Numerical methods for singular boundary value problems involving the p-laplacian

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Abstract. In this work, we are concerned about a singular boundary value problem for a nonlinear ordinary differential equation involving the N-dimensional p-laplacian. This equation may be considered as a generalized Emden-Fowler equation and arises in models of fluid mechanics, elasticity theory and other fields of physics. The main feature of the considered boundary value problem is that it has two singularities at the endpoints of the considered interval.We analyze the asymptotic behavior of the solutions near these singularities and propose computational methods that take this behavior into consideration. Numerical examples are presented and dicussed.

Keywords: singular boundary value problem, p-laplacian, upper and lower solutions, one-parameter families of solutions, asymptotic expansions
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1. INTRODUCTION

The mathematical modelling of various phenomena of mechanics and physics leads to the following Dirichlet problem for a quasi-linear elliptic equation:

$$\begin{array}{ll} -\mathrm{div}\left(|\nabla g|^{p-2}\nabla g\right) = f(|x|,g), & x \in B; \\ g(x) > 0, & x \in B; \\ g(x) = 0, & x \in \partial B, \end{array} \tag{1.1}$$

where B is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 1$, $p > 1$.

The differential operator on the left-hand side of (1.1) is the N-dimensional p-laplacian, $\triangle_p g$, which reduces to the classical laplacian when $p = 2$ and, for $p \neq 2$, is used in nonlinear models of physical phenomena, as for example problems arising in non-newtonian fluid mechanics: the case $1 < p < 2$ corresponds to pseudoplastic fluids and the case $p > 2$ to dilatant fluids [1].

Similar situations, when the generalization of classical linear problems leads to nonlinear ones, involving the p-laplacian, also occur in elasticity theory, when modelling the deformation of nonlinear membranes [2]. Eigenvalue problems for equations of the type (1.1) are also subject of study (see, for example, [2], [3]). Note that equation (1.1) not only contains a singularity due to the p-laplacian on the left-hand side, but may also have singularities due to the form of the right-hand side, like in case studied in [4]. A three-point problem for a related integro-differential equation was studied in [5]. One is

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often interested in radial solutions of problem (1.1), that is, solutions that depend only on $|x|$. In that case, problem (1.1) reduces to a boundary value problem for an ordinary differential equation:

$$\begin{gathered}-r^{1-N}\left(r^{N-1}|g'|^{p-2}g'\right)' = f(r,g), \qquad r \in (0,1);\\ g(r) > 0, \qquad r \in (0,1);\\ g'(0) = 0, g(1) = 0,\end{gathered} \tag{1.2}$$

where $N \geq 1$ is the space dimension. We shall assume that $f : (0,1) \times (0,\infty) \rightarrow \mathbb{R}$ is continuous in the considered domain and has singularities at $y = 0$ or $r = 0$.

The existence and the properties of solutions to problem (1.2), under different assumptions on the function f, were investigated by many authors. The existence and uniqueness of solution in the case $p = 2, N = 1$, with $f(r,g) = -\frac{1}{n} r g^n, n < 0$ was proved by Nachman and Callegari [6], in connection with a boundary layer problem in fluid mechanics. In [7], [8],[9] and [1] the authors also considered the case $p = 2$, but with $f(r,g) = ar^{\sigma}g^n$, $\sigma > -1$, $a > 0$. In these works the asymptotic behavior of the solutions near the singularity at $r = 1$ has been analyzed and numerical methods were introduced, which take into account this behavior. Moreover, upper and lower solutions were determined and, in some particular cases, an explicit formula for the exact solution was obtained.

The case $p \neq 2$ has been studied by F. Wong. In [1], this author considered the BVP (1.2) with $N = 1, p \geq 2$ and obtained sufficient conditions for the existence of at least one positive solution. In [1], the same author has studied a more general equation, where the right-hand side depends also on g', and derived conditions for the uniqueness of positive solution to the corresponding BVP. Uniqueness results were also obtained by Y. Naito in [1], who studied the case when the right-hand side of equation has the form $f(r,g) = p(r)q(g)$, $N = 1$ and $p > 1$.

