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COMPUTATIONAL APPROACHES TO SINGULAR FREE BOUNDARY PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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INTRODUCTION

Free boundary problem: find a real $M > 0$ and a positive $y \in C^2((0, M)) \cap C^1([0, M])$:

$$(|y'|^{m-2}y')' + \frac{N-1}{x}|y'|^{m-2}y' + f(y) = 0, \quad 0 < x < M,$$
$$y'(0) = 0, \quad y(M) = y'(M) = 0, \quad M > 0.$$

N is the space dimension ($N \geqslant 2$), $m > 1$, $f(y) = ay^q - by^p$, $p < q$, $a, b > 0$.

FIRST VARIABLE SUBSTITUTION

$$z = x/M \longrightarrow \begin{array}{l} y''(z) + \frac{1}{m-1}\frac{N-1}{z}y'(z) + \frac{1}{m-1}\lambda f(y)/|y|^{m-2} = 0, \quad 0 < z < 1, \\ y'(0) = 0, \quad y(1) = y'(1) = 0, \quad \lambda = M^m \text{ is an eigenvalue} \end{array}$$

DIFFERENT TYPES OF SINGULARITIES

- **A.** If $m \leqslant 2$ and $p \geqslant m/2 - 1$: the solution is smooth at both endpoints;
- **B.** If $m \leqslant 2$ and $p < m/2 - 1$: the solution is smooth at $z = 0$ and singular at $z = 1$;
- **C.** If $m > 2$ and $p \geqslant m/2 - 1$: the solution is singular at $z = 0$ and smooth at $z = 1$;
- **D.** If $m > 2$ and $p < m/2 - 1$, the solution is singular at both endpoints.

SECOND VARIABLE SUBSTITUTION

For each case, there is a variable substitution which transforms the solution of the problem into a smooth function (see [2]): $t = \left(1 - (1-z)^{\frac{k_2}{2}}\right)^{\frac{k_1}{2}}$:

For case B: $k_1 = 2, k_2 = \frac{m}{m-1-p}$.

For case C: $k_1 = \frac{m}{m-1}$, $k_2 = 2$.

For case D: $k_1 = \frac{m}{m-1}$ and $k_2 = \frac{m}{m-1-p}$.

NUMERICAL EXAMPLE 1

ORIGINAL PROBLEM

$$y''(z) + \frac{4}{z}y'(z) + 2\lambda\left(y(z) - \frac{1}{\sqrt{y(z)}}\right)\sqrt{|y'(z)|} = 0, \quad y'(0) = y'(1) = y(1) = 0.$$

The solution is singular at $z = 1$. When solving the problem in this formulation, the algorithm does not converge.

MODIFIED PROBLEM

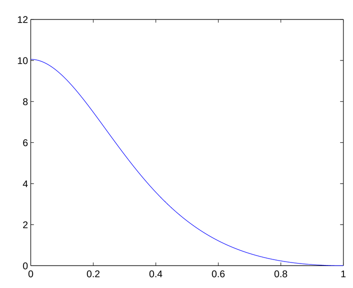
Applying the smoothing transformation $t = 1 - (1-z)^{\frac{3}{4}}$ yields

$$\frac{3\sqrt{3}}{16\sqrt{1-t}}y''(t) + \left(\frac{\sqrt{3}}{16(1-t)^{\frac{3}{2}}} + \frac{\sqrt{3}}{(1-t)^{\frac{1}{6}}\left(1-(1-t)^{\frac{4}{3}}\right)}\right)y'(t) + \lambda\sqrt{|y'(t)|}\left(y(z) - \frac{1}{\sqrt{y(t)}}\right) = 0,$$
$$y'(0) = y'(1) = y(1) = 0.$$

The solution of the new equation is smooth in the whole interval.

intervals	error	rate	error λ	rate λ
101	1.4159e-3	-	7.7077e-4	-
201	3.7705e-4	1.9089	2.0018e-4	1.9450
401	9.8751e-5	1.9329	5.1658e-5	1.9542
801	2.5368e-5	1.9608	1.3163e-5	1.9725
1601	6.2294e-6	2.0258	3.2199e-6	2.0315

Errors and convergence rate of the collocation method with one Gaussian point in each subinterval.



Numerical solution of the modified problem.

