A New Numerical Algorithm for the Neural Field Equation in the Two-dimensional Case

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The purpose of the present work is to describe a new efficient numerical method for Neural Field Equations (NFE), targeted directly for the application in a Computational Neuroscience and Cognitive Robotics context. We consider the NFE in the form

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_\Omega K(|\bar{x}-\bar{y}|)S(V(y,t-\tau(\bar{x},\bar{y}))d\bar{y}, \qquad (1)$$

$$t \in [0,T], \quad \bar{x} \in \Omega \subset \mathbb{R}^2,$$

where the unknown $V(\bar{x},t)$ is a continuous function $V : \Omega \times [0,T] \to \mathbb{R}$, I, K and S are given functions; c is a constant. We search for a solution V of this equation which satisfies the initial condition

$$V(\bar{x},t) = V_0(\bar{x},t), \qquad \bar{x} \in \Omega, \quad t \in [-\tau_{max}, 0], \qquad (2)$$

where $\tau_{max} = \max_{\bar{x},\bar{y}\in\Omega} \tau(\bar{x},\bar{y})$. Here τ is a delay depending on $\bar{x}$ and $\bar{y}$ (as a particular case, we also consider the case $\tau \equiv 0$).

Equation (1) without delay was introduced first by Wilson and Cowan [5], and then by Amari [1], to describe excitatory and inhibitory interactions in populations of neurons. $V(\bar{x},t)$ represents the post-synaptic neuronal membrane potential at instant t and position $\bar{x}$. The function I represents external sources of excitation and S describes the dependence between the firing rate of the neurons and their membrane potential (typically, it is a function of sigmoidal type). The kernel function $K(|\bar{x}-\bar{y}|)$ gives the connectivity between neurons in positions $\bar{x}$ and $\bar{y}$. The delay $\tau(\bar{x},\bar{y})$ takes into consideration the time spent by an electrical signal to travel between positions $\bar{x}$ and $\bar{y}$.

We introduce an implicit scheme with second order discretization in time. The scheme is shown to be stable under mild restrictions on the stepsize.

The system of nonlinear equations arising at each time step is solved by the fixed point method, which converges fast, for a sufficiently small stepsize.

In the two-dimensional case, the required computational effort to solve equations (1) grows very fast as the discretization stepsize in space is reduced, and therefore special attention has to be paid to the creation of effective methods. In our work, we use Gaussian quadratures for the integration in space, and combine this with low-rank methods, as those discussed in [6], when the kernel is approximated by polynomial interpolation. This enables a significant reduction of the dimensions of the matrices, without affecting the final accuracy of the method.

Comparing with other computational methods, available in the literature (see [3] and [4]) , this algorithm has the advantage that it uses an implicit method in time, which allows using larger time stepsizes, while preserving second order convergence. This reduces the computational effort when considering large intervals in time. The algorithm was extensively tested: first with examples where the exact solution is known (to test the convergence and accuracy); then with real world examples from Neuroscience, like those considered in [3] and [4]. In a near future, we intend to apply it to problems of Cognitive Robotics, where the neural field dynamics also play an important role (see [2]).

References

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