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Numerical solution of a nonuniquely solvable Volterra integral equation using extrapolation methods

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Abstract

In this work the numerical solution of a Volterra integral equation with a certain weakly singular kernel, depending on a real parameter μ, is considered. Although for certain values of μ this equation possesses an infinite set of solutions, we have been able to prove that Euler's method converges to a particular solution. It is also shown that the error allows an asymptotic expansion in fractional powers of the stepsize, so that general extrapolation algorithms, like the E-algorithm, can be applied to improve the numerical results. This is illustrated by means of some examples. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider the second kind Volterra integral equation

$$L_\mu u(t) = g(t), \quad t \in [0, T], \tag{1.1}$$

with

$$L_\mu u(t) := u(t) - \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)\,ds, \tag{1.2}$$

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where $\mu > 0$ and g is a given function. The above equation can arise in connection with some heat conduction problems with mixed-type boundary conditions [1,8]. A simple example is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}, \quad 0 \leqslant x \leqslant l, \tag{1.3}$$

with the conditions

$$u(x, -\infty) = 0, \tag{1.4}$$

$$\frac{\partial u}{\partial x}(0,t) - u(0,t) = \phi_1(t), \tag{1.5}$$

$$-\frac{\partial u}{\partial x}(l,t) - u(l,t) = \phi_2(t). \tag{1.6}$$

By expressing the solution in terms of single layer potentials (see e.g. [24, pp. 462–465]), a system of two coupled integral equations is obtained. Under certain conditions on the parameter l, this system can be simplified yielding two independent equations with the form

$$F(t) + \int_0^t \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\ln(t/s)}}\frac{1}{s}F(s)\,\mathrm{d}s = H(t), \quad t \in [0,T]. \tag{1.7}$$

Following the approach used in [9], the above equation can be transformed into one of the type of (1.1). We note that a more complex problem leading to a system of equations of the type of (1.7) was considered by Bartoshevich [1]. The analytical study of this and other related equations has been pursued by several authors ([25,23,17,26,18,13]). In particular, it was proved in [26] that (1.1) has a unique solution in the continuity class $C^m[0,T]$ if g is in $C^m[0,T]$ and $\mu > 1$. However, if $0 < \mu \leqslant 1$, then (1.1) has a family of solutions in $C[0,T]$, of which only one has C^1 continuity [13] (compare also [6] for the study of a certain homogeneous Volterra integral equation of the second kind).

For the numerical solution of (1.1), in the case $\mu > 1$, certain classes of product integration methods were studied. We note that the kernel of (1.1), that is, the function $p(t,s) = s^{\mu-1}/t^{\mu}$, is integrable with respect to s in $[0,T]$. For this reason we say that p is a weakly singular function. The numerical solution of Volterra integral equations with weakly singular kernels of the form $q(t,s) = (t-s)^{-\alpha}, 0 < \alpha < 1$, has been considered by many authors. In particular, methods based on generalized Newton–Cotes formulae combined with product integration techniques were studied in [19,7,15]. For a comprehensive study of numerical methods for Volterra integral equations we refer to [5]. Among more recent works we refer to [11] and the references therein. It should be noted that, unlike the Abel-type kernel q, p does not satisfy $\int_0^t p(t,s)\,\mathrm{d}s \to 0$ as $t \to 0^+$. Moreover, all the iterated kernels associated with p are unbounded. Owing to these properties, the classical arguments in the convergence analysis of discretization methods for weakly singular equations are not applicable. In [26] approximations to the solution of (1.7) of orders one and two were obtained with the Euler and trapezoidal methods, respectively. In [9] a fourth order Hermite-type collocation method was applied to (1.1) and, recently, the analysis and construction of higher order product integration methods has been investigated in [8]. In [18] we were concerned with the use of a low order method in conjunction with extrapolation procedures. Eq. (1.1) was transformed into a new one, so that the two equations were equivalent away from the origin. Under appropriate conditions, although for a certain g the solution of the original equation could be unbounded at the origin, the corresponding solution of the new equation was smooth. Euler's method was then used to approximate the solution of the new equation and first order convergence was proved. Moreover, under certain conditions, the error was shown to allow an asymptotic error expansion in integer powers of the stepsize which permitted the use of Richardson's extrapolation. Extrapolation methods based on the midpoint rule have been studied in [14] for nonlinear Volterra equations with smooth kernels (see also [12,4] and the references therein). The idea of solution regularization transformations was used in [21,22,10] with Abel-type equations and, recently, in [11], for both continuous and Abel-type kernels.

In the present paper we further develop the work carried out in [18]. We shall be mainly concerned with the case when $0<\mu\leqslant 1$. Although for these values of μ Eq. (1.1) has a family of solutions in $C[0,T]$, we have been able to prove that Euler's method converges to a certain particular solution. This work is organized as follows. In Section 2 we present some definitions and preliminary results on existence and uniqueness of solution. We also prove a smoothness property which will be used in the convergence analysis. In Section 3 the application of Euler's method is considered and a general convergence result is obtained for all $\mu>0$, under the assumption that $g\in C^1[0,T]$. The case when g is not sufficiently smooth is dealt with by using an appropriate transformation of the equation. Section 4 is devoted to a detailed error analysis and, in the case $0<\mu<1$, convergence of order μ is proved. Furthermore, using a sharper analysis than the one in [18], we obtain new asymptotic error expansions for the case $\mu>1$. In particular, if $1<\mu\leqslant 2$ and $g\in C^2[0,T]$ with $g'(0)\neq 0$, then the asymptotic error expansion will contain fractional powers of h and powers of h multiplied by $\ln h$. We point out that, although only the case when $1<\mu\leqslant 2$ has been considered, the same argument can in principle be employed to deal with higher values of μ. Such expansions will permit the use of more general extrapolation procedures, like the E-algorithm, and some numerical examples illustrate their application in Section 5.

2. Smoothness of solutions

We start by presenting some definitions and lemmas. For $T>0$ and m a nonnegative integer, $V_m[0,T]$ denotes the normed space of the real valued functions f such that $f\in C^m[0,T]$ with

$$\|f\|_m := \max_{0\leqslant j\leqslant m}\max_{t\in[0,T]}|f^{(j)}(t)|. \tag{2.1}$$

Definition 2.1. For $T>0$, $\beta\geqslant 0$ and $m\geqslant 0$, $V_{m,\beta}[0,T]$ denotes the normed space of real valued functions f such that $t^\beta f\in V_m[0,T]$ with

$$\|f\|_{m,\beta} := \|t^\beta f\|_m = \max_{0\leqslant j\leqslant m}\max_{t\in[0,T]}\left|\frac{\mathrm{d}^j}{\mathrm{d}t^j}(t^\beta f(t))\right|. \tag{2.2}$$

Definition 2.2. Let $m\geqslant 1$. Then $f\in V_m^0[0,T]$ if and only if $f\in V_m[0,T]$ and $f(0)=f'(0)=\cdots=f^{(m-1)}(0)=0$. If $m=0$, we have $V_0^0[0,T]=V_0[0,T]$.

Definition 2.3. Let $m\geqslant 1$ and $\beta\geqslant 0$. Then $f\in V_{m,\beta}^0[0,T]$ if and only if $t^\beta f\in V_m^0[0,T]$. If $m=0$, we have $V_{0,\beta}^0[0,T]=V_{0,\beta}[0,T]$.

The following result concerning existence and uniqueness of solution can be found in [18].

