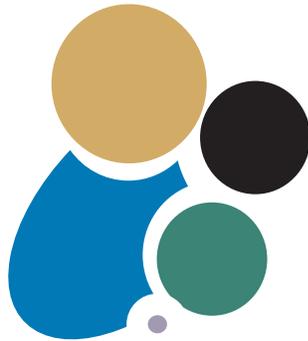


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# NUMERICAL METHODS FOR A NONUNIQUELY SOLVABLE VOLTERRA INTEGRAL EQUATION

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**Abstract.** *We shall be concerned with the following Volterra integral equation*

$$y(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} y(s) ds + g(t), \quad \mu > 0,$$

*which can arise from certain heat conduction problems with time dependent boundary conditions (see (M. Bartoshevich, 1975),(T. Diogo et al., 2000)). If  $\mu > 1$  and  $g$  belongs to  $C^m[0, T]$  then the above equation has a unique solution in  $C^m[0, T]$ . An interesting feature of the equation is that it loses uniqueness in the case  $0 < \mu < 1$ . It was shown in (W. Han, 1994) that the equation has a family of solutions in  $C[0, T]$ , one of which has  $C^1$  continuity and all other solutions have infinite gradient at  $t = 0$ . In previous works we have dealt with the approximation of the smooth solution. Here we present numerical methods which enable us to obtain approximations to any of the infinite class of solutions. Some numerical examples are given which illustrate the performance of the methods employed.*

**Keywords:** *Volterra integral equation, singular kernel, product integration methods, multiple solutions*

## 1. INTRODUCTION

We consider a class of singular Volterra integral equations of the form

$$u(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s) ds + g(t), \quad t \in (0, T], \quad (1.1)$$

where  $\mu > 0$  and  $g$  is a given function. This equation has been the subject of several works (see, for example, (T.Tang et al., 1987), (T. Diogo et al., 2000), (P. Lima and T. Diogo, 1997), (P. Lima and T. Diogo, 2002)) An interesting feature of the above equation is the unusual nature of its kernel. For values of  $\mu > 1$ , the kernel is singular only at  $t = 0$ . In this case a smooth forcing function  $g$  leads to a smooth solution  $y$ . However, if  $0 < \mu < 1$ , there is a singularity at  $t = 0$  and at  $s = 0$  for any positive values of  $t$ . It turns out that in this case the equation has an infinite set of solutions.

The following explicit representation for the solutions of (1.1) is given in (W. Han, 1994).

**Lemma 1.1.** (a) If  $0 < \mu \leq 1$  and  $g \in C^1[0, T]$  (with  $g(0) = 0$  if  $\mu = 1$ ) then (1.1) has a family of solutions  $u \in C[0, T]$  given by the formula

$$u(t) = c_0 t^{1-\mu} + g(t) + \gamma + t^{1-\mu} \int_0^t s^{\mu-2} (g(s) - g(0)) ds, \quad (1.2)$$

where

$$\gamma := \begin{cases} \frac{1}{\mu-1} g(0), & \text{if } \mu < 1, \\ 0, & \text{if } \mu = 1, \end{cases} \quad (1.3)$$

and  $c_0$  is an arbitrary constant. Out of the family of solutions there is one particular solution  $u \in C^1[0, T]$ . Such a solution is unique and can be obtained from (1.2) by taking  $c_0 = 0$ .

(b) If  $\mu > 1$  and  $g \in C^m[0, T]$ ,  $m \geq 0$ , then the unique solution  $u \in C^m[0, T]$  of (1.1) is

$$u(t) = g(t) + t^{1-\mu} \int_0^t s^{\mu-2} g(s) ds. \quad (1.4)$$

We note that (1.4) can be obtained from (1.2) with  $c_0 = 0$ . Indeed, it follows from (1.2) that

$$c_0 = \lim_{t \rightarrow 0^+} t^{\mu-1} u(t), \quad (1.5)$$

and this limit is zero when  $\mu > 1$ .

