Numerical Solution of the Two-Dimensional Neural Field Equation

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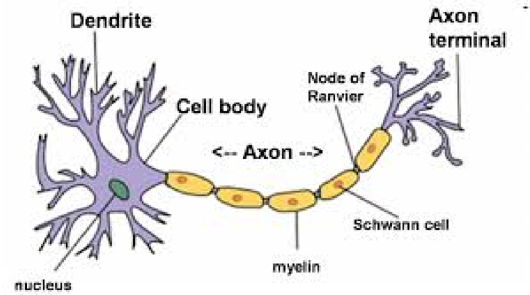
OUTLINE OF THE TALK

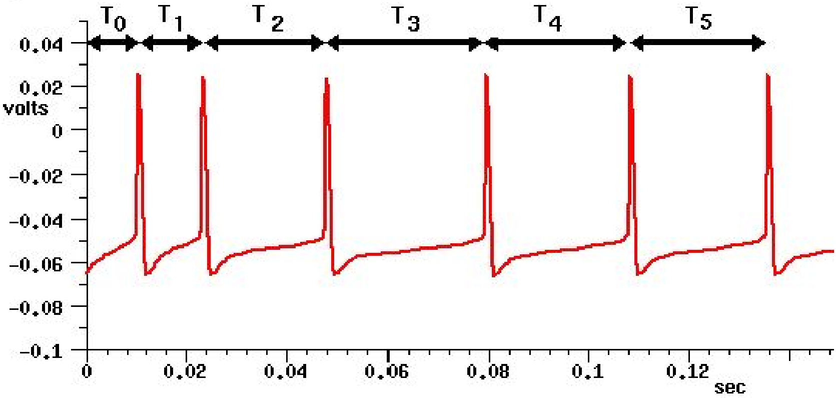
INTRODUCTION -THE HUMAN BRAIN

According to a lower estimate from 2009, the human nervous system contains 0.89×10^{11} neurons, which are connected by about 10^{15} synapses.



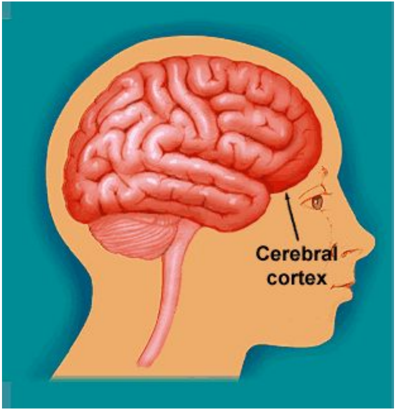
COMMUNICATION BETWEEN NEURONS

The change of voltage in the cell membrane of a neuron results in a voltage spike called an action potential, which triggers the release of other neurotransmitters. That is, neurons communicate with each other by firing.



THE CEREBRAL CORTEX

The cerebral cortex is the brain's outer layer of neural tissue in humans and other mammals.
It plays a key role in controlling memory, attention, perception, awareness, thought, language and other important processes.
The cortex of a human is about 2-4 mm thick and contains about one fifth of all the neurons.



MATHEMATICAL MODELS

- Hodgkin and Huxley, 1952 - the most successful mathematical model describing the mechanism of ion currents and voltage changes in the neuron membrane. It consists of a system of 4 ordinary differential equations.
- FitzHugh-Nagumo Equations, 1962 - the Hodgkin-Huxley system was reduced to a system of two equations; it describes the propagation of impulses in the nervous system.
- Neural Field Equations - introduced by Wilson and Cowan, in 1972, and Amari, in 1977. The main idea of the Neural Field Models is to treat the cortex as a continuous space and describe the spatiotemporal dynamics of the neural interactions.

APPLICATIONS OF NEURAL FIELDS

- In Neuroscience - interpretation of experimental data, including information obtained from EEG, fMRI and optical imaging.
- In Robotics - the architecture of autonomous robots, able to interact with other agents in solving a mutual task, is strongly inspired by the processing principles and the neuronal circuitry in the primate brain.