Lemma 2.1. *Let* $\beta \geqslant 0$ *and* $\mu > 1 + \beta$. (i) *If the function* g *in* (1.1) *belongs to* $V_{m,\beta}[0,T]$ *then* (1.1) *has a unique solution* $u \in V_{m,\beta}[0,T]$. (ii) *If* $g \in V^0_{m,\beta}[0,T]$, *then* (1.1) *has a unique solution* $u \in V^0_{m,\beta}[0,T]$.

A further result was proved by [13] which includes the case when $0 < \mu \leqslant 1$.

Lemma 2.2. (a) *If* $0 < \mu \leqslant 1$ *and* $g \in V_1[0,T]$ (*with* $g(0) = 0$ *if* $\mu = 1$) *then* (1.1) *has a family of solutions* $u \in V_0[0,T]$ *given by the formula*

$$u(t) = c_0 t^{1-\mu} + g(t) + \gamma + t^{1-\mu} \int_0^t s^{\mu-2}(g(s) - g(0))\,\mathrm{d}s, \tag{2.3}$$

where

$$\gamma := \begin{cases} \dfrac{1}{\mu - 1} g(0) & \text{if } \mu < 1, \\ 0 & \text{if } \mu = 1, \end{cases} \tag{2.4}$$

and c_0 *is an arbitrary constant. Out of the family of solutions there is one particular solution* $u \in V_1[0,T]$. *Such a solution is unique and can be obtained from* (2.3) *by taking* $c_0 = 0$.

(b) *If* $\mu > 1$ *and* $g \in V_m[0,T]$, $m \geqslant 0$, *then the unique solution* $u \in V_m[0,T]$ *of* (1.1) *is*

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)\,\mathrm{d}s. \tag{2.5}$$

We note that (2.5) *can be obtained from* (2.3) *with* $c_0 = 0$. *Indeed, it follows from* (2.3) *that*

$$c_0 = \lim_{t \to 0^+} t^{\mu-1} u(t), \tag{2.6}$$

and this limit is zero when $\mu > 1$.

We now obtain a result which characterizes the particular solution referred to in part (a) of the previous lemma.

Theorem 2.1. *Consider Eq.* (1.1) *with* $g \in V_1[0,T]$.

(a) *If* $\mu > 0$ *and* $\mu \neq 1$ *then* (1.1) *has a unique solution of the form*

$$u(t) = u_0 + z(t), \tag{2.7}$$

where $z(t) = O(t)$, *as* $t \to 0^+$, *and* u_0 *is a constant.*

(b) *If* $\mu = 1$ *and* $g(0) = 0$, *then* (1.1) *has a unique solution which satisfies*

$$u(t) = z(t), \tag{2.8}$$

where $z(t) = O(t)$, *as* $t \to 0^+$.

Proof. The ideas of this proof will be used in the discretization of (1.1) in the next sections. Consider first the case when $0<\mu \leqslant 1$. Writing

$$g(t) = g(0) + g_1(t), \tag{2.9}$$

this suggests splitting (1.1) into two equations

$$L_\mu y(t) = g(0) \tag{2.10}$$

and

$$L_\mu z(t) = g_1(t). \tag{2.11}$$

Then the solution u of (1.1) can be represented in the form

$$u(t) = y(t) + z(t), \tag{2.12}$$

where y and z are the solutions of (2.10) and (2.11), respectively. It follows from (2.3) that the general solution y is given by the formula

$$y(t) = c_0 t^{1-\mu} + u_0, \tag{2.13}$$

where

$$u_0 := \begin{cases} \dfrac{\mu}{\mu - 1} g(0) & \text{if } \mu < 1, \\ 0 & \text{if } \mu = 1. \end{cases} \tag{2.14}$$

An expression for $z(t)$ can also be obtained from (2.3) and, using the fact that $g_1(0) = 0$, we get

$$z(t) = d_0 t^{1-\mu} + g_1(t) + t^{1-\mu} \int_0^t s^{\mu-2} g_1(s)\,\mathrm{d}s, \tag{2.15}$$

where

$$d_0 = \lim_{t\to 0^+} t^{\mu-1} z(t) = \lim_{t\to 0^+} t^{\mu} \frac{z(t)}{t}. \tag{2.16}$$

We note that if $z(t)/t$ is continuous at $t = 0$ then $d_0 = 0$, in which case we obtain the (unique) particular solution of (2.11) which is in $V_1[0,T]$. Setting $c_0 = 0$ in (2.13), we may conclude that

$$u(t) = u_0 + g_1(t) + t^{1-\mu} \int_0^t s^{\mu-2} g_1(s)\,\mathrm{d}s \tag{2.17}$$

is a particular solution to (1.1) of the form (2.7) and it remains to show that $z(t) = O(t)$. It follows from (2.15), with $d_0 = 0$,

$$\begin{aligned} \lim_{t\to 0^+} \frac{z(t)}{t} &= g'(0) + \lim_{t\to 0^+} t^{-\mu} \int_0^t s^{\mu-2} g_1(s)\,\mathrm{d}s \\ &= g'(0) + \lim_{t\to 0^+} \int_0^1 \tau^{\mu-1} \frac{g_1(t\tau)}{t\tau}\,\mathrm{d}\tau \\ &= g'(0) + \lim_{t\to 0^+} \frac{g_1(t\xi)}{t\xi} \int_0^1 \tau^{\mu-1}\,\mathrm{d}\tau \\ &= g'(0)\left(1 + \frac{1}{\mu}\right). \end{aligned} \tag{2.18}$$

We have thus proved that $z(t) = O(t)$, as $t \to 0^+$. In the case $\mu = 1$ we require that $g(0) = 0$. Then we have $u_0 = 0$ and we may rewrite (2.17) as

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s)\, \mathrm{d}s. \tag{2.19}$$

Therefore, we have obtained a particular solution of form (2.8) which satisfies $u(t)=O(t)$, as $t \to 0^+$.

Finally, we note that the same analysis applies to the case when $\mu > 1$, by using the fact that $c_0 = 0$ in (2.3). □

Corollary 2.1. *Let α be a real number such that $g(t) = t^\alpha \bar{g}(t)$ in* (1.1), *with $\bar{g} \in V_1[0, T]$.*

(a) *If $\mu + \alpha > 0$ and $\mu + \alpha \neq 1$ then* (1.1) *has a unique solution of the form*

$$u(t) = u_0 t^\alpha + \bar{z}(t), \tag{2.20}$$

where $\bar{z}(t) = O(t^{\alpha+1})$, as $t \to 0^+$, and v_0 is a constant.

(b) *If $\mu + \alpha = 1$ and $\bar{g}(0) = 0$, then*

$$u(t) = \bar{z}(t), \tag{2.21}$$

where $\bar{z}(t) = O(t^{\alpha+1})$, as $t \to 0^+$.

Proof. The above result can be easily obtained by applying Theorem 2.1 to the equation

$$L_{\mu+\alpha} v(t) = \bar{g}(t), \tag{2.22}$$

where

$$\bar{g}(t) := t^{-\alpha} g(t) \tag{2.23}$$

and

$$v(t) := t^{-\alpha} u(t). \qquad \square \tag{2.24}$$

Remark 2.1. It may happen that, although $\mu > 1$, we might have $\mu + \alpha < 1$. In this case, Eq. (1.1) has a unique solution but the auxiliary equation (2.22) has an infinite set of solutions (according to Lemma 2.2). Therefore equality (2.24) is valid only for the particular solution of (2.22) which belongs to $V_1[0, T]$.

