Труды международной конференции

Крымская осенняя математическая школа-симпозиум 2007

УДК: 517.924.4, 51-73; MSC2000: 34B16, 34B40

N.B. Konyukhova, P.M. Lima, M.L. Morgado and M.B. Soloviev

SINGULAR NONLINEAR BOUNDARY VALUE PROBLEM FOR THE BUBBLE–TYPE OR DROPLET–TYPE SOLUTIONS IN NONLINEAR PHYSICS MODELS

Introduction

For a second–order nonlinear ordinary differential equation (ODE), a singular boundary value problem (BVP) is investigated which arises in hydromechanics and nonlinear field theory when static centrally symmetric bubble–type (droplet–type) solutions are sought. Being defined on a semi–infinite interval $0 < r < \infty$, this ODE, with a polynomial nonlinearity of the third order with respect to a desired function, possesses a regular singular point as $r \to 0$ and an irregular one as $r \to \infty$. Using some results for singular Cauchy problems (CPs) and stable initial manifolds (SIMs), we give the restrictions to the parameters for correct mathematical statement of the above singular nonlinear BVP, solving as well an accompanying problem concerning the transfer of the boundary condition from a singular point into a close regular one. Due to a certain variational approach and some results for so–called ground state problem, the necessary and sufficient conditions for existence of bubble–type or droplet–type solutions are discussed (in the form of additional restrictions to the parameters) and some estimates are obtained.

For a bubble model in the modern theory of nonhomogeneous or two–phase fluids with the equations of state depending on the derivatives, the singular nonlinear BVP under consideration has been posed and partially studied in [1] including numerical simulation of the problem (some preliminary results have been announced also in [2]–[4]).

In the present work we give briefly some results concerning a more complete and accurate theoretical analysis of this BVP and its applications. We don't present here the numerical methods and computational results. The detailed analytical–numerical investigation of the above singular nonlinear BVP, including physical interpretation of the numerical results, is assumed to be published in [5].

1. Statement of the problem and preliminary remarks

We study a singular nonlinear BVP of the form [1]:

$$\rho'' + \frac{N-1}{r}\rho' = 4\lambda^2(\rho+1)\rho(\rho-\xi), \qquad 0 < r < \infty, \tag{1}$$

$$|\lim_{r\to 0+}\rho(r)| < \infty, \qquad \lim_{r\to 0+} r\rho'(r) = 0, \tag{2}$$

$$\lim_{r\to\infty}\rho(r) = \xi, \qquad \lim_{r\to\infty}\rho'(r) = 0. \tag{3}$$

Here all the variables are real, N, λ and ξ are the parameters, $\lambda > 0$ (the multiple $4\lambda^2$ may be omitted by change of the variable r).

The following preliminary assertions are evident enough (for some details, see [1], [5]).

Proposition 1. 1) For any fixed $N, \xi \in \mathbb{R}$, the singular nonlinear BVP (1)–(3) is solvable: it has at least one constant solution

$$\rho(r,\xi) \equiv \rho_\xi = \xi; \tag{4}$$

in addition, the accompanying singular problem (1), (2), considered separately, has yet at least two constant solutions independent of ξ:

$$\rho(r) \equiv \rho_{\rm tr} = 0, \qquad \rho(r) \equiv \rho_- = -1; \tag{5}$$

when $\xi : \xi(\xi+1) > 0$, there are three constant solutions to Eq.(1), defined by (4) and (5), where $\rho_- < \rho_{\rm tr} < \rho_\xi$, for $\xi > 0$, and $\rho_\xi < \rho_- < \rho_{\rm tr}$, for $\xi < -1$.

2) For fixed $N \geq 2$ and $\xi : \xi(\xi+1) > 0$, any solution to the singular nonlinear BVP (1)–(3) satisfies the restrictions:

$$-1 < \rho(r,\xi) \leq \xi \quad \forall r \in \mathbb{R}_+, \qquad if \quad \xi > 0, \tag{6}$$

$$\xi \leq \rho(r,\xi) < 0 \qquad \forall r \in \mathbb{R}_+, \qquad if \quad \xi < -1; \tag{7}$$

for $N : 1 < N < 2$, the same is valid with the replacement of the conditions (2) by the conditions

$$|\lim_{r\to 0+} \rho(r)| < \infty, \qquad \lim_{r\to 0+} \rho'(r) = 0. \tag{8}$$

3) For any $N, \xi \in \mathbb{R}$, BVPs (1)–(3) and (1), (8), (3) are invariant with respect to the transformation

$$\rho_{\rm new} = -\rho - 1, \qquad \xi_{\rm new} = -\xi - 1. \tag{9}$$

For integer $N \geq 2$, the operator in the left–hand side of Eq.(1) is the radial part of the N–dimensional Laplace operator relating to the centrally symmetric solutions.

Definition 1. For fixed integer $N \geq 2$ and $\xi : \xi(\xi+1) > 0$, let $\rho(r,\xi)$ be a monotone solution of BVP (1)–(3), lying in the domain (6) or (7), respectively, and different from (4). For $\xi > 0$, if there exists $R(\xi) > 0$ such that $\rho(R(\xi),\xi) = 0$, then we say that $\rho(r,\xi)$ defines a (hyper)spherical interface (a wall) of a bubble, or simply a bubble with a radius $R_b = R(\xi)$; for $\xi < -1$, if there exists $R(\xi) > 0$ such that $\rho(R(\xi),\xi) = -1$, then we say that $\rho(r,\xi)$ describes a droplet with a radius $R_d = R(\xi)$.

We will use Definition 1 formally for any $N > 1$.

1.1. Bubbles and Droplets in the Capillary Fluid Models.

Concerning the physical models in this subsection, see, e.g., [6]– [9] and references therein.

In the second gradient theory (or shell–like theory), for nonhomogeneous or two–phase capillary fluids (fluid – fluid, fluid – vapor, fluid – gas, etc.), an additional term, depending on the gradient of density $\nabla\rho$, is added to the classical expression $E_0(\rho)$ for the volume free energy:

$$E(\rho, |\nabla\rho|^2) = E_0(\rho) + \frac{\gamma}{2}|\nabla\rho|^2, \qquad \gamma = \text{const} > 0.$$

Under isothermal process, the action functional

$$J(\rho,\vec{v}) = \int_{t_1}^{t_2}\int_{\mathbb{R}^N} \left[\rho\frac{|\vec{v}|^2}{2} - E(\rho,|\nabla\rho|^2)\right] dx^N dt$$

is introduced where $\vec{v}(\vec{r},t)$ is the vector–velocity of the particles of the medium, $N = 2, 3$. Taking into account that the law of conservation of mass has to be true in any domain $\Omega \subset \mathbb{R}^N$ (which leads to the equation $\rho_t + \text{div}(\rho\vec{v}) = 0$) and using the D'Alembert–Lagrange principle in order to find a solution of $\delta J = 0$ with the above–mentioned constraint (that implies a complex problem on a conditional extremum), the corresponding vector system of partial differential equations (PDEs), with respect to $\vec{v}(\vec{r},t)$ and $\rho(\vec{r},t)$, has been deduced.

