

# Numerical modelling of a functional differential equation with deviating arguments using a collocation method

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**Abstract.** This paper is concerned with the approximate solution of a functional differential equation of the form:

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-1) + \gamma(t)x(t+1). \quad (1)$$

We search for a solution  $x$ , defined for  $t \in [-1, k], (k \in \mathbf{N})$ , which takes given values on the intervals  $[-1, 0]$  and  $(k-1, k]$ . Continuing the work started in [10], we introduce and analyse some new computational methods for the solution of this problem which are applicable both in the case of constant and variable coefficients. Numerical results are presented and compared with the results obtained by other methods.

**Keywords:** Mixed-type functional differential equation, method of steps, collocation method, splines

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## INTRODUCTION

The present paper is devoted to the solution of functional differential equations with both delayed and advanced arguments. Such equations are often referred in the literature as mixed type functional differential equation (MTFDE) or forward-backward equations. The analysis of this type of equation has begun comparatively recently and is less developed compared with other classes of functional equation. Many important questions remain open. Interest in MTFDEs is motivated by problems in optimal control (see [8]) and applications also arise in nerve conduction [2], economic dynamics [9] and travelling waves in a spacial lattice [1]. Some problems about the decomposition of solutions of MTFDEs were investigated in the works of Mallet-Paret and Verduyn-Lunel [5], [6].

In [3], the authors consider a particular case of equation (1) (the autonomous case), when  $\alpha, \beta, \gamma$  are constants,  $\alpha = 0$ . An even more particular case was considered in [4], with  $\alpha = 0, \beta = 1, \gamma = 1$ . Our approach here is based on some of the ideas used in those works, in particular the *method of steps*, which we shall now recall and apply to our case.

Our goal is to compute a particular solution of equation (1) which satisfies

$$x(t) = \begin{cases} \varphi_1(t), & \text{if } t \in [-1, 0], \\ f(t), & \text{if } t \in (k-1, k], \end{cases} \quad (2)$$

where  $\varphi_1$  and  $f$  are smooth real-valued functions, defined on  $[-1, 0]$  and  $(k-1, k]$ , respectively ( $1 < k \in \mathbf{N}$ ). In order to analyse and solve this boundary value problem (BVP) we consider an initial value problem (IVP), with the conditions:

$$x(t) = \varphi(t), \quad t \in [-1, 1], \quad (3)$$

where the function  $\varphi$  is defined by

$$\varphi(t) = \begin{cases} \varphi_1(t), & \text{if } t \in [-1, 0], \\ \varphi_2(t), & \text{if } t \in (0, 1], \end{cases} \quad (4)$$

This reformulation provides the basis for both analytical and numerical construction of solutions using ideas based on Bellman's method of steps for solving delay differential equations. One solves the equation over successive intervals of length unity. Assuming that  $\gamma(t) \neq 0, \forall t \geq 0$ , equation (1) can be rewritten in the form

$$x(t+1) = a(t)x'(t) + b(t)x(t-1) + c(t)x(t), \quad (5)$$

where  $a(t) = \frac{1}{\gamma(t)}$ ,  $b(t) = -\frac{\beta(t)}{\gamma(t)}$ ,  $c(t) = -\frac{\alpha(t)}{\gamma(t)}$ . Using formula (5), we can construct a smooth solution of equation (1) on any interval  $[1, k]$ , starting from its definition on  $[-1, 1]$  by formula (4). In particular, provided that  $a, b, c \in C^1([0, 3])$ , we obtain the following expressions for the solution in the first two intervals:

$$\begin{aligned} x(t) &= a(t-1)\varphi_2'(t-1) + b(t-1)\varphi_1(t-2) + c(t-1)\varphi_2(t-1), \quad t \in (1, 2]; \\ x(t) &= a(t-1)a(t-2)\varphi_2''(t-2) + [a(t-1)(a'(t-2) + c(t-2)) + c(t-1)a(t-2)]\varphi_2'(t-2) + \\ &+ [c'(t-2)a(t-1) + c(t-1)c(t-2) + b(t-1)]\varphi_2(t-2) + [a(t-1)b(t-2)]\varphi_1'(t-3) + \\ &+ [a(t-1)b'(t-2) + c(t-1)b(t-2)]\varphi_1(t-3), \quad t \in (2, 3]; \end{aligned} \quad (6)$$

Remark that these formulas reduce to the corresponding formulas of Table 1 in [3], if we set  $c(t) \equiv 0$ ,  $a(t) \equiv a$ ,  $b(t) \equiv b$ .