## 2. Numerical methods

Let us reformulate equation (1.1) in the following way. We choose some fixed real number  $\epsilon > 0$ . Substituting  $t$  by  $t + \epsilon$  in (1.1) gives

$$u(t + \epsilon) = \frac{1}{(t + \epsilon)^\mu} \int_0^\epsilon s^{\mu-1} u(s) ds + \int_\epsilon^{t+\epsilon} \frac{s^{\mu-1}}{(t + \epsilon)^\mu} u(s) ds + g(t + \epsilon),$$

or, equivalently,

$$u(t + \epsilon) = \frac{I_\epsilon}{(t + \epsilon)^\mu} + \int_0^t \frac{(s + \epsilon)^{\mu-1}}{(t + \epsilon)^\mu} u(s + \epsilon) ds + g(t + \epsilon), \quad (2.1)$$

where

$$I_\epsilon := \int_0^\epsilon s^{\mu-1} u(s) ds, \quad (2.2)$$

The idea is to evaluate  $I_\epsilon$  exactly for a chosen exact solution and then apply a numerical method to (2.1). We note that the kernel of this new equation is regular in  $\{(t, s) : 0 \leq s \leq t \leq T - \epsilon\}$ . The use of standard numerical schemes for regular second kind integral equations will be investigated elsewhere. In this work we are concerned with product integration quadrature methods.

Let  $X_h := \{t_i = ih + \epsilon, 0 \leq i \leq N\}$  be an uniform grid of the interval  $[\epsilon, T]$ , with stepsize  $h$ . Setting  $t = nh$  in (2.1) we get

$$u(t_n) = \frac{I_\epsilon}{t_n^\mu} + \int_0^{t_n} \frac{(s + \epsilon)^{\mu-1}}{t_n^\mu} u(s + \epsilon) ds + g(t_n) \quad (2.3)$$

In the Euler's method we approximate  $u(s + \epsilon)$  by  $u(t_j)$  on each subinterval  $[jh, (j + 1)h]$ . Defining

$$D_j := \int_{jh}^{(j+1)h} (s + \epsilon)^{\mu-1} ds = (t_{j+1}^\mu - t_j^\mu) / \mu, \quad j = 0, 1, \dots, n-1, \quad (2.4)$$

we obtain the algorithm

$$u_n^h = \frac{I_\epsilon}{t_n^\mu} + \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} D_j u_j^h + g(t_n) \quad (2.5)$$

In the Trapezoidal method we use a piecewise linear approximation for  $u$ , that is

$$u(s + \epsilon) \simeq u((j+1)h + \epsilon)(s - jh) / h + u(jh + \epsilon)((j+1)h - s) / h, \quad s \in [jh, (j+1)h] \quad (2.6)$$

We obtain the algorithm

$$u_n^h = \frac{I_\epsilon}{t_n^\mu} + \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} (D_j^1 u_{j+1}^h - D_j^2 u_j^h) / h + g(t_n), \quad (2.7)$$

where

$$D_j^1 = \int_{jh}^{(j+1)h} (s + \epsilon)^{\mu-1} (s - jh) ds, \quad D_j^2 = \int_{jh}^{(j+1)h} (s + \epsilon)^{\mu-1} (s - (j+1)h) ds$$

can be evaluated analytically. Starting with an initial value  $u_0^h = u(\epsilon)$ , the above algorithms give approximate values  $u_n^h$  of  $u(t_n)$ .

### 3. Convergence Analysis

In the case  $\epsilon = 0$  the methods considered in the previous section reduce to the traditional product integration methods. In particular, for the product Euler's method we have proved the following result (see (P. Lima and T. Diogo, 2002)).

**Theorem 3.1** Consider equation (1.1) with  $0 < \mu \leq 1$  and  $g \in C^1[0, T]$ . Then the approximate solution obtained by the product Euler's method (2.5), with ( $\epsilon = 0$ ), converges to the particular solution of (1.1) which is in  $C^1[0, T]$ . Moreover, if  $g \in C^2[0, T]$  and  $g'(0) = 0$  we have first order convergence. However, if  $g'(0) \neq 0$  we have convergence of order  $\mu$ .

The two following convergence results for the methods of section 2, in the case when  $\epsilon \neq 0$ , can be found in (T. Diogo et al., 2003).

**Theorem 3.2** Consider equation (1.1) with  $0 < \mu \leq 1$  and  $g \in C^1[0, T]$ . Let  $\epsilon \neq 0$  be fixed in the equivalent equation (2.1) and assume the integral  $I_\epsilon$  has been evaluated exactly for a chosen particular solution (corresponding to a certain value of the parameter  $c_0$ ). Then the approximate solution obtained by the product Euler's method (2.5) converges with order one to that particular exact solution.

