Iterative methods for a singular boundary-value problem

P.M.Lima and M.P.Carpentier
C.M.A. and Departamento de Matemática, Instituto Superior Técnico
Av.Rovisco Pais,1049-001 LISBOA
PORTUGAL
e-mail : plima@math.ist.utl.pt, mcarp@math.ist.utl.pt

May 9, 2002

Abstract

We consider a second-order non-linear ordinary differential equation of the form

$$y'' = \frac{1}{q} x\, y^q, \quad 0 \leq x < 1$$

where $q < 0$, with the boundary conditions

$$y'(0) = y(1) = 0.$$

This problem arises in boundary layer equations for the flow of a power-law fluid over an impermeable, semi-infinite flat plane. We show that classical iterative schemes, such as the Picard and Newton methods, converge to the solution of this problem, in spite of the singularity of the solution, if we choose an adequate initial approximation. Moreover, we observe that these methods are more efficient than the methods used before and may be applied to a larger range of values of q. Numerical results for different values of q are given and compared with the results obtained by other authors.

Keywords: Singular boundary-value problem, Newton method, Picard method, subsolution, supersolution, finite-difference scheme.

AMS Classification:65L12;65B05.

1 Introduction

The ordinary differential equations of the form

$$y'' + cx^p y^q = 0, \tag{1.1}$$

where c, p and q are real constants, are known as the Emden-Fowler equations and have been profusely studied since the beginning of this century (see [?] for an extensive bibliography and historical developments).

The interest in these equations may be explained by the large number of mathematical models in physics, mechanics and chemistry where they arise.

For example, the Thomas-Fermi equation, which is used in the description of the atomic stucture, is a particular case of (**??**), when $p = -1/2$, $q = 3/2$ and $c = -1$. Different sets of boundary conditions are imposed, corresponding to the case of neutral or ionized atoms. Iterative schemes and discretization methods for the numerical approximation of these boundary-value problems have been proposed by Luning and Perry [**?**], Mooney [**?**]-[**?**] and Lima [**?**].

The most common discretization methods, such as finite-difference schemes, do not provide accurate results in the case of the Thomas-Fermi equation because of the singularity at the origin, which makes the convergence of these methods too slow. In order to accelerate the convergence, the authors of [**?**]-[**?**] and [**?**] have introduced new methods, taking into account the singularity. These methods have been applied later to a wider class of singular boundary-value problems, which are also particular cases of (**??**), with $-2 < p < 0$, $q > 1$ and $c < 0$ (see [**?**],[**?**]).

In the mentioned problems, the singularity results from the negative exponent of x. Singular problems for the Emden-Fowler equations occur also when q is negative. Such problems were considered in [**?**]. They arise, for example, in fluid mechanics, describing the flow of a power-law fluid over an impermeable flat plate. The boundary layer equation in this case may be reduced to the form [**?**]

$$y" - \frac{x}{q} y^q = 0, \tag{1.2}$$

with $q < 0$. The case $q = -1$ corresponds to a newtonian fluid and the non-newtonian case is divided into two subcases: $-1 < q < 0$ (dilatant fluids) and $q < -1$ (pseudoplastic fluids). In all the cases, we search for the solution of (**??**) which satisfies the boundary conditions

$$y'(0) = y(1) = 0. \tag{1.3}$$

A constructive proof of the existence of solution for problem (**??**-**??**) in the case $-1 < q < 0$ is given by Luning and Perry in [**?**], where an iterative scheme for the approximation of the solution is proposed. The case of pseudoplastic fluids ($q < -1$) was studied by Nachman and Callegari [**?**], who gave a proof of existence and uniqueness of a nonnegative continuous solution, valid for any negative value of q.

In [**?**], O'Regan proved a theorem for a more general class of singular boundary-value problems, which guarantees the existence of a $C[0,1] \cap C^2(0,1)$ solution to (**??**-**??**) in the case $-2 < q < 0$. The existence of a positive solution of this problem in $C[0,1] \cap C^2[0,1)$, for any negative q, also follows from a general result, recently obtained by Wong and Lian [**?**].

The main purpose of the present paper is to introduce efficient iterative methods for obtaining accurate numerical approximation of (**??**)-(**??**), for any negative q.

Using the results of Mooney and Roach [**?**] and Mooney [**?**], in Section 2 we introduce two iterative schemes (of Picard and Newton type) and analyse their convergence to the solution of (**??**)-(**??**), for different values of q. In Section 3, we present numerical results, analyse the error and compare the considered methods with the method introduced by Luning and Perry [**?**].

2 Monotone iterative schemes

In [?], Mooney and Roach have proposed monotone iterative methods for the solution of nonlinear convex boundary-value problems and stated sufficient conditions for the convergence of these methods. These methods were reviewed and generalized by Mooney in [?]. Let us recall some results of these works that we shall apply to the numerical approximation of the problem (??) - (??). We shal consider a particular case of the boundary value problem (2.2)-(2.3) in [?], with the form

$$y''(x) = \lambda f(x,y), \quad x \in]0,1[, \tag{2.1}$$

$$y'(0) = y(1) = 0, \tag{2.2}$$

where $\lambda \in \mathbf{R}$ and f is a function defined on

$$\Omega = \{(x,\phi(x)) : \; x \in]0,1[, \phi(x) \in C^2(]0,1[), \phi(x) \geq \theta(x)\},$$

where $\theta(x) \equiv 0$ on $[0,1]$. The Picard iterates for the problem (??)-(??), according to Sec. 3 of [?], are the solutions of the linear boundary value problem

$$y''_{n+1}(x) + \lambda k^2(x) y_{n+1}(x) = \lambda \left(f(x,y_n) + k^2(x) y_n(x) \right), \quad x \in]0,1[, \tag{2.3}$$

$$y'_{n+1}(0) = y_{n+1}(1) = 0, \quad n = 0,1,\ldots \tag{2.4}$$

where $k^2(x)$ is a positive continuous function on $]0,1[$, such that

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{y=\phi} + k^2(x) > 0, \quad \forall (x,\phi) \in \Omega. \tag{2.5}$$