Continuing this process, we can extend the solution to any interval, provided that the initial function  $\varphi$  and the variable coefficients are smooth enough functions. We shall now formulate this result in more precise terms.

**Theorem 1.** Let  $x$  be the solution of problem (1),(4), where  $\alpha, \beta, \gamma \in C^{2L}([-1, 2L+1])$ ,  $\gamma(t) \neq 0$ ,  $t \in [-1, 2L+1]$ ,  $\varphi_1(t) \in C^{2L}([-1, 0])$ ,  $\varphi_2(t) \in C^{2L}([0, 1])$  (for some  $L \in \mathbf{N}$ ). Then there exist functions  $\delta_{i,l}$ ,  $\varepsilon_{i,l}$ ,  $\bar{\delta}_{i,l}$ ,  $\bar{\varepsilon}_{i,l} \in C([-1, 2L+1])$ ,  $l = 1, \dots, L$ ,  $i = 0, 1, \dots, 2l$ , such that the following formulas are valid:

$$\begin{aligned} x(t) &= \sum_{i=0}^{2l-1} \delta_{i,l}(t)\varphi_1^{(i)}(t-2l) + \sum_{i=0}^{2l-1} \varepsilon_{i,l}(t)\varphi_2^{(i)}(t-2l+1), \quad t \in [2l-1, 2l]; \\ x(t) &= \sum_{i=0}^{2l} \bar{\varepsilon}_{i,l}(t)\varphi_2^{(i)}(t-2l) + \sum_{i=0}^{2l-1} \bar{\delta}_{i,l}(t)\varphi_1^{(i)}(t-2l-1), \quad t \in [2l, 2l+1]; \quad l = 1, \dots, L. \end{aligned} \quad (7)$$

This theorem can be proved by induction. Moreover, closed formulas for the coefficients  $\delta_{i,l}$ ,  $\bar{\delta}_{i,l}$ ,  $\varepsilon_{i,l}$ ,  $\bar{\varepsilon}_{i,l}$  can be derived (as done in [3] for the autonomous case). However, due to restrictions in space, we omit these details in the present paper.

As a corollary of Theorem 1, we can obtain the following existence result for the IVP (5), (3).

**Theorem 2.** The solution to the problem (5), (3) with  $\varphi(t) \in C^\infty([-1, 1])$ ,  $\alpha(t) \in C^\infty([-1, 1])$ ,  $\beta(t) \in C^\infty([-1, 1])$  and  $\gamma(t) \in C^\infty([-1, 1])$  exists and is differentiable if and only if  $\varphi^{(n+1)}(0) = \lim_{t \rightarrow 0} D^n(\alpha(t)\varphi(t) + \beta(t)\varphi(t-1) + \gamma(t)\varphi(t+1))$ , for  $n = 0, 1, 2, \dots$  (where  $D^n$  denotes the  $n$ -th derivative with respect to  $t$ ).

The relationship between the IVP (5),(3) and the BVP (1),(2) is complex. While it is straightforward to determine the BVP corresponding to a given IVP, the inverse problem is both ill-posed and highly unstable. Indeed, it may not be possible to solve a given BVP using this method (or, at all). One must study the associated numerical methods with an awareness of the dangers inherent in this observation.

In this paper, we continue the study started in [3] and [10] Our goal is to develop new numerical approaches to the solution of the problem (1),(2) and to provide a comparative analysis of the numerical results.

## NUMERICAL METHODS

We now give the outline of some numerical algorithms (for details see [10]).

**Algorithm based on collocation method.** We search for an approximate solution of (1),(2) in the form

$$x^{(N)}(t) = x_0(t) + \sum_{j=1}^{N-k} C_j x_j(t), \quad t \in [-1, k] \quad (8)$$

where  $x_0$  is an initial approximation of the solution;  $\{x_j\}_{1 \leq j \leq N-k}$  is a set of basis functions.