For a medium in the equilibrium state, there is one PDE with respect to $\rho(\vec{r})$ which can describe the formation of microscopical bubbles in a nonhomogeneous or two–phase fluid, e.g., vapor inside one liquid. When centrally symmetric solutions are sought, depending on the radial variable r in the polar coordinates in $\mathbb{R}^N$, the physical model leads to the following singular nonlinear BVP:

$$\gamma\left(\rho'' + \frac{N-1}{r}\rho'\right) = \mu(\rho) - \mu(\rho_l), \qquad r \in \mathbb{R}_+, \tag{10}$$

$$\lim_{r\to 0+} \rho'(r) = 0, \qquad \lim_{r\to\infty} \rho(r) = \rho_l > 0, \tag{11}$$

Here the function $\mu(\rho) = (dE_0/d\rho)(\rho)$ is the chemical potential of the medium, ρ_l is the density of the external liquid. Whenever a strictly increasing solution $\rho(r)$ to the problem (10), (11) exists, for some $\rho(0) = \rho_v$, $0 < \rho_v < \rho_l$, then ρ_v is the density of the gas at the center of the bubble and the solution $\rho(r)$ determines an increasing mass density profile.

For the simplest model, suggested in [1], $\mu(\rho)$ is a third–degree polynomial on ρ with three distinct real roots:

$$\mu(\rho) = 4\alpha(\rho - \wp_1)(\rho - \wp_2)(\rho - \rho_l), \qquad 0 < \wp_1 < \wp_2 < \rho_l, \qquad \alpha > 0.$$

In this case, the qualitative behavior of $\mu(\rho)$ is analogous to the real chemical potentials for the Van der Waals and other fluids near the critical temperature. Introducing the normalized values $\rho_{\text{new}} = (\rho - \wp_2)/(\wp_2 - \wp_1)$, $\lambda = \sqrt{\alpha/\gamma}(\wp_2 - \wp_1) > 0$,

$$\xi = (\rho_l - \wp_2)/(\wp_2 - \wp_1) > 0, \tag{12}$$

and writing ρ instead of ρ_{new}, we obtain the relations: $\rho_v = \rho(0)\,[\wp_2 - \wp_1] + \wp_2$,

$$\widetilde{\mu}(\rho) = \mu(\rho(\wp_2 - \wp_1) + \wp_2)/[\gamma(\wp_2 - \wp_1)] = 4\lambda^2(\rho + 1)\rho(\rho - \xi). \tag{13}$$

As a result, we obtain the singular nonlinear BVP (1)–(3) (with more accurate statement of the boundary conditions in the singular points).

If the bubble–type solutions exist, many important physical properties depend on them: the gas density inside the bubble, the bubble radius, surface tension, interface thickness, etc.

Besides the bubbles there are the droplet models when the density of the medium inside the object is higher than outside: $0 < \rho_l < \rho_v$, $0 < \rho_l < \wp_1 < \wp_2$; in this case (12) implies $\xi < -1$.

1.2. **Bubbles in the Models of Nonlinear Field Theory and Cosmology.** A different physical model, described below, is widely studied in the nonlinear field theory and relativistic cosmology where the bubbles can be treated as the classical patterns of the elementary particles or the domains in the universe, respectively (see, e.g., [10], Chapter 12, [11], Chapter 5). In the $(N+1)$–dimensional Minkowski space with the coordinates $x_0 = t, x_1, \ldots, x_N$ $(N \geq 1)$, let us consider the scalar neutral field $\rho(\vec{r}, t)$ with the Lagrangian

$$L = \frac{1}{2}\left(\frac{\partial\rho}{\partial t}\right)^2 - \frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial\rho}{\partial x_j}\right)^2 - W(\rho).$$

Here $W(\rho)$ is the Higgs–type self–action potential of the field,

$$W(\rho) = \lambda^2(\rho - w_1)(\rho - w_2)(\rho - \xi)^2, \quad w_{1,2} = [-(\xi + 2) \mp \sqrt{2(\xi + 2)(1 - \xi)}]/3, \tag{14}$$

and, since $w_{1,2}$ must be real, this implies

$$-2 \leq \xi \leq 1. \tag{15}$$

Here and further we use a system of units with $c = \hbar = 1$ where c is the speed of light in vacuum and $\hbar$ is the Plank constant.

The Lagrange–Euler PDE takes the form

$$\frac{\partial^2\rho}{\partial t^2} - \Delta_{\text{N}}\rho + \frac{dW}{d\rho}(\rho) = 0, \qquad \vec{r} \in \mathbb{R}^N, \qquad t \in \mathbb{R}, \tag{16}$$

where Δ_{N} is the N–dimensional Laplace operator and $dW/d\rho$ coincides with the right–hand side of Eq.(13). If $\rho(\vec{r},t)$ is a solution of Eq.(16) for $t \geq t_0$ and $E(t_0) = E_0 < \infty$, where

$$E(t) = \int\limits_{\mathbb{R}^N} \left[\frac{1}{2} \left(\frac{\partial \rho}{\partial t} \right)^2 + |\nabla \rho|^2/2 + W(\rho) \right] d^N x$$

is the energy integral, then $E(t) \equiv E_0$, $t \geq t_0$.

Definition 2. Let $\rho \equiv \rho_{vm}$ be a constant solution of Eq.(16), so that $(dW/d\rho)(\rho_{vm}) = 0$. We say that ρ_{vm} is a true vacuum of the field iff it is a point of absolute minimum of $W(\rho)$, otherwise it is a false vacuum.

For Eq.(16), if we look for a (hyper)bubble as a stationary centrally symmetric solution with a finite energy, then we obtain again the nonlinear singular BVP (1)–(3) (here the bubbles are called sphalerons). Moreover, from (15) and Definition 1, we have two admissible intervals for the values of ξ:

$$0 < \xi < 1; \tag{17}$$

$$-2 < \xi < -1. \tag{18}$$

The well–known mechanistic interpretation of the model is the following (in detail, see [10], Chapter 12). Due to the substitution (9), we can consider only the case (17) when the formulas (14) determine the Higgs–type potential $W(\rho)$ with two distorted vacua $\rho_- = -1$ and $\rho_\xi = \xi$ (a true vacuum and a false one, respectively). Let us interpret r as the time variable and ρ as a "coordinate" and introduce the potential $\widetilde{W}(\rho) \equiv -W(\rho)$ and the force $\widetilde{F}(\rho,\xi) = -(d\widetilde{W}/d\rho)(\rho,\xi)$. Then we can rewrite Eq.(1) in the form

$$\rho'' + (N-1)\rho'/r = \widetilde{F}(\rho,\xi), \quad r > 0, \tag{19}$$

and we can treat Eq.(19) as the second Newton law for a "point mass" (a unit mass "ball") in the field of potential forces where the term $(N-1)\rho'/r$ plays the role of depending on time friction. When $N > 1$, due to the friction term, we obtain the restriction (17) as a necessary condition for a bubble–type solution to exist.