In a recent work, Jin, Yin and Wang [1], using the method of lower and upper solutions, have investigated the existence of solutions to problem (1.2) in the case $N \geq 2$, $p > 1$.

In [1], we have investigated the problem (1.2) in the case $N = 1$, for $p > 1$ and $f(r,g) = ar^{\sigma}g^n$, where $a > 0$. The values of σ and n for which the existence of solution is guaranteed depend on p. According to the Existence Theorem of [1], a sufficient condition for the existence of solution is $\sigma \geq 0$, $n - p + 1 < 0$, $p \geq 2$. On the other hand, uniqueness of solution in this case is guaranteed by Theorem 2.3 of [1]. The numerical results obtained in [1] suggest the existence of solution in a more general case: $\sigma > -1$, $n < -1$, $p > 1$.

In the present paper, we will continue the work started in [1] and extend it to the case $N \geq 1$. As remarked by Naito in [1], the case $N > 1$ can be reduced to the case $N = 1$ by means of the variable substitution $t = r^{\frac{p-N}{p-1}}$, if $(p \neq N)$, and $t = logr$, if $p = N$. In the case $p > N$, we obtain a new BVP on the interval $[0,1]$, which can be studied and solved numerically by the methods introduced in [1]. However, in the case $p < N$, this leads to a new BVP in an unbounded domain, so that in can be more convenient to study the problem in the original variables, what we will do here.

As it happens in the case $N = 1$, when $N \geq 1$ this problem has a singularity at $r = 1$. Actually, since $n < 0$, f is unbounded when $g = 0$. Therefore, the second derivative of

g has a discontinuity at this point. Moreover, the first derivative has also a discontinuity at $r = 1$, if $n \leq -1$. On the other hand, the problem has a singularity at $r = 0$, if $\sigma < 0$, $p > 2$ or $N > 1$. This will be discussed in the next sections.

Singular problems involving the p-laplacian but with a different type of boundary conditions were analysed in [1]. Here we will follow a similar approach to the treatment of singularities.

The outline of this paper is as follows. In Section 2 we will analyze the asymptotic behavior of the one-parameter family of solutions which satisfy the condition $\lim_{r\to 0^+} g'(r) = 0$. In Section 3, we will investigate the asymptotic behavior of the solutions of equation (1.2) that satisfy $\lim_{r\to r_0^-} g(r) = 0$ (with $r_0 > 0$). In Section 4, we will obtain lower and upper solutions for the considered problem. We will also show that in certain cases the exact solution can be found in explicit form. In Section 5 we will describe the shooting method have we used to approximate the solution and present some numerical solutions obtained by this method. Finally, in Section 6, we discuss the numerical results and write the main conclusions of the work.

2. BEHAVIOR OF THE SOLUTION IN THE NEIGHBORHOOD OF $r = 0$

Consider the Cauchy problem:

$$r^{1-N}(r^{N-1}|g'(r)|^{p-2}g'(r))' = ar^{\sigma}g^{n}(r), \quad 0 < r < r_0, \tag{2.1}$$

$$\lim_{r\to 0+} g'(0) = 0. \tag{2.2}$$

Assume that in (2.1)

$$p > 1, \sigma > -1, n < 0, N \geq 1, a < 0. \tag{2.3}$$

As pointed out in the introduction, equation (2.1) has a singularity at $r = 0$ when $\sigma < 0$, $p > 2$ or $N > 1$. In this section we will analyze the asymptotic behavior of the solutions of the problem (2.1),(2.2) when $r \to 0+$. Our main result is resumed in the following theorem.

