$ . We can prove that  $\lim_{k \rightarrow \infty} \alpha_k \varepsilon_k = 0$ ,  $\lim_{k \rightarrow \infty} |\nabla u_k - \nabla U_{\varepsilon_k, P_k}|_2 = 0$  and  $P_k \in \partial\Omega$ , for large  $k$ .

At this point, using the ideas of [2], we follow the argument in [7], which applies with no modification. We show  $\Psi_{\alpha_k}(u_k) > S/2^{\frac{2}{N}}$ , for large  $k$ . This is impossible. Therefore  $\alpha_0$  is finite.  $\square$

*Remark 2.* – The functional behavior behind this inequality is also present, for example, in the Dirichlet problem for  $-\Delta u - au + \alpha u^{1/3} = u^{7/3}$  in  $\Omega \subset \mathbb{R}^5$ , with  $0 < a < \lambda_1(-\Delta, H_0^1(\Omega))$ . In this case  $s := [t(u)]^{2/3}$  is the solution of a cubic equation,  $(|\nabla u|_2^2 - a|u|_2^2)s + \alpha|u|_{4/3}^{4/3} - |u|_{10/3}^{10/3}s^3 = 0$ .

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