3. Numerical solutions

In this section we consider the application of Euler's method to Eq. (1.1). The case when $\mu > 1$ was studied in [18]. Assuming $g \in V_0[0, T]$, the Euler's method was shown to converge to the unique solution u of (1.1). Here we shall give a general convergence result valid for all values of $\mu > 0$ under the assumption that $g \in V_1[0, T]$. In the case $0 < \mu \leqslant 1$, we have seen that (1.1) has a family of solutions of which only one is in the continuity class C^1 (cf. Theorem 2.1). Here we show that the approximate solution obtained by Euler's method converges to this particular solution.

Let $X_h := \{t_i = ih,\ 0 \leqslant i \leqslant N\}$ be an uniform grid with stepsize $h := T/N$. Consider in $\mathbb{R}^{N+1}$ the maximum norm

$$\|v^h\| := \max_{0 \leqslant i \leqslant N} |v_i| \tag{3.1}$$

and define the linear restriction operator $r^h : V_0[0,T] \to \mathbb{R}^{N+1}$ by

$$(r^h f(t))_i := f(t_i), \quad 0 \leqslant i \leqslant N. \tag{3.2}$$

Let us associate with the operator L_μ defined by (1.2) the linear discrete operator $L_\mu^h : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ such that

$$(L_\mu^h v^h)_k := \begin{cases} v_0^h \frac{\mu-1}{\mu}, & k = 0, \\ v_k^h - t_k^{-\mu} \sum_{i=0}^{k-1} D_i v_i^h, & 1 \leqslant k \leqslant N, \end{cases} \tag{3.3}$$

where

$$D_0 := \int_{t_0}^{t_1} s^{\mu-1}\,\mathrm{d}s = h^\mu/\mu, \qquad D_i := \int_{t_i}^{t_{i+1}} s^{\mu-1}\,\mathrm{d}s = h^\mu \frac{(i+1)^\mu - i^\mu}{\mu}, \quad 1 \leqslant i \leqslant N-1. \tag{3.4}$$

Then Euler's method for (1.1) is given by

$$L_\mu^h u^h = r^h g, \tag{3.5}$$

with initial value $u_0^h = u(0) = u_0$, where u_0 is given by (2.14). In the convergence analysis it will be convenient to consider the following equivalent approach based on the ideas of splitting (1.1) into two equations (2.10) and (2.11). We seek an approximation u^h to the solution of (1.1) in the form

$$u^h = y^h + z^h, \tag{3.6}$$

where y^h and z^h are approximate solutions of Eqs. (2.10) and (2.11), respectively. Then we consider the following discretization of Eq. (2.10):

$$L_\mu^h y^h = g_0, \tag{3.7}$$

where each entry y_k^h of the vector y^h is an approximation to $y(t_k)$, $k = 1, \ldots, N$, and $g_0 = (g(0), \ldots, g(0))^{\mathrm{T}}$. An initial value is needed and, if $\mu \neq 1$, we take

$$y_0^h = g(0)\frac{\mu}{\mu-1} = u_0. \tag{3.8}$$

Then (3.7) has only one solution given by

$$y_k^h = g(0)\frac{\mu}{\mu-1}, \quad k = 1, 2, \ldots, N. \tag{3.9}$$

If $\mu = 1$, we require that $g(0) = 0$ and set $y_k^h = 0$, $k = 0, 1, \ldots, N$. In order to obtain an approximation to the solution z of (2.11), we instead consider the following equation:

$$L_{\mu+1}\phi(t) = G(t), \tag{3.10}$$

where G and ϕ are the continuous functions

$$G(t) := \frac{g_1(t)}{t}, \quad \phi(t) := \frac{z(t)}{t}. \tag{3.11}$$

Now let ϕ^h be the solution of

$$L^h_{\mu+1}\phi^h = r^h G, \tag{3.12}$$

where each component ϕ^h_k of the vector ϕ^h is an approximation to $\phi(t_k)$, $k = 1, 2, \dots, N$. Here we take

$$\phi^h_0 = \frac{\mu+1}{\mu} \lim_{t\to 0^+} \frac{g_1(t)}{t} = g'(0)\frac{\mu+1}{\mu}. \tag{3.13}$$

Then, using (3.12), we obtain that $t_k\phi^h_k$ will be an approximation to $z(t_k)$, that is, we set

$$z^h_k = t_k\phi^h_k, \quad k = 1, 2, \dots, N. \tag{3.14}$$

We now prove the following convergence result.

Theorem 3.1. *Consider Eq.* (1.1) *with* $g \in V_1[0,T]$. *If* $\mu > 0$ *and* $\mu \neq 1$ *then the approximate solution* u^h *defined by* (3.6) *converges to the particular solution of* (1.1) *which has the form* (2.7). *If* $\mu = 1$ *and* $g(0) = 0$, *then the approximate solution* u^h *defined by* (3.6) *converges to the particular solution of* (1.1) *which has the form* (2.8).

Proof. Let us first consider the case $\mu \neq 1$ and let u be the particular solution referred to in Theorem 2.1. That is, u is given by (2.17) and, using (2.12) and (3.6), we may write

$$\|r^h u - u^h\| \leqslant \|r^h y - y^h\| + \|r^h z - z^h\|. \tag{3.15}$$

From (3.9) it follows immediately that

$$\|r^h y - y^h\| = 0. \tag{3.16}$$

In order to prove that $\|r^h z - z^h\| \to 0$, as $h \to 0^+$, we first note that, since $z(t) = t\phi(t)$, we have

$$(r^h z - z^h)_k = t_k(r^h\phi - \phi^h_k). \tag{3.17}$$

We follow the ideas of Theorem 3.2 of [18] to prove that

$$\|L^h_{\mu+1}(r^h\phi) - r^h(L_{\mu+1}\phi)\| \to 0. \tag{3.18}$$

We have

$$(L^h_{\mu+1}(r^h\phi) - r^h(L_{\mu+1}\phi))_k = t_k^{-\mu}\sum_{i=0}^{k-1}\left(\int_{t_i}^{t_{i+1}} s^{\mu-1}\phi(s)\,\mathrm{d}s - D_i\phi(t_i)\right), \quad 1 \leqslant k \leqslant N. \tag{3.19}$$

Using the mean value theorem for integrals yields

$$\int_{t_i}^{t_{i+1}} s^{\mu-1}\phi(s)\,\mathrm{d}s = \phi(\xi_i)\int_{t_i}^{t_{i+1}} s^{\mu-1}\,\mathrm{d}s. \tag{3.20}$$

with $\xi_i \in (t_i, t_{i+1})$, $0 \leqslant i \leqslant k-1$. Substituting (3.20) into (3.19), we obtain

$$(L^h_{\mu+1}(r^h\phi) - r^h(L_{\mu+1}\phi))_k = t_k^{-\mu}\sum_{i=0}^{k-1} D_i(\phi(\xi_i) - \phi(t_i)), \quad 1 \leqslant k \leqslant N, \tag{3.21}$$

which yields, after taking modulus,

$$|(L^h_{\mu+1}(r^h\phi) - r^h(L_{\mu+1}\phi))_k| \leqslant t_k^{-\mu} \max_{0 \leqslant i \leqslant k-1} |\phi(\xi_i) - \phi(t_i)| \sum_{i=0}^{k-1} D_i, \quad 1 \leqslant k \leqslant N. \tag{3.22}$$

