Continuous representations of groupoids

Rogier Bos[*]

Abstract

We introduce unitary representations of continuous groupoids on continuous fields of Hilbert spaces. We investigate some properties of these objects and discuss some of the standard constructions from representation theory in this particular context. An important rôle is played by the regular representation. We discuss to what extend Schur's Lemma and the Peter-Weyl theorem hold in this context. We conclude by analysing the relationship of continuous representations of G and continuous representations of the Banach $*$-category $\hat{L}^1(G)$ and the C^*-category $C^*(G, G)$.

AMS subject classification: Primary: 22A30. Secondary: 46L08, 46L80.

Contents

[*]Institute for Mathematics, Astrophysics and Particle Physics; Radboud University Nijmegen, The Netherlands, `Email: R.Bos@math.ru.nl`, this work is part of the research programme of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)

1 Introduction

The purpose of this paper is to study some of the basic theory of continuous representations in the context of groupoids.

These objects occur at several subjects of mathematics. Let us mention a few. The parallel transport associated to a flat connection on a vector bundle is a representation of the fundamental groupoid of the base space. Also the monodromy associated to holomorphic extensions of functions is a representation of the fundamental groupoid. Another place where they occur is as vector bundles over an orbifold, since these correspond to representations of the groupoid representing the orbifold. Related to this is the fact that for a group acting on a space, equivariant vector bundles over that space correspond to representations of the associated action groupoid. Apart from these examples, we shall also be interested in representations of families of groups and representations of gauge groupoids.

Some work on representations of groupoids on vector bundles was initiated by Westman in [27, 28]. We shall look at representations not only on continuous vector bundles, but on continuous fields of Hilbert spaces. Continuous fields of Hilbert spaces were introduced and studied by Dixmier and Douady [6]. They play an important rôle in noncommutative geometry, as they occur as Hilbert C^*-modules of commutative C^*-algebras (cf. Theorem 2.43). Moreover, they are a rich source of noncommutative C^*-algebras, which are obtained as the algebra of adjointable endomorphisms of such modules (cf. below Example 2.44). A reason why we not only consider representations on continuous vector bundles is the following. One should note that the regular representation of a groupoid $G \rightrightarrows M$ with Haar system is defined on a continuous field of L^2 functions on the target fibers. Even for very simple étale groupoids this is not a locally trivial field (consider e.g. the family of groups $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R})\backslash\{(-1,0)\} \to \mathbb{R}$).

But there is another reason for our interest in representations of groupoids on continuous fields of Hilbert spaces. In the procedure of geometric quantization of a family of Kähler manifolds one construct a continuous fields of Hilbert spaces. The symmetry of a smooth family of smooth manifolds is descibed by a Lie groupoid action. In [2, 3] we introduce a notion of *Hamiltonian* groupoid action, generalizing the notion of Hamiltonian group action. We show how such a Hamiltonian action on a family of Kähler manifold induces a representation of the groupoid on the continuous field of Hilbert spaces obtained by geometric quantization. The interest in representations on continuous fields is particularly clear if one would formulate this type of geometric quantization in terms of groupoid equivariant KK-theory (cf. [2, 3, 14]).

Section 3 introduces representations of groupoids on continuous fields of Hilbert spaces. We discuss several notions of continuity of representations and show how they relate. Then we treat two examples, namely the regular representation of a groupoid and representations of continuous families of groups. In the last part of this section we "embed" the theory of continuous groupoid representations in the theory of group representations. We discuss the topological group of global bisections of a groupoid and give a theorem that explains which representations of this group correspond to representations of the groupoid. Hence one could view the representation theory of groupoids as a way to understand some specific representations of certain 'infinite-dimensional' groups.

Section 4 treats harmonic analysis in the case of groupoids. We prove an analogue of Schur's Lemma and two versions of the Peter-Weyl Theorem. As one will see, proofs of theorems in representation theory of groupoids heavily rely on the representation theory of groups. The differences mostly arise in dealing with the global topology of the groupoid and its orbit foliation. The last part of this section discusses the representation rings of a groupoid and the relation of those rings to the K-theory of the reduced C^*-algebra of the groupoid.

In the spirit of Dixmier (cf. [5]) one would like to relate the representations of the groupoid to the representations of some operator algebra. Instead of an operator algebra we introduce a category of operators, a Banach $*$-category, which turns out to be most suitable in our situation. We relate the representations of a groupoid $G \rightrightarrows M$ to the representations of the continuous Banach $*$-category $\hat{L}^1(G)$. Let us mention that representations of groupoids were also studied by J. Renault [21]. But one should note that the representations discussed there are measurable representations on measurable fields of Hilbert spaces. These behave quite differently from continuous representations as studied in the present thesis. Renault discusses the relation of these measurable representations to representations of the C^*-algebra of $G \rightrightarrows M$. Section 5 discusses a continuous analogue of Renault's theorem that gives a bijection between measurable representations of $G \rightrightarrows M$ and non-degenerate representations of the Banach algebra $L^1(G)$. We construct a bijection between continuous representations of $G \rightrightarrows M$ and continuous non-degenerate representations of the Banach $*$-category $\hat{L}^1(G)$.

For this paper to be a more or less independent read we have added some preliminaries.

2 Preliminaries

2.1 Groupoids

This section contains an introduction in (continuous) groupoids with emphasis on examples.

Definition 2.1. A **groupoid** is a (small) category in which all arrows are invertible. In other (and considerably more) words, a groupoid is a septuple

$$(G_0, G_1, s, t, m, u, i)$$

consisting of

(i) a set of objects G_0;

(ii) a set of arrows G_1;

(iii) a source map $s : G_1 \to G_0$;

(iv) a target map $t : G_1 \to G_0$;

(v) an associative partial multiplication

$$m : G_2 := G_1 \,{}_s\!\times_t G_1 \to G_1, (g,h) \mapsto m(g,h) =: g\,h,$$

satisfying $s(h\,g) = s(g)$ and $t(h\,g) = t(h)$ for all $(h,g) \in G_2$, where

$$G_1 \,{}_s\!\times_t G_1 = \{(g,h) \in G_1 \times G_1 \mid s(g) = t(h)\}$$

denotes a fibered product;

(vi) a unit map $u : G_0 \to G_1$ such that $u(x) =: 1_x$ is a left unit for $G^x := t^{-1}(x)$ and a right unit for $G_x := s^{-1}(x)$ for all $x \in G_0$, i.e. $u(x)\,g = g$ for all $g \in G^x$ and $g\,u(x) = g$ for all $g \in G_x$;

(vii) an inverse map $i : G_1 \to G_1$ such that $g^{-1} := i(g)$ is a 2-sided inverse of g for all $g \in G$, i.e. $g^{-1}\,g = u(s(g))$ and $g\,g^{-1} = u(t(g))$.

We shall denote a groupoid not by the septuple $(G_0, G_1, s, t, m, u, i)$, but simply by $G_1 \rightrightarrows G_0$.

Definition 2.2. A groupoid is **continuous** if G_1 and G_0 are topological spaces and the maps s, t, m, u and i are continuous.

Example 2.3 (Groupoids from a space)**.** Suppose X is a space. Then the **trivial groupoid** $X \rightrightarrows X$ is a continuous groupoid consisting of just one unit arrow $u(x)$ for every $x \in X$.

Also, one can consider the **pair groupoid** $X \times X \rightrightarrows X$. It has source map $s(y,x) := x$ and target map $t(y,x) := y$ for every pair $(y,x) \in X \times X$. Multiplication is given by $(z,y)(y,x) = (z,x)$, the unit map is $u(x) = (x,x)$ and inversion is defined by $(y,x)^{-1} = (x,y)$.

Every **equivalence relation** $R \subset X \times X$ is a continuous groupoid $R \rightrightarrows X$ in the subspace topology.

Suppose $G \rightrightarrows G_0$ is a groupoid. We use the notation $G_x^y := G_x \cap G^y$ for $x, y \in G_0$. The set G_x^x has the structure of a group and is called the **isotropy group** of G at $x \in G_0$. The set $t(G_x) = s(G^x) \subset G_0$ is the **orbit** through $x \in G_0$. A groupoid is **transitive** if G_0 consists of one orbit. To any groupoid $G \rightrightarrows G_0$ one associates the **orbit relation groupoid** $R_G := (t \times s)(G) \rightrightarrows G_0$. This is a particular example of an equivalence relation on G_0. It is a continuous groupoid if s and t are open. The **orbit set of a groupoid** $G \rightrightarrows G_0$ is the set of orbits of G, denoted by G_0/G. If $G \rightrightarrows G_0$ is continuous, then the orbits, isotropy groups and orbit set have an induced topology and the latter is called the **orbit space of a groupoid**.

Obviously, a pair groupoid $X \times X \rightrightarrows X$ is transitive. For an equivalence relation $R \subset X \times X$ as a groupoid $R \rightrightarrows X$, orbits correspond to equivalence classes.

Example 2.4. Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of a locally compact space X. We use the notation $U_{ij} := U_i \cap U_j$ for $i, j \in I$. Consider the groupoid

$$\coprod_{i,j\in I} U_{ij} \rightrightarrows \coprod_{i\in I} U_i,$$

where the source map is the inclusion $U_{ij} \to U_j$ and the target map is the inclusion $U_{ij} \to U_i$. Composition $U_{ij}\ {}_s\times_t U_{jk} \to U_{ik}$ is $(x, y) \mapsto x\,(= y)$. The unit is the identity $U_i \mapsto U_{ii} = U_i$. The inverse map is the identity $U_{ij} \to U_{ji}$. This groupoid is called the **cover groupoid** associated to the cover $\mathcal{U}$ of X. This cover groupoid is an **étale groupoid**, which means that $s : G \to X$ and $t : G \to X$ are local homeomorphisms.

Example 2.5 (Groups and actions)**.** Any topological group H can be seen as a continuous groupoid $H \rightrightarrows pt$ over a one-point set pt.

Suppose H acts from the left on a space X. Then one can construct the **action groupoid** $H \ltimes X \rightrightarrows X$, with $(H \ltimes X)_1 = H \times X$ and $(H \ltimes X)_0 = X$, $s(h, x) := x$, $t(h, x) := h \cdot x$, $(h', h \cdot x)(h, x) = (h'\,h, x)$, $u(x) := (e, x)$ and $(h, x)^{-1} = (h^{-1}, h \cdot x)$. This groupoid is continuous if the action is continuous.

A continuous groupoid $G \rightrightarrows G_0$ is **proper** if $t \times s : G \to G_0 \times G_0$ is a proper map. In particular, an action groupoid $G = H \ltimes X \rightrightarrows X$ is proper iff the action of H on X is proper. The action groupoid is transitive iff the action is transitive. The isotropy group G_x^x corresponds with the isotropy group of the action of H at $x \in X$.

Example 2.6 (The symmetry of a map)**.** The symmetries of an object X in a category $\mathcal{C}$ are given by the group of automorphisms $\mathrm{Aut}_{\mathcal{C}}(X)$ of the object. The group $\mathrm{Aut}_{\mathcal{C}}(X)$ is in general 'very large' and one instead studies morphisms $H \to \mathrm{Aut}_{\mathcal{C}}(X)$ for smaller groups H.

What is the symmetry of a map $f : X \to Y$ in $\mathcal{C}$? An automorphism of $f : X \to Y$ in the category of arrows in $\mathcal{C}$ consists of pair of automorphisms $\phi \in \mathrm{Aut}_{\mathcal{C}}(X)$ and $\psi \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $\psi \circ f = f \circ \phi$. Suppose $\mathcal{C}$ is a category of sets, i.e. $\mathcal{C} \subset$ **Sets**. If f is surjective, then the automorphism ϕ of X fixes the automorphism ψ of Y, hence the automorphisms of f form a subgroup of the automorphisms of X. Since Y is a set, any automorphism (ϕ, ψ) of f decomposes as a family of isomorphisms $\{\phi_y : f^{-1}(y) \to f^{-1}(\psi(y))\}$. If $\psi(y) = y$, then ϕ_y is called an **internal symmetry of the map**, else it is called an **external symmetry** (cf. [25]). The union of all internal and external symmetries has the structure of a groupoid $\mathrm{Aut}(f) \rightrightarrows Y$. Indeed, one defines

$$\mathrm{Aut}_{\mathcal{C}}(f) := \bigcup_{y,y'\in Y} \mathrm{Iso}_{\mathcal{C}}(f^{-1}(y), f^{-1}(y')),$$

with obvious structure maps. We call $\mathrm{Aut}_{\mathcal{C}}(f) \rightrightarrows Y$ the **automorphism groupoid of a map** f. Again, the groupoid $\mathrm{Aut}_{\mathcal{C}}(f) \rightrightarrows Y$ is in general 'very large' and one instead studies morphisms $G \to \mathrm{Aut}_{\mathcal{C}}(f)$ for smaller groupoids $G \rightrightarrows Y$, which are called **groupoid actions** of $G \rightrightarrows Y$ on the map f.

Example 2.7. Suppose X is a topological space. The set of homotopy classes of paths $\gamma : [0, 1] \to X$ form a groupoid $\pi_1(X) \rightrightarrows X$, called the **fundamental**

groupoid. The source map is defined by $s([\gamma]) := \gamma(0)$ and the target map is $t([\gamma]) := \gamma(1)$. Composition is induced by concatenation of paths

$$\gamma' \cdot \gamma(t) := \begin{cases} \gamma(2t) & \text{if } t \leq 1/2 \\ \gamma'(2t-1) & \text{if } t > 1/2 \end{cases}$$

The unit $u(x)$ at $x \in X$ is defined by the constant path $[x]$ and the inverse is defined by $[\gamma]^{-1} := [\gamma^{-1}]$, where $\gamma^{-1}(t) := \gamma(1-t)$.

As a generalization of this, consider the fundamental groupoid $\pi_1(f) \rightrightarrows X$ of a surjective continuous map $f : X \to Y$, which consists of homotopy classes of paths restricted to the fibers $f^{-1}(y)$ for $y \in Y$.

We always assume G_0 to be Hausdorff in this text. A continuous groupoid $G \rightrightarrows G_0$ is **Hausdorff** if G is a Hausdorff space. This is not always the case. For example, consider the projection to the first coordinate axis $p_1 : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$. Then $\pi_1(p_1) \rightrightarrows \mathbb{R}$ is not Hausdorff. The orbit space G_0/G is also not Hausdorff in many examples (independently of G being Hausdorff or not). But, if $G \rightrightarrows G_0$ is proper, then G_0/G is Hausdorff.

Example 2.8 (Gauge groupoids)**.** Suppose H is a topological group and $\pi : P \to X$ a continuous principal H-bundle over a space X. Consider the groupoid $P \times_H P \rightrightarrows X$, where $P \times_H P$ is the orbit space of the diagonal action of H on $P \times P$. It obtains a groupoid structure from the pair groupoid $P \times P \rightrightarrows P$. Indeed, $s := \pi \circ pr_2$ and $t := \pi \circ pr_1$; composition $[p, q][q', r] = [p, q][q, h \cdot r] = [p, h \cdot r]$, where $p, q, q', r \in P$, $\pi(q) = \pi(q')$ and $h \in H$ is unique such that $h \cdot q' = q$. The unit is defined by $u(x) =: [p, p]$ for any $p \in \pi^{-1}(x)$ and the inverse is given by $[p, q]^{-1} := [q, p]$ for all $p, q \in P$. This groupoid is called the **gauge groupoid** of $P \to X$. There is a canonical action of $P \times_H P \rightrightarrows X$ on $\pi : P \to X$, given by $[p, q] \cdot q = p$ for $p, q \in P$.

In particular, the universal covering space $\tilde{X} \to X$ of a connected space X is a principal $\pi_1(X, x)$-bundle, where $\pi_1(X, x)$ is the fundamental group of X at a fixed point $x \in X$. The gauge groupoid $\tilde{X} \times_{\pi_1(X,x)} \tilde{X} \rightrightarrows X$ associated to $\tilde{X} \to X$ is isomorphic to the fundamental groupoid $\pi_1(X) \rightrightarrows X$.

Suppose $E \to M$ is a continuous complex rank n vector bundle. Let $F_E := \mathrm{GL}_M(\mathbb{C}^n, E) \to M$ denote the frame bundle of $E \to M$, i.e. the fiber $\mathrm{GL}_M(\mathbb{C}^n, E)_m$ at m equals the space $\mathrm{GL}(\mathbb{C}^n, E_m)$ of invertible linear maps $\mathbb{C}^n \to E_m$. This is a principal $\mathrm{GL}(\mathbb{C}^n)$-bundle. The associated gauge groupoid $F_E \times_{\mathrm{GL}(\mathbb{C}^n)} F_E \rightrightarrows M$ is obviously isomorphic to $\mathrm{GL}(E, E) \rightrightarrows M$, the **general linear groupoid** of E, where

$$GL(E, E)^n_m := Gl(E_m, E_n),$$

the space of invertible linear maps $E_m \to E_n$ for all $m, n \in M$. There are canonical actions of $\mathrm{GL}(E, E) \rightrightarrows M$ on $E \to M$ and on $F_E \to M$.

Let g be a hermitian metric on $E \to M$. Then we can analogously construct the unitary frame bundle $F_E^U := U(\mathbb{C}^n, E) \to M$ and the **unitary groupoid** $U(E) := U(E, E) \rightrightarrows M$ of E.

Gauge groupoids are transitive. Conversely, any transitive groupoid is isomorphic to the gauge groupoid of a principal bundle. Indeed, suppose $G \rightrightarrows X$ is a continuous transitive groupoid. Choose an $x \in X$. The source map $s : G^x \to X$ is a left principal bundle for the left action of G_x^x by multiplication. One can form the gauge groupoid $G_x \times_{G_x^x} G_x \rightrightarrows X$ of this bundle. It is easy to show that $G \cong G_x \times_{G_x^x} G_x$.

Example 2.9 (Families of groups, bundles of groups and group bundles)**.** Suppose $G \rightrightarrows X$ is continuous groupoid. The union of isotropy groups $\bigcup_{x\in X} G_x^x \subset G$ is a continuous groupoid over X in the subspace topology. It is denoted by $I_G \rightrightarrows X$ and is called the **isotropy groupoid** of G. Note that $s(g) = t(g)$ for all $g \in I_G$. Groupoids $G \rightrightarrows G_0$ for which $s(g) = t(g)$ for all $g \in G$ are called **families of groups**. If $G_1 \to G_0$ is a family of groups and a fiber bundle (as a space), then it is called a **bundle of groups**.

Given a space X and a topological group H, we can construct the basic example, $X \times H \to X$, a trivial family of groups over X. We call a continuous family of groups $G \to X$ a **group bundle** if for every point $x \in X$ there exists a neighborhood $U \ni x$, a topological group H and an isomorphisms of families of groups $G|_U \to H \times U$ (with the obvious notion of morphism).

A group bundle with fiber H can be constructed from a principal H-bundle $P \to X$. Indeed, consider the associated bundle $P \times_H H \to X$ where H acts on itself by conjugation. The multiplication is defined by $[p, h] \cdot [p, h'] := [p, h\, h']$. One easily sees that this is well-defined. Idem dito for the unit map $u(x) = [p, e]$ for $x \in X$, $e \in H$ the unit and any $p \in P$. The inverse is also obtained from the inverse of H, $[p, h]^{-1} = [p, h^{-1}]$. The bundle of groups $P \times_H H \to X$ is isomorphic to the isotropy groupoid of $P \times_H P \rightrightarrows X$.

Definition 2.10. A **morphism of continuous groupoids**

$$(G \rightrightarrows G_0) \to (H \rightrightarrows H_0)$$

is a pair of continuous maps $\phi_1 : G \to H$ and $\phi_0 : G_0 \to H_0$ that commutes with the structure maps, i.e. $s \circ \phi_1 = \phi_0 \circ s$, $\phi_1(g \cdot g') = \phi_1(g) \cdot \phi_1(g')$, etcetera.

2.2 Haar systems

This section contains an introduction to Haar systems on groupoids based on [21, 22].

Suppose X and Y are locally compact spaces and $p : Y \to X$ is a continuous surjection. A **continuous family of Radon measures on** $p : Y \to X$ is a family of Radon measures $\{\nu_x\}_{x\in X}$ on Y such that

- the support of ν_x is a subset of $p^{-1}(x) =: Y_x$ and
- for every function $f \in C_c(Y)$ the function

$$x \mapsto \int_{y\in Y_x} f(y)\, \nu_x(dy)$$

is continuous $X \to \mathbb{C}$.

Suppose $G \rightrightarrows X$ is locally compact, second countable continuous groupoid.

Definition 2.11. A **left Haar system on** $G \rightrightarrows X$ is a continuous family of Radon measures $\{\lambda^x\}_{x\in X}$ on $t : G \to X$ that is left-invariant, i.e. for all $x, y \in X$, $h \in G_x^y$, and $f \in C_c(G)$,

$$\int_{g\in G^x} f(hg)\lambda^x(dg) = \int_{g\in G^y} f(g)\lambda^y(dg).$$

There is an analogous notion of right Haar system.

Example 2.12. Suppose X is a locally compact space. Trivial counting measures $\{\lambda^x\}_{x\in X}$ form a Haar system on the trivial groupoid $X \rightrightarrows X$. If ν is a Radon measure on X, then $\{\nu^x := \nu\}_{x\in X}$ is a Haar system on the pair groupoid $X \times X \rightrightarrows X$.

Example 2.13. If H is a locally compact group and κ a left Haar measure on H. Then κ is a Haar system on $H \rightrightarrows pt$. Suppose H acts on a locally compact space X. Then $\{\lambda^x := \kappa\}_{x\in X}$ forms a left Haar system on the action groupoid $H \ltimes X \rightrightarrows X$.

Suppose $p : P \to X$ is a left principal H-bundle. Suppose $x \in X$ and $\phi : P|_U \to U \times H$ is a local trivialization of $P \to X$ on a neighborhood U of x. The obvious Haar system on $U \times H \to U$ can be pushed forward to $P|_U$, that is $\kappa^x := (\phi^{-1})_*\kappa$. Since κ is left H-invariant this unambiguously defines a continuous family of H-invariant Radon measures on $p : P \to X$. Suppose ν is a Radon measure on X. We define a continuous family of Radon measures on $p \circ pr_2 : P \times P \to X$ by

$$\tilde{\lambda}^x := \int_{y\in X} \kappa^y \times \kappa^x \, \nu(dy),$$

which is H-invariant under the diagonal action of H and hence descends to a left Haar system $\{\lambda^x\}_{x\in X}$ on the gauge groupoid $P \times_H P \rightrightarrows X$.