Theorem 2.1. *Under the restrictions (2.3), the Cauchy problem (2.1)-(2.2) has, in the neighborhood of $r = 0$, a one parameter family of solutions that can be represented by:*

$$g_1(r,b) = C_1 - br^k\left[1 - \frac{bn(N+\sigma)}{2C_1(p(1+N+\sigma)-N)}r^k + o\left(r^k\right)\right], \tag{2.4}$$

where $k = \frac{p+\sigma}{p-1}$, $C_1 = \left(\frac{-a(p-1)^{p-1}}{b^{p-1}(p+\sigma)^{p-1}(\sigma+N)}\right)^{-\frac{1}{n}}$ and $b > 0$ is the parameter.

Proof. In the neighborhood of $r = 0$, let us look for a solution of (2.1)−(2.2) in the form

$$\begin{aligned} g(r) &= C_1 - C_2 r^k [1 + o(1)] \\ g'(r) &= -kC_2 r^{k-1}[1 + o(1)], \\ g''(r) &= -k(k-1)C_2 r^{k-2}[1 + o(1)], \quad r \to 0^+. \end{aligned} \tag{2.5}$$

Substituting in (2.1) we obtain

$$\lim_{r\to 0^+} k^{p-1}\left((k-1)(p-1)+N-1\right)C_2^{p-1}r^{k(p-1)-p-\sigma}C_1^{-n}\left[1-\frac{C_2}{C_1}r^k\right]^{-n}=a, \tag{2.6}$$

which implies $k=\frac{p+\sigma}{p-1}$, $k-1=\frac{\sigma+1}{p-1}$, and with $C_2=b>0$ we have $C_1=\left(\frac{-a(p-1)^{p-1}}{b^{p-1}(p+\sigma)^{p-1}(\sigma+N)}\right)^{-\frac{1}{n}}$.

Substituting $g(r)$ by $C_1-C_2r^k\left[1+y(r)\right]$ in (2.2), we obtain the Cauchy problem in y

$$\frac{p-1}{k}\left[1+y+\tfrac{r}{k}y'\right]^{p-2}\left[(k-1)k(1+y)+2kry'+r^2y''\right]+(N-1)\left(1+y+\tfrac{r}{k}y'\right)^{p-1}=$$
$$\left((k-1)(p-1)+N-1\right)\left[1-\tfrac{b}{C_1}r^k(1+y)\right]^n, \tag{2.7}$$

$$y(0)=\lim_{r\to 0^+} ry'(r)=0, \tag{2.8}$$

where $r=0$ is a regular singular point of (2.7).

For each $b\neq 0$, this problem has a particular solution that can be represented by

$$y_{par}(r,b)=\sum_{l=0,j=0,l+j\geq 1}^{+\infty} y_{l,j}(b)\,r^{l+j\frac{p+\sigma}{p-1}},$$

where $0<r\leq\delta(b),\delta(b)\geq 0$ and the coefficients $y_{l,j}$, depending on b, may be determined substituting y_{par} in (2.7), which gives in the case $l=0,\ j=1$

$$y_{0,1}=-\frac{bn(N+\sigma)}{2C_1(p(1+N+\sigma)-N)}.$$

Writing out the leading linear homogeneous terms of (2.7) for solutions that satisfy (2.8) we obtain the equation

$$r^2y''+\frac{p(3+\sigma)-2+(p-1)(N-1)}{p-1}ry'+\frac{(p+\sigma)(\sigma+N)}{p-1}y=0,\quad r\to 0^+ \tag{2.9}$$

whose characteristic exponents are $\lambda_1=-\frac{p+\sigma}{p-1}<0$ and $\lambda_2=-(\sigma+N)<0$. Hence, the linear homogeneous problem (2.9) with the initial conditions (2.8) does not have any nontrivial solution and, according the arguments of [1], for each $b\neq 0$ the problem (2.7),(2.8) has no other solution than $y_{par}(r,b)$. □

From Theorem 2.1 we can conclude that if g is a solution of problem (2.1)-(2.2), then g'' will have a discontinuity at $r=0$ whenever $k<2$, that is, when $\sigma<p-2$.