The Newton iterative method for the same problem is defined by the linear equations

$$y''_{n+1}(x) = \lambda \left(f(x,y_n) + \left.\frac{\partial f(x,y)}{\partial y}\right|_{y=y_n} (y_{n+1} - y_n) \right), \quad x \in]0,1[, \tag{2.6}$$

with the boundary conditions (??).

In the particular case of the problem (??) - (??), we have $\lambda = \frac{1}{q}$ and $f(x,y) = x\, y^q$, with $q < 0$. Hence f is unbounded on Ω (it is singular as $y \to 0$) and the results on the convergence of the iterative processes (??) and (??), obtained by Mooney [?] may not be directly applied to this problem. However, we shall try to prove that these iterative methods may be used to approximate the solution of the considered problem, if we start with a convenient initial approximation.

Let $\bar{y}(x)$ be a function such tat $\bar{y}(x) \in C^2(]0,1[)$ and $0 < \bar{y}(x) < y(x)$, $\forall x \in]0,1[$, where y is the exact solution of the nonlinear problem. Then if we set

$$k^2(x) = -q\, x\, \bar{y}^{q-1},$$

the condition (??) will be satisfied, for $\phi > \bar{y}$ and we may write the iterative scheme (??) for our problem as

$$y''_{n+1}(x) - \, x\, \bar{y}(x)^{q-1} y_{n+1}(x) = \frac{1}{q} x \left[y_n(x)^q - q \bar{y}(x)^{q-1} y_n(x) \right], \quad x \in]0,1[. \tag{2.7}$$

In the same way, the Newton iterative process for the problem (??)- (??) is defined by the equations

$$y''_{n+1}(x) - y_n(x)^{q-1} x y_{n+1}(x) = \frac{1}{q} x y_n(x)^q (1-q), \quad x \in]0,1[. \tag{2.8}$$

In order to apply these methods to the considered problem, we must find adequate initial approximations y_0, which may be subsolutions or supersolutions of the problem.

According to the definition given in [?], a function $y \in C^2(]0,1[)$ is said to be a *subsolution* of (??)-(??) if it satisfies

$$-y'' + \frac{1}{q} x y^q \leq 0, \tag{2.9}$$

$$y'(0) \leq 0; \quad y(1) = 0;$$

by reversing the signs of the inequalities, a *supersolution* is defined.

For the problem (??)-(??), due to the singularity of f, it is not trivial to find a subsolution or a supersolution. For example, the zero function may not be considered a subsolution because $f(x,y)$ is not defined as $y \equiv 0$.

As a first step of the analysis of the considered iterative methods for this problem, we shall look for its subsolutions and supersolutions.

2.1 Subsolutions and Supersolutions

2.1.1 The case $0 > q \geq -1$

In this case, we shall look for a subsolution $\bar{y}(x)$, which is differentiable at $x = 1$. Moreover, we shall look for a function $\bar{y}$ which satisfies the boundary conditions $\bar{y}'(0) = \bar{y}(1) = 0$ and, acording to (??), $\bar{y}''(0) = 0$. The simplest function which satisfies these conditions has the form

$$\bar{y}(x) = B(1 - x^3), \tag{2.10}$$

where B is a real constant such that the condition (??) is satisfied. Substituting (??) into (??) and simplifying, we may conclude that $\bar{y}$ will be a subsolution of (??), if

$$B \leq \left(\frac{-1}{6q}\right)^{\frac{1}{1-q}}.$$

2.1.2 The case $q < -1$

In this case, if we choose as initial approximation a function of the form (??), the equations of the Picard and Newton iterates will not in general be solvable. So, we have decided to look for subsolutions and supersolutions in the more general form

$$\bar{y}(x) = B(1 - x^3)^\gamma, \tag{2.11}$$

where γ is an adjustable parameter ($0 < \gamma < 1$). If we substitute the right-hand side of (??) into (??) we obtain

$$-B(1-x^3)^{\gamma-2}\, x \left(\gamma(\gamma-1)9x^3 - 6\gamma(1-x^3)\right) + \frac{1}{q}\, x\, B^q\, (1-x^3)^{\gamma q} \leq 0, \; 0 < x < 1. \tag{2.12}$$

Equating the exponents of $(1 - x^3)$ in (??) gives us

$$\gamma - 2 = \gamma q,$$

from where $\gamma = \frac{2}{1-q}$. Solving (??) with this value of γ, we may conclude that a function with the form (??) is a subsolution of (??)-(??) if B satisfies the conditions:

$$
\begin{aligned}
&B \leq \left(\frac{1}{9\gamma(\gamma-1)q}\right)^{\frac{1}{1-q}}, \ if\ 0<\gamma\leq \tfrac{1}{3};\\
&B \leq \left(\frac{1}{-6\gamma q}\right)^{\frac{1}{1-q}}, \ if\ \tfrac{1}{3}\leq\gamma<1.
\end{aligned}
\tag{2.13}
$$

