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NUMERICAL SIMULATIONS IN TWO-DIMENSIONAL NEURAL FIELDS

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The main idea of the neural field models in Mathematical Neuroscience is to treat the cortical space as continuous. Since the number of neurons and synapses is extremely high even in a small piece of cortex, this idea appears naturally as a first approximation to model the neural activity. This approach was first developped in 70's by Wilson and Cowan [5] and Amari [1]; it leads to integro-differential equations (or systems of them), which may be written in the form:

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_\Omega K(\|\bar{x}-\bar{y}\|_2)S(V(\bar{y},t))d\bar{y}, \quad (1)$$

$$t \in [0,T], \bar{x} \in \Omega \subset \mathbb{R}^2;$$

- $V(\bar{x},t)$ - the membrane potential in point x at time t;
- I - external sources of excitation;
- S - dependence between the firing rate of the neurons and their membrane potentials (sigmoidal or Heaviside function);
- $K(\|\bar{x}-\bar{y}\|_2)$ - connectivity between neurons at $\bar{x}$ and $\bar{y}$.

Initial Condition: $V(\bar{x},0) = V_0(\bar{x}), \quad \bar{x} \in \Omega.$

Numerical algorithms for the approximation of the Neural Field Equation in two dimensions have been proposed by Faye and Faugeras [3] , Hutt and Rougier [4]. Here we propose a new numerical approach, based on the use of an implicit second order scheme and Gaussian quadrature.

Time Discretisation

We begin by rewriting equation (1) in the form

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \kappa(V(\bar{x},t)) \quad (2)$$

$$t \in [0,T], \bar{x} \in \Omega \subset \mathbb{R}^2,$$

where

$$\kappa(V(\bar{x},t)) = \int_\Omega K(|\bar{x}-\bar{y}|)S(V(\bar{y},t))d\bar{y}. \quad (3)$$

Let h_t be the stepsize in time. We define

$$t_i = ih_t, \quad i = 0,...,M, \quad T = h_t M.$$

Moreover, let $V_i(\bar{x}) = V(t_i,\bar{x}), \quad \forall \bar{x} \in \Omega, \quad i = 0,...,M$. We shall approximate the partial derivative in time by the backward difference

$$\frac{\partial}{\partial t}V(\bar{x},t_i) \approx \frac{3V_i(\bar{x}) - 4V_{i-1}(\bar{x}) + V_{i-2}(\bar{x})}{2h_t}, \quad (4)$$

which gives a discretisation error of the order $O(h_t^2)$, for sufficiently smooth V. By substituting (4) into (2) we obtain the implicit scheme

$$c\frac{3U_i - 4U_{i-1} + U_{i-2}}{2h_t} = I_i - U_i + \kappa(U_i), \quad i = 2,...,M, \quad (5)$$

where U_i approximates the solution of (2).

Space Discretisation

Assume that Ω is a rectangle: $\Omega = [-1,1]\times[-1,1]$. Introduce a uniform grid of points (x_i, x_j), such that $x_i = -1 + ih$, $i = 0,...,n$, where h is the discretisation step in space. In each subinterval $[x_i, x_{i+1}]$ we introduce k Gaussian nodes: $x_{i,s} = x_i + \frac{h}{2}(1+\xi_s)$, $i = 0,1,\dots n-1$, where ξ_s are the roots of the k-th degree Legendre polynomial, $s = 1,...,k$. Using a Gaussian quadrature formula to evaluate the integral, we obtain the finite-dimensional approximation of $\kappa(U)$. This discretisation provides an accuracy order of $O(h^{2k})$.

$$(\kappa^h(U^h))_{mu,lv} = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{s=1}^{k}\sum_{t=1}^{k}\tilde{w}_s\tilde{w}_t$$
$$\times K(\|(x_{mu},x_{lv}) - (y_{is},y_{jt})\|_2)S((U^h)_{is,jt}). \quad (6)$$

After discretization we obtain the following system of nonlinear equations:

$$U^h - \lambda\kappa^h(U^h) = f^h, \quad (7)$$

where $\kappa^h(U^h)$ is defined by (6) and $(f^h)_{is,jt} = f(x_{is},x_{jt})$. To solve (7), which is a system of N^2 nonlinear equations, we use the fixed point method

Efficiency and Rank Reduction

In order to improve the efficiency of the numerical method, we apply the following technique, proposed in [6]. Assuming that the function V is sufficiently smooth, we can approximate it by an interpolating polynomial of a certain degree. Our approach for reducing the matrices rank in our method consists in replacing the solution V_i by its m-th degree interpolating polynomial at the Chebyshev nodes in Ω. If V_i is sufficiently smooth, this produces a very small error and yields a very significant reduction of computational cost. Actually, when computing the vector $\kappa^h(U^h)$ (see formula (6)) we have only to compute $m^2 << N^2$ components, one for each Chebyshev node.