*First Step.* From formulae (3)-(4) it follows that if a solution, constructed by the method of steps, belongs to  $C^n((l-1, l])$  (for a certain  $l \geq 1, n \geq 1$ ), then it also belongs to  $C^{n-1}((l, l+1])$ . Therefore, since we want  $x^{(N)}$  to be at least continuous on  $[-1, k]$  (for a certain  $k \geq 2$ ), we require that  $x_0$  belongs to  $C^k((-1, l])$ . Having this in mind, we define  $x_0$  on  $[-1, 1]$  in the following way:

$$x_0(t) = \begin{cases} \varphi_1(t), & t \in [-1, 0]; \\ P_{2k}(t) = a_0 + a_1 t + \dots + a_{2k} t^{2k}, & t \in [0, 1]. \end{cases} \quad (9)$$

where  $a_0, a_1, \dots, a_{2k}$  are computed in order to satisfy smoothness conditions (according to Theorem 2).

*Second Step.* With the purpose of constructing a set of basis functions on  $[-1, k]$ , we define a grid of stepsize  $h$  on this interval. Let  $h = 1/N$  (where  $N \in \mathbb{N}$ ) and  $t_i = ih$ ,  $i = -N, -N+1, \dots, kN$ . We first define each basis function  $x_j, j = 1, \dots, N-k$ , on the interval  $[-1, 1]$ , requiring that it satisfies the following conditions: 1)  $x_j \in C^{k-1}[-1, 1]$ , 2)  $x_j(t)$  is different from zero only in  $(t_{j-k+1}, t_{j+k-1})$ , 3)  $x_j(t)$  is a polynomial of degree  $k$  on each interval  $[t_i, t_{i+1}]$ ,  $i = 1, 0, \dots, N+1$ . In the case  $k = 3$ , for example, we obtain the well-known cubic B-splines.

The basis functions are then extended to the interval  $[-1, k]$  using recurrence formulae, analogous to (7).

Note that all the basis functions  $x_j, j = 1, \dots, N-k$  are identically 0 on  $[-1, 0]$ . This assures that the approximate solution  $x^{(N)}$ , defined by (8), satisfies  $x^{(N)}(t) = x_0(t) = \varphi_1(t)$ ,  $\forall t \in [-1, 0]$ , so that the first of boundary conditions (2) is satisfied by  $x^{(N)}$ , for any choice of the  $C_j$  coefficients.

*Third Step.* Finally, we compute the coefficients  $C_j, j = 1, \dots, N-k$  of the expansion (8) from the condition that  $x^{(N)}$  approximates  $f$  on the interval  $(k-1, k]$ . The collocation method is used with this purpose.

The coefficients are obtained from the condition

$$x^{(N)}(t_{(k-1)N+i}) = x_0(t_{(k-1)N+i}) + \sum_{j=1}^{N-k} C_j x_j(t_{(k-1)N+i}) = f(t_i); \quad i = 1, \dots, N-k. \quad (10)$$

( $x^{(N)}$  coincides with  $f$  at the grid points on  $[k-1, k]$ ).

Equations (10) form a linear system with a  $(N-k) \times (N-k)$  band matrix. This system can be solved by standard methods.

**Reducing to an ODE.** We search for a solution of (1), (2) in the form

$$x(t) = x_0(t) + \bar{x}(t), \quad t \in [-1, k] \quad (11)$$

where  $x_0$  is defined by (9).

For the sake of simplicity, in this section we shall restrict ourselves to the autonomous case (though this approach can be applied to the non-autonomous one as well).

First of all note that  $\bar{x}(t) \equiv 0, \forall t \in [-1, 0]$  (otherwise,  $x$  does not satisfy the first boundary condition). Therefore, if we define  $\bar{x}(t)$  on  $[0, 1]$ , we can extend it to the whole interval  $[0, k]$  using the method of steps. Let us denote as  $u(t)$  the restriction of  $\bar{x}(t)$  to the interval  $[0, 1]$ :

$$u(t) \equiv \bar{x}(t), \forall t \in [0, 1]. \quad (12)$$

We will obtain  $\bar{x}$  as the solution of a BVP for an ordinary differential equation of  $(k-1)$ -th order.

From the first of formulas (7), taking into account that  $\bar{x}(t) \equiv 0$ , for  $t \in [-1, 0]$ , we obtain

$$\bar{x}(t) = au'(t-1) + cu(t-1), \quad t \in [1, 2], \quad (13)$$

where  $a$  and  $b$  are defined as in (5). In the same way,  $\bar{x}(t)$  can be extended to any unit interval using formulas (7).