NEURAL FIELD EQUATION

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_\Omega K(\|\bar{x}-\bar{y}\|_2)S(V(\bar{y},t))d\bar{y}, \quad (1)$$

$$t \in [0,T], \bar{x} \in \Omega \subset \mathbb{R}^2;$$

Initial Condition: $V(\bar{x},0) = V_0(\bar{x}), \qquad \bar{x} \in \Omega.$

- $V(\bar{x},t)$ - the membrane potential in point x at time t;
- $I(\bar{x},t)$ - external sources of excitation;
- $S(x)$ - dependence between the firing rate of the neurons and their membrane potentials (sigmoidal or Heaviside function);
- $K(\|\bar{x}-\bar{y}\|_2)$ - connectivity between neurons at $\bar{x}$ and $\bar{y}$.

TIME DISCRETIZATION

Rewrite equation (1) in the form

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \kappa(V(\bar{x},t)) \quad (2)$$

$$t \in [0, T], \bar{x} \in \Omega \subset \mathbb{R}^2,$$

where

$$\kappa(V(\bar{x},t)) = \int_{\Omega} K(|\bar{x}-\bar{y}|)S(V(\bar{y},t))d\bar{y}. \quad (3)$$

h_t - stepsize in time.

$$t_i = ih_t, \quad i = 0, ..., M, \quad T = h_t M.$$

Let $V_i(\bar{x}) = V(t_i, \bar{x}), \quad \forall x \in \Omega, \quad i = 0, ..., M$. We approximate the partial derivative in time by the backward difference

$$\frac{\partial}{\partial t}V(\bar{x},t_i) \approx \frac{3V_i(\bar{x}) - 4V_{i-1}(\bar{x}) + V_{i-2}(\bar{x})}{2h_t}, \quad (4)$$

EXISTENCE OF SOLUTION OF THE FREDHOLM EQUATION

Does the nonlinear Fredholm equation have a solution? How to compute it?

$$U_i(\bar{x}) - \lambda\kappa(U_i) = f_i(\bar{x}), \qquad \bar{x} \in \Omega \tag{5}$$

where $\lambda = \frac{2h_t}{2h_t+3c}$,

$$f_i(\bar{x}) = \left(1 + \frac{2h_t}{3c}\right)^{-1} \left(I_i + \frac{c}{h_t} 2U_{i-1}(\bar{x}) - \frac{c}{2h_t} U_{i-2}(\bar{x})\right), \tag{6}$$

$\bar{x} \in \Omega$. Define the iterative process:

$$U_i^{(\nu)}(\bar{x}) = \lambda\kappa\left(U_i^{(\nu-1)}(\bar{x})\right) + f_i(\bar{x}) = G\left(U_i^{(\nu-1)}(\bar{x})\right), \tag{7}$$

$\bar{x} \in \Omega, \nu = 1, 2, \ldots$. For a sufficiently small step size h_t the function G is contractive and equation (5) has a unique solution in a certain set Y; the sequence $U_i^{(\nu)}$ defined by (7) converges to this solution, for any initial guess $U_i^{(0)} \in Y$.

SPACE DISCRETIZATION

Assume that Ω is a rectangle: $\Omega = [-1,1] \times [-1,1]$. Introduce a uniform grid of points (x_i, x_j), such that $x_i = -1 + ih$, $i = 0, ..., n$, where h is the discretisation step in space. In each subinterval $[x_i, x_{i+1}]$ we introduce k Gaussian nodes: $x_{i,s} = x_i + \frac{h}{2}(1 + \xi_s)$, $i = 0, 1, \ldots n-1$, where ξ_s are the roots of the k-th degree Legendre polynomial, $s = 1, ..., k$. Using a Gaussian quadrature formula to evaluate the integral, we obtain the finite-dimensional approximation of $\kappa(U)$. This discretisation provides an accuracy order of $O(h^{2k})$.

$$(\kappa^h(U^h))_{mu,lv} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{s=1}^{k} \sum_{t=1}^{k} \tilde{w}_s \tilde{w}_t$$
$$\times K(\|(x_{mu}, x_{lv}) - (y_{is}, y_{jt})\|_2) S((U^h)_{is,jt}). \tag{8}$$

By replacing κ with κ_h in equation (5) we obtain the following system of nonlinear equations:

$$U^h - \lambda \kappa^h(U^h) = f^h, \tag{9}$$

where $\kappa^h(U^h)$ is defined by (8) and $(f^h)_{is,jt} = f(x_{is}, x_{jt})$.