3. THE SINGULARITY AT $r = r_0$

In this section we will investigate the behavior of the solution in the neighborhood of the right endpoint, where the solution vanishes.We shall denote this point by r_0. Let us consider the singular Cauchy problem:

$$-r^{1-N}(r^{N-1}\left|g'(r)\right|^{p-2}g'(r))' = ar^{\sigma}g^{n}(r), \quad 0 < r < r_0, \tag{3.1}$$

$$g(r_0) = \lim_{r\to r_0^-}[(r_0-r)g'(r)] = 0. \tag{3.2}$$

As we shall see, the asymptotic behavior of the solutions when $r \to r_0^-$ may change significantly, depending on the value of n, as described in the following theorem.

Theorem 3.1. *Assume that in (3.1) $p > 1$, $\sigma > -1$, $n < -1$, $r_0 > 0$, $a < 0$, $N \geq 1$. The Cauchy problem (3.1)-(3.2) has, in the neighborhood of the singular point $r = r_0$, a one parameter family of solutions that can be represented by:*

$$g(r,c) = \begin{cases} g_2(r,c), & if \quad n < -1 \quad n \neq -2-\frac{1}{p-1}; \\ g_3(r,c), & if \quad n = -2-\frac{1}{p-1}; \end{cases} \tag{3.3}$$

as $r \to r_0^-$, where c is the parameter,

$$g_2(r,c) = \left(\frac{ar_0^{\sigma}(p-1-n)^p}{p^{p-1}(p-1)(1+n)}\right)^{\frac{1}{p-1-n}}(r_0-r)^{\frac{p}{p-1-n}}$$
$$\left[1-\frac{p(-1+p-n)(-1+N)-(p-1)p(1+n)\sigma}{2(-1+p)(-1+p-n)(-1-n+p(2+n)r_0}(r_0-r)+c(r_0-r)^{\frac{-p(1+n)}{p-1-n}}+O((r_0-r)^{1+\mu})\right], \tag{3.4}$$

$$\mu = \min\left\{1, -\frac{p(1+n)}{p-1-n}\right\};$$

$$g_3(r,c) = \left(\frac{ar_0^{\sigma}(p-1-n)^p}{p^{p-1}(1+n)(p-1)}\right)^{\frac{1}{p-1-n}}(r_0-r)^{\frac{p}{p-n-1}}[1+c(r_0-r)[1+b_1(r_0-r)+\cdots]+$$
$$(r_0-r)\ln\left(\frac{r_0-r}{r_0}\right)\left[c_0(r_0-r)+c_1(r_0-r)^2+\cdots\right]+(r_0-r)^2\ln^2\left(\frac{r_0-r}{r_0}\right)[d_0+\cdots]+\cdots\Big], \tag{3.5}$$

where the coefficients c_o, c_1, d_0 may be determined by formal substitution of (3.5) into (3.1).

Proof. This theorem is similar to the Proposition 2 of [1], where the case $N = 1$ is considered. Details of the proof can be found there. Here we shall only give its outline. It can be easily proved that in the neighborhood of $r = r_0$, a solution of problem (3.1)-(3.2) may be given by

$$g(r) = C(r_0-r)^k[1+o(1)], \quad r \to r_0^-, \tag{3.6}$$

where $k = \frac{p}{p-1-n} > 0$, $k(k-1) = \frac{p(1+n)}{(p-1-n)^2} < 0$ and $C = (\frac{ar_0^{\sigma}(p-1-n)^p}{p^{p-1}(p-1)(1+n)})^{\frac{1}{p-1-n}}$.