Example 2.14. Suppose $p : G \to X$ is a locally compact continuous family of groups. By a classical result there exists a left Haar measure on each group $G_x := p^{-1}(x)$, unique up to multiplication by a positive constant. Renault proves that there is a specific choice of measures λ^x on G_x for $x \in X$ such that they form a Haar system if and only if p is open. Indeed, one should construct a continuous function $F : G \to \mathbb{R}$ that is compactly supported on the fibers and that satisfies $0 \leq F \leq 1$ and $F \circ u = 1$. Then the measures λ^x should be chosen such that $\int_{G_x} F\lambda^x = 1$ for every $x \in X$.

For example, consider a group bundle $p : G \to X$ on a space X with fibers isomorphic to a fixed compact group K. We can take $F = 1$. Then by the above procedure the measure λ^x has to come from the normalized Haar measure on K for each $x \in X$.

Example 2.15. Suppose $G \rightrightarrows M$ is a *Lie* groupoid. There exists a Haar system on $G \rightrightarrows M$. Indeed, one easily sees that there exists a strictly positive smooth density ρ on the manifold $\mathcal{A}(G) = u^*(T^tG)$. This can be extended to a G-invariant density $\tilde{\rho}$ on T^tG. Then we define a Haar system on $G \rightrightarrows M$ by

$$\lambda^x(f) := \int_{G^x} f\tilde{\rho},$$

for all $f \in C_c(G)$.

Suppose $G \rightrightarrows X$ is a locally compact groupoid endowed with a left Haar system $\{\lambda^x\}_{x\in X}$. Then s and t are open maps and the orbit relation groupoid $R_G \rightrightarrows X$ is a continuous locally compact groupoid. Suppose the isotropy groupoid $I_G \to X$ is endowed with a Haar system $\{\lambda^x_x\}_{x\in X}$ (not necessarily related to the Haar system on

$G \rightrightarrows X$). These measures induce a left G-invariant continuous family of measures $\{\lambda_x^y\}_{(y,x)\in R_G}$ on $t \times s : G \to R_G$ by

$$\lambda_x^y := (l_g)_*\lambda_x^x,$$

for some $g \in G_x^y$ (independence of the choice g follows from left invariance of the Haar system on I_G).

Proposition 2.16. *(cf. [22]) If $\{\lambda^x\}_{x\in X}$ is a left Haar system on $G \rightrightarrows X$ and $\{\lambda_x^x\}_{x\in X}$ is a left Haar system on $I_G \rightrightarrows X$, then there exists a left Haar system $\{\nu^x\}_{x\in X}$ on $R_G \rightrightarrows X$ such that for all $x \in X$ there is a decomposition*

$$\lambda^x = \int_{(y,x)\in R_G} \lambda_x^y \, \nu^x(d(y,x)).$$

Obviously, one can go the other way around: given a left Haar system on $I_G \rightrightarrows X$ and on $R_G \rightrightarrows X$, one forms a left Haar system on $G \rightrightarrows X$, using the same formula.

Example 2.17. Suppose $p : P \to X$ is a left principal H-bundle for a locally compact group H. Recall the Haar system $\{\lambda^x\}_{x\in X}$ on $P \times_H P \rightrightarrows X$ that we defined in Example 2.13 given a Haar measure κ on H and a Radon measure ν on X. One can apply the above Proposition 2.16 with the continuous family of measures $\{\lambda_x^x := \psi_*^x\kappa\}_{x\in X}$ on $I_{P\times_H P} \to X$, where $\psi^x : H \hookrightarrow P \times_H H \xrightarrow{\cong} I_{P\times_H P}$ is the inclusion of H at the fiber of $I_{P\times_H P} \to X$ at x. One obtains the Haar system $\{\nu^x := \nu\}_{x\in X}$ on $R_G = X \times X \rightrightarrows X$ in the decomposition.

Example 2.18. Suppose a locally compact group H acts on locally compact space X. Given a Haar measure κ on H we constructed a Haar system on $G := H \ltimes X \rightrightarrows X$ in Example 2.13. Suppose we have constructed a continuous family of measures on the family $I_G \to X$ of isotropy groups of the action using Example 2.14. Applying Proposition 2.16 one obtains a measure ν_x on each orbit $G\,x$ such that the decomposition of Proposition 2.16 holds.

Definition 2.19. Suppose $G \rightrightarrows X$ is a groupoid endowed with a Haar system $\{\lambda_x\}_{x\in X}$. A **cutoff function for** $G \rightrightarrows X$ is a function $X \to \mathbb{R}_{\geq 0}$ such that

- the support of $(c \circ s)|_{t^{-1}K}$ is compact for all compact sets $K \subset M$;
- for all $x \in X$, $\int_{G^x} c(s(g))\lambda^x(dg) = 1$.

A cutoff for $G \rightrightarrows X$ exists iff $G \rightrightarrows X$ is proper (cf. [23]). Cutoff functions are useful in averaging processes. If G^x is compact for all $x \in X$, then one can simply take $c(x) = 1/\lambda^x(G^x)$.

2.3 Continuous fields of Banach and Hilbert spaces

This section contains an introduction to continuous fields of Banach spaces and continuous fields of Hilbert spaces. In contrast with the previous section, this one does contain some proofs. This is because ingredients of the proofs are needed in the paper. Most of this material of the first section can be found in [6].

Suppose X is a locally compact Hausdorff space.

Definition 2.20. A **continuous field of Banach spaces over** X is a family of Banach spaces $\{\mathcal{B}_x\}_{x\in X}$ and a space of sections $\Delta \subset \prod_{x\in X} \mathcal{B}_x$, such that

(i) the set $\{\xi(x) \mid \xi \in \Delta\}$ equals $\mathcal{B}_x$ for all $x \in X$.

(ii) For every $\xi \in \Delta$ the map $x \mapsto \|\xi(x)\|$ is in[1] $C_0(X)$.

(iii) Δ is locally uniformly closed, i.e. if $\xi \in \prod_{x\in X} \mathcal{B}_x$ and for each $\varepsilon > 0$ and each $x \in X$, there is an $\eta \in \Delta$ such that $\|\xi(y) - \eta(y)\| < \varepsilon$ on a neighborhood of x, then $\xi \in \Delta$.

Remark 2.21. By composing the map $x \mapsto \|\xi(x)\|_{\mathcal{B}_x}$ with the norm on $C_0(X)$ one obtains a norm

$$\|\xi\| = \sqrt{\sup_{x\in X} \|\xi(x)\|^2_{\mathcal{B}_x}}$$

on Δ. From (iii) it follows at once that Δ is complete in this norm.

There is a subclass of these continuous fields which has our special interest.

Definition 2.22. A **continuous field of Hilbert spaces over** X is a family of Hilbert spaces $\{\mathcal{H}_x\}_{x\in X}$ and a space of sections $\Delta \subset \prod_{x\in X} \mathcal{H}_x$ that form a continuous field of Banach spaces.

Example 2.23. Suppose $p : E \to X$ is continuous complex vector bundle endowed with a Hermitian metric $g : E \times E \to \mathbb{C}$. Then $(\{E_x\}_{x\in X}, \Gamma_0(E))$ is a continuous field of Hilbert spaces.

Example 2.24. Suppose $\mathcal{H}$ is a fixed Hilbert space and X a topological space. Then $(\{\mathcal{H}\}_{x\in X}, \Gamma_0(X \times \mathcal{H})$ is a (trivial) continuous field of Hilbert spaces.

Remark 2.25. In the case of a continuous field of Hilbert spaces, the condition (ii) in Definition 2.20 can be replaced by the requirement that for any $\xi, \eta \in \Delta$ the map $x \mapsto \langle \xi(x), \eta(x)\rangle_{\mathcal{H}_x}$ is in $C_0(X)$. The field is called **upper (lower) semi-continuous** if $x \mapsto \|\xi(x)\|$ is just upper (lower) continuous for every $\xi \in \Delta$.

Lemma 2.26. *If $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ is a continuous field of Banach spaces, then Δ is a left $C_0(X)$-module.*

Proof. Suppose $f \in C_0(X)$ and $\xi \in \Delta$. Let $\varepsilon > 0$ and $x \in X$ be given. Define

$$V_x := \{y \in X \mid |f(x) - f(y)| < \frac{\varepsilon}{\|\xi(x)\| + 1} \text{ and } |\|\xi(x)\| - \|\xi(y)\|| < 1\}$$

Then, for $y \in V_x$

$$\|f(y)\xi(y) - f(x)\xi(y)\| < \frac{\varepsilon}{\|\xi(x)\| + 1}\|\xi(y)\| < \varepsilon.$$

Since $f(x)\xi \in \Delta$, we conclude by (iii) that $f\xi \in \Delta$. □

Actually Δ is a Banach C^*-module as we shall see in Section 2.5.

[1] $f \in C_0(X)$ iff f is continuous and for every $\varepsilon > 0$ there exists a compact set K such that $f(x) < \varepsilon$ if $x \notin K$

Lemma 2.27. *If $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ is a continuous field of Banach spaces, then there is a topology on the total space $\mathcal{B} := \coprod_{x\in X} \mathcal{B}_x$ such that Δ equals the set of continuous sections $\Gamma_0(\mathcal{B}) := \{\xi \in \Gamma(\mathcal{B}) \mid \|\xi\| \in C_0(X)\}$.*

Proof. For each $\varepsilon > 0$, $V \subset X$ open and $\xi \in \Delta$, we define

$$U(\varepsilon, \xi, V) := \{h \in \mathcal{B} \mid \|h - \xi(p(h))\| < \varepsilon \text{ and } p(h) \in V\},$$

where $p : \mathcal{B} \to X$ is the projection of the total space on the base. One easily sees that these sets form a basis for a topology on $\mathcal{B}$. Indeed, suppose that $U(\varepsilon_1, \xi_1, V_1)$ and $U(\varepsilon_2, \xi_2, V_2)$ are two of them and $h \in \mathcal{B}$ lies in the intersection. By (i) there is a $\xi \in \Delta$ such that $\xi(x) = h$, where $x = p(h)$. Let $\varepsilon_i' = \varepsilon_i - \|h - \xi_i(x)\|$ for $i = 1, 2$. Choose any $\varepsilon > 0$ such that $\varepsilon < \varepsilon_i'$ for $i = 1, 2$. Define

$$V := \{x \in V_1 \cap V_2 \mid \|\xi(x) - \xi_i(x)\| < \varepsilon_i - \varepsilon \text{ for } i = 1, 2\}.$$

Then $U(\varepsilon, \xi, V) \subset U(\varepsilon_1, \xi_1, V_1) \cap U(\varepsilon_2, \xi_2, V_2)$.

Suppose $\xi \in \prod_{x\in X} \mathcal{B}_x$ is a continuous section. Let $\varepsilon > 0$ and $x \in X$ be given. Define $h := \xi(x)$. There is a $\xi' \in \Delta$ such that $\xi'(x) = h$. Let V be any open neighborhood of x, then $W := \xi^{-1}U(\varepsilon, \xi', V)$ is open and on W we have $\|\xi' - \xi\| < \varepsilon$. By (iii) we conclude that $\xi \in \Delta$.

Conversely, suppose $\xi \in \Delta$. Let $U(\varepsilon, \eta, V)$ be an open set in $\mathcal{B}$, then

$$\begin{aligned} \xi^{-1}U(\varepsilon, V, \eta) &= p(U(\varepsilon, \eta, V) \cap \xi(V)) \\ &= \{x \in X \mid \|\xi(x) - \eta(x)\| < \varepsilon\} \end{aligned}$$

Note that $\xi - \eta \in \Delta$, hence $x \mapsto \|\xi(x) - \eta(x)\|$ is continuous. We conclude that the above set is open, so that $\xi \in \Gamma_0(\mathcal{B})$. □

Remark 2.28. As a short notation we often denote a continuous field of Banach spaces $(\{\mathcal{B}_x\}_{x\in X}, \Delta_{\mathcal{B}})$ by $(\mathcal{B}, \Delta)$.

Lemma 2.29. *For any continuous field of Banach spaces $(\mathcal{B}, \Delta)$ the map $\|.\| : \mathcal{B} \to \mathbb{R}_{\geq 0}$ is continuous.*

Proof. Suppose $h \in \mathcal{B}_x$ for certain $x \in X$. Given $\varepsilon > 0$, take a $\xi \in \Delta$ such that $\xi(x) = h$ and

$$V := \|\xi\|^{-1}(\|h\| - \varepsilon/2, \|h\| + \varepsilon/2).$$

This is an open set, since $\|\xi\| : X \to \mathbb{R}_{\geq 0}$ is continuous. So, $h' \in U(\varepsilon/2, \xi, V)$, with $h' \in \mathcal{B}_{x'}$ implies

$$|\|h'\|_{x'} - \|h\|_x| \leq \|h' - \xi(x')\| + |\|\xi(x')\|_{x'} - \|h\|_x| \leq \varepsilon,$$

which finishes the proof. □

Definition 2.30. A **morphism** $\Psi : (\mathcal{B}^1, \Delta^1) \to (\mathcal{B}^2, \Delta^2)$ **of continuous fields of Banach spaces** is a family of bounded linear maps $\{\Psi_x : \mathcal{B}_x^1 \to \mathcal{B}_x^2\}_{x\in X}$ such that the induced map $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ on the total spaces satisfies

$$\{\Psi \circ \xi \mid \xi \in \Delta^1\} \subset \Delta^2$$

and

$$x \mapsto \|\Psi_x\|$$

is a locally bounded map.

Here $\|\Psi_x\|$ is the operator norm of Ψ_x,

$$\|\Psi_x\| := \sup_{\|h\|_{\mathcal{B}^1_x}=1} \|\Psi_x(h)\|_{\mathcal{B}^2_x}.$$

The first condition has to be satisfied only on a dense subset of Δ^1 ([6], Proposition 5).

Lemma 2.31. *The map $\Psi : \mathcal{B}^1 \to \mathcal{B}^2$ is continuous iff Ψ is a morphism of continuous fields of Banach spaces.*

Proof. "$\Leftarrow$" Suppose $h \in U(\epsilon_2, \xi_2, V_2) \subset \mathcal{B}_2$ and $p(h) = x$. By (i), there is a $\xi_1 \in \Delta_1$ such that $\xi_1(x) = h$. Since $\Psi(\xi_1) \in \Delta_2$, the set defined by

$$V_1 := \{y \in X \mid \|\Psi(\xi_1) - \xi_2\|(y) < \varepsilon/2\} \cap V_2$$

is open. Let $f : X \to \mathbb{R}$ be a locally bounded function such that $\|\Psi(\xi)\| < f\|\xi\|$ for all $\xi \in \Delta$. Let $V_1' \subset V_1$ be a small enough neighborhood of x such that f has a supremum K on V_1', then

$$\Psi(U(\frac{\varepsilon_2}{2K}, \xi_1, V_1') \subset U(\varepsilon_2, \xi_2, V_2)$$

Indeed, for any $h' \in U(\frac{\varepsilon_2}{2K}, \xi_1, V_1')$ with $p(h') = y$ we have

$$\begin{aligned} \|\Psi(h') - \xi_2(y)\| &= \|\Psi(h') - \Psi(\xi_1(y)) + \Psi(\xi_1(y)) - \xi_2(y)\| \\ &\leq \|\Psi(h' - \xi_1(y))\| + \|\Psi(\xi_1(y)) - \xi_2(y)\| \\ &= K\|h' - \xi_1(y)\| + \tfrac{\varepsilon_2}{2} \\ &= K\tfrac{\varepsilon_2}{2K} + \tfrac{\varepsilon_2}{2} = \varepsilon_2. \end{aligned}$$

"$\Rightarrow$" $\Psi(\Delta_1) \subset \Delta_2$ by Lemma 2.27. Let $x \in X$ be any element. By continuity $\Psi^{-1}(U(1, 0, X))$ is open, so it contains an open neighborhood $U(\varepsilon, 0, V)$, where V is an open neighborhood of x. Hence, $\|\Psi\|$ is bounded on V. □

The map $\Psi : (\mathcal{B}^1, \Delta^1) \to (\mathcal{B}^2, \Delta^2)$ is an (isometric) isomorphism of continuous fields of Banach spaces if all the Ψ_x are (isometric) isomorphisms and $\Psi(\Delta^1) = \Delta^2$. In fact, one can replace the second condition by $\Psi(\Lambda) \subset \Delta^2$ for a dense subset $\Lambda \subset \Delta^1$ ([6], Proposition 6).

Let $(\{\mathcal{B}_x\}_{x \in X}, \Delta)$ be a continuous field of Banach spaces over X and $J : Y \to X$ a continuous map. Define the **pullback continuous field** $J^*(\{\mathcal{B}_x\}_{x \in X}, \Delta)$ as follows. The fiber $(J^*\mathcal{B})_y$ at $y \in Y$ is the Banach space $\mathcal{B}_{J(y)}$. The space of sections $J^*\Delta$ is the closure of the linear space generated by elements of the form $f \cdot J^*\xi$ for all $\xi \in \Delta$ and $f \in C_0(Y)$ in the usual norm (cf. Remark 2.21), which takes the form

$$\|f \cdot J^*\xi\| := \sqrt{\sup_{y \in Y} |f(y)|^2 \, \|\xi(y)\|^2_{\mathcal{B}_{J(y)}}}$$

on generators. The continuous field thus obtained is denoted by $(J^*\{\mathcal{B}_x\}_{x \in X}, J^*\Delta)$.

2.4 Dimension and local pseudo-trivializations

This section contains a characterization of (uniformly) finite-dimensional continuos fields of Hilbert spaces.

The **dimension of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ over X is the supremum of the function

$$\dim : X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, x \mapsto \dim(\mathcal{H}_x).$$

A continuous field of Hilbert spaces is **uniformly finite-dimensional** if it has finite dimension. One should distinguish between uniformly finite-dimensional and **finite-dimensional** continuous fields, which means that each fiber is finite dimensional.

Example 2.32. Consider the field over $\mathbb{R}$ with $\mathcal{H}_x := \mathbb{C}^n$ if $x \in [-n, -n+1) \cup (n-1, n]$ for all $n \in \mathbb{N}$ and $\mathcal{H}_0 = 0$. The topology on the field comes from the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ on the first n coordinates. The inner product on each fiber is the standard Hermitian metric on $\mathbb{C}^n$. This field is finite-dimensional, but not uniformly finite-dimensional.

Definition 2.33. A continuous field $(\mathcal{H}, \Delta)$ is **locally trivial** if for every $x \in X$ there exist a neighborhood $U \ni x$, a Hilbert space $\mathcal{H}'$ and an isomorphism of continuous fields $\mathcal{H}|_U \to U \times \mathcal{H}'$.

Example 2.34. Locally trivial finite-dimensional continuous fields of Hilbert spaces are known as complex vector bundles with Hermitian metric (cf. Example 2.23 and 2.24).

Obviously a continuous field is not always locally trivial, see e.g. Example 2.32 or

Example 2.35 (Winding stair)**.** Consider the continuous field over $\mathbb{R}^2$ defined as follows. Fix any $d \in \mathbb{N}$ (the number of stairs). For $\vec{x} \in \mathbb{R}_{\geq 0} \times \{0\}$ let $\mathcal{H}_{\vec{x}} = 0$. For $\vec{x}$ in the sector between angles $i\, 2\pi/d$ and $(i+1)\, 2\pi/d$ (including the positive halfline at $(i+1)\, 2\pi/d$, unless $i = d-1$) take $\mathcal{H}_{\vec{x}} = \mathbb{C}^{i+1}$, where $i = 0, \ldots d-1$. Again, the topology on the field comes from the inclusion $\mathbb{C}^i \hookrightarrow \mathbb{C}^{i+1}$ on the first i coordinates. The inner product on each fiber is the standard Hermitian inner product on $\mathbb{C}^i$. Obviously, the field is not locally trivial at the origin.

Therefore, we introduce the notion of local pseudo-trivializations.

Definition 2.36. A **local pseudo-trivialization** of a continuous field $(\mathcal{H}, \Delta)$ on an open set $U \subset X$ is a family of open sets $\{U_i\}_{0 \geq i \geq \dim(\mathcal{H}|_U)}$, such that

$$\bigcup_{0 \geq i \geq \dim(\mathcal{H}|_U)} U_i = U$$

and

$$\mathcal{H}|_U \simeq \sum_{0 \geq i \geq \dim(\mathcal{H}|_U)} U_i \times \mathbb{C}^i.$$

Such local pseudo-trivializations shall be useful in Section 3.4.

Lemma 2.37. *Suppose $(\mathcal{H}, \Delta)$ is a uniformly finite-dimensional continuous field of Hilbert spaces over X. Then for any $x \in X$ there exists a neighborhood U_x of x such that $(\mathcal{H}, \Delta)$ admits a local pseudo-trivialization on U_x.*

Proof. We prove by induction on the dimension d of $(\mathcal{H}, \Delta)$. Suppose $d = 1$. If $\mathcal{H}_x \neq 0$, then there exists a $\xi \in \Delta$ such that $\xi(x) \neq 0$. By continuity, $\xi(y) \neq 0$ for y in a neighborhood U of x. Obviously, the map $U \times \mathbb{C} \to \mathcal{H}|_U$ given by $(y, z) \mapsto z\,\xi(y)$ is a local (pseudo-)trivialization.

If $\mathcal{H}_x = 0$, then we proceed as follows. For every $y \in X$ for which with $\mathcal{H}_y \neq 0$ there exists a section ξ_y such that $\xi_y(y)$ spans $\mathcal{H}_y$. Since X is locally compact, there exists an open neighborhood U (with compact closure) and a subset $\{y_i\}_{i\in I} \subset X$ such that $\{\mathrm{supp}(\xi_{y_i})\}_{i\in I}$ forms a locally finite cover of $U \cup \mathrm{supp}(\mathcal{H})$. For any subsequence $\{y_j\}_{j\in J}$ converging to x, we add the $\lim_{j\to J} \xi_{y_j} \in \Delta$ to the set of sections indexed by I. By continuity the sum $\xi(y) := \sum_{i\in I} \xi_{y_i}(y)$ still spans $\mathcal{H}_y$ for y in a small enough open neighborhood U' of x. Again, the map $U' \times \mathbb{C} \to \mathcal{H}|_U$ given by $(y, z) \mapsto z\,\xi(y)$ is a local (pseudo-)trivialization.