FIXED POINT METHOD

We obtain a system of N^2 nonlinear equations.

1. Is this system solvable?
2. Does the solution U^h of this system converge in some sense to U_i, as $h \to 0$?
3. How can we estimate the error $E_i^h = \|U^h - U_i\|$?

To answer the first question we use the fixed point theorem. Consider the iterative process:

$$U^{h,(m)} = \lambda \kappa^h(U^{h,(m-1)}) + f^h = G^h(U^{h,(m-1)}), \tag{10}$$

$m = 1, 2, \ldots$. It can be shown that G^h is contractive, if

$$h_t < \frac{3c}{2K_{\max} S_{\max}}. \tag{11}$$

CONVERGENCE AND COMPUTATIONAL IMPLEMENTATION

It may be proved that there exists such a constant $\tilde{M}$ that

$$\|U_i - U^h\|_\infty \le \tilde{M} h^{2k}. \qquad (12)$$

Stopping criterium for the iterative method:

$$\|U^{h,(n)} - U^{h,(n-1)}\|_\infty < \epsilon,$$

for some given ϵ. In all the computed examples the number of iterations in the inner cycle is not very high (3-4, in general). For the initial guess, we use the Euler method:

$$U^{h,(0)} = U_{i-1} + \frac{h_t}{c}(I_i - U^h_{i-1} + \kappa^h(U^h_{i-1})). \qquad (13)$$

EFFICIENCY AND RANK REDUCTION

In order to improve the efficiency of the numerical method, we apply the following technique.
Assuming that the function V is sufficiently smooth, we can approximate it by an interpolating polynomial of a certain degree. As it is known from the theory of approximation, the best approximation of a smooth function by an interpolating polynomial of degree m is obtained if the interpolating points are the roots of the Chebyshev polynomial of degree m.
Our approach for reducing the matrices rank in our method consists in replacing the solution V_i by its interpolating polynomial at the Chebyshev nodes in Ω. If V_i is sufficiently smooth, this produces a very small error and yields a very significant reduction of computational cost. Actually, when computing each layer of the solution we have only to compute m^2 components, one for each Chebyshev node on $[-1,1] \times [-1,1]$, instead of N^2. Choosing m much smaller than N, we thus obtain a significant computational advantage.

NEURAL FIELD EQUATION WITH DELAY

According to many authors, realistic models of neural fields must take into account that the propagation speed of neuronal interactions is finite, which leads to NFE with delays of the form

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_{\Omega} K(|\bar{x}-\bar{y}|)S(V(y,t-\tau(\bar{x},\bar{y}))d\bar{y}, \tag{14}$$

$t \in [0,T]$, $\quad \bar{x} \in \Omega \subset \mathbb{R}^2$;
$\tau(\bar{x},\bar{y}) > 0$ - delay, depending on the spatial variables.
v-constant propagation speed; then $\tau(\bar{x},\bar{y}) = \|\bar{x}-\bar{y}\|_2/v$.
Initial condition in the delay case:
$V(\bar{x},t) = V_0(\bar{x},t), \qquad \bar{x} \in \Omega, \quad t \in [-\tau_{max}, 0]$, where
$\tau_{max} = \max_{\bar{x},\bar{y}\in\Omega} \tau(\bar{x},\bar{y})$ (maximal delay).
How to take into account the delay in the computations? Note that $t - \tau_{max} \leq t - \tau(\bar{x},\bar{y}) \leq t$. Therefore, $V(t-\tau(\bar{x},\bar{y}))$ depends on the values of the solution at different instants in the past.