Supersolutions of the problem may be found in the same way, if we solve the inequation (??) with the reverse sign. In this case, we shall obtain the following conditions on B:

$$
\begin{aligned}
&B \geq \left(\frac{1}{-6\gamma q}\right)^{\frac{1}{1-q}}, \ if\ 0<\gamma\leq \tfrac{1}{3};\\
&B \geq \left(\frac{1}{9\gamma(\gamma-1)q}\right)^{\frac{1}{1-q}}, \ if\ \tfrac{1}{3}\leq\gamma<1.
\end{aligned}
$$

Note that a subsolution of the form (??) with B satisfying (??) is valid also for $\gamma=-q=1$ and in this case coincides with the subsolution obtained before. However the results obtained for supersolutions are not applicable to the case $\gamma=1$. Finally, in the case $\gamma=\frac{1}{3}$, which corresponds to $q=-5$, if we choose $B=\left(\frac{1}{-6\gamma q}\right)^{\frac{1}{1-q}}=10^{-1/6}$, we may conclude that $\bar{y}(x)$ will be a subsolution, as well as a supersolution of the problem. This means that in this case we have obtained an exact solution of (??)-(??) in the form

$$y(x)=\bar{y}(x)=10^{-1/6}(1-x^3)^{\frac{1}{3}}. \tag{2.14}$$

This result will be used to evaluate the error of the numerical approximations in the next section.

2.2 Convergence of the Iterative Methods

Based on the iterative methods (??) and (??), we have constructed numerical algorithms that converge to the solution of the nonlinear equation for a large set of values of q (up to $q=-10$, in the case of the Newton method). However, the proof of the convergence was obtained only for the case $q>-1$. So we shall treat this case separately, as we did above.

2.2.1 $0>q>-1$

We shall begin by proving that the iterative processes (??) and (??) are well-defined, that is, we must show that if we take as y_0 a subsolution or a supersolution of the form (??), and as $\bar{y}$ a subsolution of the same form, the corresponding boundary-value problems will have an unique positive solution, for any n. This fact is not trivial, since the right-hand side and the coefficient of y_{n+1} of the considered equations have non-integrable singularities at $x=1$. Linear boundary-value problems of this kind have been studied by Kiguradze and Lomatidze [?], who have established sufficient conditions for the existence and uniqueness of solution. In particular, in our analysis we shall use theorem 4.2' of [?], which may be furmulated as follows.

Let us consider a linear diffeential equation of the form

$$u''=p_1(t)u+p_2(t)u'+p_0(t),\quad t\in]a,b[; \tag{2.15}$$

where p_0, p_1 and p_2 are known functions, with boundary conditions of the form

$$u(a+) = \alpha, \quad \lim_{t \to b-} \frac{u'(t)}{\sigma(p_2)(t)} = \beta \tag{2.16}$$

where $\alpha, \beta \in R$ and $\sigma(p)(t) = \exp\left(\int_{a+b/2}^{t} p(\tau)d\tau\right)$. The problem (**??**) - (**??**) has a unique solution if the integrals

$$\int_0^1 t|p_i(t)|dt, \quad i = 0, 1. \tag{2.17}$$

are convergent and there exist such constants $\lambda \in [0, 1[$, and $l_i \in [0, \infty[$, $i = 1, 2$, that

$$\begin{gathered}\int_0^1 \frac{ds}{l_1 + l_2 s + s^2} \geq \frac{1}{1-\lambda} \\ t^{2\lambda} p_1(t) \geq -l_1, \quad t^{\lambda}(p_2(t) + \frac{\lambda}{t}) \geq -l_2; \quad 0 < t < 1.\end{gathered} \tag{2.18}$$

Note that the boundary-value problems (**??**)-(**??**) and (**??**)-(**??**) may be reduced to the form (**??**)-(**??**) with $[a, b] = [0, 1]$ if we introduce the variable substitution $x = 1 - t$ and we set

$$\begin{gathered}p_0(t) = \frac{1}{q}(1-t)\left[(y_n(1-t))^q - q\left(\bar{y}(1-t)\right)^{q-1}\; y_n(1-t)\right] \\ p_1(t) = \left(\bar{y}(1-t)\right)^{q-1}(1-t); \quad p_2(t) \equiv 0.\end{gathered} \tag{2.19}$$

in the case of the Picard iterates; and

$$\begin{gathered}p_0(t) = \frac{1}{q}(1-t)\left(y_n(1-t)\right)^q(1-q) \\ p_1(t) = \left(y_n(1-t)\right)^{q-1}(1-t); \quad p_2(t) \equiv 0.\end{gathered} \tag{2.20}$$

in the case of the Newton iterates. Moreover, the conditions (**??**) are satisfied by the functions of our problem if we choose $\lambda = l_2 = 0$, $l_1 = 1$. On the other hand, if y_0 and $\bar{y}$ are functions of the form (**??**), we have $p_0 = O(t^q)$ and $p_1 = O(t^{q-1})$ as $t \to 0$, both in the case of the Newton and the Picard method. Therefore, if $q > -1$ the integrals (**??**) are convergent for the equations (**??**) and (**??**) with $n = 0$. This proves the existence and uniqueness of the first iterate in both methods. In order to prove the differenciability and positivity of the first iterate the following lemmas are needed.