Remark 1. For $\xi = 1$, we obtain from (14) the Higgs potential

$$W(\rho) = W_H(\rho) = \lambda^2(\rho^2 - 1)^2 \tag{20}$$

with two degenerate true vacua $\rho_\pm = \pm 1$. Eq.(16) with the Higgs potential (20) has the exact one–dimensional solutions which are called the domain walls. In general, they move with constant velocity v, $0 < v < 1$, perpendicular to their surface: $\rho_{w\pm}(\vec{r}-\vec{r}_0, t-t_0, \vec{n}, v) = \pm\tanh\left(\lambda\sqrt{2}\left[(\vec{n},\vec{r}-\vec{r}_0) \pm v(t-t_0)\right]/\sqrt{1-v^2}\right)$, where $\vec{n}$ is the unit vector of the direction of wave propagation; the stationary solutions correspond to $v = 0$. The bubbles can be treated approximately as curved domain walls. However it is well known that Eq.(16) with the potential (20) has no stationary N–dimensional bubble–type solutions for $N \geq 2$ (concerning dynamical problems for the bubble–type solutions to Eq.(16), some results were obtained in [12], [13], for $N = 3$ and $\xi = 1$).

2. Associated singular nonlinear Cauchy problems

Definition 3. We say that the singular nonlinear BVP (1)–(3) is correctly formulated on $\mathbb{R}_+$ iff both the local singular problem (1), (2) at the point $r = 0$ and the singular CP (1), (3) at infinity have one–parameter sets of solutions.

Further, we use the classification of singular points for linear and nonlinear ODEs according to [14].

2.1. **The regular singularity at zero.** Eq.(1) has a regular singularity as $r \to 0$ and the limit conditions (2) are equivalent to the following ones (with unknown parameter ρ_0):

$$\lim_{r\to 0+} \rho(r) = \rho_0, \qquad \lim_{r\to 0+} r\rho'(r) = 0. \tag{21}$$

For the linearized CP in the neighborhood of $r = 0$, characteristic exponents are $\nu_1 = 0$ and $\nu_2 = 2 - N$. Then, from Theorem 5 in [15] and the results of Section 30 in [16], the following assertion is valid.

Proposition 2. For any fixed $N \geq 2$ and $\xi, \rho_0 \in \mathbb{R}$, the singular nonlinear CP (1), (21) has a unique solution $\rho(r, \rho_0)$ in the class $C[0, r_0] \bigcap C^1(0, r_0]$, for some $r_0 > 0$. This solution is a holomorphic function at the point $r = 0$ represented by the series

$$\rho(r, \rho_0) = \rho_0 + \sum_{k=1}^{\infty} \rho_{2k}(\rho_0) r^{2k}, \qquad 0 \leq r \leq r_0, \tag{22}$$

where the coefficients ρ_{2k}, depending on the parameter ρ_0, are determined by the recurrence formulas:

$$\rho_2(\rho_0) = (2\lambda^2/N)\, \rho_0(\rho_0 + 1)(\rho_0 - \xi), \tag{23}$$

$$\begin{aligned}\rho_{2k}(\rho_0) = \Big\{ \textstyle\sum_{m=0}^{k-1} \Big[\rho_{2k-2m-2} \Big(\sum_{l=0}^{m} \rho_{2l}\rho_{2m-2l} + (1-\xi)\rho_{2m} \Big) \Big] - \xi\rho_{2k-2} \Big\} \times \\ \times 2\lambda^2/[k(2k + N - 2)], \qquad k = 2, 3, \ldots.\end{aligned} \tag{24}$$

For $N : 1 < N < 2$, the formulas (22)–(24) define also the solution to the singular nonlinear CP (1), (21) but which is unique only in the class $H[-r_0, r_0]$ of the holomorphic functions at the point $r = 0$; however, as a solution to Eq.(1) satisfying the conditions

$$\lim_{r\to 0+} \rho(r) = \rho_0, \qquad \lim_{r\to 0+} \rho'(r) = 0, \tag{25}$$

it is unique in the class $C^1[0, r_0]$.

Corollary 1. *For any fixed $N > 1$ and $\xi \in \mathbb{R}$, the local singular problem* (1), (2) *has a one–parameter set of solutions $\rho(r, \rho_0)$. Each solution of this set is a holomorphic function, at the point $r = 0$, represented by the series* (22)–(24)*, so that it satisfies conditions* (8) *(three constant solutions, defined by* (4) *and* (5)*, belong to this set and correspond to the values of ρ_0 equal ξ, 0 or -1, respectively).*

Taking into account Propositions 1, 2 and Definition 1, we can deduce the following necessary condition.

Corollary 2. *For fixed $N > 1$ and $\xi : \xi(\xi + 1) > 0$, let $\rho_0 = \rho_0(\xi)$ be such that the function $\rho(r, \xi) = \rho(r, \rho_0(\xi))$ from the set* (22)–(24)*, being extended to the right, is a bubble–type, for $\xi > 0$ (resp., a droplet–type, for $\xi < -1$), solution to the singular nonlinear BVP* (1)–(3)*. Then $\rho_0(\xi)$ satisfies the condition*

$$-1 < \rho_0(\xi) < 0. \tag{26}$$

Moreover, for the corresponding solution $\rho(r, \xi)$ satisfying the restrictions (6) *(resp.,* (7)*), the conditions* (2) *and* (8) *are equivalent and $\rho''(0, \xi) > 0$ (resp., $\rho''(0, \xi) < 0$).*

Remark 2. For any fixed $N > 1$ and $\xi \in \mathbb{R}$, depending on the choice of $\rho_0(\xi) \in \mathbb{R}$, the solutions of the singular nonlinear CP (1), (21) have different behavior: there are the singular blow–up, bounded monotone and oscillating solutions. The needed bubble–type or droplet–type solution, i.e., existing globally and satisfying (3), separates the set of blow–up solutions from the set of oscillating ones and is called a separatrix solution (for a detailed analysis, see [5]; it follows also from the mechanistic interpretation of the model).