NUMERICAL EXAMPLE 1 (NO DELAY)

Main purpose: to test experimentally the convergence of the method and measure the error.

$$K(|\bar{x}-\bar{y}|) = \exp\left(-\lambda(x_1-y_1)^2 - \lambda(x_2-y_2)^2\right),$$

where $\lambda \in \mathbb{R}^+$; $S(x) = \tanh(\sigma x)$, $\sigma \in \mathbb{R}^+$.

$$I(x,y,t) = -\tanh\left(\sigma \exp\left(-\frac{t}{c}\right)\right) b(\lambda,x,y),$$

$$b(\lambda,x_1,x_2) = \int_{-1}^{1}\int_{-1}^{1} K(x_1,x_2,y_1,y_2)dy_1 dy_2 =$$
$$= \frac{\pi}{4\lambda}\left(\mathrm{Erf}(\sqrt{\lambda}(1-x_1)) + \mathrm{Erf}(\sqrt{\lambda}(1+x_1))\right) \times$$
$$\left(\mathrm{Erf}(\sqrt{\lambda}(1-x_2)) + \mathrm{Erf}(\sqrt{\lambda}(1+x_2))\right),$$

Erf - Gaussian error function.

- Gaussian nodes: $k = 4$; Space discretisation: $m = 12, N = 24$.
- Time discretisation: $h_t = 0.01, 0.02$.
- Equation parameters: $\lambda = \sigma = c = 1$.

Initial condition : $V_0(\bar{x}) \equiv 1$.

Exact solution: $V(\bar{x},t) = \exp(-\frac{t}{c})$.

EXAMPLE 1: NUMERICAL RESULTS

t	$e_i(0.01)$	$e_i(0.02)$	$e_i(0.02)/e_i(0.01)$
0.02	$6.66E-5$		
0.03	$7.24E-5$		
0.04	$7.46E-5$	$2.66E-4$	3.57
0.05	$7.56E-5$		
0.06	$7.61E-5$	$2.91E-4$	3.82
0.07	$7.65E-5$		
0.08	$7.69E-5$	$3.01E-4$	3.91
0.09	$7.72E-5$		
0.10	$7.76E-5$	$3.06E-4$	3.94

$e_i(h_t) = \|V_i - U_i\|$- error norm.

The ratios are close to 4, which confirms second order convergence.

EXAMPLE 2 (WITH DELAY)

K - as in the previous example;
External input:

$$I(x_1, x_2, t) = -\exp\left(-\frac{t}{c}\right)\beta(\lambda, \mu, x_1, x_2),$$

where $\beta(\lambda, \mu, x_1, x_2) =$

$$\int_0^1 \int_0^1 \exp\left(-\lambda((x_1 - y_1)^2 + (x_2 - y_2)^2) - \mu(y_1^2 + y_2^2)\right) dy_1 dy_2.$$

Firing rate function: $S(x) = x$.
Propagation speed: v=1.
Initial condition:

$$V_0(x_1, x_2) = \exp(-\mu(x_1^2 + x_2^2)),$$

$\forall \bar{x} \in \Omega, t \in [-\tau_{max}, 0]$.

EXAMPLE 2 : NUMERICAL RESULTS

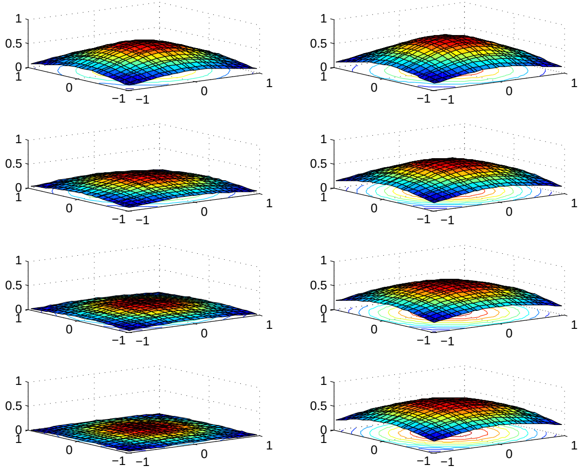


Figure: Plots of the solution without delay (left) and with delay (right), at $t = 0.5, 1.0, 1.5, 2.0$

EXAMPLE 3 (MEXICAN HAT CONNECTIVITY)

Connectivity Kernel:

$$K(r) = \frac{1}{\sqrt{2\pi\xi_1^2}} \exp\left(-\frac{r^2}{2\pi\xi_1^2}\right) - \frac{A}{\sqrt{2\pi\xi_2^2}} \exp\left(-\frac{r^2}{2\pi\xi_2^2}\right),$$