Suppose $(\mathcal{H}, \Delta)$ has dimension d. If $\mathcal{H}_x \neq 0$, then there exists a $\xi \in \Delta$ such that $\xi(x) \neq 0$. Again, then $\xi(y) \neq 0$ for y in a neighborhood U of x. Hence $\mathcal{H}|_U \simeq \mathrm{span}\,\xi|_U \oplus \mathcal{H}'$ for some continuous field $(\mathcal{H}', \Delta')$ over U. The field $(\mathcal{H}', \Delta')$ has dimension $d-1$, so by the induction hypothesis there is an isomorphism

$$\phi : \sum_{i=0}^{d-1} U_i' \times \mathbb{C}^i \to \mathcal{H}'|_{U'}$$

on an open neighborhood U' of x. Hence, an isomorphism

$$\sum_{i=1}^{d} U_{i-1}' \times \mathbb{C}^i \to \mathcal{H}|_{U'}$$

is given by

$$(y, \vec{z}) \mapsto \phi(y, z_1, \ldots, z_{i-1}) + z_i\,\xi(y).$$

If $\mathcal{H}_x = 0$, then we construct a local section ξ on a neighborhood of x as in the case $d = 1$. Proceed as above. □

Corollary 2.38. *A continuous field $(\mathcal{H}, \Delta)$ over a compact space X is uniformly finite-dimensional iff Δ is finitely generated over $C_0(X)$.*

Lemma 2.39. *For a uniformly finite-dimensional continuous field of Hilbert spaces over X the dimension is a lower semi-continuous function*

$$\dim : X \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}.$$

That is, $\dim : X \to \mathbb{Z}_{\geq 0}$ *has a local minimum at every point.*

Proof. Suppose $(\mathcal{H}, \Delta)$ is such a continuous field of Hilbert spaces and $x \in X$. Choose sections $\xi_j^x \in \Delta$ for $j = 1, \ldots, \dim(\mathcal{H}_x)$, such that $\{\xi_j^x(x)\}_{j=1}^{\dim(\mathcal{H}_x)}$ forms a basis of $\mathcal{H}_x$. Let V_x be the set on which their images stay linearly independent and non-zero. This set is open, since, for a local pseudo-trivializations ϕ,

$$x \mapsto \det(\phi^*\xi_1^x \mid \ldots \mid \phi^*\xi_{\dim \mathcal{H}_x}^x) = \det((\langle \phi^*\xi_k^x, \phi^*\xi_l^x \rangle)_{kl})$$

is continuous. Indeed, this last expression is a polynomial in $\langle \phi^*\xi_k^x, \phi^*\xi_l^x \rangle$ for $1 \leq k, l \leq j$ which are continuous . □

2.5 Banach/Hilbert C^*-modules

This section discusses Banach and Hilbert C^*-modules. We discuss the relation between Banach and Hilbert C^*-modules over commutative C^*-algebras and continuous fields of Banach and Hilbert spaces. In particular, this gives rise to a way to construct such fields.

Let A be a C^*-algebra and A^+ the set of **positive elements** in A, i.e. elements of the form $a\,a^*$ for some $a \in A$.

Definition 2.40. A **left Banach A-module** is a Banach space Δ that has a left A-module structure $A \to \mathcal{B}(\Delta)$ and a linear map $\|\cdot\| : \Delta \to A^+$ such that for all $\xi, \eta, \chi \in \Delta$ and $a \in A$:

(i) the norm on Δ satisfies $\|\xi\|_\Delta = \sqrt{\|(\|\xi\|^2)\|_A}$,

(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,

(iii) $\|a\xi\| = |a|\|\xi\|$, where $|a| := \sqrt{a^* a}$,

(iv) $\|\xi\| = 0$ iff $\xi = 0$.

Definition 2.41. A **left Hilbert A-module** is a Banach space Δ that has a left A-module structure $A \to \mathcal{B}(\Delta)$ and a sesquilinear pairing $\langle\cdot,\cdot\rangle : \Delta \times \Delta \to A$ such that for all $\xi, \eta, \chi \in \Delta$ and $a \in A$:

(i) the norm on Δ satisfies $\|\xi\|_\Delta = \sqrt{\|\langle\xi,\xi\rangle\|_A}$,

(ii) $\langle\xi, \eta + \chi\rangle = \langle\xi,\eta\rangle + \langle\xi,\chi\rangle$,

(iii) $\langle\xi, a\eta\rangle = a\langle\xi,\eta\rangle$,

(iv) $\langle\xi,\eta\rangle = \langle\eta,\xi\rangle^*$,

(v) $\langle\xi,\xi\rangle > 0$ iff $\xi \neq 0$.

The pairing is also called the A-valued inner product. Obviously, every Hilbert A-module is a Banach A-module in the A-valued norm $\|\xi\|_\Delta = \sqrt{\langle\xi,\xi\rangle}$. There is an analogous notion of right Hilbert A-module. A Hilbert A-module is called **full** if the image of the A-valued inner product is dense in A.

Example 2.42. Suppose A is a C^*-algebra. Then A is a left A-module under left multiplication. The A-valued inner product is given by $\langle a, b\rangle := a^*\,b$ for all $a, b \in A$.

A **morphism of Banach A-modules** is a bounded linear operator $\Psi : \Delta_1 \to \Delta_2$ that intertwines the A-action. In the case that $A = C_0(X)$ for a locally compact space X, the boundedness of Ψ implies that $\|\Psi\|$ is a locally bounded map $X \to \mathbb{R}$.

Theorem 2.43. *There is an equivalence of categories of continuous fields of Banach (respectively Hilbert) spaces and left Banach (respectively Hilbert) $C_0(X)$-modules.*

Proof. (sketch, for a full proof see [6] §4), Suppose $(\mathcal{B}, \Delta)$ is a continuous field of Banach spaces. Then Δ is a $C_0(X)$-module, as proven in Lemma 2.26. Its completeness as a Banach space follows immediately from locally uniform completeness. This is one direction of the correspondence.

For the other direction, suppose Λ is a Banach $C_0(X)$-module. Define, for all $x \in X$

$$N_x := \{h \in \Lambda \mid \|h\|(x) = 0\}$$

and $\mathcal{B}_x := \Lambda/N_x$. Denote the projection by $\pi_x : \Lambda \to \Lambda/N_x$. Define the space of sections by

$$\Delta := \{\xi_\lambda := (x \mapsto \pi_x(\lambda)) \mid \lambda \in \Lambda\}.$$

We check that this is indeed a continuous field of Banach spaces.

(i) $\{\xi_\lambda(x) \mid \xi_\lambda \in \Delta\} = \Lambda/N_x$ trivially;

(ii) $x \mapsto \|\xi_\lambda(x)\| = \|\lambda\|(x)$ is by definition continuous;

(iii) suppose $\lambda \in \prod_{x\in X} \Lambda/N_x$ and suppose λ is locally uniformly close to sections in Δ. We want to show that this implies $\lambda \in \Delta$. Since Λ is complete as a Banach space it suffices to show globally uniformly close to a section in Δ. This one shows using a partition of unity argument. We omit the details.

If one begins with a Banach $C_0(X)$-module Λ, then produces a continuous field of Banach spaces, and from that again constructs a Banach $C_0(X)$-module, one trivially recovers Λ, up to isomorphism.

On the other hand, from a continuous field $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ one obtains the Banach $C_0(X)$-module Δ and once again this gives rise to a continuous field $(\{\Delta/N_x\}_{x\in X}, \Delta)$. An isomorphism $\Delta/N_x \to \mathcal{B}_x$ is given by $[\xi] \mapsto \xi(x)$. □

The well-known Serre-Swan theorem states that for (locally) compact Hausdorff spaces X there exists an equivalence of categories between finitely generated projective Hilbert $C(X)$-modules and locally trivial finite-dimensional continuous fields of Hilbert spaces (i.e. finite rank vector bundles) over X. Indeed, as mentioned, finitely generated Hilbert $C(X)$-modules Δ correspond to uniformly finite-dimensional continuous fields. Moreover, one can show that Δ being projective corresponds to the field being locally trivial.

Example 2.44. Suppose $\pi : Y \to X$ is a continuous surjection endowed with a continuous family of Radon measures $\{\nu_x\}_{x\in X}$ (cf. Section 2.2). For any $p \in \mathbb{R}_{\geq 1}$ consider the norm on $C_c(Y)$ given by

$$\|f\|_p := \sup_{x\in X} \|f|_{Y_x}\|_{L^p(Y_x,\nu_x)}.$$

Define $\Delta^p_\pi(Y)$ to be the closure of $C_c(Y)$ with respect to this norm. One easily sees that this is a Banach $C_0(X)$-module with $C_0(X)$-valued norm given by

$$\|f\|(x) := \|f|_{Y_x}\|_{L^p(Y_x,\nu_x)} = \left(\int_{Y_x} |f(y)|^p \nu_x(dy)\right)^{1/p}.$$

The continuous field associated to this Banach $C_0(X)$-module is denoted by

$$(\hat{L}^p_\pi(Y), \Delta^p_\pi(Y))$$

. The fiber at $x \in X$ equals $L^p(Y_x, \nu_x)$.

If $p = 2$, one obtains a Hilbert $C_0(X)$-module and hence a continuous field of Hilbert spaces. The $C_0(X)$-valued inner product is given on $C_c(Y)$ by

$$\langle f, f' \rangle (x) := \langle f|_{Y_x}, f'|_{Y_x} \rangle_{L^2(Y_x,\nu_x)} = \int_{Y_x} \overline{f(y)} f'(y) \nu_x(dy).$$

3 Continuous representations of groupoids

3.1 Continuous representations of groupoids

In this section we introduce continuous representations of groupoids on continuous fields of Hilbert spaces. As far as we know this notion as we define it does not appear anywhere in the literature. We should mention the work of Westman [28, 27] though, who restricts himself to representations of locally trivial groupoids on vector bundles. Furthermore, there is a preprint by Amini [1], which treats continuous representations on Hilbert bundles, which is rather different from the notion of continuous field of Hilbert spaces as we use it. It seems as though his article does not give full attention to the 'continuity-issues' involved.

As for representations of groups there are several forms of continuity for such representations. We consider "normal", weak and strong continuity and in Section 3.2 also continuity in the operator norm. All these forms of continuity can be compared, cf. Lemma 3.6, Lemma 3.7 and Lemma 3.14, generalizing similar results for groups (cf. e.g. [8]). In Definition 3.8 we introduce the notion of a morphism of representations and we show in Proposition 3.10 that any representation of a proper groupoid is isomorphic to a unitary representation, generalizing a similar result for compact groups.

Let M be a locally compact space and $G \rightrightarrows M$ a continuous groupoid.

Definition 3.1. A **bounded representation** of $G \rightrightarrows M$ on a continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m \in M}, \Delta)$ over M is a family of invertible bounded operators

$$\{\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}\}_{g \in G}$$

satisfying

(i) $\pi(1_m) = id_{\mathcal{H}_m}$ for all $m \in M$,

(ii) $\pi(gg') = \pi(g)\pi(g')$ for all $(g, g') \in G_2 = G\,{}_t\!\times_s G$,

(iii) $\pi(g^{-1}) = \pi(g)^{-1}$ for all $g \in G$ and

(iv) $g \mapsto \|\pi(g)\|$ is locally bounded.

We denote such a representation by a triple $(\mathcal{H}, \Delta, \pi)$. Recall from Lemma 2.27 that $\mathcal{H}$ can be endowed with a topology such that the sections Δ equals the set of continuous sections $\Gamma_0(\mathcal{H})$ of the projection $\mathcal{H} \to M$ onto the base space M.

Definition 3.2. A representation $(\mathcal{H}, \Delta, \pi)$ is **strongly continuous** if the map

$$g \mapsto \pi(g)\xi(s(g))$$

is continuous $G \to \mathcal{H}$ for all $\xi \in \Delta$. A representation is **weakly continuous** if the map

$$g \mapsto \langle \pi(g)\xi(s(g)), \eta(t(g)) \rangle$$

is continuous $G \to \mathbb{C}$ for all $\xi, \eta \in \Delta$. A representation $(\pi, \mathcal{H}, \Delta)$ is **continuous** if

$$\Psi : (g, h) \mapsto \pi(g)h$$

is a continuous map $G\,{}_s\!\times_p \mathcal{H} \to \mathcal{H}$. The representation is **unitary** if the operators $\{\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}\}_{g \in G}$ are unitary.

For any $\xi, \eta \in \Delta^\pi$ we use the notation $\langle \xi, \pi\eta \rangle$ for the map $G \to \mathbb{C}$ given by

$$g \mapsto \langle \xi(t(g)), \pi(g)\eta(s(g)) \rangle \, ,$$

which we call a **matrix coefficient**.

Condition (iv) of Definition 3.1 is perhaps somewhat strange at first sight. The following Example 3.3, Lemma 3.4 and Example 3.5 should clarify it. Moreover, recall that for morphism Ψ of continuous fields the map $m \mapsto \|\Psi_m\|$ has to be locally bounded too, cf. Definition 2.30.

Example 3.3. A simple example shows that $g \mapsto \|\pi(g)\|$ is not always continuous. Consider the groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$, with a continuous representation on a field given by the trivial representation on $\mathbb{C}$ at each $x \in \mathbb{R}$ except in 0, where it is the zero representation. In this case, the norm of π drops from 1 to 0 at 0.

Lemma 3.4. *For any continuous representation $(\mathcal{H}, \pi, \Delta)$ the map $g \mapsto \|\pi(g)\|$ is lower semi-continuous $G \to \mathbb{R}$.*

Proof. Using the above definition and Lemma 2.29 we know that the map $(g, h) \mapsto \|\pi(g)h\|$ is continuous $G_s \times_p \mathcal{H} \to \mathbb{R}_{\geq 0}$. For any $g \in G$, let $\varepsilon > 0$ be given. Let $h' \in \mathcal{H}_{s(g)}$ be such that

$$|\|\pi(g)h'\| - \|\pi(g)\|| < \varepsilon/2.$$

by continuity there exists an open neighborhood $U \subset G_s \times_p \mathcal{H}$ of (g, h') such that $(g'', h'') \in U$ implies

$$|\|\pi(g'')h''\| - \|\pi(g)h'\|| < \varepsilon/2.$$

Take $V := pr_1(U) \subset G$. Then $g'' \in V$ implies, for an $h'' \in pr_2(U)$ such that $(g'', h'') \in U$, one has

$$\|\pi(g'')\| \geq \|\pi(g'')h''\| > \|\pi(g)h'\| - \varepsilon/2 > \|\pi(g)\| - \varepsilon,$$

and we are done. □

The function $g \mapsto \|\pi(g)\|$ is locally bounded if, for example, $(\mathcal{H}, \Delta)$ is uniformly finite-dimensional.

Example 3.5. A counterexample of a continuous representation of a proper groupoid where $g \mapsto \|\pi(g)\|$ is not locally bounded $G \to \mathbb{R}$, even though the restriction to G_m is bounded for each m, is as follows.

Consider the trivial bundle of groups $[0,1] \times \mathbb{Z}/2\mathbb{Z} \rightrightarrows [0,1]$. Define a continuous field of Hilbert spaces over $[0,1]$ by $\mathcal{H}_0 := \mathbb{C}^2 =: \mathcal{H}_1$ and $\mathcal{H}_x := \mathbb{C}^{2\,n}$ if $x \in [\frac{1}{n+1}, \frac{1}{n})$ for all $n \in \mathbb{N}$. The topology on the field is obtained from the inclusions $\mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2(n+1)}$ given by $\vec{v} \mapsto (0, \vec{v}, 0)$. Define, for every $n \in \mathbb{N}$ and $x \in [\frac{1}{n+1}, \frac{1}{n})$,

$$\pi(x,-1) := \mathrm{diag}'(1/n, \ldots, 1/2, 1, 1, 2, \ldots n),$$

where diag$'$ denotes the matrix filled with zeros except the diagonal from the upper right corner to the lower left corner, where the above sequence is filled in. Furthermore, $\pi(0,-1) :=$ diag'$(1,1)$. This representation is strongly continuous, but

$$\|\pi(x,-1)\| = n \text{ if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

Hence $g \mapsto \|\pi(g)\|$ is not locally bounded at $(0,-1)$.

Lemma 3.6. *If a representation $(\pi, \mathcal{H}, \Delta)$ is strongly continuous, then it is weakly continuous. The converse implication holds if the representation is unitary.*

Proof. Suppose $(\pi, \mathcal{H}, \Delta)$ is strongly continuous. Suppose $\xi, \eta \in \Delta$ and $g \in G$. Write $n = t(g)$. Let $\varepsilon > 0$ be given. Let $\xi' \in \Delta$ be a section satisfying $\xi'(n) = \pi(g)\xi(s(g))$. Choose a neighborhood $U \subset M$ of n such that $n' \in U$ implies $|\langle \eta(n'), \xi'(n')\rangle_{\mathcal{H}_{n'}} - \langle \eta(n), \xi'(n)\rangle_{\mathcal{H}_n}| < \varepsilon/2$. This is possible since $\langle \eta, \xi'\rangle$ is continuous on M. Since π is strongly continuous there exists an open set $V \subset G$ containing g such that for all $g' \in V$ one has $t(g') \in U$ and

$$\|\pi(g')\xi(s(g')) - \xi'(t(g'))\|_{\mathcal{H}_{t(g')}} < \varepsilon/(2 \sup_{n' \in U} \|\eta(n')\|).$$

Hence, for all $g' \in V$

$$\begin{aligned}
&|\langle \eta(t(g')), \pi(g')\xi(s(g'))\rangle_{\mathcal{H}_{t(g')}} - \langle \eta(n), \xi'(n)\rangle_{\mathcal{H}_n}| \\
&\leq |\langle \eta(t(g')), \pi(g')\xi(s(g'))\rangle_{\mathcal{H}_{t(g')}} - \langle \eta(t(g')), \xi'(t(g'))\rangle_{\mathcal{H}_{t(g')}}| \\
&\quad + |\langle \eta(t(g')), \xi'(t(g'))\rangle_{\mathcal{H}_{t(g')}} - \langle \eta(n), \xi'(n)\rangle_{\mathcal{H}_n}| \\
&< \|\eta(t(g'))\|\varepsilon/(2 \sup_{n' \in U} \|\eta(n')\|) + \varepsilon/2 \leq \varepsilon.
\end{aligned}$$

The converse implication is proven as follows. Suppose $(\pi, \mathcal{H}, \Delta)$ is weakly continuous and unitary. Let $U(\varepsilon, \eta, V)$ be a neighborhood of $\pi(g)\xi(s(g))$ in $\mathcal{H}$ for a given $g \in G$ and $\xi \in \Delta$, where $\eta \in \Delta$ satisfies $\eta(t(g)) = \pi(g)\xi(t(g))$. We compute for any $g' \in G$,

$$\begin{aligned}
&\|\eta(t(g')) - \pi(g')\xi(s(g'))\|_{\mathcal{H}_{t(g')}} \qquad (3.1)\\
&= |\langle \eta(t(g')), \eta(t(g'))\rangle - \langle \eta(t(g')), \pi(g')\xi(s(g'))\rangle \\
&\quad - \langle \pi(g')\xi(s(g')), \eta(t(g'))\rangle + \langle \pi(g')\xi(s(g')), \pi(g')\xi(s(g'))\rangle|^{1/2} \\
&\leq (|\langle \eta(t(g')), \eta(t(g'))\rangle - \langle \eta(t(g')), \pi(g')\xi(s(g'))\rangle| \\
&\quad + |\langle \xi(s(g')), \xi(s(g'))\rangle - \langle \pi(g')\xi(s(g')), \eta(t(g'))\rangle|)^{1/2} \qquad (3.2)
\end{aligned}$$

By weak continuity we can choose a neighborhood $W_g \subset G$ of g such that $g' \in W_g$ implies

$$|\left\langle \eta(t(g')), \pi(g')\xi(s(g'))\right\rangle - \langle \eta(t(g)), \pi(g)\xi(s(g))\rangle| < \varepsilon.$$

Since t is open and $\eta \in \Delta$, we can choose an open neighbor hood $W'_g \subset W_g$ of g such that

$$|\left\langle \eta(t(g')), \eta(t(g'))\right\rangle - \langle \eta(t(g)), \eta(t(g))\rangle| < \varepsilon$$

Hence the first two terms of Equation (3.2) are smaller than 2ε. Analogously, the last two terms of Equation (3.2) are also smaller than 2ε, which finishes the proof. □

Lemma 3.7. *If a representation $(\pi, \mathcal{H}, \Delta)$ is continuous, then it is strongly continuous. The converse holds if π is unitary.*

Proof. Suppose $(\pi, \mathcal{H}, \Delta)$ is continuous. Suppose $g \in G$ and $\xi \in \Delta$. There exists an open neighborhood $U(\varepsilon, \eta, V) \subset \mathcal{H}$ of $\pi(g)\xi(s(g))$ such that $\eta(t(g) = \pi(g)\xi(s(g))$. Then, by continuity of π there exists a neighborhood $W_g \subset G_s \times_p \mathcal{H}$ of g such that $g' \in W_g$ implies $\pi(W_g) \subset U(\varepsilon, \eta, V)$. Now, define a subset of G

$$W_G := \{g' \in G \mid (g', \xi(s(g'))) \in W_g\}.$$

This set is open since it equals $s^{-1}\xi^{-1}p_2(W_g) \cap p_1(W_g)$. If $g' \in W_G$, then

$$\|\eta(t(g')) - \pi(g')\xi(s(g'))\| < \varepsilon.$$

Conversely, suppose $(\pi, \mathcal{H}, \Delta)$ is strongly continuous and unitary. Suppose $(g, h) \in G_s \times_p \mathcal{H}$. Let $U(\varepsilon, \eta, V)$ be an open neighborhood of $\pi(g)h$ with $\eta(t(g)) = \pi(g)h$ as usual. Let ξ be any section in Δ such that $\xi(s(g)) = h$. Then by strong continuity there exists an open set $V_g \subset G$ such that $g' \in V_g$ implies $\|\eta(t(g')) - \pi(g')\xi(s(g'))\| < \varepsilon$. Define the set

$$W_{g,h} := \{(g', h') \in G_s \times_p \mathcal{H} \mid \|h' - \xi(s(g'))\| < \varepsilon, g' \in V_g\}.$$

It is easily seen to be open and $(g', h') \in W_{g,h}$ implies

$$\begin{aligned} \|\eta(t(g')) - \pi(g')h'\| &\leq \|\eta(t(g')) - \pi(g')\xi(s(g'))\| + \|\pi(g')\xi(s(g')) - \pi(g')h'\| \\ &< \varepsilon + \|\pi(g')\|\|\xi(s(g')) - h'\| < 2\varepsilon, \end{aligned}$$

which finishes the proof. □

Definition 3.8. A **morphism of continuous (unitary) representations**

$$(\mathcal{H}^1, \Delta^1, \pi_1) \to (\mathcal{H}^2, \Delta^2, \pi_2)$$

of a groupoid is a morphism $\Psi : (\mathcal{H}^1, \Delta^1) \to (\mathcal{H}^2, \Delta^2)$ of continuous fields of Hilbert spaces (cf. Definition 2.30) that intertwines the groupoid representations

$$\begin{array}{ccc} \mathcal{H}^1_{s(g)} & \xrightarrow{\pi_1(g)} & \mathcal{H}^1_{t(g)} \\ \Big\downarrow{\scriptstyle \Psi_{s(g)}} & & \Big\downarrow{\scriptstyle \Psi_{t(g)}} \\ \mathcal{H}^2_{s(g)} & \xrightarrow[\pi_2(g)]{} & \mathcal{H}^2_{t(g)}. \end{array}$$

Example 3.9. The trivial representation of a groupoid $G \rightrightarrows M$ is given by the continuous field $(\mathcal{H}, \Delta)$ that has fiber $\mathbb{C}$ over each $m \in M$ and a map $\pi : G \to U(M \times \mathbb{C}) \cong M \times U(\mathbb{C}) \times M$,

$$g \mapsto (t(g), 1, s(g)).$$

We give another example of a continuous unitary representation of a groupoid. For any continuous function $f : G \to \mathbb{R}$ we can construct the representation

$$\pi_f : g \mapsto (t(g), e^{2\pi i (f(t(g)) - f(s(g)))}, s(g)).$$

These representation are all isomorphic. Indeed, for $f, g : G \to \mathbb{R}$,

$$m \mapsto e^{2\pi i (f(m) - g(m))}$$

is an isomorphism $(\mathcal{H}, \Delta, \pi_g) \to (\mathcal{H}, \Delta, \pi_f)$. In particular all these representations are isomorphic to π_0, which is the trivial representation.