Lemma 2.1. Let u be the solution of (**??**)-(**??**), where $\alpha = \beta = 0$, $[a, b] = [0, 1]$ and p_0, p_1 are given by (**??**) and (**??**), with $q > -1$. Then if $y_n(1-t)$ and $\bar{y}(1-t)$ are differentiable at $t = 0$ and $y_n(1) = \bar{y}(1) = 0$, then $u(t)$ is also differentiable at $t = 0$.

Proof. In order to prove this, we shall use the representation of the solution by means of the Green function. According to lemma 2.5 of [**?**], the considered problem has a Green function with the form

$$g(t, \tau) = \begin{cases} -\frac{1}{u_2(0)}\, u_1(\tau)u_2(t), & if \quad \tau < t; \\ -\frac{1}{u_2(0)}\, u_1(t)u_2(\tau), & if \quad \tau \geq t, \end{cases}$$

where u_1 and u_2 are solutions of the homogeneous equation, associated to (**??**), which satisfy the conditions

$$\begin{gathered}u_1(0+) = 0, \quad u_1'(0+) = 1; \\ u_2(1-) = 1, \quad u_2'(1-) = 0.\end{gathered}$$

Then we may represent the solution of (??)-(??) in the form

$$u(t)=\int_0^1 g(t,\tau)p_0(\tau)d\tau=-\frac{1}{u_2(0)}u_2(t)\int_0^t u_1(\tau)p_0(\tau)d\tau-\frac{1}{u_2(0)}u_1(t)\int_t^1 u_2(\tau)p_0(\tau)d\tau. \tag{2.21}$$

Differentiating both sides of (??) and leting $t\to 0+$, we obtain

$$\lim_{t\to 0+} u'(t)=-\frac{1}{u_2(0)}\lim_{t\to 0+}\left(u_2'(t)\int_0^t u_1(\tau)p_0(\tau)d\tau+u_1'(t)\int_t^1 u_2(\tau)p_0(\tau)d\tau\right) \tag{2.22}$$

From the conditions it follows that $y_n(1-t)=O(t)$ and $\bar{y}(1-t)=O(t)$ as $t\to 0$, hence $p_0(t)=O(t^q)$ and since $q>-1$ the limit in the right-hand side of (??) is finite. This concludes the proof of the lemma. □

If $q>-1$ and $y_0=\bar{y}=B(1-x^3)$ then the conditions of lemma 2.1 are satisfied for equations (??) and (??) when $n=0$. Hence $y_1(x)$ is differentiable at $x=1$.

Lemma 2.2. Let p_0 and p_1 be functions such that $p_0(t)<0, p_1(t)>0$ and let us consider the problem (??) with the boundary conditions $u(0)=u'(1)=0$. If this problem has a solution u, this solution is positive on $]0,1[$.

Proof. From the boundary condition $u(0)=0$ and lemma 2.1 it follows that, for some $\epsilon>0$, either $u'(t)>0$ and $u(t)>0$, or $u'(t)<0$ and $u(t)<0$, $\forall t\in]0,\epsilon[$. If $u'<0$ and $u<0$ on $]0,\epsilon[$, then u' will decrease on the whole interval $]0,1]$, because u'' is negative while $u\le 0$. This is not possible, because $u'(1)=0$. Hence $u'>0$ and $u>0$ on $]0,\epsilon[$. Suppose now that u has at least one zero on $]0,1[$ and let t_1 be the smallest zero of u. Then $u'(t_1)<0$ (u is decreasing at t_1) and $u(t)<0$ on $]t_1,t_1+\epsilon_1[$. But $u''(t)$ is negative while $u\le 0$ and therefore u' will decrease on $[t_1,1]$, which is not compatible with the condition $u'(1)=0$. Hence $u(t)\ne 0$ for $t\in]0,1[$ and this concludes the proof of the lemma. □

Since the coefficients p_0 and p_1, given by (??) and (??), satisfy the conditions of lemma 2.2, the first iterate y_1 is positive. The existence and uniqueness of the Picard and Newton iterates, for $n>1$ follows from lemma 2.1 and lemma 2.2, using induction. Actually suppose that there exists a positive function $y_n(x)$, such that $y_n(x)=O(1-x)$, as $x\to 1-$. Then by Theorem 4.2' of [?] there exists a unique solution of each boundary-value problem (??) and (??), which is the new iterate y_{n+1} of the corresponding method. Then by lemma 2.1 and lemma 2.2 we conclude that $y_{n+1}(x)=O(1-x)$, as $x\to 1-$, and y_{n+1} is positive.

On the other hand, the monotonicity and boundedness of the sequences of Picard and Newton iterates may be proved in the same way as in [?], since the function $f(x,y)=-\frac{1}{q}xy^q$ on the right-hand side of the equations (??) and (??) satisfies the following conditions

$$R1'. f(x,y)>0,\ \ \forall y\ge\bar{y}\ \ (positivity)$$

$$R2'. \frac{\partial f(x,y)}{\partial y}\in C(\Omega_0)\ (differenciability)$$

$$R3'. \frac{\partial^2 f(x,y)}{\partial y^2}>0,\ \ \forall(x,y)\in\Omega_0\ \ (convexity)$$

where

$$\Omega_0=\{(x,\phi(x)):\ x\in]0,1[,\phi(x)\in C^2(]0,1[),\phi(x)\ge\bar{y}(x)\}.$$

These conditions are analogous to those imposed in [?], with Ω replaced by Ω_0. The positivity lemma, which is used by Mooney and Rooney [?] to prove the monotonicity, may be substituted in our case by Lemma 2.1. Finally, the convergence of the Picard and Newton iterates may be proved using the monotonicity and boundedness, as done, for example, in Theorem 3.1 of [?].