2.2. **The irregular singularity at infinity.** Eq.(1) has an irregular singularity as $r \to \infty$. Due to the classical results of [16] with their detailing in [17] and some results of [15], concerning the singular CPs, we obtain

Proposition 3. For any fixed $N \in \mathbb{R}$ and $\xi : \xi(\xi+1) > 0$, the singular nonlinear CP at infinity (1), (3) has a one–parameter family of solutions. For $N = 1$, this family is represented by a convergent exponential Lyapunov series

$$\rho(r,b) = \xi + \sum_{k=1}^{\infty} B_k b^k \exp(-k\varkappa r), \qquad r \geq r_\infty, \tag{27}$$

where b is a parameter, $|b\exp(-\varkappa r_\infty)|$ is small,

$$\varkappa = 2\lambda\sqrt{\xi(\xi+1)} > 0, \tag{28}$$

and the independent of b constants B_k are defined by the recurrent formulas:

$$B_1 = 1, \qquad B_2 = (1+2\xi)/[3\xi(1+\xi)], \tag{29}$$

$$B_k = \left[(1+2\xi)\sum_{l=1}^{k-1} B_l B_{k-l} + \sum_{m=2}^{k-1}\sum_{l=1}^{m-1} B_l B_{k-m} B_{m-l}\right] \Big/ \Big/ [(k^2-1)\xi(\xi+1)], \qquad k = 3,4,\ldots, \tag{30}$$

For $N \neq 1$, the one–parameter family of solutions to the singular nonlinear CP (1), (3) is represented by a convergent exponential Lyapunov series

$$\rho(r,b) = \xi + \sum_{k=1}^{\infty} C_k(r) b^k r^{-k(N-1)/2} \exp(-k\varkappa r), \qquad r \geq r_\infty, \tag{31}$$

where b is a parameter, $|b r_\infty^{-(N-1)/2} \exp(-\varkappa r_\infty)|$ is small, $\varkappa$ is defined by (28) and the independent of b coefficient functions $C_k(r)$ are the solutions to the recurrent sequence of singular linear CPs:

$$\begin{gathered} C_1'' - 2\varkappa C_1' - (N-1)(N-3)C_1/(4r^2) = 0, \qquad r \geq r_\infty, \\ \lim_{r\to\infty} C_1(r) = 1, \qquad \lim_{r\to\infty} C_1'(r) = 0; \end{gathered} \tag{32}$$

$$C_2'' - [4\varkappa + (N-1)/r]C_2' + [3\varkappa^2 + (N-1)(2\varkappa/r + 1/r^2)]C_2 = 4\lambda^2(2\xi+1)C_1^2, \ r \geq r_\infty,$$

$$\lim_{r\to\infty} C_2(r) = B_2, \qquad \lim_{r\to\infty} C_2'(r) = 0;$$

$$\begin{gathered} C_k'' - [2k\varkappa + (k-1)(N-1)/r]C_k' + \{(k^2-1)\varkappa^2 + (N-1)[k(k-1)\varkappa/r + \\ +((k^2-2k)(N-1)+2k)/(4r^2)]\}C_k = \\ = 4\lambda^2\left[(2\xi+1)\sum_{l=1}^{k-1} C_l C_{k-l} + \sum_{m=2}^{k-1}\sum_{l=1}^{m-1} C_l C_{k-m} C_{m-l}\right], \qquad r \geq r_\infty, \\ \lim_{r\to\infty} C_k(r) = B_k, \qquad \lim_{r\to\infty} C_k'(r) = 0, \qquad k = 3,4,\ldots, \end{gathered}$$

where B_k are defined by (29), (30). Each coefficient function $C_k(r)$ is defined uniquely, by recurrence, and, for large values of r, is represented in the form of an asymptotic series in negative powers of r:

$$C_k(r) \sim B_k + \sum_{s=1}^{\infty} c_s^{(k)}/r^s, \qquad k = 1,2,\ldots, \tag{33}$$

where the coefficients $c_s^{(k)}$ may be defined by the substitution of (33) in the above equations for $C_k(r)$. In particular, for $k = 1$, the following formulas are valid from (32):

$$c_1^{(1)} = (N-1)(N-3)/(8\varkappa), \qquad c_{s+1}^{(1)} = c_s^{(1)}[-s(s+1) + (N-1)(N-3)/4]/[2\varkappa(s+1)],$$

$s = 1,2\ldots$. (The constant solution (4) belongs to the family (31) with $b = 0$.)

The next necessary condition follows from Propositions 1, 3 and Definition 1.

Corollary 3. *For fixed $N \in \mathbb{R}$ and $\xi : \xi(\xi+1) > 0$, let $b = b(\xi)$ be such that the function $\rho(r,\xi) = \rho(r, b(\xi))$ from the set* (31)*, being extended to the left, is a bubble–type, for $\xi > 0$, or a droplet–type, for $\xi < -1$, solution to the singular nonlinear BVP* (1)–(3)*. Then $b(\xi)$ satisfies the conditions:*

$$b(\xi) < 0, \quad \textit{if} \quad \xi > 0; \qquad b(\xi) > 0, \quad \textit{if} \quad \xi < -1.$$

2.3. **Correct statement of the singular nonlinear BVP: restrictions on the parameters.** From Corollary 1, Proposition 3 and Definition 3, we obtain

Corollary 4. *For any fixed $\xi : \xi(\xi+1) > 0$ and $N \geq 2$, the nonlinear singular BVP* (1)–(3) *is correctly formulated on $\mathbb{R}_+$ and is equivalent to the singular nonlinear BVP* (1), (8), (3)*; for $N : 1 < N < 2$, the same is valid with the a priori requirement that a solution to the BVP* (1)–(3) *is a holomorphic function at the point $r = 0$.*

Then to solve the original nonlinear singular BVP (1)–(3) with fixed $N > 1$ and $\xi : \xi(\xi+1) > 0$, we must find from the set (31) a particular solution of Eq.(1) which also satisfies (2), i.e., belongs to the family (22)–(24). Alternatively, we can look for a particular solution of the set (22)–(24) which additionally fulfills (3), i.e., belongs to the family (31).

3. Analytic stable initial manifolds and the boundary condition transfer

In this section, we use the idea of the boundary condition transfer from infinity to a finite point in order to get the equivalent nonlinear BVP on a finite interval (in general, with a variable right boundary), which has both theoretical and practical interest. For a history of the problem and the initial concepts of the boundary condition transfer from the singular points of ODE systems, see, e.g., the survey [18]. Some existence theorems about the differential and analytic SIMs, for nonlinear ODEs, can be found in [19], Chapter XIII. In what follows, we use the results of [20] taking into account as well the more general results of [21]–[23] to describe SIMs for nonlinear ODE systems. (For $N = 1$, we have a simpler autonomous ODE and the corresponding assertions can be obtained by different ways.)

