

# Extrapolation methods for a nonlinear weakly singular Volterra integral equation

Teresa Diogo\*, Pedro Lima\* and Magda Rebelo†

\*CEMAT-Departamento de Matemática, Instituto Superior Técnico, UTL, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

†CEMAT and Departamento de Matemática, Faculdade de Ciências e Tecnologia, Monte da Caparica, 2829-516 Caparica, Portugal

**Abstract.** In this work we consider a nonlinear Volterra integral equation with weakly singular kernel. An asymptotic error expansion for the explicit Euler's method is obtained and this allows the use of certain extrapolation procedures. The performance of the extrapolation method is illustrated by several numerical examples.

**Keywords:** Nonlinear Volterra integral equations, Asymptotic expansions, Extrapolation  
**PACS:** 02.60.Nm

## 1. INTRODUCTION

This work is concerned with the numerical solution of the nonlinear Abel-type integral equation

$$y(t) = 1 - \frac{\sqrt{3}}{\pi} \int_0^t \frac{s^{\frac{1}{3}} y^4(s)}{(t-s)^{\frac{2}{3}}} ds, \quad 0 < t \leq T, \quad (1)$$

which arises in a heat transfer problem studied by Lighthill (1950). Equation (1) has a unique continuous solution  $y(t), t \in [0, T]$ , with the properties:

(a) for  $t \in [0, R)$ , with  $R \simeq 0.070784$ ,  $y$  admits the following representation

$$y(t) = 1 - 1.460998t^{2/3} + 7.249416t^{4/3} + 46.460t^2 + 332.9t^{8/3} + \dots; \quad (2)$$

(b)  $y \in C^{n,2/3}[\varepsilon, T]$ ,  $0 < \varepsilon < R$ , where

$$C^{n,2/3}[\varepsilon, T] \equiv \left\{ y(t) \in C^n(\varepsilon, T) \text{ and } |y^{(n)}(t)| \leq C(t-R)^{-\frac{2}{3}-(n-1)}, t \in (R, T], \right. \\ \left. \text{for some constant } C > 0 \right\}. \quad (3)$$

When numerical methods based on piecewise polynomials on uniform meshes are applied to equation (1), the optimal orders of convergence are not obtained (see e.g. [2], [3] and [6]). It has been proved that product integration and spline collocation methods have  $2/3$  global order of convergence, independently of the degree of the approximating polynomials used; as  $t$  increases the errors present order  $4/3$ , except in the case of the explicit Euler's method, which has order one. The low convergence orders of standard numerical methods applied to equation (1) can be explained by the singular behavior of its solution near the origin. In particular, it was shown in [2] that  $y(t) = 1 + u_1 t^{\frac{2}{3}} + O(t^{\frac{4}{3}})$ , as  $t \rightarrow 0$ , but  $y$  is continuously differentiable for  $t$  away from the origin. In this work we shall investigate the use of an extrapolation algorithm in order to improve numerical results obtained with low order methods.

Extrapolation techniques for a class of nonlinear Volterra integral equations with weakly singular kernels of a general form have been considered in [5]. A transformation of variables was first applied so that the solution of the new equation was sufficiently smooth. Then a computational method based on the modified trapezoidal rule was proposed and an extrapolation procedure was used to improve the numerical results.

It happens that our equation (1) is included in the general form treated in [5]. However we note that our approach here is different from the one described in [5] since we treat the equation directly without previously employing a

smoothing transformation. In the present work our purpose is to introduce a new computational method for equation (1), by combining the explicit Euler's method with an extrapolation procedure. Our analysis will be based on the derivation of an asymptotic error expansion.