If in (3.1)-(3.2) we perform the variable substitution

$$g(r) = C(r_0 - r)^k [1 + y(r)], \tag{3.7}$$

we obtain a nonlinear Cauchy problem in y, wich has a regular singular point at $r = r_0$. The characteristic exponents of thhis new problem are $\lambda_1 = -1 < 0$ and $\lambda_2 = -\frac{p(1+n)}{p-1-n} > 0$. Then we have to consider two different cases:

1. If $n \neq -2 - \frac{1}{p-1}$, the difference between λ_1 and λ_2 is not integer and we look for a particular solution y_{par} of the nonlinear Cauchy problem in the form a series of powers of $(r - r_0)$. Substituting this series into (3.7) we obtain the expression of g_2.
2. If $n = -2 - \frac{1}{p-1}$, the difference between λ_1 and λ_2 is an integer and we look for a particular solution y_{par} of the nonlinear Cauchy problem in the form a series whose terms are products of powers of $(r_0 - r)$ by powers of $\ln\left(\frac{r_0-r}{r_0}\right)$. Substituting this series into(3.7) we obtain the expression of g_3.

□

4. LOWER AND UPPER SOLUTIONS

Let us now consider again the boundary value problem (2.1),(2.2),(3.2) with $p > 1$, $\sigma > -1$, $n < -1$, $N \geq 1$, $a < 0$.

Definition 4.1. We shall say that $h(r)$ is a lower (resp. upper) solution of the problem (2.1),(2.2),(3.2) if $h(r) \in C[0, r_0] \cap C^2(0, r_0)$ and satisfies

$$\begin{array}{l} -r^{1-N}(r^{N-1}|h'(r)|^{p-2}h'(r))' + ar^{\sigma}h^n(r) \leq 0 \ \text{(resp. } \geq 0), \\ 0 < r < r_0, \quad h'(0) \leq 0 \ \text{(resp. } \geq 0), \qquad h(r_0) = 0. \end{array} \tag{4.1}$$

We want upper and lower solutions to verify

$$\begin{array}{l} \lim\limits_{r \to 0^+} h'(r) = 0, \\ \lim\limits_{r \to r_0^-} h(r) = 0, \\ \lim\limits_{r \to r_0^-} \frac{h(r)}{g(r)} = \alpha_1 \neq 0, \end{array}$$

where the last condition means that the upper and lower solutions are asymptotically equivalent to the exact solution in the neighborhood of the singular point $r = r_0$. Taking into account the asymptotic behavior of the solutions at $r = 0$ and $r = r_0$, studied in the previous sections, we will look for lower and upper solutions in the form

$$h(r) = B\left(r_0^k - r^k\right)^{\frac{p}{p-1-n}}, \tag{4.2}$$

where $k = \frac{\sigma+p}{p-1}$ and B is a positive constant. Note that in the case $p = 2$ we obtain the upper and lower solutions, introduced in [1]. We have $k > 1$, which follows from $\sigma > -1$

and $p > 1$. We want to define B in such a way that h satisfies equation (2.1), that is, h must verify

$$-r^{1-N}(r^{N-1}|h'(r)|^{p-2}h'(r))' = ah(r)^n r^\sigma \tag{4.3}$$

With this purpose, we note that

$$r^{1-N}(r^{N-1}|h'(r)|^{p-2}h'(r))' = \frac{1}{Bkp}(-1+p)r^\sigma(r_0^k - r^k)^{\frac{pn}{p-1-n}}\left(\frac{Bkp}{p-1-n}\right)^p q(r) \tag{4.4}$$

where

$$q(r) = \frac{(1+n)(p-N)+p(\sigma+N)}{p-1}r^k + \frac{(1+n-p)(\sigma+N)}{p-1}r_0^k = \beta r^k + \alpha, \qquad r \in [0, r_0]. \tag{4.5}$$

On the other hand, from (4.2) we have

$$h(r)^n = B^n\left(r_0^k - r^k\right)^{\frac{pn}{p-1-n}} \tag{4.6}$$

By substituting (4.4) and (4.6) into (4.3) we obtain

$$-\frac{1}{Bkp}(-1+p)\left(\frac{Bkp}{p-1-n}\right)^p q(r) + aB^n = 0. \tag{4.7}$$