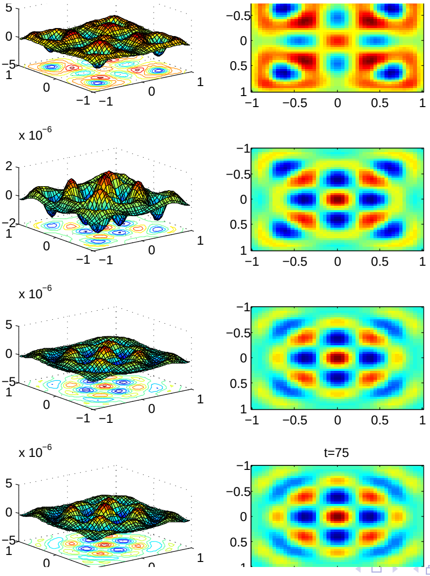
where A, ξ_1, ξ_2 - given positive numbers.
External input: $I \equiv 0$. Firing rate function: $S(x) = \frac{2}{1+e^{-\mu x}}, \quad \mu > 0.$
Propagation speed: no delay, $v = 1$.
Initial condition:

$$V_0(x_1, x_2) \equiv 0.01,$$

$\forall \bar{x} \in \Omega, t \in [-\tau_{max}, 0]$.

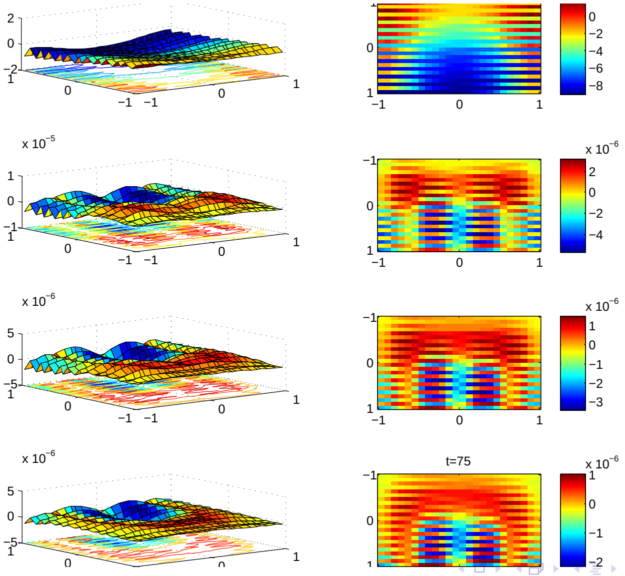
EXAMPLE 3 : NUMERICAL RESULTS

$\xi_1 = 0.1$, $xi_2 = 0.2$, $A = 1$; $\mu = 45$; no delay; time=20, 30, 40, 50.



EXAMPLE 3 : NUMERICAL RESULTS

$\xi_1 = 0.1$, $xi_2 = 0.2$, $A = 1$; $\mu = 45$; $v = 1$; time$=20, 30, 40, 50$.



EXAMPLE 4 (HEXAGONAL PATTERN)

Connectivity Kernel:

$$K(\bar{x}, \bar{y}) = K_0 \sum_{i=0}^{2} \cos(\bar{k}_i . \bar{x} - \bar{y}) \exp\left(-\frac{\|\bar{x} - \bar{y}\|}{\sigma}\right),$$

where $\bar{k}_i = k_c(\cos(\phi_i), \sin(\phi_i))$, $\phi_i = i\pi/3$ and K_0, k_c and σ are positive constants. External input: $I(\bar{x}, t) = I_0 + \frac{1}{\pi\sigma_I^2} \exp(-\bar{x}^2/\sigma_I^2)$, with $I_0 = 2$ and $\sigma_I = 0.2$. Firing rate function: $S(x) = \frac{2}{1+\exp(-5.5(x-3))}$.
Propagation speed: $v = 10$.
Initial condition:

$$V_0(x_1, x_2) = 2.00083,$$

$\forall \bar{x} \in \Omega, t \in [-\tau_{max}, 0]$.