Proposition 3.10. *If $G \rightrightarrows M$ is a proper groupoid endowed with a Haar system $\{\lambda_m\}_{m \in M}$ (cf. Section 2.2), then any continuous representation $(\mathcal{H}, \Delta, \pi)$ is isomorphic to a unitary representation.*

Proof. Suppose $(\mathcal{H}, \Delta, \pi)$ is a non-zero continuous representation of G. Let $c : M \to \mathbb{R}_{>0}$ be a cutoff function (cf. Definition 2.19), with t and s interchanged). This exists, since $G \rightrightarrows M$ is proper. Define an inner product $\langle ., . \rangle^{new}$ on $\mathcal{H}$ by the following description: for all $m \in M$ and $h, h' \in \mathcal{H}_m$,

$$\langle h, h' \rangle^{new}(m) := \int_{G_m} \langle \pi(g)h, \pi(g)h' \rangle \, c(t(g)) \lambda_m(dg).$$

This inner product is G-invariant, since the Haar system and t are right invariant. It gives rise to a new topology on $\mathcal{H}$. The isomorphism is the identity on $\mathcal{H}$, which is easily seen to be continuous. Indeed, let $h \in \mathcal{H}$ and let $U(\varepsilon, \xi, V) \ni h$ be an open set in $\mathcal{H}$ with respect to the old norm. Then there exists a an open set V' such that $V' \subset V$, $p(h) \in V'$ and $g \mapsto \|\pi(g)\|$ is bounded on $t^{-1}V' \cap \mathrm{supp}(c \circ t)$. Since $c \circ t$ has compact support on each s-fiber, the function

$$m' \mapsto \int_{g \in G_{m'}} \|\pi(g)\| c(t(g)) \lambda_{m'}(dg)$$

is bounded on V'. Hence we can set

$$\delta := \frac{\varepsilon}{\sup_{m \in V'} \int_{g \in G_{m'}} \|\pi(g)\| c(t(g)) \lambda_{m'}(dg)}.$$

Then $h' \in U(\delta, \xi, V')$ (in the old topology) implies

$$\begin{aligned} \|h' - \xi(m')\|^{new}_{m'} &= \int_{G_{m'}} \|\pi(g)(h' - \xi(m'))\| c(t(g)) \lambda_{m'}(dg) \\ &\leq \int_{G_{m'}} \|\pi(g)\| c(t(g)) \lambda_{m'}(dg) \|h' - \xi(m')\| \\ &\leq \varepsilon, \end{aligned}$$

which proves the continuity of the identity map.

The proof that the inverse (also the identity) is continuous proceeds similarly. One uses that

$$\begin{aligned}\|h' - \xi(m')\| &= \int_{G_{m'}} \|h' - \xi(m')\| c(t(g))) \lambda_{m'}(dg) \\ &= \int_{G_{m'}} \|\pi(g^{-1})\pi(g)(h' - \xi(m'))\| c(t(g)) \lambda_{m'}(dg) \\ &= \sup_{g \in G_{m'}} \|\pi(g)\| \int_{G_{m'}} \|\pi(g)(h' - \xi(m'))\| c(t(g)) \lambda_{m'}(dg)\end{aligned}$$

and local boundedness of $g \mapsto \|\pi(g)\|$. This finishes the proof. □

A representation $(\mathcal{H}, \Delta, \pi)$ is **locally trivial** if the continuous field $(\mathcal{H}, \Delta)$ is locally trivial. In [24] locally trivial representations of a groupoid $G \rightrightarrows M$ are called G-vector bundles. Representations of transitive groupoids are locally trivial.

3.2 Continuity of representations in the operator norm

In this section we go through quite some effort to define a suitable topology on the set of bounded linear operators $\{P : \mathcal{H}_m \to \mathcal{H}_n\}_{n,m \in M}$ for a continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m \in M}, \Delta_{\mathcal{H}})$. This is done not only to be able to consider representations which are continuous in the operator topology, but the lower semi-continuous field of Banach spaces thus obtained also plays a crucial rôle in Section 5. At first reading one could consider skipping the proofs.

Let $(\{\mathcal{H}_m\}_{m \in M}, \Delta_{\mathcal{H}})$ be a continuous field of Hilbert spaces over M. Consider the continuous field of Banach spaces over $M \times M$ whose fiber at (n, m) is given by the bounded linear operators $\mathcal{H}_m \to \mathcal{H}_n$, i.e. $\mathcal{B}(\mathcal{H}, \mathcal{H})_{(n,m)} := \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$. This is indeed a Banach space with the norm

$$\|P\| = \sup_{h \in \mathcal{H}_m, \|h\|_{\mathcal{H}_m} = 1} \|P(h)\|_{\mathcal{H}_n}.$$

We define a space of sections $\Delta_{\mathcal{B}}$ of the field to consist of those maps $(n, m) \mapsto P(n, m)$ in $\prod_{(n,m) \in M \times M} \mathcal{B}(\mathcal{H}, \mathcal{H})$ such that

(i) for every $m \in M$ and $h \in \mathcal{H}_m$

$$n \mapsto P(n, m)h$$

is in $\Delta_{\mathcal{H}}$,

(ii) for every $n \in M$ and $\xi \in \Delta_{\mathcal{H}}$ the map

$$m \mapsto P(n, m)\xi(m)$$

is continuous $M \to \mathcal{H}_n$,

(iii) The map $(n, m) \mapsto \|P(n, m)\|$ locally bounded, and

(iv) P is adjointable, which means that there exists a $P^* : R \to \mathcal{B}(\mathcal{H}, \mathcal{H})$, satisfying (i), (ii) and (iii), such that for all $\xi, \eta \in \Delta_{\mathcal{H}}$ one has $(\eta, P\xi) = (P^*\eta, \xi)$, more concretely: for all $(n, m) \in R$

$$\langle \eta(n), P(n,m)\xi(m)\rangle_{\mathcal{H}_n} = \langle P^*(m,n)\eta(n), \xi(m)\rangle_{\mathcal{H}_m}.$$

Lemma 3.11. *The pair* $(\{\mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)\}_{(n,m)\in M\times M}, \Delta_{\mathcal{B}})$ *is a lower semi-continuous field of Banach spaces.*

Proof. First, we prove lower semi-continuity of the norm of a section $P \in \Delta_{\mathcal{B}}$. This follows from the fact that the map

$$(n, m, h) \mapsto \|P(n,m)h\|_{\mathcal{H}_n}$$

is a continuous map $M \times M \times_p \mathcal{H} \to \mathbb{R}$, analogously to the proof of Lemma 3.4. This last statement is proven as follows. Let $\varepsilon > 0$ be given. Suppose $(n, m, h) \in M \times M \times_p \mathcal{H}$. There exists a $\xi \in \Delta_{\mathcal{H}}$ such that $\xi(m) = h$. Then by condition (i), (ii), (iii) and continuity of $\|\xi\|$, there exists a neighborhood $W \in M \times M \times_p \mathcal{H}$ such that for any $(n', m', h') \in W$ the map $\|P\|$ is bounded on W and we have

$$\begin{aligned}
&|\|P(n',m')h'\| - \|P(n,m)h\||\\
&\quad\leq |\,\|P(n',m')h'\| - \|P(n,m')h'\|\,| + |\,\|P(n,m')h'\| - \|P(n,m')\xi(m')\|\,|\\
&\qquad+|\,\|P(n,m')\xi(m')\| - \|P(n,m)\xi(m)\|\,|\\
&\quad\leq \varepsilon + \|P(n,m')\|\varepsilon + \varepsilon.
\end{aligned}$$

Next, we prove that for every $P \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ and every $\varepsilon > 0$ there exist a $Q \in \Delta_{\mathcal{B}}$ such that $\|Q(n,m) - P\| < \varepsilon$. Suppose $P \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ and let $\varepsilon > 0$ be given. Let $\xi_1, \ldots, \xi_k \in \Delta_{\mathcal{H}}$ be such that for any $h \in \mathcal{H}_m$

$$\|h - \sum_{i=1}^{k} \langle \xi_i(m), h\rangle\, \xi_i(m)\| < \varepsilon.$$

Let $\eta_1, \ldots, \eta_l \in \Delta_{\mathcal{H}}$ be such that for any $h \in \mathcal{H}_n$

$$\|h - \sum_{i=1}^{l} \langle \eta_i(n), h\rangle\, \xi_i(n)\| < \varepsilon.$$

Define, for $(n', m') \in M \times M$,

$$Q(n',m')h := \sum_{i=1}^{k}\sum_{j=1}^{l} \left\langle \xi_i(m'), h\right\rangle \langle \eta_j(n), P\xi_i(m)\rangle\, \eta_j(n')$$

One easily checks that $Q \in \Delta_{\mathcal{B}}$. Furthermore,

$$\begin{aligned}
&\|Ph - Q(n,m)h\|\\
&\quad\leq \|Ph - \textstyle\sum_{i=1}^{k} \langle \xi_i(m), h\rangle\, P\xi_i(m)\| + \|\textstyle\sum_{i=1}^{k} \langle \xi_i(m), h\rangle\, P\xi_i(m)\\
&\qquad- \textstyle\sum_{i=1}^{k}\sum_{j=1}^{l} \langle \xi_i(m'), h\rangle \langle \eta_j(n), P\xi_i(m)\rangle\, \eta_j(n')\|\\
&\quad< \|P\|\varepsilon + \varepsilon.
\end{aligned}$$

The last step is to show that $\Delta_{\mathcal{B}}$ is locally uniformly closed. Suppose

$$Q \in \prod_{(n,m) in M\times M} \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m).$$

Suppose that for all $\varepsilon > 0$ and all $(n,m) \in M \times M$ there is a $Q' \in \Delta_{\mathcal{B}}$ such that

$$\|Q(n',m') - Q'(n',m')\| < \varepsilon$$

on a neighborhood V of (n,m). We shall now show that this implies $Q \in \Delta_{\mathcal{B}}$. Indeed, let $\varepsilon > 0$ be given and suppose $n \in M$. Then there exist Q' and V as above. Define $U := p_1(V)$. Then $n' \in U$ implies, for any $h \in \mathcal{H}_m$, that

$$\|Q(n',m)h - Q'(n',m)h\| \leq \|Q(n',m) - Q'(n',m)\|\|h\| < \varepsilon\|h\|.$$

Hence $n \mapsto \|Q(n,m)h\|$ is continuous. In a similar way one proves condition (ii) for Q which finishes the proof. □

We shall see in Lemma 5.2 that $\mathcal{B}(\mathcal{H},\mathcal{H})$ is a so-called lower semi-continuous Fell bundle over $M \times M \rightrightarrows M$ and therefore a (full) lower semi-continuous C^*-category over M. The collection of sets

$$\{U(\varepsilon, \xi, V) \mid \xi \in \Delta_{\mathcal{B}}, \varepsilon > 0, V \subset M \times M \text{ open}\},$$

as defined in Lemma 2.27 for a continuous field of Banach spaces, is in general a subbasis for the topology on $\coprod_{(n,m)\in M\times M} \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$, instead of a basis. Since the field is not continuous in general, we do not have $\Delta = \Gamma_0(M \times M, \mathcal{B}(\mathcal{H},\mathcal{H}))$. Consider the restriction of the total space $\mathcal{B}(\mathcal{H},\mathcal{H})$ to the unitary operators, i.e.

$$U^{op}(\mathcal{H}) := \coprod_{(n,m)\in M\times M} U(\mathcal{H}_m, \mathcal{H}_n),$$

endowed with the subspace topology.

Lemma 3.12. *The total space $U^{op}(\mathcal{H})$ is a continuous groupoid over M.*

Proof. We show that the composition $\mathcal{B}(\mathcal{H},\mathcal{H})^{(2)} \to \mathcal{B}(\mathcal{H},\mathcal{H})$ is a continuous map. First note that for every $(P,Q) \in \mathcal{B}(\mathcal{H},\mathcal{H})^{(2)}$ the inequality $\|PQh\| \leq \|P\|\|Qh\|$ implies

$$\|PQ\| \leq \|P\|\|Q\|.$$

Suppose that $m,n,p \in M$, $P_2 \in \mathcal{B}(\mathcal{H}_p, \mathcal{H}_n)$, $P_1 \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ and $U(\varepsilon, Q, V)$ is an open neighborhood of P_2P_1 such that $Q(p,m) = P_2P_1$. There are $Q_1, Q_2 \in \Delta_{\mathcal{B}}$ such that $Q_1(n,m) = P_1$ and $Q_2(p,n) = P_2$. Choose $\varepsilon_i > 0$ and an open subset $V_i \subset M$ such that $P_i' \in U(\varepsilon_i, Q_i, V_i)$ implies $\|P_i'\|\varepsilon_i < \varepsilon/3$ for $i = 1,2$. Furthermore, note that by condition (i), for each $m' \in M$ and $h \in \mathcal{H}_{m'}$ the map $n' \mapsto Q_1(n',m')h$ is in $\Delta_{\mathcal{B}}$. Hence by condition (ii) the map for each $p', m' \in M$ the map $n' \mapsto Q_2(p',n')Q_1(n',m')$ is continuous. The map $(p',m') \mapsto Q_2(p',n')Q_1(n',m')$ is easily seen to be continuous too. Hence we can shrink V_1 and V_2 such that $(p',n',m') \in V_2 \times_M V_2$ implies

$$|\,\|Q_2(p',n)Q_1(n,m') - Q_2(p',n')Q_1(n',m')\|\,| < \varepsilon/3.$$

Define $Q \in \Delta_{\mathcal{B}}$ by $Q(p', m') := Q_1(p', n)Q_2(n, m')$ Suppose

$$(P_2', P_1') \in U(\varepsilon_2, Q_2, V_2)_s \times_t U(\varepsilon_1, Q_1, V_1),$$

then

$$\begin{aligned}\|P_2'P_1' - Q(p', m')\| &= \|P_2'P_1' - Q_2(p', n)Q_1(n, m')\| \\ &\leq \|P_2'P_1' - Q_2(p', n')P_1'\| + \|Q_2(p', n')P_1' - Q_2(p', n')Q_1(n', m')\| \\ &\quad + \|Q_2(p', n')Q_1(n', m') - Q_2(p', n)Q_1(n, m')\| \\ &< \|P_2' - Q_2(p', n')\|\,\|P_1'\| + \|Q_2(p', n')\|\,\|P_1' - Q_1(n', m')\| + \varepsilon/3 \\ &< \varepsilon_2\|P_1'\| + \|Q_2(p', n')\|\varepsilon_2 + \varepsilon/3 < \varepsilon.\end{aligned}$$

Proving that the other structure maps are continuous is similar, but easier. □

Definition 3.13. A representation $(\pi, \mathcal{H}, \Delta)$ is **continuous in the operator norm** if the map

$$G \to \mathcal{B}(\mathcal{H}, \mathcal{H}), g \mapsto \pi(g)$$

is continuous. If G is unitary, then the representation is continuous if

$$G \to U^{op}(\mathcal{H}), g \mapsto \pi(g)$$

is a continuous map of groupoids.

Lemma 3.14. *A representation is continuous if it is continuous in the operator norm. The converse implication is true if the representation Δ^π is finitely generated over $C_0(M)$ and unitary.*

Proof. Suppose $(g, h) \in G_s \times_p \mathcal{H}$ and let $n = t(g)$ and $m = s(g)$. Suppose $U(\varepsilon, V, \xi)$ is a neighborhood of $\pi(g)h$, with $\xi(n) = \pi(g)h$. Let $Q \in \Delta_{\mathcal{B}}$ be any section with $Q(n, m) = \pi(g)$, which exists since $(\mathcal{B}(\mathcal{H}, \mathcal{H}), \Delta_{\mathcal{B}})$ is a lower semi-continuous field of Banach spaces. Let $\eta \in \Delta_{\mathcal{H}}$ be a section such that $\eta(m) = h$. By the conditions (i), (ii) and (iii) above there exists a neighborhood $S \subset M \times M$ of (n, m) such that for all $(n', m') \in S$

$$\|\xi(n') - Q(n', m)h\| < \varepsilon/4,$$

the function $\|Q\|$ is bounded on S and

$$\|Q(n', m)\eta(m) - Q(n', m')\eta(m')\| < \varepsilon/4.$$

Define

$$\begin{aligned}\delta &:= \tfrac{\varepsilon}{4\sup_{(n',m)\in S}\|Q(n',m')\|}, \\ W' &:= U(\delta, \eta, p_2(S)), \\ K &:= \sup_{h' \in W'} \|h'\|,\end{aligned}$$

and

$$W := \pi^{-1}(U(Q, \frac{\varepsilon}{4K}, S)),$$

where $p_2 : M \times M \to M$ is the projection on the second entry. We claim that $(g', h') \in W_s \times_p W'$ implies $\pi(g')h' \in U(\varepsilon, V, \xi)$. Indeed, suppose $(g', h') \in W_s \times_p W'$ and $m' = s(g')$, $n' = t(g')$, then

$$\begin{aligned}\|\xi(n') - \pi(g')h'\| &\leq \|\xi(n') - Q(n', m)h\| + \|Q(n', m)\eta(m) - Q(n', m')\eta(m')\| \\ &\quad + \|Q(n', m')\eta(m') - Q(n', m')h'\| + \|Q(n', m')h' - \pi(g')h'\| \\ &< \varepsilon/4 + \varepsilon/4 + \|Q(n', m')\|\delta + \|h'\|\tfrac{\varepsilon}{4K} < \varepsilon.\end{aligned}$$

We shall now prove the converse implication. Suppose $(\mathcal{H}^\pi, \Delta^\pi, \pi)$ is a strongly continuous unitary representation on a continuous field of Hilbert spaces with Δ^π finitely generated. There exist a finite set $\{\xi_i\}_{i\in I}$ of sections in Δ^π such that for each $m' \in M$ the set $\{\xi_i(m')\}_{i\in I}$ contains a (normalized) basis for $\mathcal{H}_{m'}$. Suppose $U(\varepsilon, Q, V)$ is a neighborhood of $\pi(g)$, $s(g) = m$, $t(g) = n$ and $Q(n, m) = \pi(g)$. Note that by condition (i) $n' \mapsto Q(n', m)\xi_i(m)$ is in Δ^π, so by strong continuity of π there exists an open set $U_i \subset G$ such that $g' \in U_i$ implies

$$\|\pi(g')\xi'(s(g')) - Q(t(g'), m)\xi_i(m)\| < \varepsilon/(2|I|).$$

Moreover, by condition (ii) we can shrink U_i such that $g' \in U_i$ implies that

$$\|Q(t(g'), m)\xi_i(m) - Q(t(g'), s(g'))\xi_i(s(g')\| < \varepsilon/(2|I|).$$

Hence

$$\|\pi(g')\xi'(s(g')) - Q(t(g'), s(g'))\xi_i(s(g') < \varepsilon/|I|$$

for $g' \in U_i$. Define $U := \bigcap_{i\in I} U_i$, then $g' \in U$ implies

$$\begin{aligned}&\|\pi(g') - Q(t(g'), s(g'))\| \\ &\quad = \sup_{h'\in\mathcal{H}_{s(g')}, \|h'\|=1} \|\pi(g')h' - Q(t(g'), s(g'))h'\|_{\mathcal{H}_{t(g')}} \\ &\quad < \textstyle\sum_{i\in I} \|\pi(g')\xi_i(s(g')) - Q(t(g'), s(g'))\xi_i(s(g')\|_{\mathcal{H}_{t(g')}} \\ &\quad < \textstyle\sum_{i\in I} \varepsilon/|I| = \varepsilon,\end{aligned}$$

which finishes the proof. □

From these comparison lemmas (Lemma 3.6, Lemma 3.7 and Lemma 3.14) we can conclude that for unitary representations any of these topologies are equivalent. Hence, from now on, we shall not specify which notion we mean, but only say that a unitary representation is continuous (if it is).

3.3 Example: the regular representations of a groupoid

The following example considers the regular representation. In a different form it was studied by Renault (cf. [21]), but he considered $L^2(G)$ as a measurable field of Hilbert spaces. We are interested in representations on continuous fields of Hilbert spaces. Therefore, the statement of Proposition 3.15 is actually new. It generalizes the analogous statement for groups.

Suppose a continuous groupoid $G \rightrightarrows M$ is endowed with a left Haar system.

Proposition 3.15. *The* ***left regular representation*** *of a continuous groupoid* $G \rightrightarrows M$ *on* $(\hat{L_t^2}(G), \Delta_t^2(G))$ *(cf. Example 2.44) defined by (continuous extension of)*

$$(\pi_L(g)f)(g') = f(g^{-1}g'),$$

for $g \in G$*,* $f \in C_c(G^{s(g)})$ *and* $g' \in G^{t(g)}$*, is a continuous unitary representation.*

Proof. Unitarity is immediate from the G-invariance of the Haar system.