Neural Field Equation with Delay

According to many authors (see for example [3]), realistic models of neural fields must take into account that the propagation speed of neuronal interactions is finite, which leads to NFE with delays of the form

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_\Omega K(|\bar{x}-\bar{y}|)S(V(y,t-\tau(\bar{x},\bar{y}))d\bar{y}, \quad (8)$$

$t \in [0,T], \bar{x} \in \Omega \subset \mathbb{R}^2$, where $\tau(\bar{x},\bar{y}) > 0$ is a delay, depending on the spatial variables. Assuming that the electrical signals propagate with a constant speed v, uniform in space, we set $\tau(\bar{x},\bar{y}) = \|\bar{x}-\bar{y}\|_2/v$. In the delay case, the initial condition has the form

$V(\bar{x},t) = V_0(\bar{x},t), \quad \bar{x} \in \Omega, \quad t \in [-\tau_{max}, 0]$, where $\tau_{max} = \max_{\bar{x},\bar{y}\in\Omega}\tau(\bar{x},\bar{y})$. The numerical algorithm used to solve equation (8) is essentially the same as described in the previous sections. The main difference results from the fact that when computing the integral on the right-hand side of (8) at instant t_i we must use the approximate solution at all instants t_{i-k}, $k = 1,...,k_{max}$, where k_{max} is the integer part of τ_{max}/h_t.

Numerical Example 1

In this example we consider a kind of neural field with spatially localized oscillations which occur in excitable neural media with short-range excitation and long-range inhibition (mexican hat connectivity), in the case of a spatially localized input. Such neural fields are known as breathers due to their oscillatory behaviour and their dynamics are analysed in [2]. A similar example was described and computed in [4]. In this case the connectivity kernel has the form:

$$K(r) = 20\frac{\exp(-r)}{18\pi} - 14\frac{\exp\left(-\frac{r}{3}\right)}{18\pi}. \quad (9)$$

The firing rate function has the the sigmoidal form

$$S(x) = \frac{2}{1+\exp(-10000(x-0.005))}$$

and the external input is given by

$$I(\bar{x},t) = 5\frac{\exp(-x^2/32 - y^2/32)}{32\pi}.$$

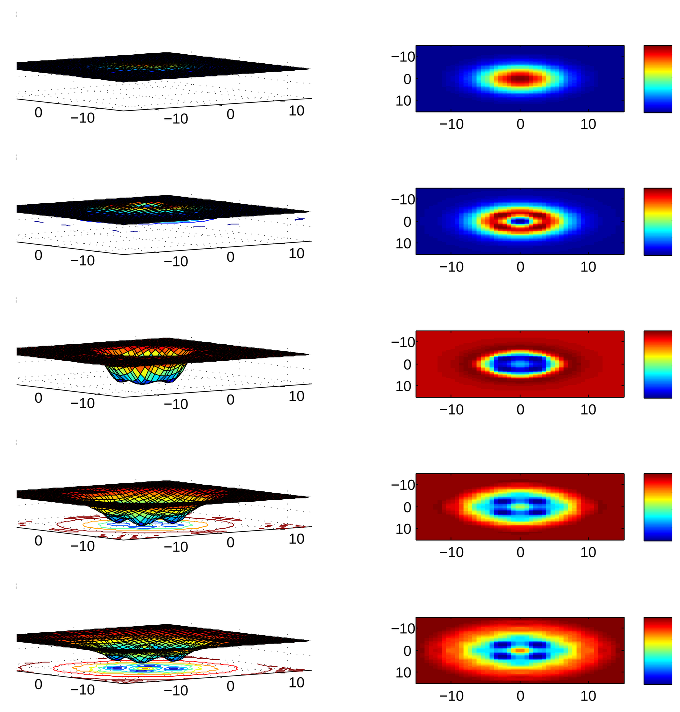

Figure: 1. Plots of the solution in the case $v = 50$ at different moments of time: $t = 0.08, 0.16, 0.24, 0.32, 0.40$.

