Numerical Approximation of a Nonlinear Boundary Value Problem for a Mixed Type Functional Differential Equation Arising in Nerve Conduction

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Abstract. This paper is concerned with the approximate solution of a nonlinear mixed type functional differential equation (MTFDE) with deviating arguments arising from nerve conduction theory. The considered equation describes conduction in a myelinated nerve axon in which the myelin totaly insulates the membrane. As a consequence, the potential change jumps from node to node. As described in [2], this process is modeled by a first order nonlinear functional-differential equation with deviated arguments. We search for a solution of this equation defined in $\mathbb{R}$, which tends to given values at $\pm\infty$. Following the approsach introduced in [15] and [9], we propose and analyse some new computational methods for the solution of this problem. Numerical results are obtained and compared with the ones presented in [2].

Keywords: Mixed-type functional differential equation, nonlinear boundary value problem, nerve conduction, method of steps, Newton method
PACS: 02.30 Ks, 02.60 Lj, 02.70 c; **MSC:** 65Q05, 34K06, 34K10, 34K28

1. INTRODUCTION

Functional differential equations of the form

$$v'(t) = F(v(t)) + \beta(t)v(t-\tau) + \gamma(t)v(t+\tau), \tag{1}$$

are known as mixed-type functional equations (MTFDE); here v is the unknown function, β, γ and $F$are known functions. The interest in this type of equation is motivated by its applications in optimal control [13], economic dynamics [14], nerve conduction [2] and travelling waves in a spatial lattice [1]. Important contributions to their analysis have appeared in the literature in the last two decades of the past century. The ill-posedness of MTFDEs has been discussed by Rustichini in 1989 [13], where he considered linear autonomous equations. The same author extended his results to nonlinear equations [14]. J. Mallet-Paret applied the Fredholm theory to obtain new results for this class of equation [10] and introduced the idea of factorization of their solutions [11]. Independently, the authors of [6] have obtained results about factorization for the linear non-autonomous case. On the other hand, Krisztin [8] has analysed the roots of the characteristic equation of linear systems of MTFDE, which has led him to important results on the qualitative behaviour of their solutions. In particular, he has shown that a MTFDE may have a nonoscillatory solution in spite of the non-existence of real roots of its characteristic equation (unlike the case of delay-differential equations).

In [3] and [4] a new approach has been developed to the analysis of these equations in the linar aotonomous case case, when F is a linear function and β, γ are constants. The authors of these papers have considered the problem of finding a differentiable solution on a certain real interval $[-1,k]$, given its values at the intervals $[-1,0]$ and $(k-1,k]$. They have concluded that in general the specification of such boundary functions is not sufficient to ensure that a solution can be found. For the case where such a solution exists they have introduced a numerical algorithm to compute it. In [15], new numerical methods were proposed for the solution of such boundary value problems. In [16] and [9] these methods were extended to the non-autonomous case (when α, β and γ are smooth functions of t). In [5], we have decomposed the solution into a growing and a decaying component, following the ideas of J. Mallet-Paret and Verduyn-Lunel [11]. By approximating each of the components separately, we partially overcome the ill-conditioning of the problem, which allowed us to improve the accuracy of the numerical results and to obtain numerical solutions on much larger intervals.

In [9] a theoretical basis has been given to the computational methods, by relating them to the existing analytical results about MTFDE and with classical results of numerical analysis.

In [17], a new algorithm has been introduced, based on the finite element method, and a comparative analysis of all these algorithms has been provided.

The present paper is concerned with a nonlinear MTFDE of the form

$$RCv'(t) = F(v(t)) + v(t-\tau) + v(t+\tau). \tag{2}$$

We are interested in a solution of (2), increasing on $]-\infty,\infty[$, which satisfies the conditions

$$\lim_{t\to\infty} v(t) = 0, \qquad \lim_{t\to\infty} v(t) = 1, \tag{3}$$

The problem (2)-(3) was analysed for the first time in [2] where its physical meaning is explained in detail. The unknown v represents the transmembrane potential at a node in a myelinated axon, in the nerve conduction model. F reflects the current-voltage model. R and C are respectively the axomatic nodal resistivity and the nodal capacity. This mathematical formulation comes from an electric circuit model which assumes the so-called pure saltatory conduction (PSC). This means that the myelin has higher resistance and lower capacitance when compared with the membrane; therefore, if the membrane is depolarized at a node, myelin tends to jump the next node and excite the membrane there.