NUMERICAL EXAMPLE 2

ORIGINAL PROBLEM

$$y''(z) + \frac{1}{z}y'(z) + \frac{\lambda}{2}\frac{y(z) - \sqrt{y(z)}}{|y'(z)|} = 0, \quad y'(0) = y'(1) = y(1) = 0.$$

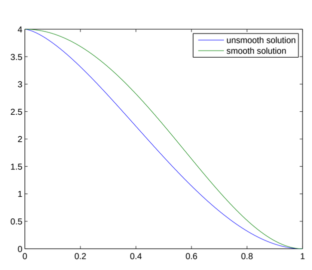
Anal. solution: $y(z) = \left(2 - 2z^{\frac{3}{2}}\right)^2$, $\lambda = 216$, $y \in C^2(0, 1]$. (non-smooth at $z = 0$)

MODIFIED PROBLEM

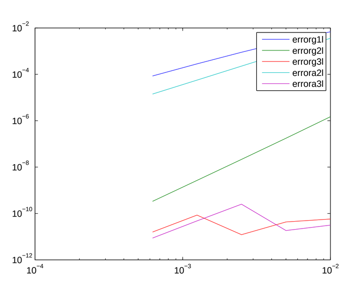
Applying the smoothing transformation $t = z^{\frac{3}{4}}$:

$$\frac{27}{32t}y''(t) + \frac{27}{32t^2}y'(t) + \lambda\frac{y(t) - \sqrt{y(t)}}{|y'(t)|} = 0, \quad y'(0) = y'(1) = y(1) = 0,$$

Anal. solution: $y(t) = \left(2 - 2t^2\right)^2$, $\lambda = 216$, $y \in C^\infty[0, 1]$.



Numerical solutions of the original and modified problem (the two functions depend on different independent variables).



Errors of the numerical results for λ (against the stepsize)

Legend for figures:

errorg1 - 1 Gaussian point, $\mu = 2$;

errorg2 - 2 Gaussian points, $\mu = 3$;

errorg3 - 3 Gaussian points, $\mu = 4$;

errora2 -2 equid. points, $\mu = 3$;

errora3 - 3 equid. points, $\mu = 4$

NUMERICAL EXAMPLE 3

ORIGINAL PROBLEM

$$y''(z) + \lambda\left(y(z) - \frac{2}{3\sqrt{y(z)}}\right) = 0, \quad y'(0) = y'(1) = y(1) = 0.$$

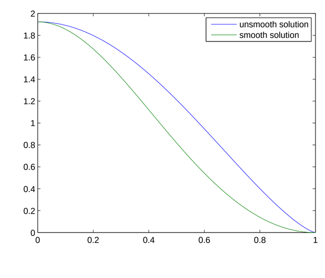
Anal. Solution: $y(z) = \left(\frac{8}{3}\right)^{\frac{2}{3}}\left(\cos\left(\frac{\pi}{2}z\right)\right)^{\frac{4}{3}}$, $\lambda = \left(\frac{2\pi}{3}\right)^2$, $y \in C^2(0, 1]$. non-smooth at $z = 0$

MODIFIED PROBLEM

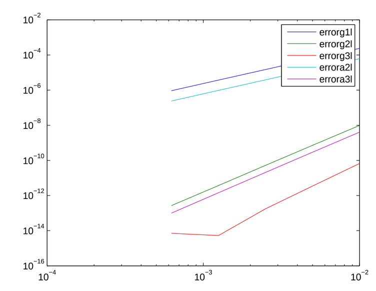
Applying the smoothing transformation $t = 1 - (1-z)^{\frac{3}{2}}$:

$$\frac{4}{9(1-t)}y''(t) + \frac{2}{9(t-1)^2}y'(t) + \lambda\left(y(t) - \frac{2}{3\sqrt{y(t)}}\right) = 0, \quad y'(0) = y'(1) = y(1) = 0,$$

Anal. sol.: $y(t) = \left(\frac{8}{3}\right)^{\frac{2}{3}}\left(\cos\left(\frac{\pi}{2}\left(1 - (1-t)^{\frac{3}{2}}\right)\right)\right)^{\frac{4}{3}}$, $\lambda = \left(\frac{2\pi}{3}\right)^2$, $y \in C^\infty[0, 1]$.



Numerical solutions of the original and modified problem (the two functions depend on different independent variables).



Errors of the numerical results for λ (against the stepsize).