We have to check that for all $\xi \in \Delta_t^2(G)$ the map $g \mapsto \pi_L(g)\xi(s(g))$ is continuous $G \to \hat{L_t^2}(G)$. Let $g \in G$. Suppose a neighborhood $U(\varepsilon, \eta, V) \subset \hat{L_t^2}(G)$ of $\pi_L(g)\xi(s(g))$ is given, where $\varepsilon > 0$, V an open set in M and $\eta \in \Delta_t(G)$ is a section satisfying $\pi_L(g)\xi(s(g)) = \eta(s(g))$. There exist $\xi', \eta' \in C_c(G)$ such that $\|\eta - \eta'\|_{\hat{L}^2} < \varepsilon/3$, $\|\xi - \xi'\|_{\hat{L}^2} < \varepsilon/3$ and $\pi_L(g)\xi'(s(g)) = \eta'(s(g))$. To continue we first need the following lemma due to A. Connes [4].

Lemma 3.16. *If* f *is a compactly supported continuous function on* $G^{(2)}$*, then the map*

$$g \mapsto \int_{h \in G^{s(g)}} f(g,h)\lambda^{s(g)}(dh)$$

is continuous on G*.*

We restate the proof for completeness.

Proof. Since $G^{(2)}$ is closed in $G \times G$, there exists a continuous and bounded extension $\bar{f}$ of f to $G \times G$ (we suppose here that G is a normal space). The map $(g, m) \to \int_{h \in G^m} f(g,h)\lambda^m(dh)$ is continuous, as is proven as follows. Let (g, m) be any element in $G \times M$ and let $\varepsilon' > 0$ be given. Since the Haar system is continuous and $\bar{f}(g', \cdot)$ converges uniformly to $\bar{f}(g, \cdot)$ for $g' \to g$. we can choose a neighborhood $W \in G \times M$ such that $(g', m') \in W$ implies

$$\left| \int_{h \in G^m} \bar{f}(g', h)\lambda^m(dh) - \int_{h' \in G^{m'}} \bar{f}(g', h')\lambda^{m'}(dh') \right| < \varepsilon'/2$$

and

$$\left| \int_{h' \in G^{m'}} \bar{f}(g', h')\lambda^m(dh') - \int_{h' \in G^{m'}} \bar{f}(g, h')\lambda^{m'}(dh') \right| < \varepsilon'/2.$$

As a consequence,

$$\begin{aligned}
&\left| \textstyle\int_{h \in G^m} \bar{f}(g', h)\lambda^m(dh) - \int_{h' \in G^{m'}} \bar{f}(g, h')\lambda^{m'}(dh') \right| \\
&\leq \left| \textstyle\int_{h \in G^m} \bar{f}(g', h)\lambda^m(dh) - \int_{h' \in G^{m'}} \bar{f}(g', h')\lambda^{m'}(dh') \right| \\
&\quad + \left| \textstyle\int_{h' \in G^{m'}} \bar{f}(g', h')\lambda^m(dh') - \int_{h' \in G^{m'}} \bar{f}(g, h')\lambda^{m'}(dh') \right| \\
&< \varepsilon'/2 + \varepsilon'/2 = \varepsilon'.
\end{aligned}$$

Restricting to $\{(g, m) \mid s(g) = m\} \subset G \times M$ gives the required result. □

Now, apply this lemma to the map

$$f(g', h') := |\xi'((g')^{-1}h') - \eta'(h')|^2.$$

As a result,

$$F(g') := \sqrt{\int_{h' \in G^{t(g')}} |\xi'((g')^{-1}h') - \eta'(h')|^2 \lambda^{t(g')}(dh')}$$

depends continuously on g'. Note that $F(g) = 0$, so that we can choose a neighborhood $U \subset G$ of g such that $F(g') < \varepsilon/3$ whenever $g' \in U$. Finally, intersect U with $t^{-1}(V)$ to obtain the required open set in G whose image is a subset of $U(\varepsilon, \eta, V)$. □

In the same way one proves that the **right regular representation** of G on $(\hat{L}^2_s(G), \Delta^2_s(G))$ given by

$$\pi_L(g)h(g') := h(g'g)$$

(where $h \in C_c(G_{s(g)})$ and $g' \in G_{t(g)}$) is strongly continuous and unitary.

Let's consider two very simple examples. Other examples will occur at other places of the text.

Example 3.17. Note that the left (and right) regular representation of the groupoid $M \rightrightarrows M$ is $m \mapsto 1_m : M \to M \times U(1)$. The left regular representation of the pair groupoid $M \times M \rightrightarrows M$

$$\pi_L : M \times M \to U(\hat{L}^2(M \times M)) \cong M \times U(L^2(M, \nu)) \times M$$

is given by

$$(m, n) \mapsto (m, 1_{L^2(M,\nu)}, n),$$

for a given Radon measure ν on M.

3.4 Example: continuous families of groups

The following example can give the reader a feeling for the issues on the global topology with continuous groupoid representations. We express the set of finite-dimensional continuous representations of a family of groups on a given continuous field of Hilbert spaces in terms of continuous sections of the family $\mathrm{Rep}^{\mathcal{H}}(G) \to M$ of the sets of finite-dimensional continuous representations of each of the groups.

Suppose H is a locally compact group. Let $\mathrm{Rep}(H)$ denote the set of non-zero continuous unitary representations of H. This set can be endowed with a topology. Indeed, one uses the Jacobson topology on the primitive spectrum of the C^*-algebra $C^*(H)$. We shall not go into the details, since there is an easier description of the case that has our interest. For $n \geq 1$, denote by $\mathrm{Rep}^n(H)$ the subspace of continuous non-zero unitary representations on $\mathbb{C}^n$ with standard inner product $\langle z, z' \rangle = \bar{z}z'$. Let $\mathrm{Rep}^0(H)$ be the set consisting of just the zero representation. Let $S(\mathbb{C}^n)$ denote the unit sphere in $\mathbb{C}^n$.

Lemma 3.18. *([5], 18.1.9) For every integer $n \geq 0$ a subbasis for the topology on $\mathrm{Rep}^n(H)$ is given by the sets*

$$U(\pi, \varepsilon, K) := \{\pi' \in \mathrm{Rep}^n(H) \mid \max_{g \in K} |\langle h', \pi(g)\, h\rangle - \langle h', \pi'(g)\, h\rangle| < \varepsilon, \forall h, h' \in S(\mathbb{C}^n)\},$$

for compact sets $K \subset H$, representations $\pi \in \mathrm{Rep}^n(H)$ and $\varepsilon > 0$.

We need the following technical notion. Suppose $p : N \to M$ is a continuous map.

Definition 3.19. We say a set $K \subset N$ is **p-open-compact** if the restriction $K \cap p^{-1}(m)$ is compact for all $m \in M$ and the image $p(K) \subset M$ is open. We say that $p : N \to M$ is **locally open-compact** if every $n \in N$ has a p-open-compact neighborhood.

Example 3.20. If $p : N \to M$ is a fiber bundle with locally compact fiber, then it is easy to show that p is locally open-compact.

Suppose $s : G \to M$ is a continuous family of groups (cf. Example 2.9). *Fix* a uniformly finite-dimensional continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$. We proceed in a few steps to define the surjection $\mathrm{Rep}^{\mathcal{H}}(G) \to M$ and endow it with a suitable topology.

(1) Choose for each $m \in M$ a group $H_m \cong s^{-1}(m)$ and an isomorphism $\psi_m : s^{-1}(m) \to H_m$, fixing the group structure at each fiber. Endow $\coprod_{m\in M} H_m$ with the topology such that

$$\coprod_{m\in M} \psi_m : G \to \coprod_{m\in M} H_m$$

is a homeomorphism. Denote the canonical projection $\coprod_{m\in M} H_m \to M$ by s'.

(2) Suppose

$$\left\{ \phi_i : \mathcal{H}|_{U_i} \hookrightarrow U_i \times \sum_{n=0}^{\dim(\mathcal{H}_{U_i})} \mathbb{C}^n \right\}_{i\in I}$$

is a local pseudo-trivialization of $(\mathcal{H}, \Delta)$ (cf. Definition 2.36). Define for all $i, j \in I$ the homeomorphism

$$\gamma_{ij} := \phi_j(\phi_i)^{-1} : \mathrm{im}(\phi_i)|_{U_i\cap U_j} \to \mathrm{im}(\phi_j)|_{U_i\cap U_j}.$$

(3) For any $i \in I$, define

$$\mathrm{Rep}^{\mathcal{H}}(G|_{U_i}) := \coprod_{m\in U_i} \mathrm{Rep}^{\dim(\mathcal{H}_m)}(H_m)$$

and denote the canonical projection by

$$p_i : \mathrm{Rep}^{\mathcal{H}}(G|_{U_i}) \to U_i.$$

(4) For each $i \in I$ the following sets form a subbasis of a topology on $\mathrm{Rep}^{\mathcal{H}}(G)|_{U_i}$: For any $\xi, \eta \in \Delta$, $V \subset \mathbb{C}$ open and $K \subset \coprod_{m\in U_i} H_m$ s'-open-compact,

$$U(\xi, \eta, K, V) := \left\{ \pi \in \mathrm{Rep}^{\mathcal{H}}(G|_{U_i}) \mid \langle \xi, \pi\eta \rangle \left(K \cap H_{p_i(\pi)}\right) \subset V \right\}.$$

(5) Define

$$\mathrm{Rep}^{\mathcal{H}}(G) := (\coprod_{i\in I} \mathrm{Rep}^{\mathcal{H}}(G|_{U_i}))/\sim,$$

where $\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})|_{U_i\cap U_j} \ni \pi_i \sim \pi_j \in \mathrm{Rep}^{\mathcal{H}}(G|_{U_j})|_{U_i\cap U_j}$ iff $\pi_j = \gamma_{ij}\pi_i\gamma_{ij}^{-1}$.

The space $\mathrm{Rep}^{\mathcal{H}}(G)$ is uniquely determined up to homeomorphism by the chosen local pseudo-trivialization of $(\mathcal{H}, \Delta)$ and isomorphisms $\{\psi_m : s^{-1}(m) \to H_m\}_{m\in M}$. One can see that $s : G \to M$ being locally open-compact implies that the topology of $\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})$ restricted to each fiber is equivalent to the topology of Lemma 3.18.

Proposition 3.21. *Suppose that $s : G \to M$ is locally open-compact family of groups. Then there is a one-to-one correspondence between continuous representations of $s : G \to M$ on $(\mathcal{H}, \Delta)$ and continuous sections of $\mathrm{Rep}^{\mathcal{H}}(G) \to M$.*

Proof. A continuous unitary representation π of G on $(\mathcal{H}, \Delta)$ corresponds to a continuous section of $\mathrm{Rep}^{\mathcal{H}}(G)$, i.e. to a family of sections $\tilde{\pi}_i : U_i \to \mathrm{Rep}^{\mathcal{H}}(G)|_{U_i}$ given by

$$\tilde{\pi}_i(m) = \phi_i \circ \pi \circ (\psi_m^{-1} \times \phi_i^{-1}).$$

These are easily seen to be compatible, i.e. $\tilde{\pi}_j = \gamma_{ij}\tilde{\pi}_i\gamma_{ij}^{-1}$. It remains to show that each $\tilde{\pi}_i$ is continuous. Consider an open set $U(\xi, \eta, K, V)$ as above. Note that

$$\begin{aligned}\tilde{\pi}^{-1}(U(\xi, \eta, K, V)) &= \{m \in U_i \mid \langle \xi, \pi\eta\rangle |_{K\cap H_m} \subset V\} \\ &= s'(K \cap \{g \in \coprod_{m\in U_i} H_m \mid \langle \xi, \pi\eta\rangle (g) \subset V\}),\end{aligned}$$

which is open since K is s'-compact and π is continuous.

A continuous section $\tilde{\pi}$ of $\mathrm{Rep}^{\mathcal{H}}(G)$ determines a continuous unitary representation by

$$\pi(g) := \phi_i^{-1} \circ \tilde{\pi}_i \circ (\psi_{s(g)}(g) \times \phi_i) \in U(\mathcal{H}_{s(g)}),$$

where $i \in I$ such that $s(g) \in U_i$. We only need to show that $\pi|_{G_{U_i}}\psi^{-1}$ is continuous. Suppose $\xi, \eta \in \Gamma_0(\mathrm{im}(\phi_i))$. Given $g \in \coprod_{m\in U_i} H_m$ and $V \subset \mathbb{C}$, let K be an s'-compact neighborhood of g and $W \subset K$ an open neighborhood of g. Consider $U(\xi, \eta, K, V)$. Define $W' := W \cap s^{-1}\tilde{\pi}^{-1}(U(\xi, \eta, K, U))$, which is open since s and $\tilde{\pi}$ are continuous. Then $g' \in W'$ implies

$$\langle \xi(s(g')), \pi(g')\eta(s(g'))\rangle = \langle \xi(s(g')), \tilde{\pi}(s(g'))(g')\eta(s(g'))\rangle \in U.$$

Moreover π is locally bounded, since $\mathcal{H}$ is uniformly finite-dimensional. This finishes the proof. □

Example 3.22. Consider a locally compact group H and a continuous principal H-bundle $\tau : P \to M$. From this we can construct a continuous bundle of groups $P \times_H H \to M$, where the action of H on H is given by conjugation. Consider a local trivialization $\{\chi_i : P|_{U_i} \to U_i \times H\}_{i\in I}$ of $P \to M$. Suppose $I = \mathbb{N}$. One can fix the group structure at each fiber of $P \times_H H \to M$ as follows: for every $m \in M$ choose the smallest $i \in I$ such that $m \in U_i$ and define

$$\psi_m : (P \times_H H)_m \to H, [p, h] \mapsto \chi_i(p)\, h\, \chi_i(p)^{-1}.$$

Given a representation $(\pi, \mathbb{C}^n) \in \mathrm{Rep}^n(H)$, one can construct a vector bundle $\mathcal{H} := P \times_\pi \mathbb{C}^n \to M$. Obviously, the trivialization of $P \to M$ gives rise to a trivialization $\{\phi_i : \mathcal{H}|_{U_i} \to U_i \times \mathbb{C}^n\}_{i\in I}$ of $\mathcal{H} \to M$, by $\phi_i([p, z]) = (\tau(p), \pi(\chi_i(p))z)$. Using these data one can form the bundle $\text{Rep}^{\mathcal{H}}(P \times_H H) \to M$ and a topology on it. A continuous section of this bundle is given by

$$\tilde{\pi}_i(m) = (h \mapsto \pi(\gamma_{ij}^{-1} h \gamma_{ij}),$$

for all $i \in \mathbb{N}$, $m \in U_i$, $h \in H$ and the smallest $j \in \mathbb{N}$ such that $m \in U_j$. This section corresponds to the representation of $P \times_H H \to M$ on $(\mathcal{H}, \Delta)$ given by $\tilde{\pi}([p,h])[p,z] = [p, \pi(h)z]$.

Remark 3.23. One can "twist" $\mathcal{H} := P \times_\pi \mathbb{C}^n$ by another continuous field $(\mathcal{H}', \Delta')$, carrying the trivial representation of $P \times_H H \to M$, to obtain a representation on $\mathcal{H} \otimes \mathcal{H}'$. A similar construction is possible for any groupoid, cf. Lemma 4.48.

3.5 Representations of the global bisections group

For the reader who prefers representation theory of groups and wonders why one should be interested in representations of groupoids at all, the next section will be of particular interest. Namely, to any continuous groupoid is associated a topological group: the group of global bisections. For a large class of continuous groupoids (the ones we call locally bisectional) we establish a bijection between the continuous representations of the groupoid on continuous fields of Hilbert spaces and a specific type of continuous representations of the group of global bisections on Banach spaces. Hence the representation theory of such groupoids can be "embedded" in the representation theory of groups. From this point of view, the groupoid offers a way to study some representations of these groups of bisections.

Suppose $G \rightrightarrows M$ is a continuous groupoid. A continuous **global bisection** is a continuous map $\sigma : M \to G$ such that $t \circ \sigma = id_M$ and $\tilde{\sigma} := s \circ \sigma : M \to M$ is a homeomorphism. Denote the set of global bisections of $G \rightrightarrows M$ by $\text{Bis}(G)$. This set has a group structure, cf. [26]. Moreover, it is even a topological group.

Lemma 3.24. $\text{Bis}(G)$ *has the structure of a topological group in the compact-open topology.*

Proof. The multiplication is given by

$$(\sigma_1 \cdot \sigma_2)(m) := \sigma_1(m)\sigma_2(\tilde{\sigma}_1(m)).$$

The unit is given by the unit section $u : M \to G$ and the inverse is defined by

$$\sigma^{-1}(m) := (\sigma(\tilde{\sigma}^{-1}(m)))^{-1}.$$

The group laws are easily checked, for example

$$\begin{aligned}(\sigma \cdot \sigma^{-1})(m) &= \sigma(m)\sigma^{-1}(\tilde{\sigma}(m)) \\ &= \sigma(m)(\sigma(\tilde{\sigma}^{-1}\tilde{\sigma}(m)))^{-1} \\ &= 1_m.\end{aligned}$$

We prove that multiplication is continuous $\text{Bis}(G) \times \text{Bis}(G) \to \text{Bis}(G)$. Suppose $\sigma_1 \cdot \sigma_2 \in U(C,V)$, where C is a compact set in M, V open in G and $U(C,V)$

the set of maps $\tau : M \to G$ that satisfy $\tau(C) \subset V$, i.e. $U(C, V)$ is in the standard subbasis of the topology on $\mathrm{Bis}(G)$. For each $m \in C$, let V_m be a neighborhood of $(\sigma_1 \cdot \sigma_2)(m) = \sigma_1(m)\sigma_2(\tilde{\sigma_1}(m))$. These V_m cover $\sigma_1 \cdot \sigma_2(C)$ which is compact by continuity of the multiplication in G and σ_1, σ_2. Let $\{V_i\}_{i\in I}$ be a finite subcover. The inverse image $m^{-1}(V_i)$ is open and contains a Cartesian product $W_i^1 \times W_i^2$ of open sets W_i^1, W_i^2 for each $i \in I$. Then $\sigma_1' \in U(C, \bigcup_{i\in I} W_i^1)$ and $\sigma_2' \in U(\tilde{\sigma_1}(C), \bigcup_{i\in I} W_i^2)$ implies $\sigma_1' \cdot \sigma_2' \in U(C, V)$. □

Example 3.25. The global bisection group of the pair groupoid $M \times M$ is the group of homeomorphisms of M.

Example 3.26. For the trivial group bundle $G \times M \to M$ (with fiber G) the group of global bisections is just the group of sections with the pointwise multiplication. In particular, if M is the circle $\mathbb{S}^1$ and G a Lie group then the group of global bisections is the loop group $C(\mathbb{S}^1, G)$ with its usual topology (cf. [19]).

Lemma 3.27. *A continuous unitary representation $(\pi, \mathcal{H}, \Delta)$ of a groupoid $G \rightrightarrows M$ canonically induces a continuous isometric representation of* $\mathrm{Bis}(G)$ *on* Δ*.*

Proof. Define the representation $\tilde{\pi}$ of $\mathrm{Bis}(G)$ by

$$(\tilde{\pi}(\sigma)\xi)(m) := \pi(\sigma(m))\xi(\tilde{\sigma}(m)),$$

where $\xi \in \Delta$, $m \in M$ and $\sigma \in \mathrm{Bis}(G)$. This representation is isometric, since π is unitary:

$$\|\tilde{\pi}(\sigma)\xi\| = \sup_{m\in M} \|\pi(\sigma(m))\xi(\tilde{\sigma}(m))\|_{\mathcal{H}_m} = \|\xi\|.$$

Continuity is proven as follows. Suppose $\varepsilon > 0$ and $\xi \in \Delta$ are given. There exists a compactly supported section $\xi' \in \Delta_c := C_c(M)\Delta$ such that $\|\xi - \xi'\| < \varepsilon/6$. Denote the support of ξ' by K. Moreover, since π is continuous and unitary it is norm continuous and hence there exists an open set $V \subset G$ such that $g, g' \in V$ implies $\|\pi(g)\xi'(s(g)) - \pi(g')\xi'(s(g'))\| < \varepsilon/3$. Now, suppose that $\sigma, \sigma' \in U(K, V)$ and $\eta \in B(\xi, \varepsilon/6)$, then

$$\sup_{m\in M} \|\pi(\sigma(m))\eta(\tilde{\sigma}(m)) - \pi(\sigma'(m))\eta(\tilde{\sigma}'(m))\| < \varepsilon,$$

which finishes the proof. □

The obtained representation of $\mathrm{Bis}(G)$ is actually $C_0(M)$**-unitary** (or C^***-unitary** with respect to $C_0(M)$), in the sense that

$$\langle \tilde{\pi}(\sigma)\xi, \tilde{\pi}(\sigma)\eta \rangle = \langle \xi, \eta \rangle$$

for all $\sigma \in \mathrm{Bis}(G)$ and $\xi, \eta \in \Delta$.

For the following result we need a technical condition on groupoids. We call a continuous groupoid $G \rightrightarrows M$ **bisectional** if

(i) every $g \in G$ is in the image of a continuous global bisection;

(ii) for all compact sets $K \subset M$ and open sets $V \subset G$, the set $\bigcup_{\sigma\in U(K,V)} \mathrm{im}(\sigma) \subset G$ is open.