Since $\phi \in V_0[0,T]$ then $\max_{0 \leqslant i \leqslant N} |\phi(\xi_i) - \phi(t_i)| \to 0$ as $h \to 0^+$ and we obtain (3.18). Next, we note that

$$L^h_{\mu+1}(\phi^h - r^h\phi) = L^h_{\mu+1}(\phi^h) - L^h_{\mu+1}(r^h\phi) = r^h(L_{\mu+1}\phi) - L^h_{\mu+1}(r^h\phi). \tag{3.23}$$

On the other hand, it can be shown that the following stability condition is satisfied:

$$\|\phi^h - r^h\phi\| \leqslant \frac{1}{\mu+1} \|r^h(L_{\mu+1}\phi) - L^h_{\mu+1}(r^h\phi)\|, \tag{3.24}$$

which, combined with (3.23), gives

$$\|r^h\phi - \phi^h\| \to 0 \quad \text{as } h \to 0^+. \tag{3.25}$$

Then from (3.17) we obtain

$$\|r^h z - z^h\| \to 0 \quad \text{as } h \to 0^+. \tag{3.26}$$

Finally, using (3.16) and (3.26) in (3.15), the desired result follows.

In the case $\mu = 1$, we note that $y(t) \equiv 0$ and $y^h_k = 0$, $k = 1,\ldots,N$. Hence $u(t) = z(t)$ and $u^h = z^h$. The convergence result concerning z^h may be proved in the same way as when $\mu \neq 1$. □

Remark 3.1. It should be noted that the case when $g \notin V_1[0,T]$ can be dealt with using Corollary 2.1. If there exists $\alpha \in \mathbb{R}$ such that $g(t) = t^\alpha \bar{g}(t)$, with $\bar{g} \in V_1[0,T]$, then Eq. (1.1) has a unique solution u of form (2.20) or (2.21), respectively, if $\mu + \alpha \neq 1$ or $\mu + \alpha = 1$. This solution is obtained by introducing the auxiliary equation (2.22)

$$L_{\mu+\alpha} v(t) = \bar{g}(t),$$

and taking $u(t) = t^\alpha v(t)$. Then we consider the following discretization of (2.22):

$$L^h_{\mu+\alpha} v^h = r^h \bar{g}, \tag{3.27}$$

where v^h is the vector with components $v^h_i := t_i^{-\alpha} u^h_i$, $i = 1,\ldots,N$. Since $\bar{g} \in V_1[0,T]$, it follows from Theorem 3.1 that u^h converges to u as $h \to 0$ in the sense that

$$\|t^\alpha u^h - r^h t^\alpha u\| \to 0, \tag{3.28}$$

provided that $\mu + \alpha \geqslant 0$.

4. Error analysis

In order to apply extrapolation methods, we need to know not only the main term of the asymptotic error expansion but also terms of higher orders. In [18], we have proved that if $g \in V^0_m[0,T]$,

$m \geqslant 1$, and $\mu > 1$, the numerical solution u^h, obtained by Euler's method, allows an asymptotic error expansion in integer powers of h

$$(r^h u - u^h)_k = \sum_{q=1}^{m-1} C_q(t_k) h^q + O(h^m), \tag{4.1}$$

where $C_q \in V^0_{m-q}[0,T]$ do not depend on h. The above expansion was the basis for applying the Richardson's extrapolation method in order to improve the numerical results.

In this section it will be shown that if $g'(0) \neq 0$ the asymptotic error expansions will contain terms of more general forms. Both cases when $0<\mu<1$ and $\mu>1$ will be considered. In particular, if $0<\mu<1$ convergence of order μ is obtained.

We shall need the following auxiliary lemma.

Lemma 4.1. *Let L^h_μ be defined by* (3.3) *and consider the following difference equation*:

$$L^h_\mu e^h = r, \tag{4.2}$$

where

$$r = (0, c, \ldots, c), \quad r \in \mathbb{R}^{N+1},\ c \in \mathbb{R}.$$

Then e^h does not depend on h and the following asymptotic equalities hold:

(a) *If $\mu = 1$, $e^h_k = c \ln k + O(1)$, as $k \to \infty$;*
(b) *If $\mu < 1$, $e^h_k = C_\mu k^{1-\mu} + O(1)$, as $k \to \infty$, where $C_\mu \in \mathbb{R}^{N+1}$ is independent of k.*

Proof. (a) If $\mu = 1$, from (3.4) we have

$$D_i = t_{i+1} - t_i = h, \quad i = 1, \ldots, N. \tag{4.3}$$

Then the difference equation (4.2) reduces to the form

$$e^h_k - \frac{h}{t_k} \sum_{i=0}^{k-1} e^h_i = r_k, \quad k = 1, \ldots, N. \tag{4.4}$$

Substituting into (4.4) the explicit form of r, we obtain $e^h_0 = 0$,

$$e^h_k = c + \frac{1}{h} \sum_{i=0}^{k-1} e^h_i = r_k, \quad k = 1, \ldots, N. \tag{4.5}$$

From (4.5) it follows that

$$e^h_k - e^h_{k-1} = \frac{c}{k}. \tag{4.6}$$

Since $e^h_0 = 0$, (4.6) implies that

$$e^h_k = c \sum_{i=0}^{k} \frac{1}{i}, \quad k = 1, \ldots, N. \tag{4.7}$$

The sum on the right-hand side of (4.7) does not depend on h and its asymptotic expansion is well known (see e.g. [16, p. 220]):

$$\sum_{i=0}^{k}\frac{1}{i}=\ln k+\gamma+O\left(\frac{1}{k}\right), \tag{4.8}$$

where γ is the so-called Euler's constant. The desired result follows from (4.7) and (4.8).

(b) Let us write (4.2) in the form

$$e_k^h-t_k^{-\mu}\sum_{i=0}^{k-1}D_ie_i^h=c,\quad k=1,\ldots,N, \tag{4.9}$$

where the weights D_i are given by (3.4). From (4.9), it follows that

$$t_k^\mu e_k^h-(t_k^\mu+D_{k-1})e_{k-1}^h=c(t_k^\mu-t_{k-1}^\mu),\quad k=1,\ldots,N, \tag{4.10}$$

Dividing both sides of (4.10) by h^μ, we obtain

$$k^\mu e_k^h-\left[(k-1)^\mu+\frac{k^\mu-(k-1)^\mu}{\mu}\right]e_{k-1}^h=c(k^\mu-(k-1)^\mu),\quad k=1,\ldots,N. \tag{4.11}$$

The last equation is a first order linear difference equation. Its solution does not depend on h and therefore we shall denote it e_k. In order to solve it using the methods described in [16, Chapter 3], let us introduce the variable substitution

$$\bar{e}_k=k^{\mu-1}e_k. \tag{4.12}$$

Then (4.11) reduces to

$$k\bar{e}_k-\left[(k-1)+\frac{k^\mu-(k-1)^\mu}{\mu(k-1)^{\mu-1}}\right]\bar{e}_{k-1}=c(k^\mu-(k-1)^\mu),\quad k=1,\ldots,N. \tag{4.13}$$

