

Mathematical Modelling and Numerical Simulations in Nerve Conduction

N.J. Ford¹, P.M. Lima² and P.M. Lumb¹

¹ *University of Chester, CH1 4BJ Chester, UK*

² *Johannes Kepler University, 4040 Linz, Austria*

njford@chester.ac.uk, plima@math.ist.utl.pt, plumb@chester.ac.uk

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Abstract: In this paper we are concerned with the numerical solution of the discrete FitzHugh-Nagumo equation. This equation describes the propagation of impulses across a myelinated axon. We analyse the asymptotic behaviour of the solutions of the considered equation and numerical approximations are computed by a new algorithm, based on a finite difference scheme and on the Newton method. The efficiency of the method is discussed and its performance is illustrated by a set of numerical examples.

1 INTRODUCTION

In the present work we analyse a functional-differential equation, sometimes known as the discrete FitzHugh-Nagumo equation, arising in nerve conduction theory. The history of this equation began in 1952, when A. Hodgkin and A. Huxley (Hodgkin and Huxley, 1952) introduced a mathematical model that describes the excitation and flow of electrical current through the surface of a giant nerve fibre from a squid. This investigation was continued in the works of FitzHugh (FitzHugh, 1960), (FitzHugh, 1962) and Nagumo (Nagumo et al., 1962).

During its evolution the FitzHugh-Nagumo equation has taken different forms. The case of myelinated axons deserves special attention.

In a myelinated nerve axon the myelin completely insulates the membrane, so that the potential change jumps from node to node (pure saltatory condition).

According to Bell (Bell, 1984) two basic properties of the myelinated axon are the following:

- it possesses threshold behaviour; this means that there are conditions which guarantee either the decay of a solution (subthreshold response) or non-decay of a solution (suprathreshold response);
- it is able to conduct pulses.

On the other hand, with the purpose of obtaining a mathematical model that can be analysed and lead to numerical solutions, some theoretical assumptions have been imposed: a) the axon is infinite in extent, b)

the Ranvier nodes are identical and uniformly spaced; c) the electric signals propagate with constant speed. These assumptions make sense when considering the propagation of signals not at the central, but at the peripheral nervous system.

The mathematical model for myelinated axons developed in (Chi et al., 1986), based on these assumptions, leads to the discrete FitzHugh-Nagumo equations:

$$v'(t) = v(t + \tau) - 2v(t) + v(t - \tau) + bv(t)(v(t) - 1)(\alpha - v(t)), \quad (1)$$

where $v(t)$ represents the potential at a Ranvier node of the axon at the moment t (in this case, the potential at the neighbouring nodes is denoted by $v(t - \tau)$ and $v(t + \tau)$); the constant τ is the time that a signal takes to be transmitted from a node to the neighbouring one (in other words, τ is inversely proportional to the propagation speed of the signal). The constant b reflects the resistance and the conductance in the nerve axon, while α is the threshold potential.

From a mathematical point of view, an important feature of equation (1) is that it contains both negative and positive deviations of the argument (delayed and advanced terms); this is the reason why it is called a mixed type functional differential equation (or an advance-delay-differential equation). Important contributions to the analysis of this type of equation have been introduced in the literature in the last two decades of the past century, by Rustichini (Rustichini, 1989a), (Rustichini, 1989b), Mallet-Paret and

Verduyn-Lunel (Mallet-Paret, 1999), (Mallet-Paret and Verduyn-Lunel, 2003). More recently, Hupkes and Verduyn-Lunel studied the behaviour of solutions to nonlinear autonomous MTFDEs in the neighbourhood of an equilibrium solution (Hupkes and Verduyn-Lunel, 2007). Based on existing insights into the qualitative behaviour of MTFDEs, the authors of (Ford and Lumb, 2009) developed a new approach to the analysis of these equations in the autonomous case. More precisely, they analysed MTFDEs as boundary value problems, that is, for a linear MTFDE they considered the problem of finding a differentiable solution on a certain real interval $[-1, k]$, $k \in \mathbb{N}$, given its values on the intervals $[-1, 0]$ and $(k-1, k]$. This approach was developed further in (Teodoro et al., 2009), where new numerical methods were proposed for the solution of such boundary value problems. In (Lima et al., 2010) and (Ford et al., 2010), these methods were extended to the non-autonomous case and new results were obtained about their numerical analysis. Once efficient computational methods had been created for the numerical treatment of linear MTFDEs, the next step was to extend these methods to the case of nonlinear ones, which includes equations of the form (1). This was done by Abell et al. (Abell et al., 2005) and then by the authors of the present paper in (Lima et al., 2013) and (Ford et al., 2014). The outline of this paper is as follows. In section 2 we investigate the asymptotic behaviour of solutions at infinity and, based on this, we replace the boundary conditions at infinity by equivalent conditions on a bounded interval. In section 3 we describe some computational methods used for numerical approximation. In section 4 we provide some numerical results and we finish with some conclusions in section 5.