The convergence results we have just obtained for $0 > q > -1$ may be expressed in the form of the following theorems, where we consider subsolutions and supersolutions of the form (??).

Theorem 2.1. Let y_0 and $\bar{y}$ be subsolutions of (??)-(??) and $y_0(x) \geq \bar{y}(x)$, $0 \leq x \leq 1$. Then the iterates (??) converge uniformly and monotonically upwards to the solution of (??)-(??).

Theorem 2.2. Let y_0 be a supersolution and $\bar{y}(x)$ be a subsolution of (??)-(??). Then the iterates (??) converge uniformly and monotonically downwards to the solution.

Theorem 2.3. Let y_0 be a subsolution of (??)-(??) . Then the iterates (??) converge uniformly and monotonically upwards to the solution.

2.2.2 $q \leq -1$

In this case, we must use as initial approximations the subsolutions and supersolutions of the form (??). Then the equations of the iterative schemes (??) and (??) may be expressed in the form

$$y''_{n+1}(x) - \frac{w(x)}{(1-x)^2} y_{n+1}(x) = \frac{1}{(1-x)^{2-\gamma}} f_n(x), \quad y \in]0,1[. \tag{2.23}$$

where w and f_n are smooth functions in a neighborhood of $x = 1$. The arguments we have used above are not applicable to prove the convergence in this case, because equation (??) does not satisfy the conditions of the theorem 4.2' of [?].This equation has a regular singular point at $x = 1$ and its solutions may be expressed in the form of a series near the singularity, using the Frobenius method. This approach will be developped in a new paper.

The numerical results that are presented in the next section suggest that the convergence results for the case $q > -1$ may be extended for $q \leq -1$.

3 Numerical results

In this section we shall present some results that we have obtained by solving numerically the problem (??)-(??). We have used three different iterative methods: the Picard scheme (??), the Newton scheme (??) and the method introduced by Luning and Perry [?], which we shall now briefly describe. If we introduce in (??) the substitution $y = (-q\lambda)^{\frac{1}{q-1}} u$, the considered equation is reduced to the following nonlinear eigenvalue problem:

$$u''(x) + \lambda\, x\, u^q(x) = 0, \quad 0 < x < 1, \tag{3.1}$$

$$u'(0) = u(1) = 0. \tag{3.2}$$

A pair (u, λ), where $u \in C[0,1] \cap C^2(0,1)$ and $\lambda \in \mathbf{R}$, is said to be a solution of this eigenvalue problem if u satisfies (??)-(??) and $u(0) = 1$. If such a pair is found, the solution of (??)-(??) may be expressed as

$$y(x) = u(x)\,(-q\lambda)^{-\frac{1}{1-q}}.$$

As proved in [?] for the case $q > -1$, the solution of (??)-(??) may be approximated by means of the following iterative scheme

$$v''_k(x) = -x\, u^q_{k-1}(x), \tag{3.3}$$

$$v_k'(0) = v_k(1) = 0,$$
$$\lambda_k = \frac{1}{v_k(0)}$$
$$u_k(x) = v_k(x)\,\lambda_k, \quad 0 < x < 1, \; k = 1, 2, \ldots \tag{3.4}$$

Starting with an initial function of the form $u(x) = (1 - x^2)^\alpha$, where α is chosen acording to q, we obtain the sequences $\{u_k\}$ and $\{\lambda_k\}$, converging, respectively, to λ and u, which form a solution of (**??**-**??**). Moreover, according to [?], the terms of these sequences satisfy the inequalities

$$u_0(x) < u_2(x) < \ldots < u_{2k}(x) < \ldots < u(x) < \ldots < u_{2k-1}(x) < \ldots u_1(x), \quad 0 < x < 1, \tag{3.5}$$

$$\lambda_2 > \lambda_4 > \ldots > \lambda_{2k} > \lambda > \lambda_{2k-1} > \ldots > \lambda_1, \quad k = 1, 2, \ldots.$$

In the present work, the linear equations arising in this iterative method, as well as in the Picard and Newton methods, were discretized by means of finite-difference schemes. We have chosen an uniform grid with stepsize $h = \frac{1}{N}$ and gridpoints $x_i = i\,h$ on the interval $[0, 1]$. Then, if $y(x, h)$ is a given function of x and h, we define the central difference operator as usual by

$$\delta^2 y(x, h) := \frac{y(x + h, h) - 2y(x, h) + y(x - h, h)}{h^2}, \tag{3.6}$$

where y is any real function on $[0, 1]$. Replacing the second derivative in the linear differential equation by (**??**), we look for approximate solutions in the form $v_{\nu+1}$ and $y_{\nu+1}(x, h)$ which satisfy the sets of equations

$$\delta^2 v_{\nu+1}(x_i, h) = -x_i u_\nu^q(x_i, h), i = 1, \ldots, N - 1, \tag{3.7}$$

in the case of the scheme (**??**)-(**??**);