In the work of Chi, Bell and Hassard [2], published more than 20 years ago, the problem (2)-(3) has been thoroughly investigated, both from the analytical and numerical point of view. A computational algorithm has been proposed and the first mumercial results (as far as we know) have been obtained for a nonlinear MTFDE. Iy is interesting to remark that by the time when this paper was published, few results were known about MTFDE, even in the linear case. The systematic analysis of this type of equation has started later, as it can be seen by the references given above.

2. PRELIMINARIES

The problem studied here arose from nerve conduction theory and it is modeled by equation (2). It is not an easy problem to solve numerically: it is advanced-retarded and is a BVP defined on $I\!R$ with τ unknown. It is assumed the pure saltatory conduction. In addition, the circuit model supposes that the nodes are uniformly spaced and electrically identical, the axon is infinite in extent and in the axon at cross-sectional variations in potential are negligible. Several models can be obtained using different current-voltage expressions. Using FitzHugh-Nagumo dynamics for the nodal membrane, without a recovery term and assuming that a supra-threshold stimulus begins a propagated axon potential and consequently travels down the axon from node to node. With adequate variable substitutions, one can get the following non-dimensional model:

$$\begin{cases} v'(t) = f(v(t)) + v(t-\tau) + v(t+\tau) - 2v(t), \quad -\infty < t < +\infty, \\ v(-\infty) = 0, \\ v(+\infty) = 1. \end{cases} \tag{4}$$

The function f is given by $f(v) = bv(v-a)(1-v)$ with a the threshold potential in the non dimensional problem $(0 < a < 1)$, τ the non dimensional time delay and b related with the strength of the ionic current density $(b > 0)$. The solution at any node should be monotone increasing. This arises from the current-voltage relation $f(v)$, once a node is turned on, it cannot return to rest potential $v = 0$.

3. OUTLINE OF THE PRESENT METHOD

The main goal of the present work is to revisit problem (2),(3). and apply to it the more recent approaches and techniques. With this purpose we extend the θ and collocation methods, developped in [4] and [15], to the case of nonlinear problems. On the other hand, the asymptotic analysis of the solutions (as $t \to \pm\infty$) allows us to transfer the boundary conditions and to reduce the present problem (4) to the equivalent problem (5) on a finite interval

$$v'(t) = f(v(t)) + v(t-\tau) + v(t+\tau) - 2v(t), \quad t \in [-L-\tau, L+\tau], \tag{5}$$

where L is a sufficiently large number. The outline of the method can be summarized as follows:

1. Using the asymptotic properties of the solutions, for large values of $|t|$, obtain approximations of the needed solution on certain intervals $[-L-\tau,-L]$ and $[L,L+\tau]$; then the boundary conditions (3) are reduced to the form

$$\begin{aligned} v(t) = \phi_0(t), \qquad & t \in [-L-\tau,-L]; \\ v(t) = \phi_1(t), \qquad & t \in [L,L+\tau]. \end{aligned} \tag{6}$$

2. Based on this approximation of the solution and on the method of steps, compute the value of τ, for each the solution of the problem exists;
3. The nonlinear equation (2) with boundary conditions (6) is reduced by the Newton method to a sequence of linear BVP;
4. The numerical solution of each linear BVP is obtained by the collocation method.

4. FINAL REMARKS

To analyse the convergence of the numerical scheme we take into account some test problems with known solutions. Numerical results are compared with the results obtained by other methods presented in [2]. A question in study is how the solution of equation (3) will be affected by changing the parameters of the problem and the deviating argument τ.

ACKNOWLEDGMENTS

One of the authors (M. F. Teodoro) acknowledges financial support from Fundação para a Ciência e Tecnologia (FCT), grant SFRH/BD/37528/2007.

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