Theorem 3.28. *Suppose $G \rightrightarrows M$ is bisectional. Then there is a bijective correspondence between continuous unitary representations of G and continuous $C_0(M)$-unitary representations of* $\mathrm{Bis}(G)$ *on a Hilbert $C_0(M)$-module satisfying*

(i) $C_0(M)$-linearity, i.e.

$$\tilde{\pi}(\sigma)(f\,\xi) = \tilde{\sigma}^*(f)\,\tilde{\pi}(\sigma)(\xi)$$

for all $\sigma \in \mathrm{Bis}(G)$, $\xi \in \Delta$ and $f \in C_0(M)$ and

(ii) locality, i.e. if $\sigma(m) = 1_m$ for some $m \in M$, then $\|\tilde{\pi}(\sigma)\xi - \xi\|(m) = 0$

Proof. Given a representation $(\tilde{\pi}, \Delta)$ of $\mathrm{Bis}(G)$ as above, define a representation $\pi : G \to U(\mathcal{H})$ as follows. Form the continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m\in M}, \Delta)$ associated to Δ (cf. Theorem 2.43). For any $g \in G$ and $h \in \mathcal{H}_{s(g)}$, define

$$\pi(g)h := (\tilde{\pi}(\sigma)\xi)(t(g)),$$

for any $\xi \in \Delta$ such that $\xi(s(g)) = h$ and $\sigma \in \mathrm{Bis}(G)$ such that $\sigma(t(g)) = g$, which exist by assumption. We now show that this definition does not depend on the choice of σ and ξ. Suppose ξ, ξ' satisfy $\xi(m) = h = \xi'(m)$. Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of sets such that $\bigcap_{i\in\mathbb{N}} U_i = \{s(g)\}$ and $\{\chi_i : U_i \to [0,1]\}$ a family of functions such that $\chi_i(s(g)) = 0$ and $\chi_i(n) = 1$ for all $n \in M\backslash U_i$. Then

$$\begin{aligned}(\tilde{\pi}(\sigma)\xi)(t(g)) - (\tilde{\pi}(\sigma)\xi')(t(g)) &= \lim_{i\to\infty} (\tilde{\pi}(\sigma)\chi_i(\xi - \xi'))(t(g)) \\ &= \lim_{i\to\infty} \chi_i(\tilde{\sigma}(t(g)))(\tilde{\pi}(\sigma)(\xi - \xi'))(t(g)) \\ &= 0,\end{aligned}$$

since $\tilde{\pi}$ is $C_0(M)$-linear and $\tilde{\sigma}(t(g)) = s(g)$.

Suppose $\sigma(m) = \sigma'(m)$ for $\sigma, \sigma' \in \mathrm{Bis}(G)$ and $m \in M$. Then, by locality, for all $\xi \in \Delta$

$$\|\tilde{\pi}(\sigma^{-1}\,\sigma')\xi - \xi\|(m) = 0,$$

and hence $(\tilde{\pi}(\sigma)\xi)(m) = (\tilde{\pi}(\sigma')\xi)(m)$.

Unitarity of π follows at once from $C_0(M)$-unitarity of $\tilde{\pi}$.

Next, we prove continuity of π. Suppose $(g, h) \in G\,{}_s\!\times_p \mathcal{H}$ and $U(\varepsilon, \eta, V)$ open neighborhood of $\pi(g)h$, where $\eta(t(g)) = \pi(g)h$. We need to construct an open neighborhood of (g, h), which maps to $U(\varepsilon, \eta, V)$. Consider

$$B(\eta, \varepsilon) := \{\xi \in \Delta \mid \|\eta - \xi\| < \varepsilon\}.$$

Let $\sigma \in \mathrm{Bis}(G)$ be such that $\sigma(t(g)) = g$, which exists since G is bisectional. Define $\xi := \bar{\pi}(\sigma)^{-1}\eta$. By continuity of $\bar{\pi}$ there exists an open neighborhood $B(\xi, \delta)$ of ξ and an open neighborhood $U(K, W)$ of σ such that $\bar{\pi}(U(K, W) \times B(\xi, \delta)) \subset B(\eta, \varepsilon)$. Since $G \rightrightarrows M$ is bisectional, there exists an open neighborhood W' of g in $\bigcup_{\sigma\in U(K,W)} \mathrm{im}(\sigma)$.

Suppose that $(g', h') \in W'\,{}_s\!\times_p U(\xi, \delta, \tilde{\sigma}^{-1}(V))$, then

$$\pi(g')h' = (\bar{\pi}(\sigma')\xi')(t(g')) \in U(\varepsilon, \eta, V),$$

for some $\sigma' \in U(K, W)$ and $\xi' \in B(\xi, \delta)$.

One easily sees that the constructions given in this proof to obtain representations of G from representations of $\mathrm{Bis}(G)$ and vice versa in the proof of the above lemma are inverses of each other. □

Remark 3.29. As an intermediate step, one can also relate the representations of $G \rightrightarrows M$ to the representation of the inverse semi-group of continuous **local bisections** of $G \rightrightarrows M$. These are continuous maps $\sigma : U \to G$ for open $U \subset M$ such that $t \circ \sigma = id_U$ and $\tilde{\sigma} := s \circ \sigma : U \to U$ is a homeomorphism.

4 Groupoid representation theory

Is there a Schur's Lemma for groupoids? Is there a Peter-Weyl theorem for groupoids? In this chapter we give answers to these questions. We discuss a way to generalize these statements, that are well-known for groups, to groupoids. It turns out that you need extra conditions on the groupoid for the statements to be true (unlike what is suggested in [1]). A crucial rôle is played by the functors that restrict representations of a groupoid to representations of its isotropy groups. This chapter shows that representation theory of groupoids is quite different from representation theory for groups, but many results can be carried over using some caution.

4.1 Decomposability and reducibility

Definition 4.1. (i) The **direct sum of a countable family of continuous fields of Hilbert spaces** $\{(\mathcal{H}^i, \Delta_i\}_{i\in I}$ is the continuous field of Hilbert spaces $(\mathcal{H}^{\oplus}, \Delta^{\oplus})$ whose fibers are given by $\mathcal{H}^{\oplus}_m := \bigoplus_{i\in I} \mathcal{H}^i_m$ and whose space of continuous sections $\Delta^{\oplus}$ is the closure of the pre-Hilbert $C_0(M)$-module of finite sums of sections $\sum_{j\in J} \xi_j$, where $J \subset I$ is a finite index set and $\xi_j \in \Delta_j$ for all $j \in J$.

(ii) The **direct sum of a countable family of continuous representations** $\{(\mathcal{H}^i, \Delta_i, \pi_i)\}_{i\in I}$ of a groupoid $G \rightrightarrows M$ is the representation of $G \rightrightarrows M$ on the direct sum of continuous fields of Hilbert spaces $(\mathcal{H}^{\oplus}, \Delta^{\oplus})$, given by the continuous extension of the map $\bigoplus_{j\in J} \pi_j : g \mapsto \sum_{j\in J} \pi_j(g)$ on finite sums by continuity.

(iii) We say that that a continuous unitary representation $(\mathcal{H}, \pi)$ of a groupoid G is **decomposable** if it is equivariantly isomorphic to a direct sum of representations of $(\mathcal{H}^1, \pi_1)$ and $(\mathcal{H}^2, \pi_2)$

$$(\mathcal{H}, \Delta) \cong \mathcal{H}^1 \oplus \mathcal{H}^2.$$

and **indecomposable** if this is not possible.

(iv) A **continuous subfield** of a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$ is a continuous field of Hilbert spaces $(\mathcal{H}', \Delta')$, such that $\mathcal{H}'_m \subset \mathcal{H}_m$ is a closed linear subspace with the induced inner product for all $m \in M$ and $\Delta' \subset \Delta$ a Hilbert $C_0(M)$-submodule.

(v) A **continuous subrepresentation** of a continuous unitary representation $(\mathcal{H}, \pi)$ of a groupoid G is a continuous subfield of $(\mathcal{H}, \Delta)$ stable under π.

(vi) A continuous unitary representation is **reducible** if it has a proper continuous subrepresentation. It is **irreducible** if it is not reducible.

Proposition 4.2. *If $(\mathcal{H}, \Delta, \pi)$ is a continuous locally trivial unitary representation and $(\mathcal{H}', \Delta', \pi')$ a locally trivial subrepresentation $(\mathcal{H}, \Delta, \pi)$, then $(\mathcal{H}, \Delta, \pi)$ decomposes as a direct sum of $(\mathcal{H}', \Delta', \pi')$ and another locally trivial subrepresentation.*

Proof. For each $m \in M$ let $\mathcal{H}''_m$ be the orthogonal complement with respect to the inner product. The family $\{\mathcal{H}''_m\}_{m \in M}$ forms a continuous field, with

$$\Delta'' := \{\xi \in \Delta \mid \xi(m) \in \mathcal{H}''_m \text{ for all } m \in M\},$$

since $\mathcal{H}$ is locally trivial. Moreover, $(\mathcal{H}'', \Delta'')$ is locally trivial too. Since π is unitary, this complement is G-invariant. □

Decomposability implies reducibility (irreducible implies indecomposable), but not vice versa. Indeed, a representation can contain a subrepresentation without being decomposable.

Example 4.3. Consider the trivial representation of $\mathbb{R} \rightrightarrows \mathbb{R}$ on $(\mathbb{R} \times \mathbb{C}, C_0(\mathbb{R}))$. It has a subrepresentation given by the continuous field of Hilbert spaces which is 0 at 0 and $\mathbb{C}$ elsewhere, with space of sections

$$C_0^0(M) := \{f \in C_0(M) \mid f(0) = 0\}.$$

This subrepresentation has no complement, since this would be a field that is $\mathbb{C}$ at 0 and zero elsewhere, whose only continuous section could be the zero section. Hence it would not satisfy condition (i) of Definition 2.20. Note that $\mathbb{R} \rightrightarrows \mathbb{R}$ is an example of a groupoid that has no continuous irreducible representations.

Definition 4.4. Define the **support of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta_{\mathcal{H}})$ by

$$\text{supp}(\mathcal{H}, \Delta) := \{m \in M \mid \mathcal{H}_m \neq 0\}.$$

This last set equals

$$\{m \in M \mid \xi(m) \neq 0 \text{ for some } \xi \in \Delta_{\mathcal{H}}\}.$$

One easily sees that for all continuous fields of Hilbert spaces $(\mathcal{H}, \Delta_{\mathcal{H}})$ the support $\text{supp}(\mathcal{H}, \Delta_{\mathcal{H}})$ is open in M.

Lemma 4.5. *(i) If the support of a continuous representation $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ of a groupoid $G \rightrightarrows M$ properly contains a closed union of G-orbits, then it is reducible.*

(ii) If the support of a continuous representation $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ of a groupoid $G \rightrightarrows M$ properly contains a clopen set of G-orbits, then it is decomposable.

Proof. Let $(\mathcal{H}, \Delta, \pi)$ be a continuous representation of $G \rightrightarrows M$. Suppose $U \subset M$ is a closed union of orbits. Define a new continuous field of Hilbert spaces by

$$\mathcal{H}'_m := \begin{cases} \mathcal{H}_m & \text{if } m \notin U \\ 0 & \text{if } m \in U \end{cases}$$

and

$$\Delta_{\mathcal{H}'} := \{\xi \in \Delta \mid \xi|_U = 0\},$$

The groupoid $G \rightrightarrows M$ represents on $(\mathcal{H}', \Delta')$ by

$$\pi'(g) := \begin{cases} \pi(g) & \text{if } s(g) \notin U \\ id_0 & \text{if } m \in U \end{cases}$$

One easily sees that $(\mathcal{H}', \Delta_{\mathcal{H}'}, \pi')$ is a continuous subrepresentation of $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$. The second statement is proved analogously. □

The representation $(\mathcal{H}', \Delta_{\mathcal{H}'}, \pi')$ is called the **restriction of** $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ **to** U^c.

Example 4.6. If a groupoid $G \rightrightarrows M$ is proper and M, then the orbits are closed. Hence an irreducible representation must consist of one orbit that is clopen, since it is the support of a continuous field and the orbit of a proper groupoid. Therefore, a space $M \rightrightarrows M$ has an irreducible representation iff it has a discrete point $m \in M$.

4.2 Schur's lemma

In the previous section we have seen that in many cases of interest the irreducible representations do exist. Therefore, we introduce the weaker notion of internal irreducibility.

Definition 4.7. A continuous representation $(\pi, \mathcal{H}, \Delta)$ of a groupoid $G \rightrightarrows M$ is called **internally irreducible**, if the restriction of π to each of the isotropy groups is an irreducible representation.

Obviously, if a representation is irreducible, then it is internally irreducible. The converse does not hold as we have seen in Example 4.3.

Example 4.8. Suppose H is a topological group, $P \to M$ a continuous principal H-bundle and (π, V) an irreducible representation of H. Then, $P \times_H V \to M$ carries a canonical internally irreducible (but reducible, if $M \neq pt$ and Hausdorff) representation of the bundle of groups $P \times_H H \to M$ (cf. Section 3.4).

Example 4.9. If M is a topological space with a non-trivial rank 2 vector bundle $E \to M$. Then $E \to M$ is not internally irreducible as a representation of $M \rightrightarrows M$, even though it might be indecomposable.

Example 4.10. A morphism of internally irreducible continuous representations is not necessarily an isomorphism or the zero map. A counterexample is given by the following: let G be the constant bundle of groups $\mathbb{R} \times U(1) \rightrightarrows \mathbb{R}$. It represents internally irreducibly on the trivial rank one vector bundle $\mathcal{H} := \mathbb{R} \times \mathbb{C}$ over $\mathbb{R}$ by scalar multiplication. The map $\Psi : (x, z) \mapsto (x, x \cdot z)$ is an equivariant adjointable map $\mathcal{H} \to \mathcal{H}$, not equal to a scalar times the identity.

Remark 4.11. What one does see in this example is that Ψ is a function times the identity on $\mathcal{H}$, namely the function $\lambda : \mathbb{R} \to \mathbb{C}, x \mapsto x$, i.e. $\psi = \lambda 1_{\mathcal{H}}$. An alternative formulation of Schur's lemma for groupoids would be that an endomorphism of an internally irreducible representation $(\mathcal{H}, \pi)$ is a function $\lambda \in C(M)$ times the identity on $\mathcal{H}$. This we shall proof under some conditions in Lemma 4.13.

Notation 4.12. For a continuous groupoid $G \rightrightarrows M$ denote

(i) the set of isomorphism classes of continuous unitary representations by $\mathrm{Rep}(G)$;

(ii) the subset of isomorphism classes of indecomposable unitary representation by $\mathrm{IdRep}(G)$;

(iii) the subset of isomorphism classes of irreducible unitary representations by $\mathrm{IrRep}(G)$. For groups H this set is known as the unitary dual and denoted by $\hat{H}$;

(iv) the set of isomorphism classes of internally irreducible unitary representations by $\mathrm{IrRep}^i(G)$.

Lemma 4.13 (Schur's Lemma for groupoids)**.** *Suppose $(\pi_i, \mathcal{H}^i, \Delta^i)$ is an internally irreducible representation for $i = 1, 2$.*

(i) every equivariant endomorphism $\Psi : \mathcal{H}^1 \to \mathcal{H}^1$ is equal to a continuous function $\lambda \in C(M)$ times the identity on E, i.e. $\psi = \lambda\, 1_{\mathcal{H}^1}$.

(ii) If $\Phi : \mathcal{H}^1 \to \mathcal{H}^2$ is a morphism of representations then Φ_m is either an isomorphism or the zero map $\mathcal{H}^1_m \to \mathcal{H}^2_m$ for all $m \in M$.

(iii) If, furthermore, $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$ is injective for every $m \in M$, then

$$\mathrm{Hom}_G(\mathcal{H}^1, \mathcal{H}^2) = \begin{cases} \textit{line bundle} & \textit{if } (\pi_1, \mathcal{H}^1, \Delta^1) \cong (\pi_2, \mathcal{H}^2, \Delta^2); \\ 0 & \textit{if } (\pi_1, \mathcal{H}^1, \Delta^1) \ncong (\pi_2, \mathcal{H}^2, \Delta^2). \end{cases}$$

The proof follows easily from the analogous statement for groups.

Example 4.14. Suppose $P \to M$ is a principal H-bundle for a group H. If $G \rightrightarrows M$ is the gauge groupoid $P \times_H P \rightrightarrows M$, then every irreducible representation is internally irreducible. Moreover, $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$ is injective for all $m \in M$. Hence Schur's Lemma holds for all representations of these groupoids. Moreover, for two representations $E_i = P \times_H V_i \to M$ of $G \rightrightarrows M$ $(i = 1, 2)$, with V_1 and V_2 isomorphic representations of H.

$$\begin{aligned} \mathrm{Hom}_G(E_1, E_2) &= \mathrm{Hom}_G(P \times_H V_1, P \times_H V_2) \\ &\cong P \times_H \mathrm{Hom}_H(V_1, V_2) \\ &\cong P \times_H \mathbb{C} \end{aligned}$$

where we used Schur's Lemma for groups in the third equation. The group H acts on $\mathrm{Hom}_H(V_1, V_2)$ by $(h \cdot \phi)v_1 = h^{-1}\, \phi(h \cdot v)$.

Example 4.15. Consider the two-sphere as a groupoid $S^2 \rightrightarrows S^2$. It is proper and all indecomposable vector bundles over S^2 have rank one. These are internally irreducible representations, but obviously $\mathrm{Res}_m : \mathrm{IrRep}^i(S^2) \to \mathrm{IrRep}(\{m\})$ is not injective for any $m \in M$. Moreover, for non-isomorphic line bundles $L_1 \to S^2$ and $L_2 \to S^2$, one has

$$\mathrm{Hom}_M(L_1, L_2) \cong L_1^* \otimes L_2 \ncong 0.$$

4.3 Square-integrable representations

In this section we define the notion of square-integrability for continuous groupoid representations. In the end, we prove that for proper groupoids, with M/G compact, unitary representations are square-integrable, generalizing an analogous result for compact groups.

Suppose $G \rightrightarrows M$ is a locally compact groupoid endowed with a Haar system $\{\lambda_m\}_{m\in M}$, which desintegrates as $\lambda_m = \int_{n\in t(G_m)} \lambda_m^n \mu_m(dn)$, for a Haar system $\{\mu_m\}_{m\in M}$ on $R_G \rightrightarrows M$ and a continuous family of measures $\{\lambda_m^n\}_{(n,m)\in R_G}$ on $t \times s : G \to M \times M$ (cf. Proposition 2.16).

Using the family $\{\lambda_m^n\}_{(n,m)\in R_G}$ one can construct the continuous field of Hilbert spaces

$$(\hat{L}^2(G), \Delta^2(G)) := (\hat{L}^2_{t\times s}(G), \Delta^2_{t\times s}(G)),$$

over R_G, cf. Example 2.44.

Example 4.16. A simple example of this is the following (also see the example following Proposition 2.16). If M is a space and μ a Radon measure on M and H a Lie group with Haar measure λ. Then the trivial transitive groupoid $M \times H \times M \rightrightarrows M$ with isotropy groups H has a Haar system $\{\lambda_m = \mu \times \lambda\}_{m\in M}$. Obviously, this decomposes as $\lambda_m = \int_{n\in M} \lambda\mu(dn)$, hence

$$(\hat{L}^2(G), \Delta^2(G)) = \left(L^2(G,\lambda) \times (M \times M), C_0(M \times M, L^2(G,\lambda))\right).$$

Definition 4.17. A map $f : G \to \mathbb{C}$ is called **$\hat{L}^2(G)$-square integrable** if the induced map

$$(m,n) \mapsto (g \mapsto f(g), G_m^n \to \mathbb{C})$$

is in $\Delta^2(G)$.

Definition 4.18. (i) The **conjugate $(\bar{\mathcal{H}}, \bar{\Delta})$ of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ is the family of Hilbert spaces is given by $\bar{\mathcal{H}}_m = \mathcal{H}_m$ as Abelian groups, but with conjugate complex scalar multiplication and the space of sections $\bar{\Delta} = \Delta$, but with conjugate $C_0(M)$-action.

(ii) The **conjugate representation** $(\bar{\mathcal{H}}, \bar{\Delta}, \bar{\pi})$ of a representation $(\mathcal{H}, \Delta, \pi)$ of $G \rightrightarrows M$ is the representation on the conjugate continuous field of Hilbert spaces $(\bar{\mathcal{H}}, \bar{\Delta})$ is given by $\bar{\pi}(g)h = \pi(g)h$, where $g \in G$ and $h \in \bar{\mathcal{H}}_{s(g)}$.

(iii) The **tensor product** $(\mathcal{H}^1 \otimes \mathcal{H}^2, \Delta^\otimes, \pi_1 \otimes \pi_2)$ **of two continuous continuous fields of Hilbert spaces** is the family of Hilbert spaces is given by $\mathcal{H}_m := \mathcal{H}_m^1 \otimes \mathcal{H}_m^2$. The space $\Delta^\otimes$ is the closure of the pre-Hilbert $C_0(M)$-module of all finite sums of sections $\sum_{j\in J} \xi_j \otimes \eta_j$ of $\xi_j \in \Delta^1$ and $\eta_j \in \Delta^2$.

(iv) The **tensor product of two representations** $(\mathcal{H}^1, \Delta^1, \pi_1)$ and $(\mathcal{H}^2, \Delta^2, \pi_2)$ of a groupoid $G \rightrightarrows M$ is the representation of $G \rightrightarrows M$ on $(\mathcal{H}^\otimes, \Delta^\otimes)$ given by linearly extending the map $(\pi_1 \otimes \pi_2)(g)(h \otimes h') = \pi(g)h \otimes \pi(g)h'$ and then extending it continuously to the closure $(\mathcal{H}^\otimes, \Delta^\otimes)$.

Definition 4.19. A continuous representation $(\pi, \mathcal{H}, \Delta)$ is **square-integrable** if the map

$$(\bar{\mathcal{H}} \otimes \mathcal{H}, \Delta^{\otimes}) \to (\hat{L^2}(G), \Delta^2(G))$$

given by

$$h_2 \otimes h_1 \mapsto (g \mapsto (h_2, \pi(g)h_1)_{\mathcal{H}_{t(g)}}$$

is a map of continuous fields of Hilbert spaces.

This means that the matrix coefficients $\langle \xi, \pi\eta \rangle$, defined by

$$(n, m) \mapsto (g \mapsto \langle \xi(n), \pi(g)\eta(m) \rangle)$$

for $\xi, \eta \in \Delta$ are $\hat{L}^2(G)$-square-integrable maps.

Example 4.20. For example, consider a topological space M. A (finite-dimensional) vector bundle $E \to M$ is a square-integrable representation of $M \rightrightarrows M$.

Example 4.21. Consider the family of continuous groups $G := (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}) \backslash (0, -1) \rightrightarrows \mathbb{R}$. One easily sees that the trivial representation $g \mapsto id_{\mathbb{C}}$ on $(\mathbb{R} \times \mathbb{C}, C_0(\mathbb{R}))$ is not square-integrable. But, note that G is not proper (although for every $m \in M$ the set $s^{-1}(m) = t^{-1}(m)$ is compact).