Proposition 4. For $N = 1$ and any fixed $\xi : \xi(\xi+1) > 0$, the values of solutions to the singular nonlinear CP (1), (3) (i.e., belonging to the family (27)) form in the phase plane of the variables (ρ, ρ'), in the neighborhood of the point $(\xi, 0)$, a one–dimensional SIM invariant with respect to r. This analytic curve, as the Lyapunov manifold of conventional stability, is given by the relation

$$\rho' = -\varkappa(\rho - \xi) + \beta(\rho - \xi), \qquad |\rho - \xi| \leq \Delta, \qquad \Delta > 0, \tag{34}$$

where $\varkappa$ is given by (28) and $\beta(z)$ is a solution to the singular nonlinear Lyapunov–type problem,

$$\frac{d\beta}{dz}\Big(-\varkappa z + \beta\Big) = \varkappa\beta + 4\lambda^2 z^2(2\xi + 1 + z), \qquad |z| \leq \Delta, \qquad \Delta > 0,$$

$$\beta(0) = \frac{d\beta}{dz}(0) = 0;$$

this problem has a unique solution in the class $C^1[-\Delta, \Delta]$ and it is a holomorphic function at the point $z = 0$:

$$\beta(z) = \sum_{k=2}^{\infty} \beta_k z^k, \quad |z| \leq \Delta,$$

$$\begin{gathered} \beta_2 = -4\lambda^2(2\xi+1)/(3\varkappa), \qquad \beta_3 = (\beta_2^2 - 2\lambda^2)/(2\varkappa), \\ \beta_k = \Big(\textstyle\sum_{m=2}^{k-1} m\beta_m\beta_{k+1-m}\Big) \Big/ [\varkappa(k+1)], \qquad k = 4, 5, \dots; \end{gathered} \tag{35}$$

this solution is represented in the explicit form

$$\beta(z) = \varkappa z\left(1 - \sqrt{1 + 8\lambda^2(2\xi+1)z/(3\varkappa^2) + 2\lambda^2 z^2/\varkappa^2}\right),$$

for all $|z| \le \Delta$ such that this formula makes sense. Thus Eq.(34) can be rewritten in the explicit form

$$\rho' = -\varkappa(\rho-\xi)\sqrt{1 + 8\lambda^2(2\xi+1)(\rho-\xi)/(3\varkappa^2) + 2\lambda^2(\rho-\xi)^2/\varkappa^2}, \qquad |\rho-\xi| \le \Delta. \quad (36)$$

Remark 3. For $\xi = \pm 1$, replacing $\varkappa$ in (36) by $\operatorname{sign}(\xi)\varkappa$, we obtain the whole Lagrange manifold generated in the phase plane by the values of the exact one–dimensional solutions $\rho_{w\pm}(r - r^0) = \pm\tanh\left(\lambda\sqrt{2}\left(r - r^0\right)\right)$, $r, r^0 \in \mathbb{R}$ (see Remark 1 concerning the planar domain walls).

Proposition 5. For any fixed $N : (N \in \mathbb{R}) \bigwedge (N \ne 1)$ and $\xi : \xi(\xi+1) > 0$, the values of solutions to the singular nonlinear CP (1), (3) (i.e., belonging to the family (31)) form in the phase plane of the variables (ρ, ρ'), in the neighborhood of the point $(\xi, 0)$, a one–dimensional SIM depending on r as on parameter. This analytic curve, as the Lyapunov manifold of conventional stability, is given by the relation

$$\rho' = -\varkappa(\rho-\xi) + \alpha(r, \rho-\xi), \qquad r \ge r_\infty, \qquad |\rho-\xi| \le \Delta, \qquad \Delta > 0, \quad (37)$$

where $\varkappa$ is given by (28). Here $\alpha(r, z)$ is analytic function at the point $z = 0$, for each $r \ge r_\infty$, $\alpha(r, 0) \equiv 0$, and is a unique solution, in this class, to the singular nonlinear CP at infinity:

$$\frac{\partial\alpha}{\partial r} + \frac{\partial\alpha}{\partial z}\Big(-\varkappa z + \alpha\Big) = \varkappa\alpha - \frac{N-1}{r}\Big(\alpha - \varkappa z\Big) + 4\lambda^2 z^2(2\xi+1+z), \;\; |z| \le \Delta, \;\; r \ge r_\infty, \quad (38)$$

$$\lim_{r\to\infty} \alpha(r, z) = \beta(z) \quad \text{uniformly on} \quad z : |z| \le \Delta, \quad (39)$$

where $\beta(z)$ is defined by Proposition 4. For large enough values of r and small enough $|z|$, this solution may be represented in two equivalent forms:

$$\alpha(r, z) \sim \beta(z) + \sum_{k=1}^{\infty} \alpha_k(z)/r^k, \qquad |z| \le \Delta, \qquad r \ge r_\infty,$$

$$\alpha(r, z) = \sum_{k=1}^{\infty} \widetilde{\alpha}_k(r) z^k, \qquad |z| \le \Delta, \qquad r \ge r_\infty.$$

Here the coefficient functions $\alpha_k(z)$ are holomorphic at the point $z = 0$ solutions to a recurrent sequence of singular linear Lyapunov–type problems whereas $\widetilde{\alpha}_k(r)$ are the solutions to a recurrent sequence of singular CPs at infinity (for a Riccati–type ODE, when $k = 1$, and for linear ODEs, when $k \ge 2$) which, for large enough values of r, have asymptotic expansions on integer negative powers of r. As a result, the expansion of the form

$$\alpha(r, z) \sim \sum_{k=1,\; m=0,\; k+m\ge 2}^{\infty} \alpha_{k,m} z^k / r^m, \qquad |z| \le \Delta, \qquad r \ge r_\infty, \quad (40)$$

is valid where the coefficients $\alpha_{k,m}$ are defined by the formal substitution of (40) in Eq.(38), $\alpha_{k,0} = \beta_k$ $(k = 2, 3, \ldots)$ where β_k are defined by (35).

In more detail, for $\alpha_k(z)$ we obtain the singular linear Lyapunov–type problems:

$$\frac{d\alpha_1}{dz}\Big(-\varkappa z + \beta(z)\Big) = \Big[\varkappa - \frac{d\beta}{dz}(z)\Big]\alpha_1 + (N-1)[\varkappa z - \beta(z)], \quad |z| \le \Delta, \quad \alpha_1(0) = 0;$$

$$\frac{d\alpha_{k+1}}{dz}\Big(-\varkappa z + \beta(z)\Big) = \left[\varkappa - \frac{d\beta}{dz}(z)\right]\alpha_{k+1} + k\alpha_k - \sum_{l=1}^{k} \alpha_l \frac{d\alpha_{k-l+1}}{dz} +$$
$$+ (N-1)[\varkappa z - \beta(z) - \alpha_k], \qquad |z| \le \Delta, \qquad \alpha_{k+1}(0) = 0, \qquad k = 1, 2, \ldots.$$

Alternatively, for $\widetilde{\alpha}_k(r)$ we have the singular CPs at infinity:

$$\widetilde{\alpha}_1' - [2\varkappa - (N-1)/r]\widetilde{\alpha}_1 + \widetilde{\alpha}_1^2 = (N-1)\varkappa/r, \quad r \ge r_\infty, \quad \lim_{r\to\infty} \widetilde{\alpha}_1(r) = 0; \quad (41)$$

$$\widetilde{\alpha}_k' - [(k+1)\varkappa - (N-1)/r]\widetilde{\alpha}_k + \sum_{m=1}^{k} m\widetilde{\alpha}_m\widetilde{\alpha}_{k+1-m} = 4\lambda^2[\delta_{k,2}(2\xi+1) + \delta_{k,3}], \quad r \geq r_\infty,$$
$$\lim_{r\to\infty} \widetilde{\alpha}_k(r) = \beta_k, \qquad k = 2, 3, \ldots,$$

where β_k are defined by the formulas (35), $\delta_{k,j}$ is the Kronecker delta.