Let us first consider the homogeneous equation associated with (4.13) and denote the general solution of this equation $\bar{e}_k^0$. We have

$$\bar{e}_k^0=d\prod_{i=0}^{k}\left[\frac{i-1}{i}+\frac{i^\mu-(i-1)^\mu}{\mu(i-1)^{\mu-1}i}\right], \tag{4.14}$$

where d is an arbitrary constant. The general term of the product (4.14) may be rewritten as

$$\frac{k-1}{k}+\frac{k^\mu-(k-1)^\mu}{\mu(k-1)^{\mu-1}k}=1-\frac{1}{k}+\frac{1}{k}\left(1+O\left(\frac{1}{k}\right)\right)=1+O\left(\frac{1}{k^2}\right). \tag{4.15}$$

Therefore, the product on the right-hand side of (4.14) is convergent. Moreover, if we consider the logarithm of this product and analyse the resulting series, using the Euler–Maclaurin theorem (see [27, pp. 32–42]) we obtain the following asymptotic expansion:

$$\bar{e}_k^0=C_\mu+O\left(\frac{1}{k}\right)\quad\text{as } k\to\infty, \tag{4.16}$$

where C_μ does not depend on k. From (4.16) and (4.12) it follows immediately that

$$e_k^0=C_\mu k^{1-\mu}+O(k^{-\mu})\quad\text{as } k\to\infty. \tag{4.17}$$

Let us now find a particular solution of the non-homogeneous equation (4.11). Using the method of variation of parameters as in [16, p. 67], it can be shown that (4.11) has a solution of the form

$$e_k^1 = e_k^0 a_k, \tag{4.18}$$

where a_k satisfies

$$e_{k+1}^0 \Delta a_k = c\frac{k^\mu - (k-1)^\mu}{k^\mu} \tag{4.19}$$

and $\Delta a_k := a_{k+1} - a_k$. Since $e_k^0 = C_\mu k^{1-\mu} + O(k^{-\mu})$ and

$$\frac{k^\mu - (k-1)^\mu}{k^\mu} = O(k^{-1}),$$

it follows from (4.19) that $\Delta a_k = O(k^{\mu-2})$. Therefore, $a_k = O(k^{\mu-1})$ and, substituting this into (4.18), we may conclude that $e_k^1 = O(1)$. Finally, using the theory of linear nonhomogeneous equations, we obtain that the general solution of (4.11) is given by

$$e_k = e_k^0 + e_k^1 = C_\mu k^{1-\mu} + O(1) + O(k^{-\mu}) = C_\mu k^{1-\mu} + O(1). \tag{4.20}$$

This concludes the proof of Lemma 4.1. □

Theorem 4.1. *Consider Eq.* (1.1) *with* $g \in V_2[0,T]$ *and* $g'(0) \neq 0$. *If* $0 < \mu < 1$ *then the approximate solution* u^h *defined by* (3.6) *satisfies the error estimate*

$$(r^h u - u^h)_k = C_\mu(t_k)h^\mu + O(h), \tag{4.21}$$

where C_μ *does not depend on* h.

Proof. Since $g \in V_2[0,T]$ and $g'(0) \neq 0$, it follows from Theorem 2.1 that there exists a solution u, such that $u \in V_2[0,T]$ and $u'(0) \neq 0$. Then we may write

$$u(t) = u_0 + at + t\hat{u}(t), \tag{4.22}$$

where $u_0 = u(0)$ is given by (2.14), $a := u'(0) \neq 0$ and $\hat{u}(t) \in V_1^0[0,T]$. We shall again make use of the decomposition (3.6). Since we have $\|r^h y - y^h\| = 0$ (cf. (3.16)), then the value of u_0 does not affect the error of the numerical solution and we may consider, without loss of generality, that $u_0 = 0$. In this case (4.22) reduces to $u(t) = at + t\hat{u}(t)$. Similarly, let us write

$$g(t) = bt + t\hat{g}(t), \quad b := g'(0). \tag{4.23}$$

Then we consider the splitting of (1.1) into the two equations

$$L_\mu(at) = bt, \quad L_\mu t\hat{u}(t) = t\hat{g}(t). \tag{4.24}$$

Associated with the last equation is

$$L_{\mu+1}\hat{u}(t) = \hat{g}(t), \tag{4.25}$$

to which corresponds the following discrete equation:

$$L_{\mu+1}^h \hat{u}^h = r^h \hat{g}. \tag{4.26}$$

Let us set

$$e_k^h := (r^h u - u^h)_k$$

and

$$\hat{e}_k^h := t_k(r^h\hat{u} - \hat{u}^h)_k.$$

Since $\mu + 1 > 1$ and $\hat{u}(t) \in V_1^0[0, T]$, then it follows that (4.1) is valid with $m = 1$, therefore

$$\hat{e}_k^h = O(h). \tag{4.27}$$

If we denote

$$\tilde{e}_k^h := e_k^h - \hat{e}_k^h, \tag{4.28}$$

then

$$L_\mu^h \tilde{e}^h = L_\mu^h r^h(at) - r^h(L_\mu at). \tag{4.29}$$

Using similar arguments to the ones of the proof of [18, Theorem 3.3], where the case $\mu > 1$ was considered, we have

$$(L_\mu^h r^h(at) - r^h(L_\mu at))_k = C_1(t_k)h, \tag{4.30}$$

where

$$C_1(t_k) = \begin{cases} 0 & \text{if } k = 0, \\ \alpha_0 a/\mu & \text{if } k > 0, \end{cases} \tag{4.31}$$

with

$$\alpha_0 = -\int_0^1 \sigma \, \mathrm{d}\sigma = -\frac{1}{2}. \tag{4.32}$$

Therefore, Eq. (4.29) may be rewritten in the form

$$L_\mu^h \tilde{e}^h = vh, \tag{4.33}$$

where the components of v are given by

$$v_k := \begin{cases} 0 & \text{if } k = 0, \\ \alpha_0 a/\mu & \text{if } k > 0. \end{cases} \tag{4.34}$$

Now we can use Lemma 4.1 to estimate $\tilde{e}^h$. Since $\mu < 1$ and $k = t_k/h$, we have

$$\tilde{e}_k^h = (C_\mu k^{1-\mu} + O(1))h = C_\mu t_k^{1-\mu} h^\mu + O(h). \tag{4.35}$$

Finally, from (4.27), (4.28) and (4.35), we obtain

$$e_k^h = C_\mu t_k^{1-\mu} h^\mu + O(h). \tag{4.36}$$

This concludes the proof of Theorem 4.1. □

To deal with the case when the solution is not sufficiently smooth, we use the approach described in Remark 3.1. If, for some $\alpha \in \mathbb{R}$, we have $\bar{g}(t) = t^{-\alpha} g(t) \in V_2[0, T]$, we replace (1.1) by the auxiliary equation (2.22) and approximate this equation by means of the discretization (3.27). An application of Theorem 4.1 gives the following corollary.

Corollary 4.1. *Let $g(t)=t^{\alpha}\bar{g}(t)$, with $\bar{g}\in V_2[0,T]$ and $\bar{g}'(0)\neq 0$, and define $u_i^h := t_i^{\alpha}v_i^h$, $i=1,\dots,N$, where v^h is the solution of* (3.27). *Then, if $\mu+\alpha<1$ we have*

$$(r^h u - u^h)_k = C_{\mu+\alpha} t_k^{1-\mu-\alpha} h^{\mu+\alpha} + O(h). \tag{4.37}$$

Therefore, the order of the main term of the error depends only on the sum $\mu+\alpha$, and not on the value of μ.