2 BOUNDARY CONDITIONS AND ASYMPTOTIC ANALYSIS

Let us rewrite equation (1) in the following form:

$$v'(t) = f(v(t)) + v(t - \tau) - 2v(t) + v(t + \tau), \quad (2)$$

where

$$f(v) = bv(v - \alpha)(1 - v); \quad (3)$$

this function is the nonlinear part of (2) and is called the current voltage function. Due to the form of f , given by (3), equation (2) has two stable equilibrium points: $v = 0$ (resting potential) and $v = 1$ (fully activated potential). Therefore, we are interested in a solution of (2), increasing on $]-\infty, \infty[$, which satisfies the boundary conditions

$$\lim_{t \rightarrow -\infty} v(t) = 0, \quad \lim_{t \rightarrow +\infty} v(t) = 1. \quad (4)$$

These conditions will be satisfied by the potential at any node. In order to guarantee uniqueness of solution, we add the condition

$$v(0) = 1/2. \quad (5)$$

We are interested in a **monotone** solution of problem (2)-(5), that is, we assume that once the signal starts propagating, the potential will increase at every node, tending to its maximal value ($v(t) = 1$). Such a solution exists for a certain value of τ , which must be computed.

Since in the next section we describe computational methods for the numerical solution of the given problem, it is desirable to study the asymptotic behaviour of the required solution at infinity, so that we can truncate the domain where this solution needs to be computed.

An extensive analysis of this behaviour has been provided in (Chi et al., 1986), so here we will just recall the main results from that paper.

Let us first consider the case where $t \rightarrow -\infty$. According to the conditions (4), $v(-\infty) = 0$, so that in order to linearise equation (2) about this point, we first use the Taylor expansion for the function f :

$$f(v) = f(0) + vf'(0) + \frac{v^2}{2}f''(0) + o(v^2) = vf'(0) + \frac{v^2}{2}f''(0) + o(v^2), \quad (6)$$

where f is given by (3).

As usual, in order to obtain a characteristic equation for (2) at $-\infty$ we replace f by the main term of its Taylor expansion and assume that v has the form

$$v_-(t) = \epsilon_- e^{\lambda(t+L)}, \quad (7)$$

where L is a sufficiently large parameter and ϵ_- is an estimate for $v_-(-L)$. In this way we obtain the equation

$$\lambda + 2 - f'(0) - 2 \cosh(\lambda\tau) = 0. \quad (8)$$

This equation has two real roots; since we are interested in a function v_- that tends to 0 at $-\infty$, we choose the positive one, which we denote by λ_+ .

The case where $t \rightarrow \infty$ can be handled in an analogous way. In this case, we have the following Taylor expansion for f :

$$f(v) = f(1) + (v-1)f'(1) + \frac{(v-1)^2}{2}f''(1) + o((v-1)^2) \quad (9)$$

Moreover, since $v(\infty) = 1$, as $t \rightarrow +\infty$, we assume that v has the form

$$v_+(t) = 1 - \epsilon_+ e^{\lambda(t-L)}, \quad (10)$$

where ϵ_+ is an estimate of $1 - v_+(L)$. In this way we obtain the characteristic equation

$$\lambda + 2 - f'(1) - 2 \cosh(\lambda\tau) = 0 \quad (11)$$

In this case we choose the negative root of the characteristic equation, which we will denote by λ_- , in order to have $v_+(t) \rightarrow 1$, as $t \rightarrow +\infty$.