## 2. ASYMPTOTIC ERROR EXPANSION

The application of the explicit Euler's method for the numerical solution of equation (1) has been analyzed in [2]. Introducing on  $[0, T]$  a uniform grid  $X_h = \{t_i = ih, 0 \leq i \leq N\}$  with stepsize  $h$ , the explicit Euler's method is defined by

$$\begin{aligned} y_0 &= 1 \\ y_i &= 1 - \frac{\sqrt{3}}{\pi} \sum_{j=0}^{i-1} W_{ij} t_j^{\frac{1}{3}} y_j^4, \quad i = 1, \dots, N, \end{aligned} \quad (4)$$

where  $y_j$  denotes an approximation of  $y(t)$  at  $t_j$  and

$$W_{ij} = \int_{t_j}^{t_{j+1}} \frac{1}{(t_i - s)^{\frac{2}{3}}} ds. \quad (5)$$

It is straightforward to show that the total error  $e_i = y(t_i) - y_i$  satisfies

$$e_i = T_i - \frac{\sqrt{3}}{\pi} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left( t_j^{\frac{1}{3}} y^4(s) - t_j^{\frac{1}{3}} y_j^4 \right) \frac{1}{(t_i - s)^{\frac{2}{3}}} ds, \quad i = 1, \dots, N, \quad (6)$$

where

$$T_i = -\frac{\sqrt{3}}{\pi} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left( s^{\frac{1}{3}} y^4(s) - t_j^{\frac{1}{3}} y^4(t_j) \right) \frac{1}{(t_i - s)^{\frac{2}{3}}} ds, \quad i = 1, \dots, N, \quad (7)$$

In order to obtain an asymptotic error expansion we need to refine the error estimates obtained in [2], in particular the results of Lemma 3.3. We will assume that  $t_i \gg h$ , that is, we will consider the error at a point away from the origin and our analysis will be divided into two parts. First, we will consider the terms of the sum (7), for which  $t_j > t_r$  and then those for which  $t_j < t_r$  ( $t_j$  close to the origin); here  $t_r$  is a gridpoint such that  $r \ll N$ .

Using the variable transformation  $s = t_j + \theta h$ ,  $\theta \in [0, 1]$ ,  $j = 0, 1, \dots, i-1$  we may write the quadrature error  $T_i$  as follows

$$T_i = -\frac{\sqrt{3}}{\pi} \sum_{j=0}^{i-1} h \int_0^1 \frac{\left( (t_j + \theta h)^{1/3} y^4(t_j + \theta h) - (t_j)^{1/3} y^4(t_j) \right)}{(t_i - t_j - \theta h)^{2/3}} d\theta = S_1(i) + S_2(i), \quad (8)$$

where

$$S_1(i) = -\frac{\sqrt{3}}{\pi} \sum_{j=0}^{r-1} h \int_0^1 \left( (t_j + \theta h)^{1/3} y^4(t_j + \theta h) - (t_j)^{1/3} y^4(t_j) \right) (t_i - t_j - \theta h)^{-2/3} d\theta, \quad (9)$$

$$S_2(i) = -\frac{\sqrt{3}}{\pi} \sum_{j=r}^{i-1} h \int_0^1 \left( (t_j + \theta h)^{1/3} y^4(t_j + \theta h) - (t_j)^{1/3} y^4(t_j) \right) (t_i - t_j - \theta h)^{-2/3} d\theta. \quad (10)$$

We can prove that there exist functions  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4 \in C([0, T])$  such that the sums  $S_1$  and  $S_2$  satisfy the following asymptotic expansions, with  $\gamma > 4/3$ :

$$S_1(i) = h \mathcal{F}_1(t_i) + h^{4/3} \mathcal{F}_2(t_i) + O(h^\gamma), \quad (11)$$

$$S_2(i) = h \mathcal{F}_3(t_i) + h^{4/3} \mathcal{F}_4(t_i) + O(h^\gamma), \quad i = r, r+1, \dots, N, \quad (12)$$

From (11) and (12) we can conclude that the quadrature error  $T_i$  of the explicit Euler's method, defined by (7), satisfies the asymptotic error expansion

$$T_i = \widetilde{\mathcal{F}}_1(t_i) h + \widetilde{\mathcal{F}}_2(t_i) h^{4/3} + O(h^\gamma), \quad (13)$$

where  $\gamma > 4/3$  and  $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2 \in C([0, T])$ . Using (13) and some results from [2] we prove the global error  $e_i = y(t_i) - y_i$ ,  $i = r, r+1, \dots, N$  satisfies

$$e_i = \widetilde{\mathcal{F}}_1(t_i)h + \widetilde{\mathcal{F}}_2(t_i)h^{4/3} - \frac{4\sqrt{3}}{\pi} \sum_{j=0}^{i-1} t_j^{1/3} y^3(t_j) W_{ij} e_j + O(h^\gamma), \quad (14)$$

where the weights  $W_{ij}$  are defined by (5) and  $\gamma > 4/3$ . By a detailed analysis of equation (14), it is possible to show that the  $e_i$  allow an asymptotic expansion, where the powers of  $h$  of the first two terms are the same as the ones in (13).