Theorem 4.2. *Consider Eq.* (1.1) *with $g\in V_2[0,T]$ and $g'(0)\neq 0$. If $1<\mu\leqslant 2$ then the approximate solution u^h defined by* (3.6) *allows an asymptotic error expansion with the form*

$$(r^h u - u^h)_k = C_1(t_k)h + C_\mu(t_k)h^\mu + O(h^2) \quad \textit{if } \mu<2, \tag{4.38}$$

and

$$(r^h u - u^h)_k = C_1(t_k)h + C_2'(t_k)h^\mu \ln h + O(h^2) \quad \textit{if } \mu=2. \tag{4.39}$$

Proof. Here we use a similar technique to the one developed by [20] for deriving asymptotic error expansions. Since $g\in V_2[0,T]$ and $\mu>1$ then the exact solution u of (1.1) is also in $V_2[0,T]$. Moreover, the following asymptotic expansion for the consistency error is valid [18]

$$(L_\mu^h r^h u - r^h L_\mu u)_k = hD_1(t_k) + h^2 D_2(t_k) + o(h^2), \tag{4.40}$$

where D_1, D_2 are continuous functious given by

$$D_1(t) = t^{-\mu}\alpha_0 \int_0^t s^{\mu-1}u'(s)\,\mathrm{d}s, \tag{4.41}$$

and

$$D_2(t) = t^{-\mu}\alpha_1 \int_0^t s^{\mu-1}u''(s)\,\mathrm{d}s + (1/t)\int_0^1 \theta_0(\sigma)B_1(\sigma)\,\mathrm{d}\sigma \tag{4.42}$$

$$\times(s^{\mu-1}u'(st)|_{s=1} - s^{\mu-1}u'(st)|_{s=0}) \quad \text{if } \mu=2, \tag{4.43}$$

$$D_2(t) = t^{-\mu}\alpha_1 \int_0^t s^{\mu-1}u''(s)\,\mathrm{d}s + (1/t)\int_0^1 \theta_0(\sigma)\gamma(0,1-\sigma)\,\mathrm{d}\sigma \tag{4.44}$$

$$\times(s^{\mu-1}u'(st)|_{s=1}) \quad \text{if } \mu<2. \tag{4.45}$$

Above $\alpha_r = \int_0^1 (-1)^{r+1}\sigma^{r+1}/(r+1)!$ and $\gamma(a,s)$ is the generalized zeta function [18]. From (4.40) we obtain

$$(L_\mu^h r^h u)_k = (r^h L_\mu u)_k + hD_1(t_k) + h^2 D_2(t_k) + o(h^2), \tag{4.46}$$

which can be rewritten in operator form as follows. Let Λ_j be the linear operators defined by

$$\Lambda_j v = D_j, \quad j=1,2, \tag{4.47}$$

and let Λ_2^h be the operator defined by

$$L_\mu^h(r^h v) = r^h L_\mu v + hr^h(\Lambda_1 v) + h^2\Lambda_2^h(r^h v). \tag{4.48}$$

Comparing (4.40) and (4.48) we see that

$$\Lambda_2^h(r^h v) = r^h(\Lambda_2 v) + o(1) \quad \text{when } h \to 0. \tag{4.49}$$

Writing the error in the form

$$(u^h - r^h u)_k = hC_1(t_k) + \tau^h(t_k), \tag{4.50}$$

where $\tau^h(t) = o(h)$ is the remainder, gives

$$(u^h)_k = (r^h u)_k + hC_1(t_k) + \tau^h(t_k). \tag{4.51}$$

Then we obtain from (4.48)

$$L_\mu^h(r^h u) = r^h(L_\mu u) + hr^h(L_\mu C_1) + hr^h\Lambda_1(u) + h^2 r^h(\Lambda_1 C_1) \tag{4.52}$$

$$+ h^2\Lambda_2^h(r^h u) + h^3\Lambda_2^h(r^h C_1) + r^h L_\mu \tau^h + hr^h\Lambda_1\tau^h + h^2\Lambda_2^h r^h\tau^h. \tag{4.53}$$

Using the fact that $L_\mu^h u^h = r^h g$ and collecting terms of equal powers of h gives

$$\Lambda_1 u + L_\mu C_1 = 0 \;\Leftrightarrow\; L_\mu C_1 = -\Lambda_1 u, \tag{4.54}$$

$$h^2\Lambda_2^h(r^h u) + h^2 r^h(\Lambda_1 C_1) + r^h L_\mu \tau^h + o(h^2) = 0. \tag{4.55}$$

Since $u \in V_2[0,T]$ then $\Lambda_1 u = D_1$ is in $V_1[0,T]$, therefore (4.54) has a unique solution in $V_1[0,T]$. We now consider Eq. (4.55) which, taking into account (4.49), can be rewritten as

$$L_\mu^h r^h\tau^h = h^2(-r^h\Lambda_1 C_1 - r^h\Lambda_2 u - o(1)). \tag{4.56}$$

We begin by noting that the functions $\Lambda_1 C_1$ and $\Lambda_2 u$ may not be defined at the origin. However it can be shown that

$$t\Lambda_1 C_1(t) =: \zeta_1(t) = \begin{cases} 0 & \text{if } t = 0, \\ \zeta_1 + O(t) & \text{if } t \neq 0, \end{cases} \tag{4.57}$$

and

$$t\Lambda_2 u(t) =: \zeta_2(t) = \begin{cases} 0 & \text{if } t = 0, \\ \zeta_2 + O(t) & \text{if } t \neq 0, \end{cases} \tag{4.58}$$

where ζ_1, ζ_2 are different from zero and do not depend on t. Then we are led to consider the equation

$$L_{\mu-1}^h r^h(t\tau^h) = -h^2(r^h(\zeta_1 + \zeta_2) + o(1)). \tag{4.59}$$

Since $0 < \mu - 1 \leqslant 1$ we are in a position to apply Lemma 4.1 which gives

$$t\tau^h = \begin{cases} C_\mu(t)h^\mu + O(h^2) & \text{if } 1 < \mu < 2, \\ C_2'(t)h^2 \ln h + O(h^2) & \text{if } \mu = 2. \end{cases} \tag{4.60}$$

Finally, substituting (4.60) into (4.50) yields (4.38) and (4.39). □

We note that the argument of the above proof extends to higher values of μ but with greater complexity of the analysis.

Corollary 4.2. *Let* $g(t)=t^\alpha\bar{g}(t)$, *with* $\bar{g}(t)\in V_2[0,T]$ *and* $\bar{g}'(0)\neq 0$, *and define* $u_i^h := t_i^\alpha v_i^h$, $i=1,\dots,N$, *where* v^h *is the solution of* (3.27). *Then, if* $1<\mu+\alpha\leqslant 2$, *we have*

$$(u^h - r^h u)_k = C_1(t_k)h + C_{\mu+\alpha}(t_k)h^{\mu+\alpha} + O(h^2) \quad \textit{if} \ \ \mu+\alpha<2, \tag{4.61}$$

and

$$(u^h - r^h u)_k = C_1(t_k)h + C_2'(t_k)h^2 \ln h + O(h^2) \quad \textit{if} \ \ \mu+\alpha=2. \tag{4.62}$$

This result may be proved in the same way as Corollary 4.1.

