Equações Diferenciais Ordinárias
LMAC, MMA
Exame - 08 de Janeiro de 2014

Apresentar todos os cálculos e justificar todas as afirmações

1. (2 valores) Show that a solution $x(t)$ of the equation

$$\ddot{x} - 4x = \sin(2t),$$

is bounded for $t > 0$ if and only if

$$2x(0) + \dot{x}(0) + \frac{1}{4} = 0.$$

Solution The associated homogeneous equation is $\ddot{x} - 4x = 0$, with general solution

$$x(t) = ae^{2t} + be^{-2t};$$

searchin a particular solution of the complete equation of the form $u(t) = c\sin(2t) + d\cos(2t)$, we get

$$\sin(2t) = \ddot{u}(t) - 4u(t) = -8c\sin(2t) - 8d\cos(2t),$$

and so $u(t) = -\frac{\sin(2t)}{8}$, and the general form for the solutions of the equation is

$$x(t) = ae^{2t} + be^{-2t} - \frac{\sin(2t)}{8},$$

which is bounded for $t > 0$ if and only if $a = 0$. As

$$x(0) = a + b, \qquad \dot{x}(0) = 2a - 2b - \frac{1}{4},$$

we have $2x(0) + \dot{x}(0) + \frac{1}{4} = 4a$ implying the equivalence stated.

2. (3 valores) Prove the following generalization of the divergence criterion:

Dulac's Criterion: Let $D \subset \mathbb{R}^2$ be a simply connected domain and $F : D \to \mathbb{R}^2$ a C^1 vector field. If there exists a C^1 function $g : D \to \mathbb{R}$ such that $div(gF)$ is never zero in D, then D does not contain any periodic orbit of the flow associated to F.

Solution The proof is identical to the one of the Divergence Criterion:: if $\gamma \subset D$ was a periodic orbit of the flow associated to $\dot{x} = F(x)$ and U the region bounded by γ, the Divergence Theorem implies that

$$\int_U div(gF) = \int_\gamma gF\dot{n},$$

where n represents a normal unit vector pointing to the exterior of U. Now, $F\dot{n} = 0$, since γ is an orbit of the flow; buts $\int_U div(gF) = 0$ leads to a contradiction, since the function under the integral is continuous and allways different from zero: the integral of a (for instance) positive function in an open domain is necessarily positive.

3. (5 valores) Consider the system

$$\dot{x} = -x(x^2 + y^2 - 2), \qquad \dot{y} = -y(x^2 + y^2 - 3x + 1).$$

a) Use the function $V(x, y) = x^2+y^2$ to justify that, for any initial condition (x_0, y_0), the corresponding solution is defined for all $t > 0$, by finding a bounded region R containing the ω-limit of every solution.

b) Determine and characterize the singularities of the system and sketch the phase portrait.

c) Prove that the system has no periodic orbits.
Hint: Start by showing that a possible periodic orbit should be contained in the first or fourth quadrant; apply Dulac's Criterion with $g(x, y) = \frac{1}{xy}$.

Solution: We have

$$\partial_t V(x, y) = -2(x^2(x^2 + y^2 - 2) + y^2(x^2 + y^2 - 3x + 1)),$$

which is less than zero in the region defined by the conditions

$$x^2 + y^2 - 2 > 0, \qquad x^2 + y^2 - 3x + 1 > 0.$$

Choosing a sufficiently big R we have $\partial_t V(x,y) < 0$ para $x^2+y^2 > R$. Therefore, for an initial condition in that region, the distance to the origin decreases along the orbit and so for t sufficiently large the orbit is contained in the disc $x^2+y^2 \leq R$, while for an initial condition in this disc, the orbit remains in it for all $t > 0$.
Thus we verify that all solutions are contained, for t sufficiently large, in the disc $x^2+y^2 \leq R$ (a compact), which implies that they are defined for all $t > 0$ (there are no “explosions in finite time”) and the ω-limit of any orbit is necessarily non empty.