Now we have obtained two representations for the solution of our problem, (7) and (10), which we shall use to approximate the solution, for $t < -L$ and $t > L$, respectively, where L is a sufficiently large number. According to the form of equation (2), L must be a multiple of the delay τ ; in our computations we have used $L = k\tau$, where $2 \leq k \leq 9$, depending on the specific problem (as discussed in Sec. 4).

These representations of the solution are used in the computational methods to replace the boundary conditions (4). In the next section we will show how this can be achieved.

3 COMPUTATIONAL METHODS

In this section we will describe and compare some computational methods that can be applied to obtain approximate solutions of the problem (2), (4), (5). Since the problem is nonlinear, some of the considered methods require initial approximations, which must be sufficiently close to the true solution, to guarantee the convergence of the iterative process. Thus we begin by presenting some preliminary results, which can help us to build a rough approximation of the solution.

3.1 Finding an Initial Approximation

We now present an approach that results from approximating the solution by a piecewise differentiable function. In this case, we search for an approximate solution of the considered problem in the form

$$v_0(t) = \begin{cases} \varepsilon_- e^{\lambda_+(t+2\tau)}, & \text{if } t < -2\tau; \\ a_0 + a_1 t + a_2 t^2, & \text{if } -2\tau \leq t < -\tau; \\ \frac{1}{2} + b_1 t + b_2 t^2 + b_3 t^3, & \text{if } -\tau \leq t < 0; \\ \frac{1}{2} + c_1 t + c_2 t^2 + c_3 t^3 & \text{if } 0 < t < \tau; \\ d_0 + d_1 t + d_2 t^2, & \text{if } \tau \leq t < 2\tau; \\ 1 - \varepsilon_+ e^{\lambda_-(t-2\tau)}, & \text{if } t \geq 2\tau; \end{cases} \quad (12)$$

From the form of v_0 it follows that this function satisfies the boundary conditions (4) and (5). We easily see that v_0 depends on 17 parameters: $\tau, \varepsilon_-, \varepsilon_+, \lambda_-, \lambda_+, a_0, a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, d_0, d_1, d_2$. These coefficients are computed from a system of 17 equations, which include

- the two characteristic equations (8) and (11);
- two equations that ensure the continuity of v_0 and v_0' at -2τ ;

- two equations that ensure the continuity of v_0 and v_0' at $-\tau$;
- two equations that ensure the continuity of v_0' and v_0'' at 0;
- two equations that ensure the continuity of v_0 and v_0' at τ ;
- two equations that ensure the continuity of v_0 and v_0' at 2τ ;
- five equations that ensure that v_0 satisfies (2) at $t = -2\tau, t = -\tau, t = 0, t = \tau$ and $t = 2\tau$.

The nonlinear system of equations can be solved by the Newton method. Note that in this case the number of equations in the system is not so high as when we apply the finite difference method, and therefore it is not so difficult to find an initial approximation for the Newton method. For example, if we know the solution for a certain set of values a, b , we can use this solution as initial approximation to solve the system and find the solution for close values of a, b . The estimates for λ_-, λ_+ and τ from above can be used as initial approximations. As follows easily from the construction, $v_0 \in C^1(\mathbb{R})$. Some examples of application of this approximation will also be given in section 4. As we shall see, though these results have relative errors which may attain 0.1, the corresponding approximate solutions have the correct qualitative behaviour and this explains that they provide good initial approximations for the more refined method we describe in the next subsection.

3.2 A Finite Difference Approach

In this section we describe a finite difference scheme for the solution of problem (2), (4), (5). This scheme has some common features with the one described in (Chi et al., 1986), but it has the advantage that it can be easily solved by the Newton iterative method, without using the continuation algorithm. As an initial approximation for the Newton method we have used the function v_0 , defined by (12).