Note that if

$$\beta = \frac{(1+n)(p-N)+p(\sigma+N)}{p-1} = 0 \tag{4.8}$$

then $q(r)$ reduces to a constant:

$$q(r) \equiv r_0^k \frac{(1+n-p)(\sigma+N)}{p-1} = \alpha. \tag{4.9}$$

In this case, that is, if $\beta = 0$, equation (4.7) takes the form

$$-\tfrac{1}{Bkp}\left(\tfrac{Bkp}{p-1-n}\right)^p r_0^k(\sigma+N)(1+n-p) + aB^n = 0 \tag{4.10}$$

Solving (4.10) with respect to B, we obtain

$$B = \left(\frac{-a(p-1-n)^{p-1}}{r_0^k(kp)^{p-1}(\sigma+N)}\right)^{\frac{1}{p-1-n}}. \tag{4.11}$$

Hence we conclude that when the condition (4.8) is satisfied the exact solution of the problem (2.1),(2.2),(3.2) has the form (4.2), with B given by (4.11). For values of n, p, σ, N which do not satisfy the condition (4.8)we cannot find an explicit formula for the exact solution. However, by analyzing the function $q(r)$, we can obtain conditions on B, under which the function h can be an upper or a lower solution, according to Definition 4.1.

Let us consider first the case $\beta > 0$. Since, by (4.7), $q(r) = \beta r^k + \alpha$, in this case we have

$$q_{max} = \max_{r \in [0,r_0]} q(r) = q(r_0) = \alpha + \beta r_0^k \tag{4.12}$$

and

$$q_{min} = \min_{r \in [0,r_0]} q(r) = q(0) = \alpha, \tag{4.13}$$

where α,β are defined by (4.9) and (4.8). Therefore, we conclude that

$$-\frac{1}{Bkp}(-1+p)\left(\frac{Bkp}{p-1-n}\right)^p q(r) \leq -\frac{1}{Bkp}(-1+p)\left(\frac{Bkp}{p-1-n}\right)^p q_{min}, \qquad \forall r \in [0,r_0]. \tag{4.14}$$

Hence, in order to satisfy the condition (4.1)for a lower solution it is enough that

$$-\frac{1}{Bkp}(-1+p)\left(\frac{Bkp}{p-1-n}\right)^p q_{min} + aB^n \leq 0. \tag{4.15}$$

Solving the inequality (4.15) with respect to B, we obtain

$$B \leq B1 = \left(\frac{-1+p}{akp}\left(\frac{kp}{p-1-n}\right)^p q_{min}\right)^{\frac{1}{n+1-p}}, \tag{4.16}$$

where q_{min} is defined by (4.13).

In a similar way, we can guarantee that the condition (4.1) for an upper solution is satisfied if

$$-\frac{1}{Bkp}(-1+p)\left(\frac{Bkp}{p-1-n}\right)^p q_{max} + aB^n \geq 0. \tag{4.17}$$

Solving this inequality with respect to B, we obtain

$$B \geq B2 = \left(\frac{-1+p}{akp}\left(\frac{kp}{p-1-n}\right)^p q_{max}\right)^{\frac{1}{n+1-p}}, \tag{4.18}$$

where q_{max} is defined by (4.12).

Consider now the case $\beta < 0$. In this case we have $q_{min} = \alpha + \beta r_0^k$ and $q_{max} = \alpha$.

Hence, using the same arguments as above, we can conclude that in this case the condition for a subsolution is satisfied if $B \leq B_2$ and the condition for an upper solution is satisfied if $B \geq B_1$.

The above results can be summarized in the following Theorem.