**Proof:**

The exact solution  $u$  satisfies

$$u(t_n) = \frac{I_\epsilon}{t_n^\mu} + \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} D_j u(t_j) + g(t_n) + \delta(h, t_n) \quad (3.1)$$

where  $\delta(h, t_n)$  is the consistency error given by

$$\delta(h, t_n) = \int_\epsilon^{t_n} \frac{s^{\mu-1}}{t_n^\mu} - \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} D_j u(t_j) \quad (3.2)$$

Setting  $e_i = u(t_i) - u_i^h$  and subtracting equations (3.1) and (2.5), we obtain

$$e_n = \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} e_j \int_{t_j}^{t_{j+1}} s^{\mu-1} ds + \delta(h, t_n), \quad n \geq 1. \quad (3.3)$$

Now

$$\frac{1}{t_n^\mu} \int_{t_j}^{t_{j+1}} s^{\mu-1} ds \leq \frac{ht_{j+1}^{\mu-1}}{t_n^\mu} \leq h \left( \frac{t_{j+1}}{t_n} \right)^\mu \frac{1}{t_{j+1}} \leq \frac{h}{\epsilon} \quad (3.4)$$

Taking modulus in (3.3) and using (3.4) yields

$$|e_n| \leq \frac{h}{\epsilon} \sum_{j=0}^{n-1} |e_j| + |\delta(h, t_n)|, \quad n \geq 1. \quad (3.5)$$

On the other hand, from (3.2) and (2.4), we have

$$\begin{aligned}
|\delta(h, t_n)| &= \frac{1}{t_n^\mu} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} s^{\mu-1} |u(s) - u(t_j)| ds \leq \frac{h}{t_n^\mu} \max_{s \in [\epsilon, T]} |u'(s)| \int_{\epsilon}^{t_n} s^{\mu-1} ds \\
&= \left(1 - \frac{\epsilon^\mu}{t_n^\mu}\right) \frac{Ch}{\mu}
\end{aligned} \tag{3.6}$$

Using (3.6) and applying a known discrete Gronwall Lemma to (3.5), we obtain the estimate

$$|e_n| \leq \left(1 - \frac{\epsilon^\mu}{t_n^\mu}\right) \frac{Ch}{\mu} e^{(T-\epsilon)/\epsilon} \tag{3.7}$$

and first order convergence follows.

**Remark 3.1** In the case of the choice  $c_0 = 0$  (that is, the  $C^1$  solution), the result of the previous theorem is an improvement over the convergence order  $k = \mu$  obtained with the traditional method ( $\epsilon = 0$ ), when  $g'(0) \neq 0$  (cf. theorem 3.1).

**Theorem 3.3** Consider equation (1.1) with  $0 < \mu \leq 1$  and  $g \in C^2[0, T]$ . Let  $\epsilon \neq 0$  be fixed in the equivalent equation (2.1) and assume the integral  $I_\epsilon$  has been evaluated exactly for a chosen particular solution (corresponding to a certain value of the parameter  $c_0$ ). Then the approximate solution obtained by the product Trapezoidal method (2.7) converges with order two to that particular exact solution.

#### 4. Allowing $\epsilon$ to vary

So far we have assumed that  $\epsilon$  remains fixed when  $h \rightarrow 0$ . In practice we may need to approximate  $I_\epsilon$  and it is natural to expect that the error in the approximation will change with  $\epsilon$ . In particular, if  $\epsilon$  is too large in comparison with  $h$  this may lead to significant initial errors. Thus it is of interest to consider  $\epsilon$  as a function of  $h$  such that  $\epsilon(h) \rightarrow 0$  as  $h \rightarrow 0$ . The following theorem shows how the error of Euler's method depends on  $h$  and  $\epsilon$  (T. Diogo et al., 2003).

**Theorem 4.1** Consider equation (1.1) with  $0 < \mu \leq 1$  and  $g \in C^1[0, T]$  satisfying  $g(0) = 0$ . Assume that the integral  $I_\epsilon$  (with  $\epsilon \neq 0$ ) in (2.1) has been evaluated exactly for a chosen particular solution (corresponding to a certain value of the parameter  $c_0$ ). Then the error in the approximate solution obtained by the product Euler's method (2.5) satisfies

$$\max_{t_n \in X_h} |e(t_n)| := \max |u(t_n) - u_n^h| \leq C \frac{h}{\epsilon} \tag{4.8}$$

where  $C$  is a constant independent of  $h$  and  $\epsilon$ .