$$\begin{array}{l} \delta^2 y_{\nu+1}(x_i, h) - \bar{y}(x_i)^{q-1} x_i y_{\nu+1}(x_i, h) = \frac{1}{q} x_i \left[x y_\nu(x_i, h)^q - q \bar{y}(x_i)^{q-1} y_\nu(x_i, h) \right], \\ i = 1, \ldots, N - 1, \end{array} \tag{3.8}$$

in the case of the Picard method; and

$$\begin{array}{l} \delta^2 y_{\nu+1}(x_i, h) - y_\nu(x_i, h)^{q-1} x_i y_{\nu+1}(x_i, h) = \frac{1}{q} x_i y_\nu(x_i, h)^q\,(1 - q), \\ i = 1, \ldots, N - 1, \end{array}$$

in the case of the Newton method. In each case, boundary conditions are replaced by the equations

$$\begin{array}{l} y_{\nu+1}(x_1, h) = y_{\nu+1}(0, h), \\ y_{\nu+1}(x_N, h) = 0. \end{array}$$

Thus we obtain systems of $N + 1$ linear equations with tridiagonal matrices, which may be solved by elementary methods.

One of the purposes of the present work was to compare the method (**??**)-(**??**) with the monotone methods, introduced in section 2. With this purpose, we have carried out some numerical experiments with different values of q. Note that the convergence of the method (**??**)-(**??**) was proved in [?] only for $q > -1$. For such values of q, following Luning and Perry, we have used the initial approximation $u_0(x) = 1 - x^2$, which has the same asymptotic behavior as the subsolution (**??**), when $x \to 1$. For $q = -1$, we have used the same initial approximation. The iterates u_0 and u_1 , for this case, are displayed in fig.1, together with the approximate solution $u(x, h)$. For $q < -1$, we have used the initial approximation

$u_0(x) = (1-x^2)^{\frac{2}{1-q}}$, which has the same asymptotic behavior as the subsolution (**??**) near the singularity. For such values of q, the first iterates of the scheme (**??**)-(**??**) give lower and upper bounds of the solution, which, in certain cases, are good approximations of it. However, begining with a certain iterate u_k (where k depends on the value of q), the inequalities (**??**) are not satisfied and the sequence of the iterates does not converge (it oscillates near the solution or goes away from it). Therefore, for such values of q, we give only numerical approximations obtained by the Picard and Newton methods.

When these last methods were used, we started with a subsolution or a supersolution of the form (**??**) (if $q > -1$) or (**??**) if ($q \leq -1$), with B satisfying the required conditions:

- for $q = -0.5$, we considered a subsolution with $B = 0.48$;
- for $q = -1$, a subsolution with $B = 0.4$;
- for $q = -2$, a subsolution with $B = 0.5$ and a supersolution with $B = 0.63$;
- for $q = -3$, a subsolution with $B = 0.57$ and a supersolution with $B = 0.63$ (see fig.2);
- for $q = -5$, a subsolution with $B = 0.675$ and a supersolution with $B = 0.7$.
- for $q = -7$, a subsolution with $B = 0.73$.

For each method and each value of q, we have iterated the scheme until the condition

$$\|y_{\nu+1}(x,h) - y_\nu(x,h)\|_2 \leq \epsilon \tag{3.9}$$

was satisfied, for a given ϵ, where we use the norm

$$\|u(x,h)\|_2 := \sqrt{\sum_{i=1}^{N} (u(x_i,h))^2}.$$

In table **??** we give the number of iterations, for different methods and values of q, when $\epsilon = 10^{-6}$. In all the examples we have started with the described above subsolution and used stepsize $h = 10^{-3}$. The function $\bar{y}$ for the Picard method was considered equal to the subsolution. If we take the mentioned above supersolutions as initial approximations for the Picard method, the number of iterations differs by no more than 1. Note that when we use the scheme (**??**) for values of q, less than -1, the condition (**??**) with the considered value of ϵ was not satisfied after 50 iterations. For $q < -5$, the condition (**??**) was not satisfied when we used the scheme (**??**), for $h = 10^{-3}$. This may be explained by the fact that the algorithm becomes numerically unstable for high values of $-q$ and small values of h. But if we use $h = 0.01$, for example, when $q = -7$, the condition (**??**) with $\epsilon = 10^{-6}$ is satisfied after 23 iterations. The Newton method has proved to be the most efficient of the considered schemes, mantaining the numerical stability and fast convergence even for $q < -5$ and $h = 1/4000$. The graphics of the approximate solutions for some different values of q are displayed in fig.3.

In order to verify the convergence of the Picard method and the accuracy of the numerical results, we have compared, for some values of q, the results obtained when we start with a subsolution with the ones obtained in the case of a supersolution. In table **??** such results are displayed for the case $q = -3$. In this case, the conditions (**??**) was satisfied with $\epsilon = 10^{-4}$.