The main result of this section may be summarized in the following theorem.

**Theorem 1** *There exist functions  $C_1$  and  $C_2 \in C([0, 1])$ , such that for  $t_i$  far from the origin, the error of the explicit Euler's method for equation (1) satisfies, with  $\gamma > 4/3$ ,*

$$e_i = C_1(t_i)h + C_2(t_i)h^{4/3} + O(h^\gamma). \quad (15)$$

### 3. THE EXTRAPOLATION METHOD

From Theorem 1 it follows that the approximate solution  $y^h$  obtained with the explicit Euler's method has an asymptotic error expansion with the form

$$y_i^h = y(t_i) + a_1 h + a_2 h^{4/3} + O(h^\gamma), \quad (16)$$

with  $\gamma > 4/3$  and  $a_1 = C_1(t_i)$  and  $a_2 = C_2(t_i)$ , where  $C_1$  and  $C_2$  are the functions defined in Theorem 1.

When the error of a discretization method allows an asymptotic error expansion including non-integer powers of  $h$ , as it is the case in (16), the E-algorithm of Brezinski can be used to improve the numerical results [1]. This algorithm is a generalization of the Richardson's extrapolation and generates a sequence  $E_k^{(n)}$  of approximations of  $y(t)$  at  $t = t_j$ , where  $t_j = h_0 j$ . The computation of  $E_k^{(n)}$  begins with

$$E_0^{(n)} = y_j^{h_n}, \quad n = 0, 1, \dots, n_{max} \quad g_{0,i}^{(n)} = g_i(n), \quad i = 1, 2, \dots, n_{max}, \quad n = 1, \dots, n_{max} - 1, \quad (17)$$

where  $y_j^{h_n}$  is an approximation of  $y(t_j)$  obtained with a mesh with stepsize  $h_n$ . Here  $g_i(n)$  are certain auxiliary sequences which depend on the specific form of the asymptotic error expansion. For example, if we have an asymptotic error expansion of the form

$$y_i^h = y_i + a_1 h^{\alpha_1} + a_2 h^{\alpha_2} + \dots + a_k h^{\alpha_k} + O(h^\gamma),$$

where  $\alpha_1 < \alpha_2 < \dots < \alpha_k < \gamma$ , then we have  $g_i(n) = h_n^{\alpha_i}$ ,  $i = 1, \dots, k$ . The recursive formulae for computing  $E_k^n$ ,  $k = 1, 2, \dots$  can be found in [1].

In the numerical examples of next section, the asymptotic expansion (16) suggested the choice of the auxiliary sequences  $g_1(n) = h_n$  and  $g_2(n) = h_n^{4/3}$ .

#### 3.1. Numerical results

The absolute errors for  $t = 0.0005$  and  $t = 0.001$  of the entries of the E-array are displayed, respectively, on the left and right hand sides of Table 1. The values  $y(t_i)$  have been computed by using the first 20 terms of the series (2). For the points  $t_i > R$ , where  $R$  is the radius of convergence of the series (2), we have taken  $E_2^{(2)}$  as the exact solution of equation (1). The absolute errors at  $t_i = 0.5$  and  $t_i = 1.0$  of the entries of the E-array are shown in Table 2, on the left and right hand sides, respectively. The results show that the accuracy is improved in each step of the extrapolation process. Moreover, they confirm the  $O(h)$  order of the first term in the error expansion and the  $O(h^{4/3})$  order of the second term.