EXAMPLE 4 (CONNECTIVITY KERNEL)

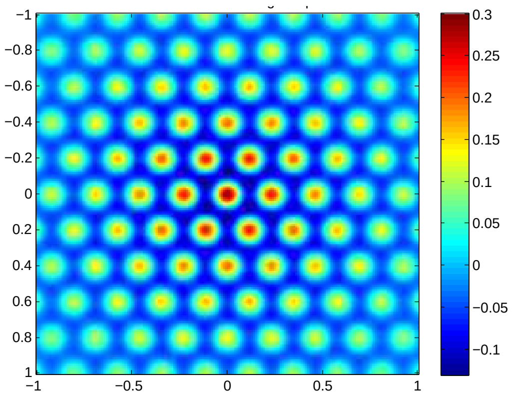


Figure: Plot of the connectivity function $K(x, y)$ of example 4.

EXAMPLE 4 (Plots)

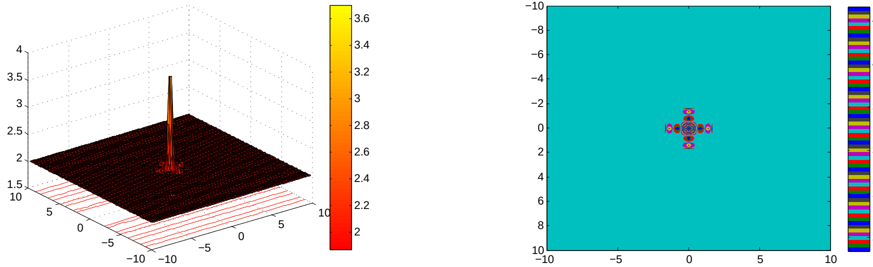
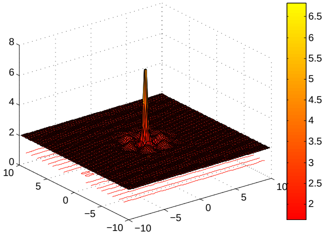


Figure: Solution at $t = 0.16$.

EXAMPLE 4 (Plots)



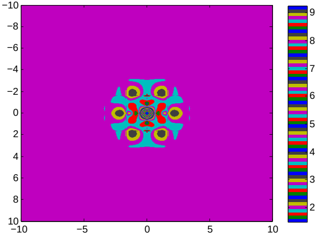


Figure: Solution at $t = 0.48$.

EXAMPLE 4 (Plots)

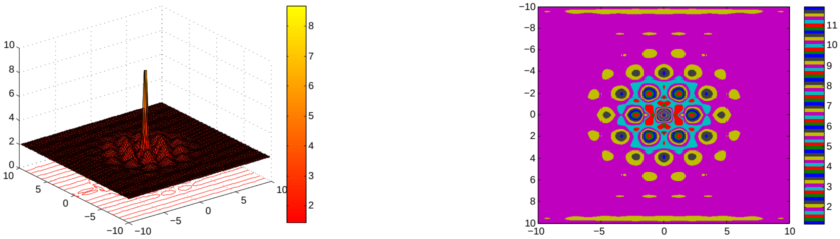
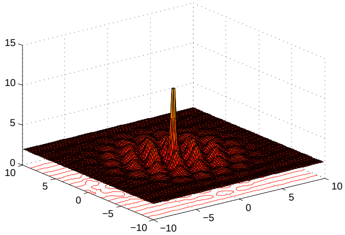


Figure: Solution at $t = 0.72$.

EXAMPLE 4 (Plots)



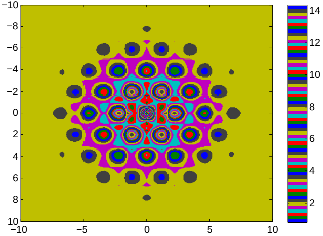


Figure: Solution at $t = 0.96$.

CONCLUSIONS AND FUTURE WORK

- A remarkable feature of our method is that we use an implicit second order scheme for the time discretisation, which improves its accuracy and stability, when compared with the available algorithms.
- To reduce the computational complexity of our method and improve its efficiency we have used an interpolation procedure which allows a drastic reduction of matrix dimensions, without a significant loss of accuracy.
- Our numerical results confirm the theoretical predictions and are in agreement with the expected behaviour of the solutions.
- In order to deal with the case of non-smooth input data, we have to introduce non-uniform meshes.
- In the future we intend to analyse a stochastic version of the model, to take into account the effect of noise.

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