In order to approximate the solution we introduce on $[-K\tau, K\tau]$ a uniform mesh with stepsize $h = \tau/N$. Let $t_i = -K\tau + ih$, $i = 0, \dots, 2KN$ be the nodes of this mesh. Here K is a sufficiently large integer so that $\varepsilon_1 = v(-K\tau)$ is comparable with h^2 (the reason for this choice will be explained below). As in (Chi et al., 1986), the first derivative is approximated by a 4-th order finite difference:

$$v'(t_i) \approx L_h(v)_i = \frac{1}{h} \left(\frac{2}{3}(v(t_{i+1}) - v(t_{i-1})) - \frac{1}{12}(v(t_{i+2}) - v(t_{i-2})) \right). \quad (13)$$

By using this approximation at each node t_i we obtain $2KN + 1$ equations of the form:

$$L_h(v)_i = v(t_i + \tau) + v(t_i - \tau) - 2v(t_i) + f(v(t_i)) + r_i^h, \quad (14)$$

where $\|r_i^h\| = O(h^4)$. Note that for $t_i > (K-1)\tau$ and $t_i < -(K-1)\tau$ equation (14) involves the value of v at one or more points that do not belong to the interval $[-K\tau, K\tau]$. In this case the boundary conditions (4) are applied, by considering the fact that v satisfies (7) or (10), when $v < -K\tau$ or $v > K\tau$, respectively. In this way, we write

$$v(-K\tau - x) = v(-K\tau)e^{-\lambda_+x}, \quad (x > 0), \quad (15)$$

$$1 - v(K\tau + x) = (1 - v(K\tau))e^{-\lambda_-x}, \quad (x > 0). \quad (16)$$

Finally, by ignoring r_i^h in (14), we obtain $(2K-2)N$ finite difference equations of the form:

$$L_h(v)_i = v_{i+N} + v_{i-N} - 2v_i + f(v_i), \quad (17)$$

$$i = N + 1, \dots, (2K-1)N + 1.$$

Here as usual v_i represents the approximate value of $v(t_i)$. For $0 \leq i < N + 1$ and $(2K-1)N + 1 < i \leq 2KN + 1$, we have modifications of equation (17) which result from applying formulae (15) or (16), respectively. This gives a system of $2KN + 1$ equations, which is then completed with the equation $v_{KN} = 1/2$, resulting from (5). Moreover, we have the characteristic equations (8) and (11), making a total of $2KN + 4$ equations. Note that the number of unknowns is also $2KN + 4$: $2KN + 1$ entries of the vector $v = (v_0, \dots, v_{2KN})$, λ_- , λ_+ and τ .

This nonlinear system is then solved by the Newton method. The Jacobian matrix for the Newton method has at most 8 nonzero entries at each row: 7 corresponding to the unknowns v_{i-N} , v_{i-2} , v_{i-1} , v_i , v_{i+1} , v_{i+2} , v_{i+N} ; one corresponding to τ .

Although the condition number of the Jacobian matrices increases with the the number of gridpoints in the scheme (as could be expected), it is remarkable that even for the finest meshes it doesn't become very large. Even in the case of $n = 2^8$ and $K = 9$, the condition numbers do not get higher than 10^5 . As we shall see below, this enables us to obtain accurate results, within a large range of values of a and b .

It is well known that the Newton method is very sensitive to the choice of initial approximation, especially in the case of large dimensional systems. In our case, as stated above, we have overcome this problem by using as initial approximation to v the function v_0 , described in the previous section, and the corresponding approximations to λ_- , λ_+ and τ .

Table 1: Estimates of τ , λ_- and λ_+ for different values of a , with $b = 15$.

a	τ_0	τ_1	λ_-	λ_+
0	0.3101	0.38029	-6.22752	5.1007
0.05	0.3461	0.43511	-5.44866	4.5111
0.1	0.3910	0.5056	-4.6909	3.9297
0.15	0.4485	0.5993	-3.95523	3.3586
0.2	0.5250	0.73001	-3.32594	2.6073
0.25	0.6318	0.92525	-2.55197	2.2691
0.3	0.7916	1.2515	-1.88069	1.66568
0.35	1.0575	1.9371	-1.2111	1.08772

Table 2: Estimates of τ , λ_- and λ_+ for different values of b , with $a = 0.05$.

b	τ_0	τ_1	λ_-	λ_+
1	1.652	1.6250	-0.4333	0.4096
5	0.6639	0.7229	-2.0677	1.8678
11	0.4325	0.5008	-4.1869	3.5574
16	0.3296	0.4227	-5.7499	4.7354
21	0.2656	0.3744	-7.1795	5.7840
51	0.1216	0.1200	-14.0692	10.6338

4 NUMERICAL RESULTS

We first compare the estimates of τ , λ_- , and λ_+ , obtained by the two methods described in Sec. 3. In both cases we consider the problem (2), (4), (5). We denote by τ_0 the estimate of τ obtained when using the approximating function v_0 , defined by (12); τ_1 stands for the value obtained by the finite difference method with $N = 64$. Table 1 contains the values of τ_0 and τ_1 , for $b = 15$, with different values of a . The values of τ_0 and τ_1 , obtained in the case $a = 0.05$, for different values of b , are displayed in Table 2.