**Remark 4.1** Formula (4.8) can be considered as an improvement of the error estimate (3.7) obtained in the proof of theorem 3.2. Both formulas show that the Euler's method has first order of convergence (for fixed  $\epsilon$ ) but (3.7) cannot be used to prove convergence when  $\epsilon$  tends to zero. The particular case when  $\epsilon$  tends to zero as  $h^\theta$  is considered in the the next corollary.

**Corollary 4.1** Let us assume that the conditions of theorem 4.1 are satisfied. If  $\epsilon = h^\theta$ ,  $0 < \theta < 1$ , then

$$\max_{t_n \in X_h} |e(t_n)| \leq C h^{1-\theta} \tag{4.9}$$

## 5. Numerical examples

In this section we consider two examples to which we have applied the methods described in the previous section.

**Example 1** If we set  $g(t) = 1 + t + t^2$  in equation (1.1), then, using (1.2), we obtain the general form of its family of solutions

$$u(t) = c_0 t^{1-\mu} + \frac{\mu}{\mu-1} + \frac{\mu+1}{\mu}t + \frac{\mu+2}{\mu+1}t^2, \quad c_0 \text{ arbitrary constant} \quad (5.1)$$

**Example 2** We set  $\mu = 0.75$  and choose  $g$  such that the general solution of (1.1) is given by

$$u(t) = c_0 t^{0.25} + t^{1/2} \sin(2\pi t), \quad c_0 \text{ arbitrary constant} \quad (5.2)$$

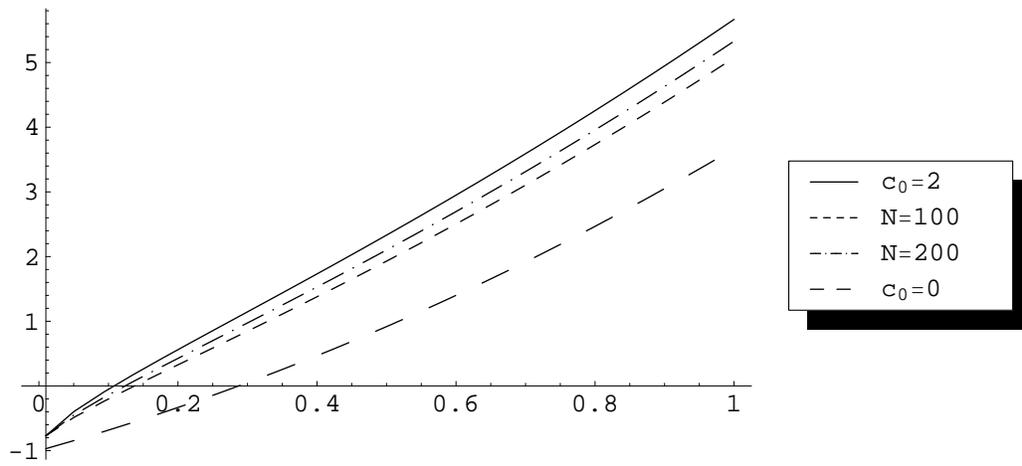
In tables 1-5 we have considered  $t \in [0, T] = [0, 1]$ . The following quantity has been used as an estimate of the convergence order

$$k := -\log_2 \left( \frac{\|r^h u - u^h\|_\infty}{\|r^{2h} u - u^{2h}\|_\infty} \right). \quad (5.3)$$