The linearization of the vector field is

$$DF(x,y) = \begin{pmatrix} -(x^2+y^2-2)-2x^2 & -2xy \\ & \\ -(2x-3)y & -(x^2+y^2-3x+1)-2y^2 \end{pmatrix};$$

The singularities of the system, with the corresponding linearization and type, are

$(0,0)$ $\quad \begin{pmatrix} 2 & 0 \\ & \\ 0 & -1 \end{pmatrix}$; saddle.

$(-\sqrt{2},0)$ $\quad \begin{pmatrix} -4 & 0 \\ & \\ 0 & -3-3\sqrt{2} \end{pmatrix}$; sink.

$(\sqrt{2},0)$ $\quad \begin{pmatrix} -4 & 0 \\ & \\ 0 & -3+3\sqrt{2} \end{pmatrix}$; saddle.

$(1,1)$ $\quad \begin{pmatrix} -2 & -2 \\ & \\ 1 & -2 \end{pmatrix}$; spiral sink.

$(1,-1)$ $\quad \begin{pmatrix} -2 & 2 \\ & \\ -1 & -2 \end{pmatrix}$; spiral sink.

Since the coordinate axes are unions of orbits of the flow, a periodic orbit would have to be contained in one of the quadrants; in

fact, since the region bounded by a periodic orbit would have to contain a singularity, they could occur only in the first or fourth quadrant.
However, using the hint, we find that

$$div(gF) = div\left(-\frac{x^2+y^2-2}{y}, -\frac{x^2+y^2-3x+1}{x}\right) = -\frac{2x}{y} - \frac{2y}{x}$$

which is never zero in those regions (it is negative in the first quadrant and positive in the fourth). So Dulac's Criterion garanties that there are no periodic orbits.

4. (5 valores Consider the system

$$\dot{x} = x - y - x(x^2 + \frac{3y^2}{2}), \qquad \dot{y} = x + y - y(x^2 + \frac{y^2}{2}).$$

a) Determine the stability type of the singularity at $(0,0)$.

b) Apply Poincaré-Bendixon's Theorem to justify the existence of at least one periodic orbit; obtain estimates for the constants $0 < a < b$ such that the periodic orbits are contained in the region $\{(x,y) : a^2 < x^2 + y^2 < b^2\}$.

c) Is the periodic orbit unique?
Hint: Compute the Divergence of the vector field.

Solution: The linearization of the vector field at the origin is $\begin{pmatrix} 1 & -1 \\ \\ 1 & 1 \end{pmatrix}$ which has eigenvalues $1 \pm i$. So it it a spiral source.
Computing $\partial_t(x^2+y^2)$, we get

$$2x^2(1 - x^2 - \frac{3}{2}y^2) + 2y^2(1 - x^2 - \frac{y^2}{2});$$

this function may be written as

$$2(x^2+y^2)(1 - x^2 - \frac{y^2}{2}) - 2x^2y^2$$

which is less than zero in the region $U = \{(x, y) : x^2 + \frac{y^2}{2} > 1\}$.
On the other hand, it may also be written as

$$2(x^2 + y^2)(1 - x^2 - \frac{3y^2}{2}) + 2y^4$$

which is positive in the region $V = \{(x, y) : x^2 + \frac{3y^2}{2} < 1\} \subset \mathbb{R}^2 \setminus U$.
So, if we consider a circle C_1 contained in U and another one C_2 contained in V, both centered at the origin, the flow will cross the first from the outside to the inside and the second from the inside to the outside.
Assuming there are not other singularities (see note below), Poincaré-Bendixon's Theorem implies that the anular region between the two circles contains a periodic orbit.

To verify uniqueness of the periodic orbit, we compute the divergence of the field:

$$div(F) = \partial_x F_1 + \partial_y F_2 = 2 - 4x^2 - 3y^2.$$

If, in the region considered above, the divergence is negative, the area is contracted by the flow, which implies uniqueness: if there were two periodic orbits (with one of them necessarily contained in the region bounded by the other, as the origin is the only singularity) the anular region bounded by them, which has positive area, is invariant under the flow.
But in the construction done before, we may take C_2 as the circle with radius $\sqrt{2/3}$; so in the region bounded by the two circles we have $div(F) = 2 - 3(x^2 + y^2) - x^2 < 0$.