Proposition 4.22. *If $G \rightrightarrows M$ is proper and M/G compact, then every unitary representation is square-integrable.*

Proof. Suppose $(\mathcal{H}, \Delta, \pi)$ is a unitary representation and $\xi, \eta \in \Delta$. Given $\varepsilon > 0$, choose $\xi', \eta' \in C_c(M)\Delta$ such that $\|\xi - \xi'\| < \varepsilon'$ and $\|\eta - \eta'\| < \varepsilon'$, where

$$\varepsilon' = \frac{\min\{\varepsilon, 1\}}{3M \max\{\|\xi\|, \|\eta\|\}}$$

and

$$M = \max_{(n,m) \in R_G} \lambda_m^n(G_m^n),$$

which exists since M/G is compact. First note that $\langle \xi', \pi\eta' \rangle$ has compact support, since $G \rightrightarrows M$ is proper. Moreover,

$$\begin{aligned} \| \langle \xi, \pi\, \eta \rangle - \langle \xi', \pi\eta' \rangle \|_{\hat{L}^2} &\leq \| \langle (\xi - \xi'), \pi\, \eta \rangle \| + \| \langle \xi', \pi\, (\eta - \eta') \rangle \| \\ &\leq \max_{(n,m) \in R_G} \lambda_m^n(G_m^n)(\|\xi - \xi'\| \|\eta\| + \|\xi'\| \|\eta - \eta'\|) \\ &\leq \varepsilon' \|\eta\| + (\|\xi\| + \varepsilon')\varepsilon' \leq \varepsilon, \end{aligned}$$

which finishes the proof. □

Proposition 4.23. *If a continuous groupoid G has the property that for all $m \in M$ the restriction map*

$$\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G_m^m)$$

is injective, then for any two non-isomorphic internally irreducible unitary square-integrable representations $(\mathcal{H}, \Delta, \pi)$, $(\mathcal{H}', \Delta', \pi')$ and $\xi, \eta \in \Delta$, $\xi', \eta' \in \Delta'$,

$$\langle \langle \xi, \pi\eta \rangle, \langle \xi', \pi'\eta' \rangle \rangle_{\hat{L}^2(G)} = 0$$

Proof. This easily follows from the version of this statement for compact groups and the invariance of the Haar system. □

4.4 The Peter-Weyl theorem I

Suppose $G \rightrightarrows M$ is a continuous groupoid endowed with a Haar system $\{\lambda_m\}_{m\in M}$, which decomposes using a continuous family of measure $\{\lambda_m^n\}_{(n,m)\in R_G}$ as in Section 4.3. Let $\mathcal{E}(G) \subset \Delta^2(G)$ denote the $C_0(R_G)$-submodule spanned by the matrix coefficients (cf. Section 4.3) of all finite-dimensional representations of $G \rightrightarrows M$.

A generalization of the Peter-Weyl theorem as we are going to prove (cf. Theorem 4.29 and Theorem 4.36) appears not to be true for all continuous groupoids. Therefore, we introduce an extra condition:

Definition 4.24. For a continuous groupoid $G \rightrightarrows M$ the restriction map

$$\mathrm{Res}_m : \mathrm{Rep}(G) \to \mathrm{Rep}(G_m^m)$$

is **dominant** if for every $m \in M$ and every continuous unitary representation (π, V) of G_m^m there exists a continuous unitary representation $(\pi', \mathcal{H}, \Delta)$ of $G \rightrightarrows M$ such that (π, V) is isomorphic to a subrepresentation of $(\pi'|_{G_m^m}, \mathcal{H}_m)$.

Example 4.25. Suppose H is a group and $P \to M$ a principal H-bundle. Since $(P \times_H P)_m^m \cong H$ and $P \times_H P \rightrightarrows M$ are Morita equivalent, $\mathrm{Res}_m : \mathrm{Rep}(P \times_H P) \to \mathrm{Rep}((P \times_H P)_m^m)$ is dominant for all $m \in M$.

Example 4.26. Suppose H is a compact connected Lie group that acts on manifold M. Consider the action groupoid $G := H \ltimes M \rightrightarrows M$.

Proposition 4.27. *The restriction map* $\mathrm{Res}_m : \mathrm{Rep}(H \ltimes M) \to \mathrm{Rep}((H \ltimes M)_m^m)$ *is dominant for all* $m \in M$.

Proof. First we note that from every representation $(\pi, V) \in \mathrm{Rep}(H)$ we can construct a representation $\tilde{\pi} : H \ltimes M \to U(M \times V)$ of $H \ltimes M \rightrightarrows M$ on $M \times V \to M$ by $\tilde{\pi}(h, m) : (m, v) \mapsto (h \cdot m, \pi(h)v)$. Note that the isotropy groups of $H \ltimes M \rightrightarrows M$ coincide with the isotropy groups of the action. These are subgroups of H, hence the question is whether every representation of a subgroup of H occurs as the subrepresentation of the restriction of a representation of H.

Suppose K is a compact Lie subgroup of H. Fix a maximal tori $T_K \subset K$ and $T_H \subset H$ such that $T_K \subset T_H$, with Lie algebras $\mathfrak{t}_K$ and $\mathfrak{t}_H$. Note that $T_K \cong \mathfrak{t}_K/\Lambda_K$ and $T_H \cong \mathfrak{t}_H/\Lambda_H$ for lattices $\Lambda_K \subset \mathfrak{t}_K$ and $\Lambda_H \subset \mathfrak{t}_H$. There is an injective linear map $M : \mathfrak{t}_K \to \mathfrak{t}_H$ that induces the inclusion $\mathfrak{t}_K/\Lambda_K \hookrightarrow \mathfrak{t}_H/\Lambda_H$. Let P_K denote the integral weight lattice of T_K and P_H the integral weight lattice of T_H. Hence $q := M^T : \mathfrak{t}_H^* \to \mathfrak{t}_K^*$ is surjective map, mapping P_H onto P_K. Hence restriction of representations $\mathrm{Rep}(T_H) \to \mathrm{Rep}(T_K)$ is surjective too, since for tori irreducible representations correspond to integral weights.

The following argument is valid if one fixes positive root systems R_K^+, R_H^+ and hence fundamental Weyl chambers C_K^+, C_H^+ in a way specified in [9]. Suppose (π_λ, V) is an irreducible representation of K corresponding to the dominant weight $\lambda \in P_K \cap C_K^+$. One can choose any integral weight $\Lambda \in q^{-1}(\lambda) \cap P_H \cap C_H^+$; this set is non-empty, since q is surjective and the positive root systems have been fixed appropriately. Let π_Λ denote the irreducible representation of H associated to Λ. Then the multiplicity of π_λ in $\pi_\Lambda|_K$ is a positive integer (not necessarily 1), as follows from the Multiplicity Formula (3.5) in [9]. This finishes the proof. □

Example 4.28. A simple, but non-Hausdorff example of a proper groupoid which has a non-dominant restriction map is defined as follows. Consider $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z} \rightrightarrows \mathbb{R}$ and identify $(x, 0)$ with $(x, 1)$ for all $x \neq 0$. Endow the obtained family of groups $(\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})/\sim \rightrightarrows \mathbb{R}$, with the quotient topology. The non-trivial irreducible representation of $\mathbb{Z}/2\mathbb{Z}$ is not in the image of $\mathrm{Res}_0 : \mathrm{Rep}(G) :\longrightarrow \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$.

We now prove a generalization of the Peter-Weyl theorem for groupoids. Consider the continuous field of Hilbert spaces $(\hat{L}^2(G), \Delta^2(G))$ associated to a groupoid $G \rightrightarrows M$. Let $\overline{\mathcal{E}(G)}$ denote the closure of $\mathcal{E}(G)$ to a Hilbert $C_0(R_G)$-module.

Theorem 4.29 (Peter-Weyl for groupoids I)**.** *If $G \rightrightarrows M$ is a proper groupoid, M/G is compact and Res_m is dominant for all $m \in M$, then*

$$\overline{\mathcal{E}(G)} = \Delta^2(G).$$

Proof. Note that G_m^m is compact so Peter-Weyl for compact groups applies. Using the dominance property

$$\overline{\{\Theta(m,m) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^m, \lambda_m^m),$$

since $(\mathcal{H}, \Delta, \pi) < (\mathcal{H}', \Delta', \pi')$, implies $\langle \xi, \pi' \eta \rangle = \langle \xi, \pi \eta \rangle$ for $\xi, \eta \in \Delta$.

Note that $l_g^* : L^2(G_m^m, \lambda_m^m) \to L^2(G_m^n, \lambda_m^n)$ is an isometry for a chosen $g \in G_n^m$. Thus $\overline{\{l_g^*(\Theta(m,m)) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^n, \lambda_m^n)$. But, for all $h \in G_m^n$ and every continuous unitary finite-dimensional representation $(\mathcal{H}, \Delta, \pi)$

$$\begin{aligned} l_g^* \langle \xi, \pi\eta \rangle (h) &= \langle \xi(t(g)), \pi(gh)\eta(s(h)) \rangle_{\mathcal{H}^\pi_{t(g)}} \\ &= \textstyle\sum_{k=1}^{\dim(\mathcal{H}_n)} \langle \xi(m), \pi(g) e_k(n) \rangle_{E_n} \langle e_k(n), \pi(h)\eta(m) \rangle_{E_m}, \end{aligned}$$

where $e_1, \ldots, e_{\dim(\mathcal{H}_n)} \in \Delta$ are sections that form a basis of $\mathcal{H}$ at n. Thus $l_g^* \langle \xi, \pi\eta \rangle$ is a linear combination of matrix coefficients $\langle e_k, \pi\eta \rangle$ restricted to G_m^n, which implies $\overline{\{\Theta(n,m) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^n)$.

Let $f \in \Delta^2(G)$ and $\varepsilon > 0$ be given, then there exists a section $\tilde{f} \in \Delta^2(G)$ with compact support K such that $\|f - \tilde{f}\| < \varepsilon/2$, where the norm is the one associated to the $C_0(M)$-valued inner product. Moreover, for all $(m,n) \in R$ there are representations $(\mathcal{H}_{m,n}, \Delta_{m,n}, \pi_{m,n})$ and sections $u_{m,n}, v_{m,n} \in \Delta_{m,n}$, such that

$$\|\tilde{f} - (u_{m,n}, \pi_{m,n} v_{m,n})\|_{L^2(G_n^m)} < \varepsilon/2.$$

Since $\pi_{m,n}, u_{m,n}$ and $v_{m,n}$ are continuous we can find an open neighborhood $S_{m,n} \subset R$, such that still

$$\|\tilde{f} - (u_{m,n}, \pi_{m,n} v_{m,n})\|_{\hat{L}^2(G)|_{S_{m,n}}} < \varepsilon/2,$$

for all $(m,n) \in R$. These $S_{m,n}$ cover K, thus there is a finite subcover, which we denote by $\{S_i\}_{i \in I}$ to reduce the indices. Denote the corresponding representations by π_i and sections by u_i and v_i for $i \in I$. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\{S_i\}$. Define $\tilde{u}_i = \sqrt{\lambda_i} u_i$ and $\tilde{v}_i = \sqrt{\lambda_i} v_i$, then

$$\phi = \sum_{i \in I} (\tilde{u}_i, \pi_i \tilde{v}_i)$$

is a finite sum of matrix coefficients and

$$\begin{aligned}\|f-g\| &\leq \|f-\tilde{f}\|+\|\tilde{f}-\phi\| \\ &\leq \varepsilon/2+\sup_{(m,n)\in R}\|\tilde{f}-\textstyle\sum_{i\in I}(\tilde{u_i},\pi_i\tilde{v_i})\|_{L^2(G_n^m)} \\ &= \varepsilon/2+\sup_{(m,n)\in R}\|\textstyle\sum_{i\in I}\lambda_i\tilde{f}-\sum_{i\in I}(\sqrt{\lambda_i}u_i,\pi_i\sqrt{\lambda_i}v_i)\|_{L^2(G_n^m)} \\ &\leq \varepsilon/2+\textstyle\sum_{i\in I}\lambda_i\sup_{(m,n)\in R}\|\tilde{f}-\sum_{i\in I}(u_i,\pi_iv_i)\|_{L^2(G_n^m)} \\ &\leq \varepsilon/2+\textstyle\sum_{i\in I}\lambda_i\varepsilon/2=\varepsilon,\end{aligned}$$

which finishes the proof. □

Example 4.30. For a space M, $\overline{\mathcal{E}(M \rightrightarrows M)} = C_0(M)$ and $\overline{\mathcal{E}(M\times M \rightrightarrows M)} = C_0(M\times M)$ as Theorem 4.29 asserts.

Example 4.31. If H is a compact group and $P\to M$ an H-principal bundle. Then, for the bundle of groups $P\times_H H\to M$ one finds (cf. Example 4.8),

$$\begin{aligned}\overline{\mathcal{E}(P\times_H H \rightrightarrows M)} &\cong \Gamma_0(P\times_H \overline{\mathcal{E}(H)}) \\ &\cong \Gamma_0(P\times_H L^2(H)) \\ &\cong \Delta^2(P\times_H H),\end{aligned}$$

where in the second line we used the Peter-Weyl theorem for the group H.

4.5 The Peter-Weyl theorem II

In this section we shall try to find a decomposition of $(\hat{L}^2(G),\Delta^2(G))$ for proper groupoids $G\rightrightarrows M$, analogous to the case of compact groups H, where one has

$$L^2(H)\cong \bigoplus_{(\pi,V)\in\hat{H}} \bar{V}\otimes V$$

H-equivariantly.

Remark 4.32. There is a seemingly relevant proposition that asserts that

Proposition 4.33. *([24], Proposition 5.25) Any locally trivial countably generated representation $(\mathcal{H},\Delta,\pi)$ of a proper groupoid $G\rightrightarrows M$ is a direct summand of the regular representation, after stabilizing, i.e. $\mathcal{H}\subset\hat{L}^2_s(G)\otimes\mathbb{H}$, G-equivariantly, where $\mathbb{H}$ denotes a standard separable Hilbert space, say $l^2(\mathbb{N})$.*

Example 4.34. The Serre-Swan theorem for vector bundles is a nice example of this. Consider the groupoid $M\rightrightarrows M$ for a compact space M. Locally trivial representations of this groupoid are vector bundles. The theorem states that any vector bundle is a direct summand of $\hat{L}^2(M)\otimes\mathbb{H}\cong M\times\mathbb{H}$. The Serre-Swan Theorem is actually somewhat stronger, since instead of $\mathbb{H}$ one could put a finite-dimensional vector space $\mathbb{C}^N$ for large enough $N\in\mathbb{N}$. This is because the projection onto the direct summand can be proven to proper in this case.

In general the direct summands will not add up to the whole of $\hat{L}^2_s(G) \otimes \mathbb{H}$. Moreover, stabilization is not something that occurs in the case of compact groups, where one simply has $L^2(H) \cong \bigoplus_{(\pi,V)\in\hat{H}} \bar{V} \otimes V$, not something involving $L^2(H) \otimes \mathbb{H}$.

The continuous field of Hilbert spaces $(\hat{L^2}(I(G)), \Delta^2(I(G)))$ is the pullback of $(\hat{L^2}(G), \Delta^2(G))$ to the diagonal $\{(m,m) \in R_G \mid m \in M\} \hookrightarrow R_G$. It carries a continuous unitary representation

$$\pi_{LR}(g)f(h) := f(g^{-1}hg),$$

where $g \in G^n_m$, $h \in G^n_n$ and $f \in L^2(G^m_m)$.

Lemma 4.35. *For any square-integrable continuous unitary representation* $(\mathcal{H}^\pi, \Delta^\pi, \pi)$ *of a groupoid* $G \rightrightarrows M$ *there is an equivariant map*

$$\Psi_\pi : (\bar{\mathcal{H}^\pi} \otimes \mathcal{H}^\pi, \Delta^{\otimes}) \to (\hat{L}^2(I(G)), \Delta^2(I(G))),$$

given by

$$h_2 \otimes h_1 \mapsto (g \mapsto (h_2, \pi(g)h_1)_{\mathcal{H}_{t(g)}}).$$

This map is a slight adaptation of the one introduced for the definition of square-integrability.

Proof. For equivariance we compute

$$\begin{aligned}\Psi(\pi(g)(h_1 \otimes h_2)) &= \Psi(\pi(g)h_1 \otimes \pi(g)h_2) \\ &= (g' \mapsto (\pi(g)h_1, \pi(g')\pi(g)h_2)) \\ &= (g' \mapsto (h_1, \pi(g^{-1})\pi(g')\pi(g)h_2)) \\ &= (g' \mapsto (h_1, \pi(g^{-1}g'g)h_2)) \\ &= \pi_{LR}(g)(g' \mapsto (h_1, \pi(g')h_2))\end{aligned}$$

which finishes the proof. □

Theorem 4.36 (Peter-Weyl for groupoids II)**.** *Suppose* $G \rightrightarrows M$ *is a proper groupoid with* s *and* t *open maps and for every* $m \in M$

$$\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$$

is bijective. Then

$$\bigoplus_{\pi\in\mathrm{IrRep}^i(G)} \Psi_\pi : \bigoplus_{\pi\in\hat{G}} (\bar{\mathcal{H}^\pi}, \bar{\Delta^\pi}) \otimes (\mathcal{H}^\pi, \Delta^\pi) \to (\hat{L}^2(I(G)), \Delta^2(I(G))) \tag{4.3}$$

is an isomorphism of representations.

Proof. Surjectivity of the map follows from Theorem 4.29. Injectivity follows from Proposition 4.23. □

Example 4.37. Consider the pair groupoid $M \times M \rightrightarrows M$ for a space M. It has just one irreducible and indecomposable continuous unitary representation, namely the trivial one $M \times \mathbb{C} \to M$. Suppose μ is a Radon measure on M. The isomorphism of continuous fields of Hilbert spaces $\hat{L}^2(I(M \times M)) \cong M \times \mathbb{C}$, is obviously $(M \times M)$-equivariant.

Example 4.38. Consider a principal H-bundle $P \to M$ for a compact group H and the associated gauge groupoid $G := P \times_H P \rightrightarrows M$. By Morita equivalence of H and $G \rightrightarrows M$, there is a bijection between unitary irreps (V, π) of H and unitary indecomposable, irreducible representations $P \times_H V \to M$ of G. Therefore, Res_m is bijective. Hence, by Theorem 4.36, one has the decomposition of formula 4.3. This is no surprise, since $I(P \times_H P) \cong P \times_H H$, where H acts on H by conjugation, hence

$$\begin{aligned}
\hat{L}^2(I(P \times_H P) &\cong P \times_H L^2(H) \\
&\cong P \times_H \bigoplus_{(\pi,V)\in\hat{H}} \overline{V^\pi} \otimes V^\pi \\
&\cong \bigoplus_{(\pi,V^\pi)\in\hat{H}} (P \times_H \overline{V^\pi}) \otimes (P \times_H V^\pi) \\
&\cong \bigoplus_{(\pi,\mathcal{H}^\pi)\in\mathrm{IrRep}^i(P\times_H P)} \overline{\mathcal{H}^\pi} \otimes \mathcal{H}^\pi.
\end{aligned}$$

This is exactly the statement of Theorem 4.36.

Remark 4.39. Only for a few (types of) groupoids the map $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G_m^m)$ is bijective for all $m \in M$. If the map is just surjective, then one could try to find a subset $\mathrm{PW}(G)$ of $\mathrm{IrRep}^i(G)$ that does map bijective to $\mathrm{IrRep}(G_m^m)$ for every $m \in M$. Then, if this set is well chosen, the decomposition of Theorem 4.36 holds with $\mathrm{IrRep}^i(G)$ replaced by $\mathrm{PW}(G)$. We call such a set a **PW-set** (or **Peter-Weyl set**) **for** $G \rightrightarrows M$.

Example 4.40. Suppose M is a space. The Peter-Weyl set for $M \rightrightarrows M$ is the trivial representation $M \times \mathbb{C} \to M$.

Example 4.41. If H is a compact group and $P \to M$ a principal H-bundle, then $G := P \times_H H \to M$ is a bundle of groups (cf. Example 2.9) and

$$\mathrm{PW}(G) := \{P \times_H V \mid (\pi, V) \in \mathrm{IrRep}(H)\}$$

is a Peter-Weyl set (cf. Example 4.8).

4.6 Representation rings and K-theory of a groupoid

Suppose $G \rightrightarrows M$ is a continuous groupoid and M/G is compact.

Definition 4.42. The set of isomorphism classes of finite-dimensional continuous unitary representations of $G \rightrightarrows M$, endowed with $\oplus$ and $\otimes$ form a unital semi-ring. Applying the Grothendieck construction one obtains the **representation ring of** $G \rightrightarrows M$, denoted by $\mathcal{R}_f(G)$. Denote the subring of locally trivial representations (projective Hilbert $C_0(M)$-modules) by $\mathcal{R}(G)$.

Example 4.43. Suppose M is a compact space. Consider the groupoid $M \rightrightarrows M$. By definition one has $K_0(M) = \mathcal{R}(M)$.

Example 4.44. Suppose H is a compact group. Consider the groupoid $G := H \rightrightarrows pt$. Then $\mathcal{R}_f(G) = \mathcal{R}(G)$ equals the usual representation ring $\mathcal{R}(H)$ of H.

Example 4.45. Suppose M is a compact space. Then for the pair groupoid $M \times M \rightrightarrows M$ one sees that $\mathcal{R}(M \times M) = \mathcal{R}_f(M \times M) \cong \mathbb{Z}$ generated by the trivial representation.

Example 4.46. suppose H is a compact group acting on a compact space M. Then the representation ring of the action groupoid $H \ltimes M \rightrightarrows M$ satisfies $\mathcal{R}(H \ltimes M) = K^0_G(M)$.

Example 4.47. One easily sees that Morita equivalent groupoids have isomorphic representation rings. Hence, for a group H and a principal H-bundle $P \to M$ one has

$$\mathcal{R}_f(P \times_H P) \cong \mathcal{R}(P \times_H P) \cong \mathcal{R}(H) \cong \mathcal{R}_f(H),$$

which generalizes the previous example.

Suppose $s, t : G \to M$ are open maps. Recall that the orbit relation of a groupoid $G \rightrightarrows M$ is denoted by $R_G \rightrightarrows M$.

Lemma 4.48. *The representation ring $\mathcal{R}_f(G)$ is a $\mathcal{R}_f(R_G)$-module via the inclusion $\mathcal{R}_f(R_G) \to \mathcal{R}_f(G)$ given by*

$$\pi_G(g) := \pi_{R_G}(t(g), s(g)).$$

Analogously, $\mathcal{R}(G)$ is a $\mathcal{R}(R_G)$-module.