NUMERICAL EXAMPLE 4

ORIGINAL PROBLEM

$$y''(z) + \frac{1}{z}y'(z) + \frac{\lambda}{2}\frac{y(z)-1}{|y'(z)|} = 0, \quad y'(0) = y'(1) = y(1) = 0.$$

(Sol. non-smooth at $z = 0$ and $z = 1$)

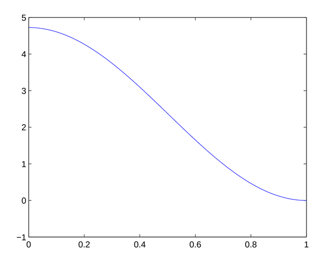
When solving the problem in this formulation, the algorithm does not converge.

MODIFIED PROBLEM

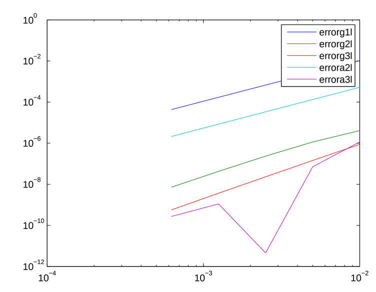
Applying the smoothing transformation $t = \left(1 - (1-z)^{\frac{3}{4}}\right)^{\frac{3}{4}}$

$$\frac{729}{2048t\left(1-t^{\frac{4}{3}}\right)}y''(z) + \left(\frac{81}{512t^{\frac{2}{3}}\left(1-t^{\frac{4}{3}}\right)^2} - \frac{243}{2048t^2\left(1-t^{\frac{4}{3}}\right)} + \frac{81}{128t^{\frac{2}{3}}\left(1-t^{\frac{4}{3}}\right)^{\frac{2}{3}}\left(1-\left(1-t^{\frac{4}{3}}\right)^{\frac{4}{3}}\right)}\right)y'(z) +$$
$$+\lambda\frac{y(z)-1}{|y'(z)|} = 0, \quad y'(0) = y'(1) = y(1) = 0,$$

which has a smooth solution on [0,1]



Numerical solution of the modified problem.



Errors of the numerical results for λ (against the stepsize.)

SUMMARY OF NUMERICAL RESULTS

Estimates of the convergence order for various examples with different methods.

	Example 1	Example 2	Example 3	Example 4
o.p., g., $\mu = 0$	no conv.	1.92–1.93	0.87 – 0.92	no conv.
m.p., g., $\mu = 2$	2	1.7	2	2
m.p., g., $\mu = 3$	no conv.	3	3.8	2.5
m.p., g., $\mu = 4$	no conv.	??	5	2.7
m.p., e., $\mu = 3$	no conv.	2	2	2
m.p., e., $\mu = 4$	no conv.	??	3.8	??

μ - degree of polynomial collocation;
o.p. - original problem; m.p. - modified problem
g. -Gaussian points; e. - equidistant points
?? means that there is a large dispersion between the estimates (effect of rounding-off errors)

CONCLUSIONS AND FUTURE WORK

- We have implemented a new numerical method for the computation of approximate solutions to singular free boundary problems in ODEs, using the open domain MATLAB code bvpsuite.
- Our numerical approach is based on smoothing variable transformations which reduce the original problem (with endpoint singularities) into a new problem, whose solution is smooth in the whole interval.
- As illustrated by the numerical examples, when solving the modified problem the performance of the collocation method is always better than if it is applied to the original one. Even in the cases in which the numerical method fails to approximate the original problem, accurate results are obtained after applying the variable transformation.
- As shown by the numerical results, the approximations obtained using the collocation method have convergence order not less than two, both for the solutions and the eigenvalues. However, it is not always possible to recover the optimal convergence order of the collocation method, as it was previously observed in the case of boundary value problems with the p-laplacian [3].
- When the collocation method is used with 1 Gaussian point, $\mu = 2$, or 2 equidistant points, $\mu = 3$, the numerical results suggest second order convergence, the same which is obtained when the finite differences method is applied (see [2]).
- By increasing the degree of the collocation polynomial μ, the accuracy of the approximations is significantly improved in most of the numerical examples. However, it is not always clear from the numerical results what is the convergence order of the method.
- In the future, we intend to carry out a detailed numerical analysis of the method to better understand its convergence behaviour.

References

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Further Reading

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