The errors of τ_0 can be obtained by comparing these estimates with τ_1 , which can be considered as the most accurate value for this purpose. Notice that the presented values of τ_1 coincide with those obtained in (Chi et al., 1986), within the given precision.

The approximate values of λ_- and λ_+ obtained by the finite difference method, with $N = 64$, are given in the two last columns of the mentioned tables.

All the estimates preserve the main characteristics of τ , as a function of a and b : they increase with a and decrease as b increases. This behaviour agrees with the physical meaning of the variables. Since a is the threshold potential, the propagation speed is supposed to decrease as a increases, and therefore we observe τ increasing. On the other hand, increasing b means a higher potential at the nodes and this leads to a greater propagation speed and the decreasing of τ .

Table 3: Estimates of $v'(0)$ for different values of b , with $a = 0.05$

b	$v'_0(0)$	$v'_1(0)$
1	0.1224	0.112695
5	0.6045	0.58339
11	1.2821	1.2774
16	1.83603	1.84116
21	2.39174	2.40116
51	5.7504	5.76174

It is worth remarking that for values of a , greater than 0.3 large discrepancies between the different estimates arise. This is connected with the numerical instability of the different methods which is observed for the values of a we considered. In particular, in the case $a \geq 0.3$, the value τ obtained by the finite difference method seems to have a larger error than the other approximations (and this does not arise elsewhere). The explanation for this may be the instability which is also visible in the graphs of Figures 4 and 5. Note that according to the available theoretical results, existence of a solution can be proved only for $a < 0.5$. The accuracy of the results is also reduced for values of b , greater than 21.

According to Keener (Keener, 1987), the discrete Nagumo equation has a ‘‘propagation failure’’ for sufficiently small coupling coefficient. When the equation is written in the form (2), a small coupling corresponds to high values of b . Thus when b is large the problem (2),(4),(5) becomes unsolvable. This explains why for high values of b estimates obtained by different methods differ significantly.

In Tables 3 and 4 numerical approximations for $v'(0)$, obtained by the two considered methods, are given for a set of values of a and b . In agreement with the notation used in Tables 1 and 2, we denote by $v'_0(0)$ (resp. $v'_1(0)$) the estimate obtained by the method, corresponding to τ_0 (resp. τ_1). In this case the differences between the estimates obtained by different methods are not so large as in the case of the evaluation of τ . Even for $a > 0.3$ or $b > 21$, these differences are not greater than 5 per cent. This suggests that the gradient of the solution at the origin is not so sensitive to computational errors as the value of τ .

In Figure 1 the absolute error of the approximate solution v_0 is plotted, in the case $a = 0.05$, $b = 15$; by absolute error of v_0 , we mean the difference $|v_0 - v_1|$, where v_1 is the finite difference approximation of the solution, obtained with $N = 256$. Note that, for all the approximations, the largest errors occur close to $t = 0$, where the solution changes faster. In this region the error can reach about 10 per cent of the solution value.

We remark that by differentiating v_0 we obtain a

Table 4: Estimates of $v'(0)$ for different values of a , with $b = 15$.

a	$v'_0(0)$	$v'_1(0)$
0	1.9171	1.9181
0.05	1.72515	1.72889
0.1	1.53326	1.53918
0.15	1.34141	1.34891
0.2	1.1496	1.1580
0.25	0.957907	0.96647
0.3	0.76624	0.774237
0.35	0.57463	0.58131

reasonable approximations of v' . The derivatives of v_0 and v_1 are plotted in Figure 2.

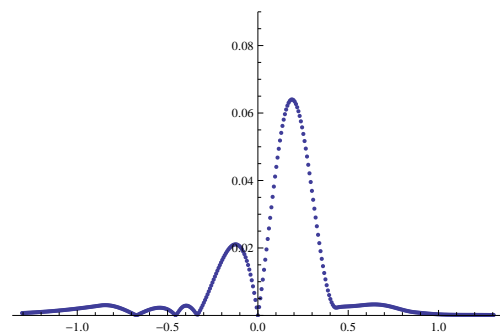


Figure 1: Absolute error of v_0 with respect to the finite difference approximation.