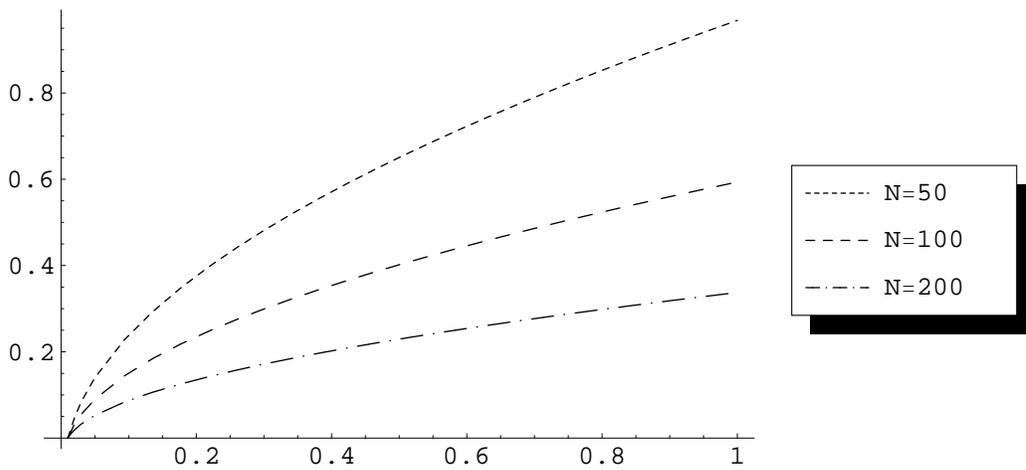
Table 1 shows the errors obtained with Euler's method (2.5) applied to example 1, with  $\mu = 0.5$  and  $c_0 = 2$ . That is, for each fixed value of  $\epsilon$ , we have used the exact value of  $I_\epsilon$  corresponding to this particular solution. We see that the errors decrease as  $\epsilon$  increases and this could be expected since we are fixing the exact solution over a larger interval. The results indicate convergence to the chosen particular solution ( $c_0 = 2$  in (5.1)) and this is clearly illustrated in figure 1, which also shows the smooth solution ( $c_0 = 0$ ). Moreover, different choices of  $\epsilon$  (Table 1) and  $\mu$  (Table 3) do not affect the expected first order of convergence. Similar results are obtained if instead of the particular solution corresponding to  $c_0 = 2$  we choose any other member of the family. As an example, we consider the smooth solution, that is, we take  $c_0 = 0$  in (5.1) and put in (2.5) the exact value of  $I_\epsilon$  for this particular solution. In Table 2 the errors and convergence rates produced with  $\epsilon = 0.02$  and  $\epsilon = 0.05$  confirm the predicted first order of convergence. For the sake of comparison, we have also applied the conventional product Euler's method (which corresponds to setting  $\epsilon = 0$ ) to the same example. The results displayed in the first column of Table 2 are in agreement with the theoretical order  $k = \mu$  given in theorem 3.1 (see also remark 3.1). An increase in order is thus obtained with  $\epsilon \neq 0$ . Figures 3 and 4 illustrate the performance of Euler's method for example 2.

Table 4 contains the errors obtained with the Trapezoidal method (2.7) for example 1, with  $\mu = 0.5$ . First we have taken  $c_0 = 2$  and convergence of order 2 to the chosen particular solution is confirmed, independently of the choice of each fixed value of  $\epsilon$ . Similarly to Euler's method (Table 2) we have also considered the smooth solution ( $c_0 = 0$ ) and the errors in the approximation are displayed in Table 5. Again the algorithm (2.7) with  $\epsilon \neq 0$  gives a higher convergence order than the conventional product Trapezoidal method (that is, when  $\epsilon = 0$ ).

Finally, Table 6 shows the behaviour of the error in solution given by the Euler's method, when  $\epsilon$  varies with  $h$ . We have taken  $\epsilon = h^\theta$ , for several choices of  $\theta$ , namely: 0.2, 0.6, 0.8. According to corollary 4.1, the expected convergence order is  $1 - \theta$  and this is confirmed by the numerical results.



**Figure 1:** Euler's method for example 1; approximate solutions ( $N = 100, 200$ ) and convergence to chosen exact solution ( $c_0 = 2$ ); the smooth solution ( $c_0 = 0$ )



**Figure 2:** Errors of approximate solutions in Euler's method, example 1 ( $c_0 = 2$ )

**Table 1:** Errors and orders for Euler's method in example 1  
Several values of  $\epsilon$  (fixed),  $c_0 = 2$

N	$\epsilon = 0.01$		$\epsilon = 0.02$		$\epsilon = 0.05$	
	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$
100	5.93E-1		3.84E-1		2.01E-1	
200	3.37E-1	0.81	2.06E-1	0.90	1.04E-1	0.96
400	1.82E-1	0.89	1.07E-1	0.94	5.26E-2	0.98
800	9.44E-2	0.94	5.47E-2	0.97	2.65E-2	0.99
1600	4.82E-2	0.97	2.76E-2	0.98	1.33E-2	0.99