All these problems are uniquely solvable by recurrence and their solutions are represented by the expansions indicated above. For example, for the solution to the singular CP (41), for large enough r, we obtain:

$$\widetilde{\alpha}_1(r) \sim \sum_{s=1}^{\infty} \alpha_{1,s}/r^s, \qquad \alpha_{1,1} = -(N-1)/2, \tag{42}$$

$$\alpha_{1,s} = \left[(N-s)\alpha_{1,s-1} + \sum_{m=1}^{s-1} \alpha_{1,m}\alpha_{1,s-m}\right] \Big/ (2\varkappa), \quad s = 2, 3, \ldots. \tag{43}$$

Corollary 5. *For fixed $N \in \mathbb{R}$ and $\xi : \xi(\xi+1) > 0$, let $\rho(r,\xi)$ be a solution to the singular nonlinear CP* (1), (3), *i.e., belonging to the set* (31). *Then the values of $\rho(r,\xi)$ belong to the SIM* (37) *for all $r \geq r_\infty$. It means that the limit boundary conditions* (3) *at infinity for the solutions to Eq.*(1) *are equivalent, for large enough r and small enough $|\rho - \xi|$, to the nonlinear relation* (37) *(for $N = 1$, it is the same as the relation* (34) *which is equivalent to* (36)*) where $\varkappa$ is defined by* (28) *and $\alpha(\rho - \xi)$ is described by Proposition 5.*

Remark 4. Corollary 5 and Proposition 5 give us the algorithm for replacement of the limit boundary conditions (3) at infinity by the equivalent nonlinear relation (37) taken in a finite point $r = r_\infty$ (nonlinear boundary condition of the third kind). For any fixed $N \in \mathbb{R}$ and $\xi : \xi(\xi+1) > 0$, the following approximations to SIM (37) are valid: for autonomous approach, coinciding with the case $N = 1$, equation of SIM has the explicit form (36); for a linear approach, we have the relation

$$\rho' = [-\varkappa + \widetilde{\alpha}_1(r)](\rho - \xi), \tag{44}$$

which defines a tangent to the SIM (37) taken in the limit stationary point $(\xi, 0)$ and depending on r as on parameter. For large enough r, $\widetilde{\alpha}_1(r)$ can be defined approximately by the asymptotic expansion (42), (43) (for $N = 1$, $\widetilde{\alpha}_1(r) \equiv 0$).

4. Restrictions on the parameters as the necessary and sufficient conditions for the solutions to exist

4.1. **The necessary conditions.** First, we use the known variational approach of [24].

Eq.(1.1) may be considered as the Lagrange–Euler equation for the functional

$$J(\rho) = \int_0^\infty \left(\frac{1}{2}\left(\frac{d\rho}{dr}\right)^2 + W(\rho)\right) r^{N-1} dr, \tag{45}$$

where $W(\rho)$ is defined by (14) which is equivalent to $W(\rho) = 4\lambda^2 \int_\xi^\rho (s+1)s(s-\xi)ds$. The functional $J(\rho)$ is considered on the class of functions for which the integral (45) converges.

Let $\rho(r)$ be a nonconstant solution of Eq.(1), for which the integral (45) converges, and $\rho_\sigma(r) = \rho(\sigma r)$ where σ is a positive parameter ($\rho_1 \equiv \rho$). Then we obtain

$$\Im(\sigma) = J(\rho_\sigma) = J_1(\rho)/\sigma^{N-2} + J_2(\rho)/\sigma^N, \tag{46}$$

where $J_j(\rho)$ $(j = 1, 2)$ are independent of σ:

$$J_1(\rho) = \int_0^\infty \frac{1}{2}\left(\frac{d\rho}{dr}\right)^2 r^{N-1} dr, \qquad J_2(\rho) = \int_0^\infty W(\rho) r^{N-1} dr.$$

Differentiating (46) with respect to σ, we obtain

$$\frac{d\Im}{d\sigma}(\sigma) = -(N-2)J_1(\rho)/\sigma^{N-1} - NJ_2(\rho)/\sigma^{N+1}. \tag{47}$$

Taking into account that the function $\rho(r)$, corresponding to the value $\sigma = 1$, is the solution of variational problem, we must put $(d\Im/d\sigma)_{\sigma=1} = 0$. Then (47) implies

$$J_2(\rho) = -[(N-2)/N]J_1(\rho) \tag{48}$$

(according to [25], the similar relations are associated both with G.H.Derrick and the other names; for some more general problems, it is known also as the Pokhozhaev identity).

For a nonconstant solution, J_1 is positive; if $W(\rho) \geq 0$ $\forall \rho \in \mathbb{R}$ then J_2 is positive as well and, for any $N \geq 2$, (48) is not satisfied. Hence the existence of at least one nonconstant solution requires that $W(\rho) < 0$, for a certain range of values of ρ, that implies from (14) the restriction

$$-2 < \xi < 1. \tag{49}$$

Concerning the singular nonlinear BVP (1), (8), (3) with $N > 1$, there is a simpler way leading to the same necessary condition for the existence of a nonconstant solution. Namely, let $\rho(r,\xi)$ be a nonconstant solution of the above BVP for which the integral

$$I = \int_0^\infty \frac{(\rho'(s,\xi))^2}{s} ds \tag{50}$$

converges. Then, multiplying Eq.(1) by ρ' and integrating between zero and r, we obtain

$$\frac{(\rho'(r,\xi))^2}{2} + (N-1)\int_0^r \frac{(\rho'(s,\xi))^2}{s} ds = W(\rho(r,\xi)) - W(\rho_0(\xi)),$$

where $\rho_0(\xi) = \rho(0,\xi)$; in particular for $r \to \infty$ we get

$$(N-1)\int_0^\infty \frac{(\rho'(s,\xi))^2}{s} ds = -W(\rho_0(\xi)). \tag{51}$$

Due to (51), we conclude that the value $W(\rho_0(\xi))$ must be negative, for any $N > 1$, that implies the restriction (49) once again.

As a result, taking into account also Propositions 2, 3 and Corollaries 1, 4, we obtain

Proposition 6. For each fixed $N \geq 2$, the restriction (49) is the necessary condition for the existence of a nonconstant solution to Eq.(1), for which the integral (45) converges. Concerning the singular nonlinear BVP (1), (8), (3) with $N > 1$, the same restriction (49) is the necessary condition for the existence of a nonconstant solution, for which the integral (50) converges; under condition $\xi(\xi+1) > 0$, connected with the correct statement of the above BVP, the restriction (17) (resp., (18)) is the necessary condition for a bubble–type (resp., a droplet–type) solution to exist.

Proposition 6, Corollary 2 and the formula (51) (in the same way as the mechanistic interpretation of the models) lead to more exact estimates for $\rho_0(\xi) = \rho(0,\xi)$ than (26).