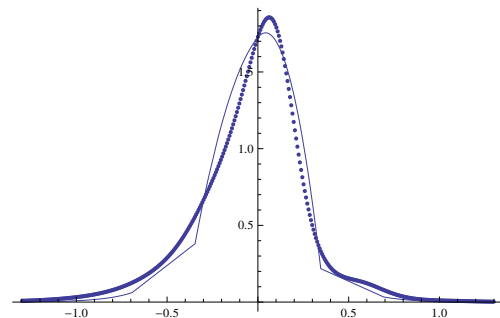


Figure 2: Approximation of the solution derivative: by the finite difference method (thick line); using v_0 (thin line).

We will now focus on the numerical results for the problem (2)-(5), obtained by the finite difference method. We have tested numerically the convergence order of the method. When the exact solution is not known, an estimate of the convergence order can be obtained from three finite difference solutions, obtained with different stepsizes: $v_{2h}, v_h, v_{h/2}$, where $h > 0$. The estimate of the convergence order is then given by

$$p = \log_2 \frac{\|v_h - v_{2h}\|}{\|v_h - v_{h/2}\|},$$

Table 5: Estimates of convergence order in the case $a = 0.05, b = 5$, with $K = 9$.

h	$\ v_h - v_{2h}\ $	p
$\tau/2^4$	$1.51E - 5$	6.22
$\tau/2^5$	$2.03E - 7$	3.981
$\tau/2^6$	$1.29E - 8$	3.994
$\tau/2^7$	$0.07E - 10$	3.998
$\tau/2^8$	$5.05E - 11$	

Table 6: Estimates of convergence order in the case $a = 0.05, b = 15$, with $K = 6$.

h	$\ v_h - v_{2h}\ $	p
$\tau/2^4$	0.015	7.42
$\tau/2^5$	$8.80E - 5$	5.11
$\tau/2^6$	$2.55E - 6$	3.97
$\tau/2^7$	$1.63E - 7$	3.99
$\tau/2^8$	$1.02E - 8$	

where the maximum norm is used. The numerical results displayed in Tables 5,6,7 suggest that the method has 4-th order of convergence, as could be expected. The value of K , which determines the interval length, depends on the value of $\varepsilon = \max(\varepsilon_-, \varepsilon_+)$ (see (7,10)). More precisely, we choose such value of K that ε^2 has the same order as h_{min}^4 , where h_{min} is the minimal stepsize. In this way, the error resulting from ignoring the remainder in the Taylor's expansions (6,9) has the same order as the discretization error of the finite difference scheme.

In order to illustrate the application of this criterion, in Table 8 we display the values of ε , for the three considered examples, with different values of K . Note, that when the chosen value of K is not large enough, there is a drop in the estimated convergence order, which is due to the error resulting from the truncation of the interval. On the other hand, the use of too large a value of K makes the method less efficient, requiring the use of large matrices, without improving the accuracy.

The large differences between the estimates of τ , obtained by the two methods, for certain values of a and b , suggest that some of the computational meth-

Table 7: Estimates of convergence order in the case $a = 0.05, b = 21$, with $K = 6$

h	$\ v_h - v_{2h}\ $	p
$\tau/2^4$	0.00041	6.518
$\tau/2^5$	$4.48E - 6$	3.973
$\tau/2^6$	$2.85E - 7$	3.992
$\tau/2^7$	$1.79E - 8$	3.998
$\tau/2^8$	$1.12E - 9$	

Table 8: Values of ε , for different K , in the three examples considered above. The value corresponding to the chosen K is written in boldface.