The value of $y(0)$ is specially important by its physical meaning, since it is needed to compute the drag of fluid imparted by a section of the plate. This value may be obtained also by solving a different equation, as it is done by Nath in [?]. In table **??** we give the

q	$(3.4)-(3.7)$	*Picard*	*Newton*
-0.5	8	7	4
-1	15	20	5
-2	$(*)$	12	5
-3	$(*)$	8	4
-5	$(*)$	43	5
-7	$(*)$	$(*)$	7

Table 1: Comparison of convergence speed of the considered iterative methods for different values of q. In each case, we give the number of iterations performed until the condition $\|y_{\nu+1}-y_\nu\|_2 \leq 10^{-6}$ was satisfed. The symbol (*) means that this condition was not satisfied after 50 iterations.

x	*lower iterate*	*upper iterate*
.00	.586733	.586737
.10	.586458	.586462
.20	.584532	.584528
.30	.579250	.579253
.40	.568803	.568806
.50	.551060	.551064
.60	.523255	.523259
.70	.481254	.481258
.80	.417658	.417662
.90	.314543	.314546

Table 2: Approximate solution in the case $q=-3$, obtained by Picard iterates, starting from a subsolution (first column) and from a supersolution (second column). The stepsize of the discretization method was $h=10^{-3}$.

q	$N=1000$	$N=2000$	$N=4000$	*Nath*
-0.5	0.5277507	0.5277537	0.5277547	
-1	0.4695723	0.4695874	0.4695943	0.4697
-2	0.5227900	0.5228433	0.5228698	0.5229
-3	0.5867342	0.5868079	0.5868446	
-5	0.6811209	0.6812065	0.6812493	0.6821
-7	0.7422381	0.7423229	0.7423652	

Table 3: The approximations of $y(0)$ for some values of q, with three different stepsizes. In the last column the corresponding values obtained by Nath [?] are displayed.

x	$N = 1000$	$N = 2000$	$N = 4000$	k
0.00	$0.17122236E-03$	$0.85554251E-04$	$0.42761965E-04$	1.07
0.10	$0.17148537E-03$	$0.85691301E-04$	$0.42831873E-04$	1.07
0.20	$0.17347438E-03$	$0.86690556E-04$	$0.43332660E-04$	1.07
0.30	$0.17901793E-03$	$0.89465639E-04$	$0.44720941E-04$	1.07
0.40	$0.19030629E-03$	$0.95111103E-04$	$0.47543810E-04$	1.07
0.50	$0.21044910E-03$	$0.10518108E-03$	$0.52578105E-04$	1.08
0.60	$0.24473988E-03$	$0.12232122E-03$	$0.61146278E-04$	1.08
0.70	$0.30409289E-03$	$0.15198628E-03$	$0.75974937E-04$	1.08
0.80	$0.41742689E-03$	$0.20862905E-03$	$0.10428833E-03$	1.08
0.90	$0.70655166E-03$	$0.35312275E-03$	$0.17651303E-03$	1.00

Table 4: The discretization error of the approximations in the case $q = -5$. $N+1$ is the number of gridpoints and k is the estimated convergence order.

approximate values of $y(0)$, obtained by the Newton method with different stepsizes and, where it is possible, we compare them with the values obtained by Nath.

So far, we did not analyse the discretization error of the considered methods. As it is well-known, the finite-difference operator (**??**) gives an approximation of second order if the fourth derivative of the unknown function is continuous in $[0,1]$. However, this condition is not fulfilled in the present case, and therefore a convergence of order less than two should be expected. In order to evaluate the convergence order empirically, we have examined the error of the results in the case $q = -5$, when the exact solution is given by (**??**). We have computed the numerical solution with $N = 1000$,$N = 2000$ and $N = 4000$. Comparing the results, we see that the error of the iterative method, when the condition (**??**) is satisfied, is negligible in comparison with the discretization error, so that we can ignore it in the error analysis. In table **??** we give the absolute error of the approximations and an estimate of the convergence order:

$$k = -log_2 \left(\frac{y(x,h) - y(x)}{y(x,2h) - y(x)} \right)$$

From this analysis, it follows that for $q = -5$, the method has approximately first order convergence in h. Similar numerical experiments for different values of q indicate that the convergence order is approximately 1, for $q \leq -1$, and between 1 and 2, when $-1 < q < 0$. A detailed error analysis of the problem was not done yet and will be the subject of future research.

4 Concluding Remarks

The convergence speed of the iterative methods, considered in this paper, depends strongly on the initial approximation y_0. We think that the fast convergence we have observed when we used functions of the type (**??**) follows from the fact that these functions have the same asymptotic behavior as the solution near the singularity. In our future work we shall analyse the asymptotic properties of the solution. Knowing these asymptotic properties, it will be possible to accelerate the convergence of the discretization schemes by means of extrapolation methods, as it was done for other singular problems. On the other hand, the use of a variable substitution may be also a way to improve the accuracy of the numerical approximation.

5 Acknowledgements

The authors would like to thank Professor William Perry for giving them the opportunnity to know the paper [?], which was a starting point for the present work. We also thank the anonymous referee for constructive criticism.

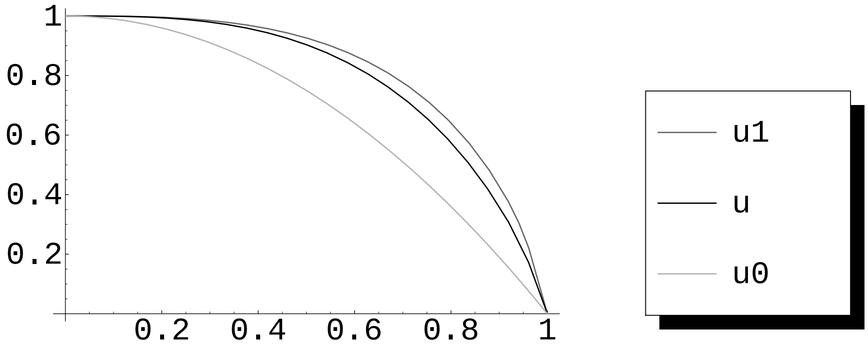

Figure 1: Iterates of the method (3.4)-(3.7) in the case $q = -1$.