In this way, we obtain

$$\bar{x}(t) = L^{k-1}u(t-k+1) := \sum_{i=0}^{k-1} \rho_{i,k} u^{(i)}(t-k+1), \quad t \in [k-1, k], \quad (14)$$

where  $\rho_{i,k} = \varepsilon_{i,k}$ , if  $k$  is even, and  $\rho_{i,k} = \bar{\varepsilon}_{i,k}$ , if  $k$  is odd. Here,  $L^{k-1}$  denotes a linear differential operator of order  $k-1$ . Since  $L^{k-1}$  is linear, from (11) we conclude that  $x$  satisfies

$$x(t) = L^{k-1}(u(t-k+1) + x_0(t-k+1)) = L^{k-1}u(t-k+1) + L^{k-1}x_0(t-k+1), \quad (15)$$

$t \in [k-1, k]$ . Now, the second boundary condition in (2) implies that  $u$  must satisfy

$$L^{k-1}u(t-k+1) + L^{k-1}x_0(t-k+1) = f(t), \quad t \in [k-1, k], \quad (16)$$

or equivalently

$$L^{k-1}u(t) = f(t+k-1) - L^{k-1}x_0(t), \quad t \in [0, 1]. \quad (17)$$

Moreover, since  $u(t) = x(t) - x_0(t)$ ,  $\forall t \in [0, 1]$ , and due to the smoothness conditions satisfied by  $x(0)$ , we conclude that  $u \in C^k([0, 1]) \cap C^{k-1}(1)$  and the following boundary conditions must be satisfied:  $u(0) = u'(0) = \dots = u^{(k)}(0) = 0$ ;  $u(1) = u'(1) = \dots = u^{(k-1)}(1) = 0$ .

For example, in the case  $k = 3$ , (17) is a second order ODE and we consider two boundary conditions:  $u(0) = 1, u(1) = 0$ . The obtained BVP can then be solved by standard numerical methods, for example, the collocation method with a basis of cubic B-splines.

**TABLE 1.** Autonomous case. Error  $\varepsilon$  on  $[0, 1]$  and estimated convergence order  $p$  obtained by collocation and  $\theta$ - method for  $k = 4$  and  $m = -0.5, 3$  ( $\varepsilon = h\|x - x^{(N)}\|_2$ ). The results for the  $\theta$ -method are obtained from [3].

Collocation method					$\theta$ - Method				
Step	$m = -0.5$		$m = 3$		Step	$m = -0.5$		$m = 3$	
$h_i$	$\varepsilon$	$p$	$\varepsilon$	$p$	$h_i$	$\varepsilon$	$p$	$\varepsilon$	$p$
$\frac{1}{16}$	$1.45e-8$	0.5	$2.27e-3$	0.5	$\frac{1}{16}$	$1.03e-5$	2.04	$2.36e-2$	2.12
$\frac{1}{32}$	$1.04e-8$	0.5	$1.54e-3$	0.5	$\frac{1}{32}$	$2.53e-6$	2.02	$5.66e-3$	2.06
$\frac{1}{64}$	$7.33e-9$	0.5	$1.09e-3$	0.5	$\frac{1}{64}$	$6.36e-7$	2.00	$1.38e-3$	2.03

## ERROR ANALYSIS

### ODE Approach

When the original problem (1) is reduced to a BVP by the method described in the previous section, we have to compute an approximate solution of equation (17) which satisfies certain boundary conditions on  $[0, 1]$ . This can be obtained, for example, by the collocation method, using a basis of splines of appropriate order. We shall restrict our error analysis to the case  $k = 3$ . In this case, we have to solve a BVP for a second order ODE and, if  $f$  is at least 4 times continuously differentiable, we have the following estimate for the error norm (see, for example, [7]):

$$\|u^{(N)}(t) - u(t)\|_{\infty} \leq B_2 h^2, \quad (18)$$

where  $u^{(N)}$  is the obtained approximate solution with  $N$  basis functions and  $B_2$  is a constant that does not depend on  $h$ . This result applies to the interval  $[0, 1]$ , where  $u$  is defined. On the interval  $[1, 2]$  this estimate is not valid and the convergence order may in principle be lower. However, as we will see in section 4, the numerical experiments indicate that second order convergence can be attained in all the domain.