K	$b = 5$	$b = 15$	$b = 21$
3	$1.29E - 2$	$1.80E - 3$	$2.50E - 3$
6	$2.30E - 4$	$4.98E - 6$	$1.91E - 6$
9	$3.96E - 6$	$1.37E - 8$	$2.63E - 9$

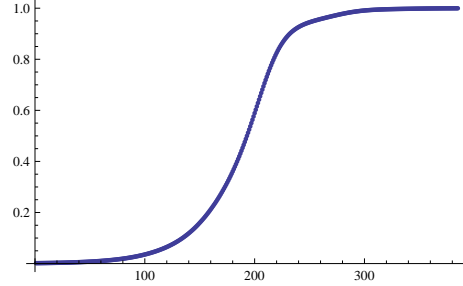


Figure 3: Graph of the numerical solution obtained by the finite difference method with $N = 64$, in the case $a = 0.1, b = 15$. Here the numerical solution preserves the smoothness of the true solution.

ods may become unstable for such values, in particular, when a is close to 0.5 or b is large. This is true, in particular, in the case of the finite difference method, as is shown by the graphs of the approximate solutions plotted in Figures 3,4, and 5.

As mentioned in subsection 3.2, the finite difference method developed in this work was inspired by the method described in (Chi et al., 1986). However, the algorithm described in that work relies on the continuation method, that is, a numerical solution is first computed for a test problem and then, by a continuous change of a parameter, a sequence of auxiliary equations is solved, until reaching the target problem. Possibly following on from this, the results reported in (Chi et al., 1986) are limited to the cases where $a \leq 0.2$ and $b \leq 20$. In our numerical experiments, the initial approximations for the Newton method were obtained by the numerical method described in subsection 3.1; this enabled us to solve the problem for a wider range of values of a and b . Moreover, our algorithm worked with stepsizes as small as 0.001, while in the case of the numerical results reported in (Chi et al., 1986) the typical stepsize was $h = 0.05$.

5 CONCLUSIONS

We have applied two computational approaches to the solution of the problem (2)-(4), analysed and compared the numerical results. The more accurate results are obtained by the finite difference method,

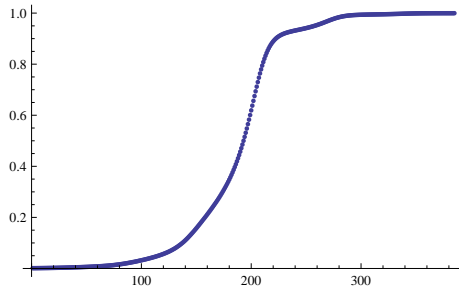


Figure 4: Graph of numerical solutions obtained by the finite difference method with $N = 64$, in the cases $a = 0.3, b = 15$. Here the effect of numerical instability is visible .

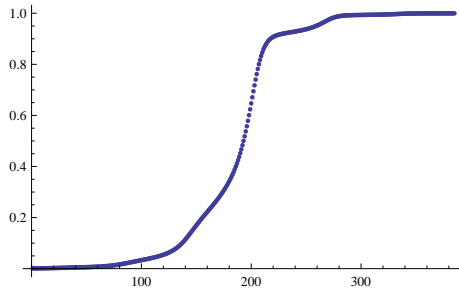


Figure 5: Graph of numerical solutions obtained by the finite difference method with $N = 64$, in the cases $a = 0.35, b = 15$ (right side). Here the effect of numerical instability is even more evident than in the previous figure.

described in Sec. 3.2. The numerical results suggest that this method has fourth order of convergence, as it could be expected. Highly accurate results can be obtained, within a reasonable computational effort, when the parameters satisfy $0 \leq a < 0.3$ and $5 \leq b \leq 51$. However, for other values of the parameters computational instability arises.

Another approximation method was discussed in Sec. 3.1 . The computational effort required by this method is very small and the algorithms are very simple. Although its accuracy is reduced, it can provide good initial approximations for the finite difference method.

The numerical results obtained in our paper confirm the main features of the considered mathematical model. In particular, it was observed that the propagation speed ($1/\tau$) increases as the threshold potential a decreases (see table 1) and as the intensity of the ionic currents (represented by b) increases (see table 2). The typical S-shaped form of the solution graphic (illustrated by fig. 7, for example) means that the potential value changes slowly when it is close to its resting or fully activated value; and changes fast, when it is close to the average value. As a consequence, the solution derivative takes its highest values when t is close to 0, and these values are particularly

high when a is small and b is large (as it follows from tables 3 and 4). Finally, the fact that the method fails to produce an acceptable solution approximation for $a > 0.3$ is not surprising, since it is known that propagation failures may occur for high values of a .

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