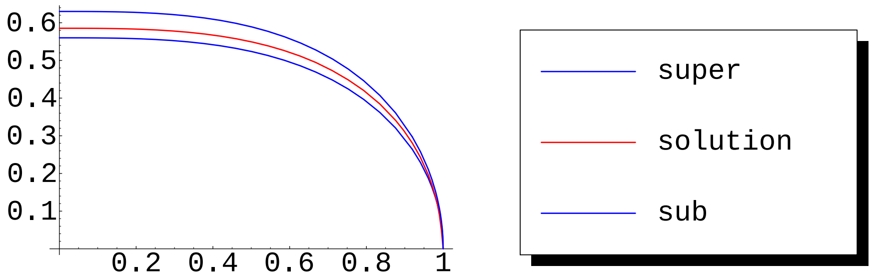

Figure 2: The solution in the case $q = -3$, between a subsolution and a supersolution.

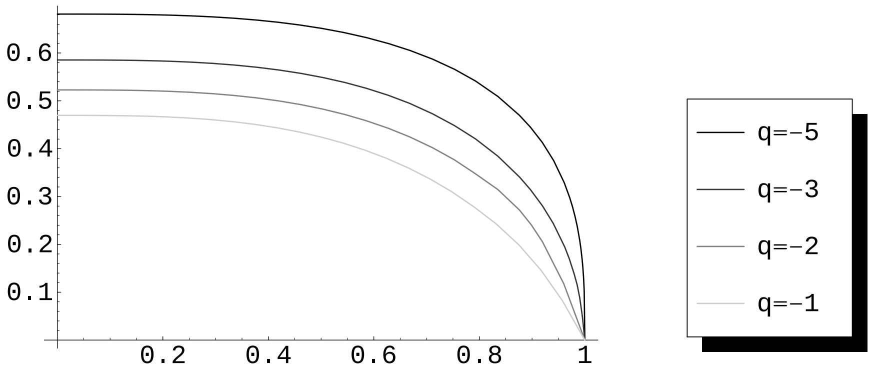

Figure 3: Graphics of the numerical solutions for some different values of q

References

[1] I.T. Kiguradze and A.G.Lomatidze, On certain boundary-value problems for second-order linear differential equations with singularities, *J.Math. Anal. Appl.* 101 (1984), 325–347.

[2] P.M. Lima, Numerical methods and asymptotic error expansions for the Emden-Fowler equations, *J. Comp. Appl. Math.* 70 (1996),245-266.

[3] P.M. Lima and M. Carpentier, Asymptotic expansions and numerical approximation of non-linear degenerate boundary-value problems (to appear in *Appl. Num. Math.*)

[4] C.D. Luning and W.L. Perry, Implementation of new iterative techniques for solution of Thomas-Fermi and Emden-Fowler equations, *J.Comp.Phys.* 38 (1980), 396–405.

[5] C.D. Luning and W.L. Perry, Positive solutions of negative exponent generalized Emden-Fowler BVP, *SIAM J. Math. Anal.* 12 (1981), 874–879.

[6] C.D. Luning and W.L. Perry, An iterative method for solution of a boundary value problem in non-newtonian fluid flow. *J. Non-Newtonian Fluid Mechanics* 15 (1984) 145–154.

[7] J.W. Mooney, A unified approach to the solution of certain classes of nonlinear boundary value problems using monotone iterations, *Nonlinear Analysis* 3 (1979),449–465.

[8] J.W. Mooney, Monotone methods for the Thomas-Fermi equation, *Quart. Appl. Math.* 36 (1978), 305–314.

[9] J.W. Mooney, Numerical schemes for degenerate boundary-value problems, *J.Phys.* A26 (1993), L413–L421.

[10] J.W. Mooney, Solution of a Thomas-Fermi problem using linear approximants, *Comp. Phys. Commun.* 76 (1993), 51–57.

[11] J.W. Mooney, Solution of Emden-type problems using accurate, efficient discretization schemes, *Comp. Phys. Commun.* 83 (1994), 245–254.

[12] J.W. Mooney and G.F.Roach, Iterative bounds for the stable solutions of convex non-linear boundary value problems, *Proc. Roy. Soc. Edinburgh* 76A (1977) 81–94.

[13] A.Nachman and A.Callegari, A nonlinear singular boundary- value problem in the theory of pseudoplastic fluids, *SIAM J. Appl.Math.* 38 (1980) 275–281.

[14] G. Nath, Initial value method for a pseudoplastic fluid in a laminar boundary layer over a porous flat plate with suction, *Trans. Amer. Soc. Mech. Eng., Series E, J. Appl. Mech.* 40 (1973) 1123–1124.

[15] D. O'Regan, Singular second-order boundary-value problems, *Nonlinear Analysis 15* (1990), 1097-1109.

[16] F.H. Wong and W.C. Lian, Positive solution for singular boundary-value problems, *Comput. Math. Appl.* 32 (1996), 41–49

[17] J. W. Wong, On the Generalized Emden-Fowler Equation, *SIAM Review* 17 (1975), 339–360.