### Method of Steps Approach

Let us now turn our attention to the method of steps approach. In this case the error analysis is much more complicated. Focusing our attention on the case  $k = 3$ , we remark that when we solve the linear system (10), we are approximating the solution  $\bar{x}$  on the interval  $[2, 3]$  by a certain function  $x^{(N)}$  which does not have the same properties as the cubic splines, defined on  $[0, 1]$ . Actually, taking into account the formula used to extend the basis functions,  $x^{(N)}$  is piecewise polynomial of third degree and it does not belong to  $C^2$  (it is just a continuous function). As far as we know, there are no available results on the convergence of collocation methods with basis functions of this kind. So the error analysis of the proposed method is still an open question that we intend to investigate in the future. The obtained numerical results, presented above, suggest that the method converges, though its convergence order is lower than in the case of the ODE approach.

## NUMERICAL RESULTS AND COMPARISON OF METHODS

In order to analyse the performance of the described numerical methods, we have considered the following MTFDE:

$$x'(t) = (m - 0.5e^{-m} - 0.5e^m)x(t) + 0.5x(t-1) + 0.5x(t+1), \quad (19)$$

with the boundary conditions  $\varphi_1(t) = e^{mt}$ ,  $t \in [-1, 0]$ ;  $f(t) = e^{mt}$ ,  $t \in (k-1, k]$ , with  $m \in \mathbb{R}$ ,  $m \neq 0$ . The exact solution is  $x(t) = e^{mt}$ . This example was also considered in [3], where the  $\theta$ - method is proposed. Table 1 presents some numerical results for the autonomous case.

**TABLE 2.** Non-autonomous case. Error  $\varepsilon$  on  $[0, k-1]$  and estimated convergence order  $p$  obtained by the ODE approach for  $k = 2, 3$  and  $m = -0.5, 3$  ( $\varepsilon = \frac{h}{k-1} \|x - x^{(N)}\|_2$ ).

ODE approach (k=2)					ODE approach (k=3)				
Step	$m = -0.5$		$m = 3$		Step	$m = -0.5$		$m = 3$	
$h_i$	$\varepsilon$	$p$	$\varepsilon$	$p$	$h_i$	$\varepsilon$	$p$	$\varepsilon$	$p$
$\frac{1}{16}$	$2.57e-8$	2.61	$2.17e-3$	2.41	$\frac{1}{16}$	$9.07e-11$	2.64	$4.34e-3$	2.45
$\frac{1}{32}$	$4.35e-9$	2.57	$3.91e-4$	2.47	$\frac{1}{32}$	$1.51e-11$	2.58	$7.72e-4$	2.49
$\frac{1}{64}$	$7.50e-10$	2.54	$6.98e-5$	2.49	$\frac{1}{64}$	$2.60e-12$	2.54	$1.37e-4$	2.50

For the non-autonomous case, we have considered the following mixed type equation:

$$x'(t) = m x(t) - e^{-m(t-2)}x(t-1) + e^{m(t-1)}x(t+1), \quad (20)$$

where  $\varphi_1(t) = e^{mt}$ ,  $t \in [-1, 0]$ ;  $f(t) = e^{mt}$ ,  $t \in [k-1, k]$ . The exact solution is  $x(t) = e^{mt}$ . This example is also considered with some different values of  $m$ . The numerical results are displayed in table 2.

## CONCLUSIONS AND FUTURE WORK

The proposed computational methods have produced accurate numerical results when applied to the solution of equations (19), (20), for the considered values of  $m$ . The algorithm based on the collocation method has convergence order  $p = 1.5$  when  $k = 4$ . The  $\theta$ -method has greater absolute error (in the 2-norm), when applied with stepsize greater than or equal to  $h = 1/64$ . However, it has a higher order of convergence ( $p = 2$ ). Finally, the method of ODE approach, based on the reduction of the MTFDE to an ODE, seems to be the most efficient, providing second order convergence and giving results with absolute error, less than  $10^{-7}$ , for  $h = 1/64$ , when  $k = 3$  and  $m = 1$ .

In the future, we intend to carry out a more detailed numerical analysis of the presented methods. We also intend to use the ODE approach to investigate the existence and uniqueness of solution of the considered problem.

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