5. Numerical examples and convergence acceleration

In this section the theoretical results of the previous sections are illustrated by means of some numerical examples. We shall consider the numerical solution of (1.1), with

$$g(t) := t^\alpha(1+t), \tag{5.1}$$

for several choices of the constants μ and α such that $\mu+\alpha>0$ and $\mu+\alpha\neq 1$. Then Corollary 2.1 is applicable with $\bar{g}(t)=t^{-\alpha}g(t)=1+t$ and it follows that (1.1) has a unique solution of the form $u(t)=t^\alpha u_0 + O\left(t^{1+\alpha}\right)$. More precisely this solution is given by

$$u(t) = t^\alpha v(t) = t^\alpha \frac{\mu+\alpha}{\mu+\alpha-1} + t^{\alpha+1}\frac{\mu+\alpha+1}{\mu+\alpha}, \tag{5.2}$$

where the expression of $v(t)$ can be obtained from (2.17). In the examples below, the numerical approximations to the solutions were computed by applying the algorithm (3.27) to the auxiliary equation (2.22). The absolute errors in the approximate solutions u^h to u are shown in Figs. 1–3 for the values $h_n = 1/2^n 1/40$, $n=0,\dots,5$ of the stepsize h. The following quantity has been used as an estimate of the convergence order

$$k := -\log_2\left(\frac{\|r^h u - u^h\|}{\|r^{2h}u - u^{2h}\|}\right). \tag{5.3}$$

In the case $\mu>1$, an asymptotic error expansion was obtained for Euler's method which contained only integer powers of the stepsize h (cf. [18]). This allowed the use of the Richardson's extrapolation

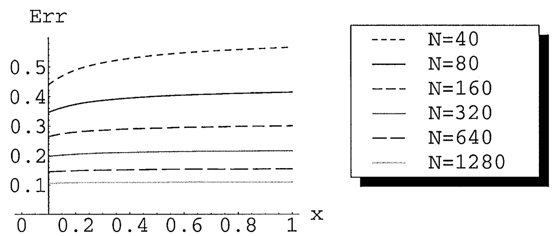

Fig. 1. Absolute error of the approximate solution in Example 5.1.

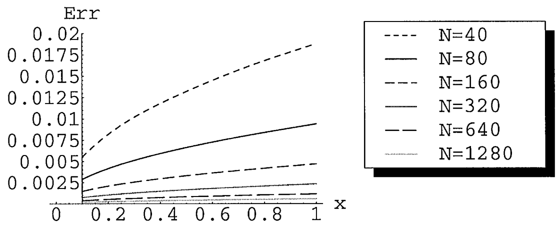

Fig. 2. Absolute error of the approximate solution in Example 5.2.

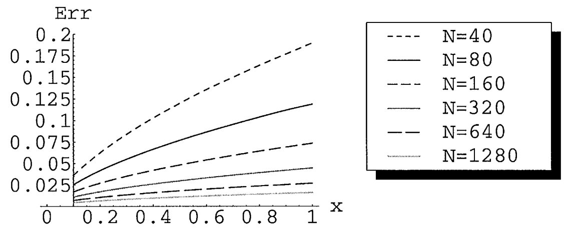

Fig. 3. Absolute error of the approximate solution in Example 5.3.

algorithm to improve the numerical results. In the examples below, the asymptotic error expansions are of the types (4.37) and (4.62). A natural way to accelerate the convergence of the numerical results is then to use the E-algorithm of Brezinski [2,3] which is a generalization of the Richardson's extrapolation. It was designed under the assumption that we know an asymptotic expansion for a given sequence S_n:

$$S_n = S + a_1 g_1(n) + a_2 g_2(n) + \cdots + a_k g_k(n), \quad n = 0,1,2,\ldots, \tag{5.4}$$

where $g_i(n)$ are predefined sequences, which satisfy the condition $g_{i+1}(n) = o(g_i(n))$, as $n \to \infty$, and the a_i are real numbers that do not depend on n. If $k+1$ terms of the sequence S_n are known, we can compute S by solving the linear system

$$S_{n+i} = S + a_1 g_1(n+i) + a_2 g_2(n+i) + \cdots + a_k g_k(n+i), \quad i = 0,1,\ldots,k. \tag{5.5}$$

Usually the terms of S_n do not satisfy (5.4) exactly (to obtain (5.4) we must ignore the remainder of the asymptotic error expansion). Therefore, the solution of (5.5) is only an approximation of S, which depends on n and k and is usually denoted by $E_k^{(n)}$. When we use the E-algorithm, instead of using Gaussian elimination or LU factorization, we use recursive formulae to compute the solution of (5.5), which are preferable due to their numerical stability. The computation of $E_k^{(n)}$ begins with

$$E_0^{(n)} = S_n, \quad n = 0,1,\ldots,n_{\max}\, g_{0,i}^{(n)} = g_i(n), \quad i = 1,2,\ldots,n_{\max}, \quad n = 1,\ldots,n_{\max}-1. \tag{5.6}$$

For $k = 1,2,\ldots,n_{\max}$ and $n = 0,1,\ldots,n_{\max}-k$ the recursive formulae are [2]:

$$\begin{aligned} E_k^{(n)} &= E_{k-1}^{(n)} + g_{k-1,k}^{(n)} \frac{E_{k-1}^{(n)} - E_{k-1}^{(n+1)}}{g_{k-1,k}^{(n+1)} - g_{k-1,k}^{(n)}}, \\ g_{k,i}^{(n)} &= g_{k-1,i}^{(n)} + g_{k-1,k}^{(n)} \frac{g_{k-1,i}^{(n)} - g_{k-1,i}^{(n+1)}}{g_{k-1,k}^{(n)} - g_{k-1,k}^{(n)}}, \quad i = k+1,k+2,\ldots,n_{\max}. \end{aligned} \tag{5.7}$$

Here $g_{k,i}^{(n)}$ are auxiliary sequences, which depend only on the terms $g_i(n)$ of the asymptotic expansion (5.4). The values of $E_k^{(n)}$ are usually displayed in a doule-entry array, called the E-array, with the form

$$\begin{array}{llll} E_0^{(0)} = S_0, & & & \\ E_0^{(1)} = S_1, & E_1^{(0)}, & & \\ E_0^{(2)} = S_2, & E_1^{(1)}, & E_2^{(0)}, & \\ E_0^{(3)} = S_3, & E_1^{(2)}, & E_2^{(1)}, & E_3^{(0)}, \\ \quad\dots & \dots & \dots & \dots \end{array} \tag{5.8}$$

The first column of this array represents the sequence whose convergence we want to accelerate (in the examples below, it will be u^{h_n}) and each subsequent column contains a new transformed sequence.

Example 5.1. Let $\alpha = -0.5$, $\mu = 1$. Then (5.2) takes the form

$$u(t) = -t^{-1/2} + 3t^{1/2}, \quad t > 0. \tag{5.9}$$

An application of Corollary 4.1 gives an asymptotic error expansion with the form

$$(r^h u - u^h)_i = C_{0.5}(t_i)h^{1/2} + O(h). \tag{5.10}$$

The absolute errors in the solution are shown in Fig. 1 and the convergence order estimate given by (5.3) is $k = 0.49$, which confirms the theoretical prediction. In order to accelerate the convergence, we have applied the E-algorithm, with the auxiliary sequences

$$g_1(n) = h_n^{1/2}, \quad g_2(n) = h_n, \quad g_3(n) = h_n^{3/2}, \quad g_4(n) = h_n^2, \quad g_5(n) = h_n^{5/2}. \tag{5.11}$$

This choice is based on the assumption that an asymptotic error expansion of the form is valid

$$(r^h u - u^h)_i = C_1(t_i)h^{1/2} + C_2(t_i)h + C_3(t_i)h^{3/2} + C_4(t_i)h^2 + C_5(t_i)h^{5/2} + O(h^3). \tag{5.12}$$

The first column of Table 1 gives the absolute error $|(u^{h_n})_i - u(t_i)|$, with i such that $t_i = 1$. Each entry of the subsequent columns represents the absolute error of the corresponding value $E_k^{(n)}$. The results show that the accuracy is significantly improved in each step of the extrapolation process and this is in agreement with the above conjecture.