Corollary 6. *For any fixed $N > 1$ and $\xi : \xi(\xi+1) > 0$, let $\rho_0(\xi)$ be such that the solution $\rho(r,\xi) = \rho(r,\rho_0(\xi))$ of the singular nonlinear CP* (1), (25), *being extended to the right, is a bubble–type, for $\xi > 0$, or a droplet–type, for $\xi < -1$, solution to the singular nonlinear BVP* (1), (8), (3). *Then $\rho_0(\xi)$ satisfies the restrictions:*

$$-1 < \rho_0(\xi) \leq w_2(\xi) = [-(\xi+2) + \sqrt{2(\xi+2)(1-\xi)}]/3 < 0, \quad \text{if} \quad 0 < \xi < 1; \tag{52}$$

$$-1 < w_1(\xi) = [-(\xi+2) - \sqrt{2(\xi+2)(1-\xi)}]/3 \leq \rho_0(\xi) < 0, \quad \text{if} \quad -2 < \xi < -1. \tag{53}$$

4.2. **The sufficient conditions.** For the singular nonlinear BVP (1), (8), (3) with $N > 1$ and $\xi : \xi(\xi+1) > 0$, the restrictions (17), (18) on the parameter ξ are not only necessary, but also sufficient conditions for the existence of a bubble–type or droplet–type solution. It follows both from the mechanistic interpretation of the model and some results relating to the so–called ground state problem applicable to the above BVP (see, e.g., [26] and references therein).

In [26], as a special case, the authors search for a nonnegative solution on $\mathbb{R}_+$ (ground state) to the following singular nonlinear BVP with $N > 1$:

$$u'' + [(N-1)/r]u' + f(u) = 0, \qquad r \in \mathbb{R}_+, \tag{54}$$

$$\lim_{r\to 0+} u(r) = \alpha > 0, \qquad \lim_{r\to 0+} u'(r) = 0, \tag{55}$$

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} u'(r) = 0. \tag{56}$$

Here $f(u)$ is a locally Lipschitz continuous on $[0,\infty)$ function and $f(0) = 0$. Then in particular the following assertion is valid (see, e.g., Theorem 2 in [15]): for any fixed $N > 1$ and $\alpha \in \mathbb{R}_+$, the singular nonlinear CP (54), (55) has a unique solution, at least on some interval $0 < r \le r_0(\alpha)$, $r_0(\alpha) > 0$.

Let us introduce the function

$$F(s) = \int_0^s f(t)dt, \tag{57}$$

which is continuous on $[0,\infty)$ and $F(0) = 0$.

The formulated below assertion follows from some more general results of [26] (see Theorem 1 and Lemma 2.1 therein).

Theorem. *Let the following conditions be fulfilled: (i) $f(u)$ is a locally Lipschitz continuous on $[0,\infty)$ function and $f(0) = 0$; (ii) for the function (57), there exists $\beta > 0$ such that $F(s) < 0$ for $0 < s < \beta$, $F(\beta) = 0$ and $f(\beta) > 0$; (iii) there is a finite value $\gamma > \beta$ such that $\gamma = \min\{s > \beta : f(s) = 0\}$. Then, for each fixed $N > 1$, there exists $\alpha \in [\beta,\gamma)$ such that the solution to the singular nonlinear CP (54), (55) is continuable on $\mathbb{R}_+$ monotone decreasing function satisfying the conditions (56), i.e., there exists a nonnegative monotone solution to the singular nonlinear BVP (54)–(56).*

For $\xi > 0$, introducing the variable substitution $\rho = \xi - u$, we obtain, from (1), (25), (3), the singular nonlinear BVP (54)–(56) where $f(u) = 4\lambda^2 u(\xi - u)(u - \xi - 1)$, $\alpha = \xi - \rho_0 > 0$ (due to (26), $\xi < \alpha < 1 + \xi$).

Then we have: $f(0) = f(\xi) = f(\xi+1) = 0$; $f(u) < 0$ $\forall u \in (0,\xi)$; $f(u) > 0$ $\forall u \in (\xi, \xi+1)$; $f(u) \to -\infty$ as $u \to \infty$. Moreover, for the function (57), we obtain $F(s) = -\lambda^2 s^2(s - s_1)(s - s_2)$, where $s_{1,2}(\xi) = \xi - w_{2,1}(\xi)$ (for $w_{1,2}(\xi)$, see (14)). Then, if the restriction (17) is valid, we have: $F(0) = F(s_1) = F(s_2) = 0$; $F(s) < 0$ $\forall s \in (0, s_1)$; $F(s) > 0$ $\forall s \in (s_1, s_2)$; $F(s) \to -\infty$ as $s \to \infty$.

As a result we obtain that the hypotheses of the above Theorem are satisfied where $\beta = s_1(\xi) = \xi - w_2(\xi)$ $(\xi < \beta < 1 + \xi)$, $\gamma = \min\{s > \beta : f(s) = 0\} = \xi + 1$, and the condition $\alpha \in [\beta,\gamma)$ implies the same restrictions on $\rho_0(\xi)$ as (52).

Using Corollary 4, the properties of substitution (9) and the above Theorem, due to [26], we finally obtain

Proposition 7. For the singular nonlinear BVP (1), (8), (3) with $N > 1$, the restriction (17) (resp., (18)) is the sufficient condition for a bubble–type (resp., a droplet–type) solution to exist, and if $\rho(r,\xi)$ is the above solution then the estimate (52) (resp., (53)) is valid where $\rho_0(\xi) = \rho(0,\xi)$.

5. Conclusions

We have described two different nonlinear physics models leading to the singular nonlinear BVP (1)–(3) and have presented very briefly some results concerning the mathematical analysis of this BVP. In particular, we have obtained the theoretical background for its numerical solution by efficient shooting methods (see [1], [5]). From the mathematical point of view, the problem under consideration is interesting by itself as an example of a singular nonlinear BVP which can be investigated in detail, both analytically and numerically.

A more complete analytical–numerical investigation of the problem will be presented in [5]. The comparative analysis for the computed physical magnitudes in [1] and [5] confirms their qualitative accordance with the expected results for the bubble and droplet models in hydromechanics and nonlinear field theory.

As a future work, it would be interesting to consider the modifications of Eq.(1) for the cases of Van der Waals and other types of fluids; the corresponding dynamical problems are more difficult, especially when nonlinear vector systems of PDEs are considered in hydromechanics (in the case of the scalar PDE (16), the dynamics of the bubbles have been investigated numerically in [12], [13], for $N = 3$ and $\xi = 1$).

Acknowledgements

N.B.Konyukhova and M.B.Soloviev acknowledge support from RFBR, projects No.05–01–00257 and No.08–01–00139; P.Lima and L.Morgado acknowledge support from FCT, project POCTI/MAT/45700/2002; L.Morgado also acknowledges support from FCT, through SFRH/BD/31513/2006.

The authors would like to thank Dr. N.V.Chemetov, calling our attention on the hydromechanics model, for his aid in the description of this problem and useful information.

References

[1] Lima, P.M., Konyukhova, N.B., Sukov, A.I. and Chemetov, N.V. *Analytical–Numerical Investigation of Bubble–Type Solutions of Nonlinear Singular Problems*// J. Comput. Appl. Math. (Elsevier Science). – 2006. – V.189. – P.260–273.