Note: We may confirm the non existence of other singularities: in the first place there are certainly no other singularities in the coordinate axes, and we may assume that both x and y are different from zero. The euations to determine the singularities may be written as

$$\begin{cases} x(1 - x^2 - \frac{3y^2}{2}) = y \\ \\ y(x^2 + \frac{y^2}{2} - 1) = x \end{cases},$$

implying, under the given hypothesis,

$$xy(1 - x^2 - \frac{3y^2}{2})(x^2 + \frac{y^2}{2} - 1) = xy \implies (1 - x^2 - \frac{3y^2}{2})(1 - x^2 - \frac{y^2}{2}) = -1,$$

This equality implies, in particular, that any other singularity would be contained in the region $\mathbb{R}^2 \setminus (U \cup V)$, bounded by two elipses; if we locate the extreme values of $g(x,y) = (1-x^2-\frac{3y^2}{2})(1-x^2-\frac{y^2}{2})$, we verify that they satisfy

$$\begin{cases} x(1-x^2-y^2) = 0 \\ \\ y(4-4x^2-3y^2) = 0 \end{cases},$$

and we conclude that the minimum of g is attained at the points $(0, \pm\sqrt{4/3})$, but $g(0, \pm\sqrt{4/3}) = -1/3$, and so, in the region under consideration, $g(x,y) > -1$.

5. (5 valores) Consider the system

$$\dot{x} = x^2 - y - 1, \qquad \dot{y} = (x-2)y.$$

a) Determine and characterize the singularities and sketch the phase portrait.

b) Identify the tangent space at $(1,0)$ to the stable manifoldof that point. Justify that one of the components of that manifold is bounded and find it's α-limit.

c) Let $\Psi(t,x)$ the flow of this equation and $\Phi(t,y)$ the flow of the equation of problem 3.
Justify if there are neighborhoods U of $(1,0)$ and V of $(0,0)$ such that the restrictions $\Psi_{|U}$ and $\Phi_{|V}$ are topologically conjugated.

Solution: The singularities, with the corresponding linearizations of the vector field, are

$$\begin{array}{lll} (-1,0) & \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix} & \text{sink} \\ (1,0) & \begin{pmatrix} 2 & -1 \\ 0 & -1 \end{pmatrix} & \text{saddle} \\ \\ (2,3) & \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix} & \text{(source} \end{array};$$

The negative eigenvalue of $DF(1,0)$ is -1 and an associated eigenvector is $(1,3)$; the tangent space to the stable manifold is thus the space generated by this vector. In other words, the stable manifold is tangent, at the point $(1,0)$ to the line $y = 3(x-1)$.
The sketch of the phase portrait suggests that, for any initial condition (x_0, y_0) in the first quadrant, the orbit, for $t < 0$ is bounded; this may be confirmed verifying, for instance that, choosing $a > 2$ sufficiently large, the triangle with vertices $(0,0)$, $(b,0)$ and (b,ab), where b is the positive root of $x^2 - ax - 1$, contains (x_0, y_0) and is negatively invariant under the flow: the only computation needed is to show that for $x \in]0, b[$ the slope of the vector field, given by $\frac{(x-2)y}{x^2-y-1}$ is smaller than a, along the line $y = ax$.

In this way, we conclude that the α-limit of the branch of the stable manifold of $(1,0)$ contained in the first quadrant is non-empty. As $(2,3)$ is the only singularity in that region, we need only to show that there are no periodic orbits there; but the phase portrait shows that, if it existed, a periodic orbit would be contained in $\{(x,y) : x > 1 \wedge y > 0\}$; but in that domain $div(F) > 0$ and so there are no periodic orbits.
The α-limit of that orbit is then $(2,3)$.

The singularities refered to in c) are both saddles; so by the Grobman-Hartman theorem, the flows are locally topologically conjugated to the corresponding linearizations

$$\dot{X} = \begin{pmatrix} 2 & -1 \\ 0, -1 & \end{pmatrix} X \qquad \dot{X} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} X;$$

as these linear vector fields are both hyperbolic and have the same stability index, thweir flows are conjugated.
Combining these conjugacies, we confirm that $\Psi_{|U}$ and $\Phi_{|V}$ are topologically conjugated.