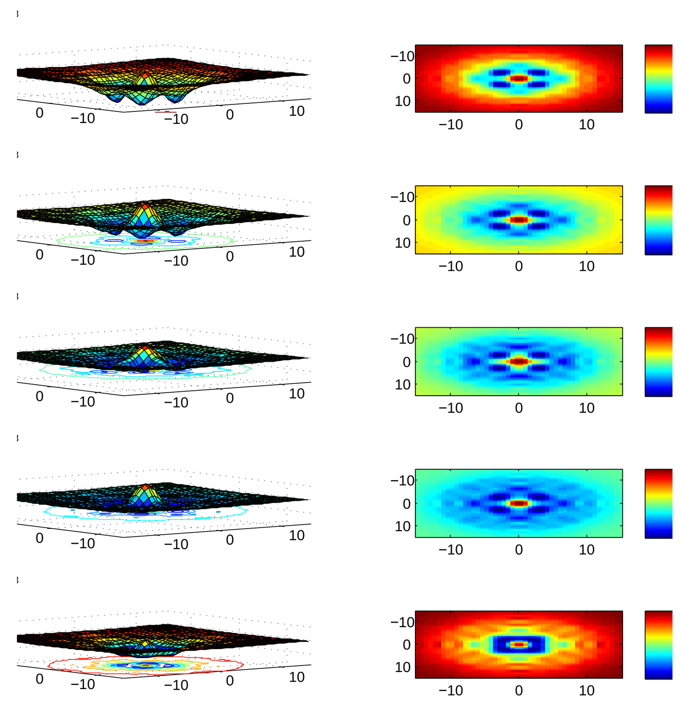

Figure: 2. Plots of the solution in the case $v = 50$ at different moments of time: $t = 0.48, 0.56, 0.64, 0.72, 0.80$.

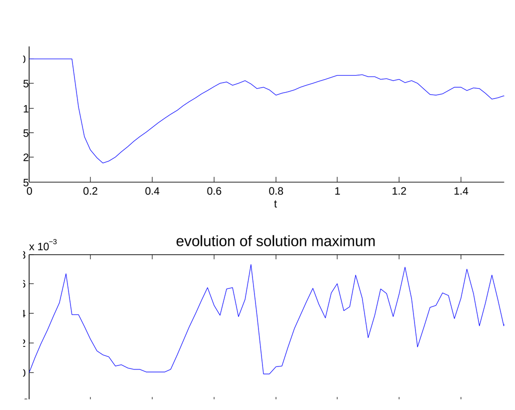

Figure: 3. Evolution of the solution maximum and minimum for the breather with $v = 50$.

Numerical Example 2

In this example (described in [3]),the firing rate function has the form

$$S(x) = \frac{2}{1+e^{-\mu x}}, \qquad x \in \mathbb{R}.$$

where $\mu \in \mathbb{R}^+$, and the connectivity function is given by

$$K(r) = \frac{1}{\sqrt{2\pi\xi_1^2}}\exp\left(-\frac{r^2}{2\pi\xi_1^2}\right) - \frac{A}{\sqrt{2\pi\xi_2^2}}\exp\left(-\frac{r^2}{2\pi\xi_2^2}\right),$$

where $r = \|x-y\|_2 = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ and $\xi_1, \xi_2, A \in \mathbb{R}^+$. We consider $c = 1$ and the external input $I(x,t)$ is 0 . Our aim is to investigate how the behaviour of the solutions depends on the equation parameters. We set the initial condition $V(0,x) \equiv 0.01$ and check whether the solution tends or not to the trivial steady state. In Fig. 4 (left and right) a 3D-plot and a contour plot of the corresponding solution are displayed, respectively, for $t = 3$, in the case $A = 1$, $\xi_1 = 0.4$, $\xi_2 = 0.2$, $\mu = 10$, with no delay. The computations were carried on the time interval $[0,3]$ with stepsize $h_t = 0.1$. The parameters of the space discretisation are $m = 12$, $N = 48$. Let x_1 be a point close to the center of Ω and x_2 be a point in the boundary of the domain. In Fig. 5 the graphs of $V(x_1,t)$ (left) and $V(x_2,t)$ (right), as functions of time, are displayed.The behaviour of the solution depends strongly on μ. For $\mu = 10$, for example, we see that after a certain time the solution becomes decreasing, both in x_1 and x_2. But for $\mu = 15$, if t is sufficiently high, the solution increases in both points. This suggests that for some value of μ, between 10 and 15, there should be a bifurcation (the zero solution becomes unstable).