[2] Lima, P.M., Konyukhova, N.B., Sukov, A.I. and Chemetov, N.V. *Mathematical Analysis and Numerical Solution of a Singular Problem in Nonlinear Physics*// Vestnik Nizhegorodskogo Universiteta im. N.I.Lobachevskogo (ser. Mathematical Modelling and Optimal Control). – 2005. –V.1. – P.162–170.

[3] Lima, P., Chemetov, N., Konyukhova, N.B. and Sukov, A.I. *Analytical–Numerical Approach to a Singular Boundary Value Problem*// Preprint 20/2003. Lisboa: Departamento de Matematica, Instituto Superior Técnico, October 2003. – 13p.

[4] Lima, P., Chemetov, N., Konyukhova, N.B. and Sukov, A.I. *Analytical–Numerical Approach to a Singular Boundary Value Problem*// Proc. of XXIV Iberian Latin–American Congress on Comput. Methods in Engineering (CILAMCE XXIV; Ouro Preto, Brazil, October 29–31, 2003). Ouro Preto/MG – Brazil, 2003. – 13 p. (CD-ROM).

[5] Konyukhova, N.B., Lima, P.M., Morgado, M.L. and Soloviev, M.B. *Bubbles and Droplets in Nonlinear Physics Models: Analysis and Numerical Simulation of Singular Nonlinear Boundary Value Problem*//Zh. Vychisl. Matem. i Matem Fiz. (in preparation).

[6] Dell'Isola, F., Gouin, H. and Seppecher P. *Radius and Surface Tension of Microscopic Bubbles by Second Gradient Theory*// C.R.Acad. Sci. Paris. –1995. – V.320(Serie IIb). – P.211–216.

[7] Dell'Isola, F., Gouin, H. and Rotoli, G. *Nucleation of Spherical Shell–Like Interfaces by Second Gradient Theory: Numerical Simulations*//Eur. J. Mech. B/Fluids. – 1996. – V.15. – P.545–568.

[8] Gavrilyuk, S.L. and Shugrin, S.M. *Media with Equations of State that Depend on Derivatives*// Prikl. Mekhan. i Tekhn. Fiz. – 1996. – V.37. – P.35–49 [J. Appl. Mech. and Tech. Phys.– 1996. – V.37. – P.177–189].

[9] Gouin, H. and Rotoli, G. *An Analytical Approximation of Density Profile and Surface Tension of Microscopic Bubbles for Van der Waals Fluids*// Mechanics Research Communications. – 1997.– V.24. – P.255–260.

[10] Rubakov V.A. *Classical Gauge Fields*//Editorial URSS, Moscow, 1999 [in Russian].

[11] Linde, A.P. *Particle Physics and Inflationary Cosmology*// Harwood Academic, Chur, Switzerland, 1990; Nauka, Moscow, 1991 [in Russian].

[12] Voronov, N.A., Kobzarev, I.Yu. and Konyukhova, N.B. *On Possibility of the Existence of New Type Mesons*// Pis'ma v ZhETF. – 1975. – V.22. – P.590–594 [JETP Lett. – 1975. – V.22.– P.290–294].

[13] Belova, T.I., Voronov, N.A., Kobzarev, I.Yu. and Konyukhova, N.B. *Particle–Like Solutions of the Scalar Higgs Equation*// Zh. Exper. Teor. Fiz. – 1977. – V.73. – P.1611–1622 [Sov.Phys. JETP. – 1977. – V.46. – P.846–852].

[14] Wasov, W. *Asymptotic Expansions for Ordinary Differential Equations*// Wiley, New York, 1965.

[15] Konyukhova, N.B. *Singular Cauchy Problems for Systems of Ordinary Differential Equations*// Zh. Vychisl. Mat. Mat. Fiz. – 1983. – V.23. – P.629–645 [U.S.S.R. Comput. Math. Math. Phys. – 1983. – V.23. – P.72–82].

[16] Lyapunov, A.M. *The General Problem of Motion Stability*// GITTL, Moscow, 1950 [in Russian].

[17] Konyukhova, N.B. *On the Behavior Inside and Outside of Stable Manifold for Certain Two–Dimensional Nonlinear Systems of Ordinary Differential Equations*// Matem. Zametki. – 1970. – V.8. – P.285–295 [Math. Notes. – 1970. – V.8. – P.632–638].

[18] Abramov, A.A., Konyukhova, N.B and Balla, K. *Stable Initial Manifolds and Singular Boundary Value Problems for Systems of Ordinary Differential Equations*// Comput. Math. Banach Center Publs. – 1984. – V.13. – P.319–351 (PWN – Polish Scient. Publishers, Warsaw, 1984) [in Russian].

[19] Coddington, E.A. and Levinson, N. *Theory of Ordinary Differential Equations*// McGraw-Hill Book Co., Inc., New York - Toronto - London, 1955.

[20] Konyukhova, N.B. *On Numerical Isolation of the Solutions Tending to Zero at Infinity of Certain Two–Dimensional Non–Linear Sets of Ordinary Differential Equations*// Zh. Vychisl. Mat. Mat. Fiz. – 1970. – V.10. – P.74–87 [U.S.S.R. Comput. Math. Math. Phys.– 1970. – V.10. – P.95–111].

[21] Konyukhova, N.B. *On the Limiting Behavior of a Bounded Solution of a System of Quasilinear of First–Order Partial Differential Evolution Equations as Time Grows to Infinity*// Differents. uravn. – 1992. – V.28. – P.1561–1573 [Diff. Equ. – 1992. – V.28. – P.1283–1293].

[22] Konyukhova, N.B. *Stationary Lyapunov Problem for a System of First–Order Quasilinear Partial Differential Equations*// Differents. uravn. – 1994. – V.30. – P.1384–1395 [Diff. Equ. – 1994. – V.30. – P.1284–1294].

[23] Konyukhova, N.B. *Stable Lyapunov Manifolds for Autonomous Systems of Non–Linear Ordinary Differential Equations*// Zh. Vychisl. Mat. Mat. Fiz.– 1994. – V.34. – P.1358–1379 [Comput. Maths Math. Phys. – 1994. – V.34. – P.1179–1195].

[24] Derrick, G.H. *Comments on Nonlinear Wave Equations as Models for Elementary Particles*// J. Math. Phys. – 1964. – V.5. – P.1252–1254.

[25] Berestycki, H. and Lions, P.-L. *Nonlinear Scalar Field Equations, I, Existence of a Ground State*// Arch. Rat. Mech. Anal. – 1983. – V.82. – P.313–345; *II, Existence of Infinitely Many Solutions*// Arch. Rat. Mech. Anal. – 1983. – V.82. – P.347–375.

[26] Gazzola,F., Serrin, J. and Tang, M. *Existence of Ground States and Free Boundary Problems for Quasilinear Elliptic Operators*// Advances in Diff. Equ. – 2000. – V.5.– P.1–30.