Theorem 4.1. *Let $\sigma > -1$, $n < -1$, $p > 1$, $a < 0$, $N \geq 1$ and let $B > 0$ be a constant. Then the function defined by (4.2) will be*

- *a lower solution of the problem (2.1),(2.2),(3.2) if B satisfies $B \leq B_1$ (when $\beta > 0$) or $B \leq B_2$ (when $\beta < 0$);*
- *an upper solution of the problem (2.1),(2.2),(3.2) if B satisfies $B \geq B_2$ (when $\beta > 0$) or $B \geq B_1$, (when $\beta < 0$).*

B_1,B_2 and β are defined by (4.16),(4.18) and (4.8), respectively.

Moreover, we see that when $(1+n)(p-N)+p(\sigma+N)=0$, $B_1=B_2=B$, where B is given by (4.11). As we have remarked above, in this case we have an exact solution in the form (4.2).

The upper and lower solutions can be used to locate the exact solution and to obtain suitable initial approximations for the application of computational methods.

5. NUMERICAL RESULTS

Let us briefly describe the numerical algorithm we used in order to obtain the approximate solutions of problem (2.1),(2.2),(3.2). For given values of a, n, r_0, σ, p, N we have considered two regular Cauchy problems

$$\begin{cases} -r^{1-N}(|g'(r)|^{p-2}g'(r)r^{N-1})'=ar^{\sigma}g^{n}(r), & 0<r<\frac{r_0}{2}, \\ g(\delta)=\overline{g}_1(\delta,b), \qquad g'(\delta)=\overline{g}_1'(\delta,b) & \end{cases} \tag{5.1}$$

$$\begin{cases} -r^{1-N}(|g'(r)|^{p-2}g'(r)r^{N-1})'=ar^{\sigma}g^{n}(r), & \frac{r_0}{2}<r<r_0, \\ g(r_0-\delta)=\overline{g}_i(r_0-\delta,c), \quad g'(r_0-\delta)=\overline{g}_i'(r_0-\delta,c), & \end{cases} \tag{5.2}$$

which we denote Problem A and Problem B, respectively. Here $i=2$ or 3, depending on n, δ is a small constant. The functions $\overline{g_1}$, $\overline{g_2}$ and $\overline{g_3}$ are computed according to the formulae (2.4) ,(3.4) and (3.5), respectively, ignoring the remainders of the series. We solve problems A and B for certain values of the parameters b and c. With this purpose we use the program NDSolve of Mathematica. Then, following the idea of the shooting method, we determine the values of b and c so that g and g' are continuous at $r=r_0/2$. This gives a system of two nonlinear equations, which is solved by the Newton's method.

When $p\le 2+\sigma$, $\sigma\ge 0$ and $N=1$, since we have no singularity at $r=0$, we solve the auxiliary problem (B) starting from $r_0-\delta$, for a certain value of the parameter c. Then we compute c by the shooting method, requiring that the boundary condition at $r=0$ is satisfied.

In our calculations we have used $\delta=0.001$.

In figure 1 we plot the numerical solutions obtained by the proposed method, as well as the corresponding upper and lower solutions of problem (2.1),(2.2),(3.2), for different values of σ, n, r_0, a, p and N.

FIGURE 1. Approximate, upper and lower solutions of the boundary value problem with $r_0=1,\sigma=0,n=-3,a=-1,p=3,N=2$. (left) and $r_0=1,\sigma=0,n=-2,a=-1,p=1.5,N=2$(right).

In table 1 we display the computed values of the solutions at $r=r_0/2$, in some cases where the exact solution is known, and the corresponding error. In all the cases we consider $r_0=1,a=-1$.

TABLE 1. Numerical solution and absolute error at $r = \frac{r_0}{2}$

σ	n	p	N	appr.value	error
0.5	-6	4	2	0.8516748647	1.20×10^{-8}
-0.5	-4	4	2	0.8016642094	2.61×10^{-8}
0	-8.5	5	3	0.9191721657	1.38×10^{-8}

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