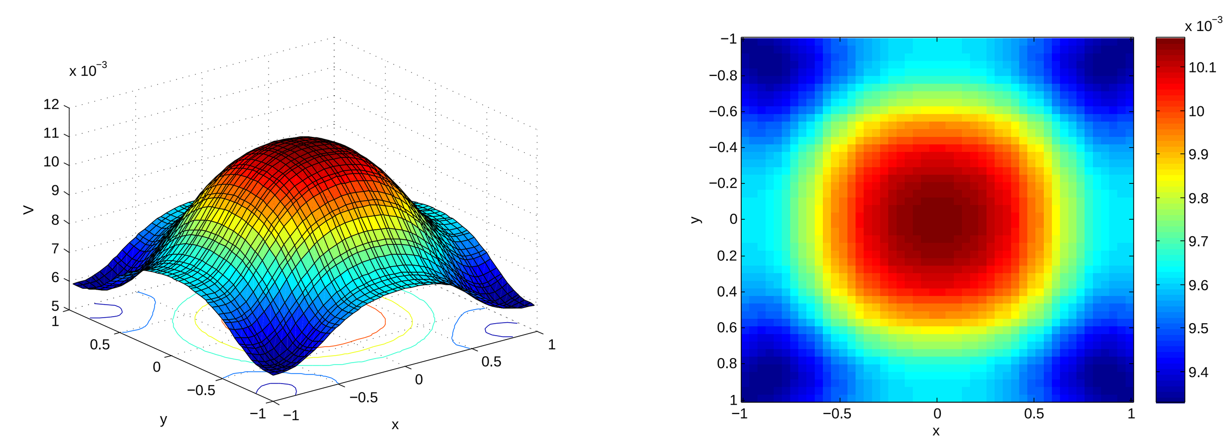

Figure: 4.3D-plot and contour plot of the solution

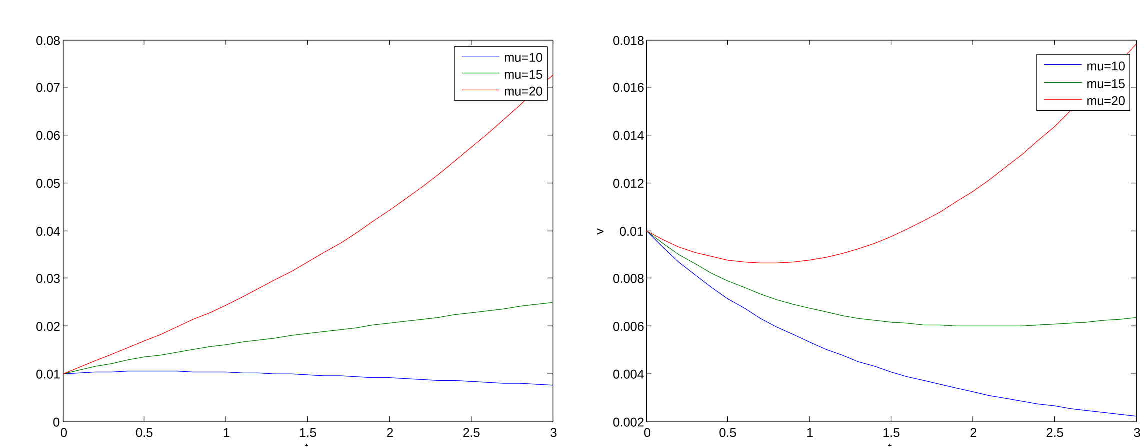

Figure: 5. Evolution of the solution at the middle (left) and at the boundary (right) of the domain, for different values of μ.

Conclusion and Future Work

- A remarkable feature of our method is that we use use an implicit second order scheme for the time discretisation, which improves its accuracy and stability, when compared with the available algorithms.
- To reduce the computational complexity of our method and improve its efficiency we have used an interpolation procedure which allows a drastic reduction of matrix dimensions, without a significant loss of accuracy.
- Our numerical results confirm the theoretical predictions and are in agreement with the expected behaviour of the solutions.
- As future work, we intend to analyse a stochastic version of the neural field equations, to take into account the effect of noise.

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