**TABLE 1.** Absolute errors of the E-algorithm. **Left:**  $t_i = 0.0005$  **Right:**  $t_i = 0.001$ .

$n$	$h_n$	$y(t_i) - E_0^{(n)}$	$y(t_i) - E_1^{(n)}$	$y(t_i) - E_2^{(n)}$	$n$	$h_n$	$y(t_i) - E_0^{(n)}$	$y(t_i) - E_1^{(n)}$	$y(t_i) - E_2^{(n)}$
0	$\frac{0.001}{80}$	$2.34e-5$	$2.81e-6$	$5.37e-8$	0	$\frac{0.001}{80}$	$5.96e-5$	$1.62e-6$	$1.74e-8$
1	$\frac{0.001}{160}$	$3.98e-5$	$1.08e-6$	$1.17e-8$	1	$\frac{0.001}{160}$	$2.90e-5$	$6.33e-7$	$3.81e-9$
2	$\frac{0.001}{320}$	$1.93e-5$	$4.22e-7$	$2.58e-9$	2	$\frac{0.001}{320}$	$1.42e-5$	$2.49e-7$	$8.38e-10$
3	$\frac{0.001}{640}$	$9.46e-6$	$1.66e-7$		3	$\frac{0.001}{640}$	$6.97e-6$	$9.82e-8$	
4	$\frac{0.001}{1280}$	$4.65e-6$			4	$\frac{0.001}{1280}$	$3.43e-6$		

**TABLE 2.** Absolute errors of the E-algorithm. **Left:**  $t_i = 0.5$  **Right:**  $t_i = 1.0$ .

$n$	$h_n$	$y(t_i) - E_0^{(n)}$	$y(t_i) - E_1^{(n)}$	$y(t_i) - E_2^{(n)}$	$n$	$h_n$	$y(t_i) - E_0^{(n)}$	$y(t_i) - E_1^{(n)}$	$y(t_i) - E_2^{(n)}$
0	$\frac{1}{400}$	$3.33e-5$	$1.30e-6$	$2.04e-7$	0	$\frac{1}{400}$	$3.33e-5$	$1.30e-6$	$2.04e-7$
1	$\frac{1}{800}$	$1.60e-5$	$6.38e-7$	$4.67e-8$	1	$\frac{1}{800}$	$1.60e-5$	$6.38e-7$	$4.67e-8$
2	$\frac{1}{1600}$	$7.68e-6$	$2.81e-7$		2	$\frac{1}{1600}$	$7.68e-6$	$2.81e-7$	
3	$\frac{1}{3200}$	$3.70e-6$	$1.12e-7$		3	$\frac{1}{3200}$	$3.70e-6$	$1.12e-7$	
4	$\frac{1}{6400}$	$1.79e-6$			4	$\frac{1}{6400}$	$1.79e-6$		

## 4. CONCLUSIONS

In this work it was shown that the application of the E-algorithm has led to an improvement of results obtained with the explicit Euler's method. The starting point for this procedure was the derivation of an asymptotic error expansion in non-integer powers of the stepsize. Our investigations suggest that the same approach can be applied to other discretization methods and other nonlinear singular equations.

## ACKNOWLEDGMENTS

This work was supported in part by Centro de Matemática e Aplicações (CEMAT) and Fundação para a Ciência e a Tecnologia (Project PTDC/MAT/101867/2008). M. Rebelo also acknowledges a Ph.D. grant SFRH/BD/32117/2006.

## REFERENCES

1. C. Brezinski and M. Redivo-Zaglia, *M. Redivo-Zaglia, Extrapolation Methods: Theory and Practice*, North-Holland, Amsterdam, 1991.
2. T. Diogo, P.M. Lima and M. Rebelo, *Numerical solution of a nonlinear Abel type Volterra integral equation*, Communications on Pure and Applied Analysis 5, 2006, pp. 277–288.
3. T. Diogo, P. M. Lima and M. S. Rebelo, *Comparative analysis of numerical methods for a nonlinear Volterra integral equation*, Hermis, V. 7, 2006, pp. 1–20.
4. J.M. Lighthill, *Contributions to the theory of the heat transfer trough a laminar boundary layer*, Proc. Roy. Soc. London 202A (1950), 359–377.
5. L. Tao and H. Yong, *Extrapolation method for solving weakly singular nonlinear Volterra integral equations of the second kind*, J. Math. Anal. Appl. 324, 2006, pp. 225–237.
6. M. Rebelo and T. Diogo, *A hybrid collocation method for a nonlinear Volterra integral equation with weakly singular kernel*, J. Comput. Appl. Math., 234 2010, pp. 2859–2869.