**Table 2:** Errors and orders for Euler's method in example 1  
Approximation of the smooth solution ( $c_0 = 0$ )

N	$\epsilon = 0$		$\epsilon = 0.02$		$\epsilon = 0.05$	
	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$
100	4.03E-1		1.82E-1		1.17E-1	
200	2.85E-1	0.496	9.65E-2	0.912	6.00E-2	0.959
400	2.02E-1	0.499	4.99E-2	0.952	3.04E-2	0.979
800	1.43E-1	0.500	2.54E-2	0.975	1.53E-2	0.989
1600	1.01E-1	0.500	1.28E-2	0.987	7.69E-3	0.995

**Table 3:** Errors and orders for Euler's method in example 1  
The effect of varying  $\mu$  ( $c_0 = 2$ ,  $\epsilon = 0.01$ )

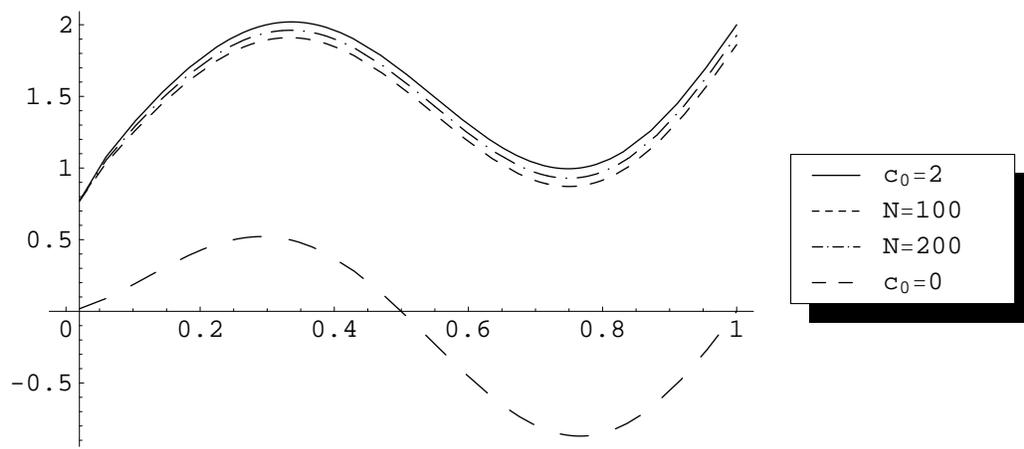
N	$\mu = 0.3$		$\mu = 0.5$		$\mu = 0.8$	
	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$
100	1.05E+0		5.93E-1		2.47E-1	
200	6.09E-1	0.78	3.37E-1	0.81	1.35E-1	0.87
400	3.32E-1	0.87	1.81E-1	0.89	7.09E-2	0.93
800	1.74E-1	0.93	9.44E-2	0.94	3.64E-2	0.96
1600	8.94E-2	0.96	4.82E-2	0.97	1.84E-2	0.98

**Table 4:** Errors and orders for the Trapezoidal method in example 1  
Several values of  $\epsilon$  (fixed),  $c_0 = 2$