Table 1
Absolute error of the entries of the E-array in Example 5.1

n	$\lvert S_n - u(1)\rvert$	$\lvert E_1^{(n)} - u(1)\rvert$	$\lvert E_2^{(n)} - u(1)\rvert$	$\lvert E_3^{(n)} - u(1)\rvert$	$\lvert E_4^{(n)} - u(1)\rvert$	$\lvert E_5^{(n)} - u(1)\rvert$
0	0.567					
1	0.415	$0.482D-1$				
2	0.301	$0.248D-1$	$0.144D-2$			
3	0.216	$0.127D-1$	$0.506D-3$	$0.240D-5$		
4	0.155	$0.642D-1$	$0.178D-3$	$0.907D-6$	$0.408D-6$	
5	0.110	$0.324D-2$	$0.628D-4$	$0.280D-6$	$0.713D-7$	$0.9918D-9$

Table 2
Absolute error of the entries of the E-array in Example 5.2

n	$\|S_n - u(1)\|$	$\|E_1^{(n)} - u(1)\|$	$\|E_2^{(n)} - u(1)\|$	$\|E_3^{(n)} - u(1)\|$	$\|E_4^{(n)} - u(1)\|$	$\|E_5^{(n)} - u(1)\|$
0	$0.188D-1$					
1	$0.940D-2$	$0.109D-4$				
2	$0.470D-2$	$0.399D-5$	$0.123D-4$			
3	$0.235D-2$	$0.269D-5$	$0.207D-5$	$0.246D-7$		
4	$0.117D-2$	$0.110D-5$	$0.388D-6$	$0.132D-8$	$0.652D-8$	
5	$0.586D-3$	$0.380D-6$	$0.775D-7$	$0.564D-10$	$0.375D-9$	$0.130D-11$

Example 5.2. In this example, we have chosen $\alpha = 0.5$, $\mu = 1.5$, so that (5.2) gives

$$u(t) = 2t^{1/2} + 3/2t^{3/2}, \quad t \geqslant 0. \tag{5.13}$$

In this case, $\mu + \alpha = 2$ and it follows from Corollary 4.2, with $m = 1$, that the approximate solution u^h has an asymptotic error expansion with the form

$$(r^h u - u^h)_i = C_1(t_i)h + C_2'(t_i)h^2 \ln h + O(h^2). \tag{5.14}$$

The absolute errors of the approximate solution are given in Fig. 2 and the estimate of the convergence order is $k = 1.0005$, as expected. The convergence accelaration was obtained again by means of the E-algorithm, using the following auxiliary sequences:

$$g_1(n) = h_n, \quad g_2(n) = h_n^2 \ln h_n, \quad g_3(n) = h_n^2, \quad g_4(n) = h_n^3 \ln h_n, \quad g_5(n) = h_n^3. \tag{5.15}$$

The absolute errors of the entries of the E-array are displayed in Table 2. As in the Example 5.1, we have assumed that the asymptotic error expansion may be extended to terms of orders higher then h^2, with the form:

$$\begin{aligned} g_{2k}(n) &= h_n^{k+1} \ln h_n, \\ g_{2k-1}(n) &= h_n^k, \quad k = 1, 2, 3, \ldots. \end{aligned} \tag{5.16}$$

As shown in Table 2, the extrapolation process based on this assumption improves the accuracy of the results. Finally, we note that Euler's method could be applied directly to Eq. (1.1). Since $\mu > 1$ and $g \in V_0^0[0,1]$, convergence follows from Theorem 3.2 of [18]. However, in order to assure first order convergence, g is required to belong to $V_1^0[0,1]$.

Example 5.3. Let $\alpha = 0.3$ and $\mu = 0.5$. In this case, we have $\mu < 1$ and, according to Corollary 2.1, the equation has a unique solution of the form

$$u(t) = u_0 t^{\alpha} + O(t^{\alpha+1}) \tag{5.17}$$

for $t \geqslant 0$. More precisely, by (5.2), this solution is

$$u(t) = -4t^{0.3} + 2.25t^{1.3}. \tag{5.18}$$

The particular solution (5.18) may be approximated by applying Euler's method to the auxiliary equation. In this case, according to Corollary 4.1, the approximate solution u^h has an asymptotic error expansion with the form

$$(r^h u - u^h)_i = C_{0.8}(t_i)h^{0.8} + O(h). \tag{5.19}$$

Table 3
Absolute error of the entries of the E-array in Example 5.3

n	$\|S_n - u(1)\|$	$\|E_1^{(n)} - u(1)\|$	$\|E_2^{(n)} - u(1)\|$	$\|E_3^{(n)} - u(1)\|$	$\|E_4^{(n)} - u(1)\|$	$\|E_5^{(n)} - u(1)\|$
0	0.190					
1	0.119	$0.240D-1$				
2	$0.737D-1$	$0.121D-1$	$0.242D-3$			
3	$0.449D-1$	$0.609D-2$	$0.695D-4$	$0.803D-7$		
4	$0.271D-1$	$0.306D-2$	$0.199D-4$	$0.300D-7$	$0.667D-7$	
5	$0.162D-1$	$0.153D-2$	$0.571D-5$	$0.146D-7$	$0.950D-8$	$0.101D-9$

The absolute errors of the approximate solution for this example are given in Fig. 3 and the estimate of the convergence order is $k = 0.73$. In order to apply the E-algorithm, we have considered the auxiliary sequences

$$g_1(n) = h_n^{0.8}, \quad g_2(n) = h_n, \quad g_3(n) = h_n^{1.8}, \quad g_4(n) = h_n^2, \quad g_5(n) = h_n^{2.8}. \tag{5.20}$$

As in the previous examples, we have assumed that the terms of the asymptotic error expansion may be extended to terms of orders higher than 2, with

$$\begin{aligned} g_{2k-1}(n) &= h_n^{\alpha+\mu+k-1}, \\ g_{2k}(n) &= h_n^k, \quad k = 1, 2, 3, \ldots . \end{aligned} \tag{5.21}$$

The absolute errors of the entries of the E-array are displayed in Table 3, showing that this assumption leads to a high performance of the E-algorithm.

6. Concluding remarks

This work has been concerned with the application of Euler's method to Eq. (1.1), in conjunction with extrapolation algorithms. Although in the case $0<\mu<1$ the equation has multiple solutions, Euler's method has been shown to converge to a particular solution of the form $u(t) = \text{const} + O(t)$. An asymptotic error expansion has been derived whose main term is Ch^μ thus showing convergence of order μ. In the case $\mu>1$ first order convergence was proved in [18]. In this work we have obtained new asymptotic error expansions under less restrictive conditions. In particular, if $1<\mu<2$, the expansion will contain a term of the form Ch^μ and, if $\mu=2$, it will also contain a term $Ch^2 \ln h$. The results of this paper show that the application of a low order method together with extrapolation procedures can produce highly accurate approximations, even in the cases of multiple or non-smooth solutions. The same approach may be used to treat a more general class of equations, for example with a kernel of the form $K(t,s)s^{\mu-1}/t^\mu$, where K is a smooth function.

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