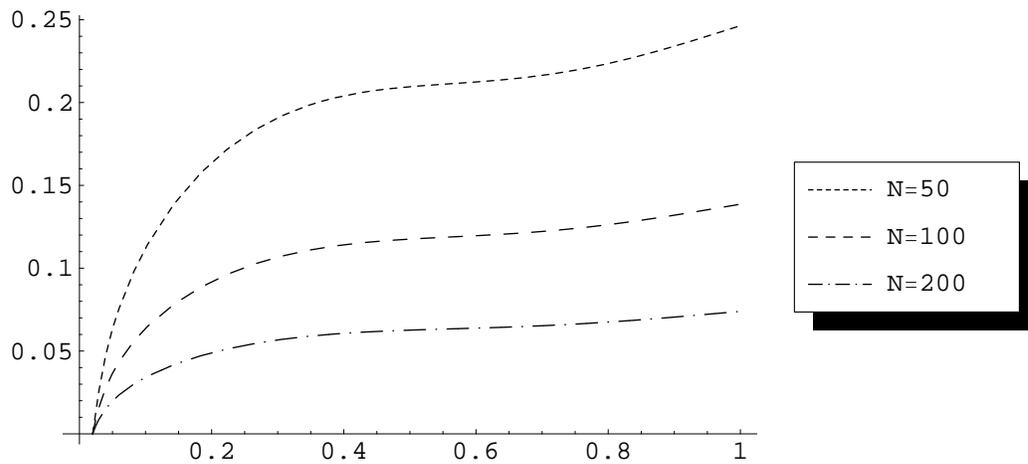
N	$\epsilon = 0.01$		$\epsilon = 0.02$		$\epsilon = 0.05$	
	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$
100	1.77E-2		4.51E-3		5.72E-4	
200	4.81E-3	1.88	1.16E-3	1.96	1.44E-4	1.99
400	1.23E-3	1.96	2.92E-4	1.99	3.60E-5	2.00
800	3.11E-4	1.99	7.30E-5	2.00	9.00E-6	2.00
1600	7.78E-5	2.00	1.83E-5	2.00	2.25E-6	2.00

**Table 5:** Errors and orders for the Trapezoidal method in example 1  
Approximation of the smooth solution ( $c_0 = 0$ )

N	$\epsilon = 0$		$\epsilon = 0.02$		$\epsilon = 0.05$	
	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$
100	1.81E-3		3.23E-4		1.74E-4	
200	6.47E-4	1.487	8.09E-4	1.996	4.35E-5	1.999
400	2.30E-4	1.491	2.02E-5	1.999	1.09E-5	2.000
800	8.18E-5	1.494	5.06E-6	2.000	2.72E-6	2.000
1600	2.90E-5	1.496	1.26E-6	2.000	6.80E-7	1.999



**Figure 3:** Euler's method for example 2; approximate solutions ( $N = 100, 200$ ) and convergence to chosen exact solution ( $c_0 = 2$ ); the smooth solution ( $c_0 = 0$ )



**Figure 4:** Errors of approximate solutions in Euler's method, example 2 ( $c_0 = 2$ )

**Table 6:** Errors and orders for Euler's method in example 1  
Varying  $\epsilon$  with  $h$

$$\epsilon = h^\theta, \quad \theta \in \{0.2, 0.6, 0.8\}, \quad T = 1 + \epsilon$$

h	$\theta = 0.2$		$\theta = 0.6$		$\theta = 0.8$	
	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$	$\ e(t_i)\ _\infty$	$k$
0.01	0.099989		0.476465		1.025023	
0.005	0.056365	0.827	0.352668	0.434	0.875238	0.228
0.0025	0.031752	0.828	0.261367	0.432	0.750982	0.221
0.00125	0.017888	0.828	0.194073	0.429	0.647112	0.215
0.000625	0.010081	0.827	0.144423	0.426	0.559551	0.210
0.000313	0.005684	0.827	0.107719	0.423	0.485150	0.206

## 6. Conclusions

This work was concerned with the solution of the singular Volterra integral equation (1.1), with an infinite family of solutions in the case  $0 < \mu < 1$ , of which only one has finite gradient

at the origin. We have introduced a technique based on splitting up the integral  $\int_0^t = \int_0^\epsilon + \int_\epsilon^t$  (with  $\epsilon$  fixed) and evaluating  $I_\epsilon = \int_0^\epsilon$  exactly for a chosen particular exact solution. Product integration quadrature methods were then applied and shown to converge to the required solution. While in previous works ((T. Diogo and P.M. Lima, 2002), (P. Lima and T. Diogo, 2002)) we have only dealt with the approximation of the smooth solution, the present methods enable us to approximate any member of the family of solutions. Moreover, in the case of approximating the smooth solution we get higher convergence orders than with the conventional product integration methods used in the referred works. So far we have assumed that the exact value of  $I_\epsilon$  is known. In future investigation we need to consider the possibility of knowing this integral only approximately and to estimate how this approximation will affect the error in the solution. It will be natural to think that if  $\epsilon$  tends to zero as a certain power of  $h$  than the error in evaluating  $I_\epsilon$  will also tend to zero. Having this in mind we have analysed the case when  $\epsilon = h^\theta$ .

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