Example 4.49. Suppose $s : G \to M$ is a continuous family of groups. Then, $\mathcal{R}_f(G)$ is a $\mathcal{R}_f(M)$-module and $\mathcal{R}(G)$ is a $K^0(M)$-modules.

For proper groupoids the representation ring is isomorphic to the K-theory of the reduced C^*-algebra of the groupoid, under some technical conditions. This was proved in [24] in a more general (twisted) setting. We give a summary of their proof. Suppose $G \rightrightarrows M$ is a proper groupoid and $c : G \to \mathbb{R}_{\geq 0}$ a cutoff function for $G \rightrightarrows M$ (cf. Definition 2.19). A bounded operator $P \in \mathcal{B}_{C_0(M)}(\Delta^2_t(G))$ on the Hilbert module $\Delta^2_t(G)$, corresponds to a family of operator $\{P_m\}_{m \in M}$. The average of P is defined by

$$P^G_m := \int_{g \in G^m} g \cdot P_{s(g)}\, c(s(g))\, \lambda^m(dg),$$

where $g \cdot P_m := \pi_L(g) P_m \pi_L(g^{-1})$.

Lemma 4.50. *([24]) The reduced C^*-algebra $C^*_r(G)$ is equal to the C^*-algebra of averaged compact operators $(\mathcal{K}_{C_0(M)}(\Delta^2_t(G)))^G$.*

One uses this result to prove:

Theorem 4.51. *([24]) If $G \rightrightarrows M$ is proper, M/G compact and $C^*_r(G) \otimes \mathbb{H}$ has an approximate unit consisting of projections, then $K_0(C^*_r(G)) \cong \mathcal{R}(G)$.*

Proof. (sketch) If $C^*_r(G) \otimes \mathcal{K}(\mathbb{H})$ has an approximate unit consisting of projections, then $K_0(C^*_r(G))$ is obtained from the semi-ring generated by projections in $C^*_r(G) \otimes \mathbb{H}$, i.e. averaged compact projections of $\hat{L}^2_t(G) \otimes \mathbb{H}$. But these correspond precisely to locally trivial unitary representations of G according to Proposition 4.33 and the Serre-Swan theorem. □

Example 4.52. Suppose M is a compact space. Then $C^*(M \rightrightarrows M) = C(M)$, and $K_0(C(M)) = K^0(M) = \mathcal{R}(M \rightrightarrows M)$. Also, for the pair groupoid one can show $C_r^*(M \times M \rightrightarrows M) \cong \mathcal{K}(L^2(M))$ (cf. [13]) and hence $K_0(C_r^*(M \times M \rightrightarrows M)) \cong K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z} \cong \mathcal{R}(M \times M \rightrightarrows M)$.

Example 4.53. For a compact group H one has

$$K_0(C_r^*(H)) \cong \mathcal{R}(H)$$

. Theorem 4.51 generalizes this statement to proper groupoids (satisfying the mentioned condition).

Example 4.54. For a principal H-bundle $P \to M$ one can prove $C_r^*(P \times_H P \rightrightarrows M) \cong C_r^*(H) \otimes \mathcal{K}(L^2(M))$, hence

$$\begin{aligned} K_0(C_r^*(P \times_H P \rightrightarrows M)) &\cong K_0(C_r^*(H) \otimes \mathcal{K}(L^2(M))) \\ &\cong K_0(C_r^*(H)) \\ &\cong \mathcal{R}(H) \cong \mathcal{R}(P \times_H P \rightrightarrows M), \end{aligned}$$

by stability of K-theory.

5 The groupoid convolution C^*-category

In [21] Renault established a bijective correspondence between representations of groupoids $G \rightrightarrows M$ on measurable fields of Hilbert spaces and the non-degenerate bounded representations of the Banach *-algebra $L^1(G)$ on Hilbert spaces, generalizing the analogous statement for groups. Since there is bijection between representations of $L^1(G)$ and representations of $C^*(G)$, which is the universal enveloping C^*-algebra of $L^1(G)$, there is a bijection between measurable unitary representations of $G \rightrightarrows M$ and bounded non-degenerate representations of $C^*(G)$.

In this section we shall prove a different generalization suitable for continuous representations of groupoids. We give a bijective correspondence between continuous representations of groupoids on continuous fields of Hilbert spaces and continuous representations on continuous fields of Hilbert spaces of the continuous Banach *-category $\hat{L}^1(G)$. Moreover, we introduce the universal enveloping C^*-category of a Banach $*$-category and use this to define the C^*-category $C^*(G,G)$ of a groupoid. As a corollary we find a bijection between representations of $C^*(G,G)$ and the continuous representations of $G \rightrightarrows M$.

5.1 Fell bundles and continuous C^*-categories

First we need some terminology. We discuss the relation between continuous Fell bundles over groupoids (cf. [29, 17, 11]) and Banach $*$-categories and C^*-categories (cf. [7]).

A **(lower semi-)continuous Fell bundle over a groupoid** G is a (lower semi-)continuous field of Banach spaces $(\{\mathcal{B}_g\}_{g\in G}, \Delta)$ over G endowed with an associative bilinear product

$$\mathcal{B}_g \times \mathcal{B}_h \to \mathcal{B}_{gh}, (P,Q) \mapsto PQ$$

whenever $(g,h) \in G^{(2)}$ and an anti-linear involution

$$\mathcal{B}_g \to \mathcal{B}_{g^{-1}}, P \mapsto P^*$$

satisfying the following conditions for all $(g,h) \in G^{(2)}$ and $(P,Q) \in \mathcal{B}_g \times \mathcal{B}_h$

(i) $\|PQ\| \leq \|P\|\|Q\|$;

(ii) $\|P^*P\| = \|P\|^2$;

(iii) $(PQ)^* = Q^*P^*$;

(iv) P^*P is a positive element of $\mathcal{B}_{1_{s(g)}}$;

(v) the image of the multiplication $\mathcal{B}_g \times \mathcal{B}_h \to \mathcal{B}_{gh}, (P,Q) \mapsto PQ$ is dense;

(vi) multiplication $m^*\mathcal{B} \to \mathcal{B}$ and involution $\mathcal{B} \to \mathcal{B}$ are continuous maps of fields of Banach spaces.

where $\mathcal{B}$ denotes the total space of $(\mathcal{B}_{m\in M}, \Delta)$ endowed with the topology given by Δ and $m^*\mathcal{B}$ the pullback of the field $\mathcal{B}$ over G along $m : G^{(2)} \to G$.

Example 5.1. Our main example will be the following. Let $(\{\mathcal{H}_m\}_{m\in M}, \Delta_{\mathcal{H}})$ be a continuous field of Hilbert spaces over M. Consider the lower semi-continuous field of Banach spaces over $M \times M$ whose fiber at (n,m) is given by the bounded linear operators $\mathcal{H}_m \to \mathcal{H}_n$, i.e. $\mathcal{B}_{(n,m)} := \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$. This field was introduced in Section 3.2.

Suppose $G \rightrightarrows M$ is a continuous groupoid with open $s,t : G \to M$ and $(\{\mathcal{H}_m\}_{m\in M}, \Delta, \pi)$ a representation of $G \rightrightarrows M$. Let $R_G \rightrightarrows M$ denote the orbit relation groupoid. Consider the pullback $(\{\mathcal{B}_{(n,m)}\}_{(n,m)\in R_G}, \Delta_{\mathcal{B}})$ of $\mathcal{B}(\mathcal{H},\mathcal{H})$ along the inclusion $R_G \hookrightarrow M \times M$.

Lemma 5.2. *The lower semi-continuous field of Banach spaces*

$$(\{\mathcal{B}_{(n,m)}\}_{(n,m)\in R_G}, \Delta_{\mathcal{B}})$$

is a lower semi-continuous Fell bundle over R_G.

Proof. The continuity of the composition was proven in the proof of Lemma 3.12. Note that $\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}$ is an isomorphism of Hilbert spaces. Hence, the properties (i), (ii), (iii),(iv) and (v) follow from the fact that these are true for $\mathcal{B}(\mathcal{H})$, where $\mathcal{H} \cong \mathcal{H}_{t(g)} \cong \mathcal{H}_{s(g)}$. □

A (lower semi-)continuous Fell bundle A over a continuous equivalence relation $R \subset M \times M$ on M is a full **(lower semi-)continuous C^*-category over M**. Leaving out the C^*-norm equality (ii) we speak of a full **(lower semi-)continuous Banach $*$-category**. Because of the denseness condition (v), it is called a *full* (lower semi-)continuous Banach (or C^*-)category over M. The continuous field $\mathcal{B}(\mathcal{H},\mathcal{H})$ over $M \times M$ is a continuous C^*-category. It is full iff $m \to \dim(\mathcal{H}_m)$ is constant.

Example 5.3. Let $G \rightrightarrows M$ be a locally compact groupoid endowed with a Haar system $\{\lambda^m\}_{m\in M}$. Suppose there exist a continuous families measures $\{\lambda^n_m\}_{(n,m)\in R_G}$ on G and $\{\mu_m\}_{m\in M}$ on M such that

$$\lambda^n = \int_{m\in s(G^n)} \lambda^n_m \mu_n(dm),$$

cf. Proposition 2.16.

Consider the continuous field of Banach spaces $(\hat{L}^1(G), \Delta^1(G)) := (\hat{L}^1_{t\times s}(G), \Delta^1_{t\times s}(G))$, cf. Example 2.44.

Lemma 5.4. *$(\hat{L}^1(G), \Delta^1(G))$ is a continuous Banach $*$-category over M, where the multiplication map $\hat{L}^1(G)^{(2)} \to \hat{L}^1(G)$ is the continuous extension of*

$$f * f'(g) := \int_{h\in G^m_k} f(gh^{-1})f'(h)\lambda^m_k(dh),$$

for all $f \in C_c(G^n_m)$ and $f' \in C_c(G^m_k)$.

Proof. (sketch) One, indeed easily checks that $\|f * g\|(n, k) < \|f\|(n, m)\|g\|(m, k)$, so this extension is well-defined. □

Definition 5.5. A **strongly continuous representation** $(\mathcal{H}, \Delta, L)$ **of a continuous Banach $*$-category** A over a space M on a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$ over M is a continuous $*$-homomorphism

$$L : A \to \mathcal{B}(\mathcal{H}, \mathcal{H}),$$

such that $a \mapsto L(a)\xi(s(a))$ is continuous $A \to \mathcal{H}$ for every $\xi \in \Delta$ (with the obvious notion of homomorphism). A strongly continuous representation of a continuous C^*-category is a representation of this as a Banach $*$-category.

One has analogous definitions for weakly continuous representations and representations continuous in the operator norm.

Definition 5.6. A representation $(\mathcal{H}, \Delta, L)$ of a Banach $*$-category A is non-degenerate if $\overline{L(A)\mathcal{H}} = \mathcal{H}$.

Suppose $(A_{(n,m)\in R}, \Delta)$ is a continuous Banach $*$-category over M. Then a **universal enveloping C^*-category** of (A, Δ) is a continuous C^*-category (B, Δ) and a morphism $A \to B$ such that any morphism of continuous Banach $*$-categories $A \to C$ factors via B. It can be constructed as the closure of the A under the image of

$$\bigoplus_{\pi\in\hat{A}} \pi : A \mapsto \bigoplus_{\pi\in\hat{A}} \mathcal{H}_\pi,$$

where $\hat{A}$ denotes the set of isomorphism classes of continuous bounded non-degenerate representations of A.

Example 5.7. Suppose $G \rightrightarrows M$ is a continuous groupoid. The **continuous C^*-category** $C^*(G, G)$ **of** $G \rightrightarrows M$ is the universal enveloping C^*-category of $(\hat{L}^1(G), \Delta^1(G))$. One easily sees that this is a continuous C^*-category, since $(\hat{L}^1(G), \Delta^1(G))$ is a continuous Banach $*$-category. Analogously to the group case, one can also introduce the reduced C^*-category of a groupoid, but we shall not need this here.

5.2 Representations of $G \rightrightarrows M$, $\hat{L}^1(G)$ and $C^*(G, G)$

We say $G \rightrightarrows M$ allows **Dirac sequences** $\{(\delta_k^g)_{k\in\mathbb{N}}\}_{g\in G}$ for the Haar system $\{\lambda_m^n\}_{(n,m)\in R_G}$, if

(i) $\delta_k^g \geq 0$ on $G_{s(g)}^{t(g)}$,

(ii) $\int_{g'\in G_m^n} \delta_k^g(g')\lambda_{s(g)}^{t(g)}(dg') = 1$ for all $k \in \mathbb{N}$,

(iii) for every open neighborhood $U \subset G_m^n$ of g and every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for $k > N$
$$\int_{g\in U^c} \delta_k^g(g')\lambda_m^n(dg') < \varepsilon.$$

Lemma 5.8. *If $(\mathcal{H}, \Delta, \pi)$ is a continuous unitary representation of $G \rightrightarrows M$, then $L_\pi : \hat{L}^1(G) \to \mathcal{B}(\mathcal{H}, \mathcal{H})$ given by*
$$f \mapsto \left((n, m) \mapsto \int_{G_m^n} f(g)\pi(g)\lambda_m^n(dg) \right)$$
is a non-degenerate strongly continuous representation of $(\hat{L}^1(G), \Delta^1(G))$ as a continuous Banach $*$-category on the continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$.

Proof. By the properties of the Bochner integral one has
$$\begin{aligned} \|\pi(f)\|(n, m) &= \| \textstyle\int_{G_m^n} f(g)\pi(g)\lambda_m^n \| \\ &\leq \textstyle\int_{G_m^n} |f(g)| \|\pi(g)\|\lambda_m^n(dg) \\ &= \textstyle\int_{G_m^n} |f(g)|\lambda_m^n(dg) \end{aligned}$$
(Note that therefore, $\|\pi(f)\| = \sup_{(n,m)\in R} \|\pi(f)\|(n, m) \leq \|f\|_{\hat{L}^1(G)}$).

We now prove that L_π is a $*$-homomorphism. Suppose $f \in C_c(G_m^k)$ and $f' \in C_c(G_k^n)$, then
$$\begin{aligned} L_\pi(f * f') &= \int_{g \int G_m^n} (f * f')(g)\pi(g)\lambda_m^n(dg) \\ &= \int_{g\in G_m^n} \int_{h\in G_m^k} f(gh^{-1})f'(h)\lambda_m^k(dh)\pi(g)\lambda_m^n(dg) \\ &= \int_{g\in G_k^n} f(g)\pi(g)\lambda_k^n(dg) \int_{h\in G_m^k} f'(h)\pi(h)\lambda_k^m(dh) \\ &= L_\pi(f)L_\pi(f'), \end{aligned}$$
by invariance of the Haar system. One easily checks that $L^\pi(f)^* = L^\pi(f^*)$.

Suppose $f \in C_c(G_m^n)$ is given. Suppose $F \in C_c(G)$ satisfies $F|_{G_m^n} = f$. Note that
$$\sup_{(n',m')\in R_G} \int_{G_m^n} \|F(g)\pi(g)\xi(s(g))\|\lambda_m^n(dg) \leq \|F\|_{\hat{L}^1(G)} \max_{m\in s(\mathrm{supp}(F))} \|\xi(m)\|.$$

Using this, one easily proves that L_π is strongly continuous.

The representation π_L is non-degenerate, since for any $m \in M$ and $h \in \mathcal{H}_m$

$$\lim_{k\to\infty} \|h - L(\delta_k^{1_m})h\| = \lim_{k\to\infty} \|h - \int_{g\in G_m^n} \delta_k^{1_m}(g)\pi(g)h\,\lambda_m^n(dg)\| = 0.$$

This finishes the proof. □

For $f \in C_c(G)$, $m \in M$ and $g, g' \in G^m$, we shall use the notation $f^g(g') := (\pi_L(g)f)(g') = f(g^{-1}g')$.

Lemma 5.9. *If* $(\mathcal{H}, \Delta, L)$ *is a strongly continuous non-degenerate representation of* $(\hat{L}^1(G), \Delta^1(G))$*, then*

$$\pi(g)(L(f)h) := L(f^g)h$$

defines a continuous unitary representation of $G \rightrightarrows M$ *on the continuous field of Hilbert spaces* $(\mathcal{H}, \Delta)$*.*

Proof. By non-degeneracy of L, the above formula defines π_L on a dense set. It extends to the whole of $\mathcal{H}$, since for all $g \in G$ and $h \in \mathcal{H}_{s(g)}$ one has

$$\begin{aligned}\|\pi(g)h\| &= \lim_{k\to\infty} \|L(\delta_g^k)h\| \\ &\leq \lim_{k\to\infty} B\|\delta_g^k\|\|h\| \\ &= B\|h\|,\end{aligned}$$

for a constant $B \in \mathbb{R} \geq 0$.

This is well-defined. Indeed, suppose $L(f)h = L(f')h'$ for $f \in L^1(G_m^n)$, $f' \in L^1(G_{m'}^n)$, $h \in \mathcal{H}_m$ and $h' \in \mathcal{H}_{m'}$. Let δ_k^g denote the translation of δ_k^m along $g \in G_m^{m'}$. One easily checks that

$$\|\delta_k^g * f - f^g\| \to 0$$

when $k \to \infty$. Then one has for all $k \in \mathbb{N}$:

$$\begin{aligned}\|L((f')g)h' - L(f^g)h\| &\leq \|L((f')^g)h' - L(\delta_k^g * f')h'\| \\ &\quad \|L(\delta_k^g * f')h' - L(\delta_k^g * f)h\| + \|L(\delta_k^g * f)h' - L(f^g)h\| \\ &\leq B\|(f')^g - \delta_k^g * f'\|\|h'\| + \|L(\delta_k^g)(L(f')h' - L(f)h)\| \\ &\quad + B\|(f)^g - \delta_k^g * f\|\|h\|.\end{aligned}$$

The second term is zero and the first and the last term go to zero as $k \to \infty$, hence $L((f')g)h' = L(f^g)h$.

π is a homomorphism. Indeed, for $(g, g') \in G^{(2)}$, $f \in L^1(G_m^{s(k)})$ and $h \in \mathcal{H}_m$ one has

$$\begin{aligned}\pi(gg')(L(f)h) &= L(f^{gg'})h \\ &= L((f^{g'})^g)h \\ &= \pi(g)L(f^{g'})h \\ &= \pi(g)\pi(g')(L(f)h).\end{aligned}$$

Furthermore, the following computation shows that $\pi(g)^* = \pi(g^{-1})$:

$$\begin{aligned}
\langle \pi(g)^* L(f)h, L(f')h' \rangle &= \langle h, L(f)^* \pi(g) L(f')h' \rangle \\
&= \langle h, L(f^*) L((f')^g)h')h' \rangle \\
&= \langle h, L(f^* * (f')^g)h' \rangle \\
&= \left\langle h, L((f^{g^{-1}})^* * f')h' \right\rangle \\
&= \left\langle h, L(f^{g^{-1}})^* L(f')h' \right\rangle \\
&= \left\langle L(f^{g^{-1}})h, L(f')h' \right\rangle \\
&= \left\langle \pi(g^{-1}) L(f)h, L(f')h' \right\rangle,
\end{aligned}$$

where the fourth step follows from equivariance of the Haar system and the fact that

$$(f^g)^*(g') = f^*(g'g^{-1}).$$

The continuity of π follows from the fact that for any $F \in C_c(G)$, representing a section of $\hat{L}^1(G) \to R_G$, and any $\xi \in \Delta$, the section $m \mapsto L(F)(m,m)\xi(m)$ is again in Δ and that $g \mapsto F(s(g), s(g))^g$ is continuous, cf. Proposition 3.15. □

Theorem 5.10. *The correspondence $\pi \mapsto L_\pi$ is a bijection between the set of continuous unitary representations of $G \rightrightarrows M$ and the set of non-degenerate strongly continuous representations of $(\hat{L}^1(G), \Delta^1(G))$.*

Proof. The inverse correspondence is given by Lemma 5.9, which we denote by $L \mapsto \pi^L$ (not to be confused with the left regular representation π_L). Given a continuous unitary representation π of G, we compute

$$\begin{aligned}
\pi^{(L^\pi)}(g)(L^\pi(f)h) &= L^\pi(f^g)h \\
&= \int_{g' \in G_m^n} f(g^{-1}g')\pi(g')h\lambda_m^n(dg') \\
&= \int_{g' \in G_m^n} f(g')\pi(g)\pi(g')h\lambda_m^p(dg') \\
&= \pi(g)(L^\pi(f)h).
\end{aligned}$$

Conversely, suppose a non-degenerate strongly continuous representation L of $\hat{L}^1(G)$ is given. Then we have

$$\begin{aligned}
L^{(\pi^L)} L(f')h &= \int_{g \in G_m^n} f(g)\pi^L(g)L(f')h\lambda_m^n(dg) \\
&= \int_{g \in G_m^n} f(g)L((f')^g)h\lambda_m^n(dg) \\
&= L(\int_{g \in G_m^n} f(g)(f')^g\lambda_m^n(dg))h \\
&= L(f * f')h = L(f)(L(f')h),
\end{aligned}$$

which finishes the proof. □

Corollary 5.11. *The bijective correspondence of Theorem 5.10 extends to a bijective correspondence between the set of continuous unitary representations of $G \rightrightarrows M$ and the set of non-degenerate strongly continuous representations of $C^*(G, G)$.*

6 Acknowledgements

The author would like to thank Gert Heckman, Peter Hochs, Klaas Landsman, Michael Mueger for helpful discussions on the topic and Maarten Solleveld for some comments on an earlier version of this paper.

References

[1] M. Amini, *Tannaka-Krein duality for compact groupoids I, II and III*, preprint, ArXiv:math.OA/0308259, math.OA/0308260, math.OA/0308261.

[2] Rogier Bos, *Geometric quantization of Hamiltonian actions of Lie algebroids and Lie groupoids*, Int. J. Geom. Methods Mod. Phys. **4** (2007), no. 3, 389–436.

[3] Rogier Bos, *Groupoids in geometric quantization*, PhD-thesis, Radboud University Nijmegen, 2007.

[4] A. Connes, *Sur la théorie non commutative de l'intégration*, Springer Lecture Notes in Math. **725** (1979), 19–143.

[5] J. Dixmier, *les C*-algebras et leurs representations*, Deuxième édition, Paris, Gauthier-Villars, 1969.

[6] J. Dixmier and A. Douady, *Champs continues d'espaces hilbertiens et de C^*-algèbres*, Bul. de la S.M.F. **19** (1963), 227–284.

[7] S. Doplicher and J. Roberts, *A new duality theory for compact groups*, Invent. Math. **98** (1989), no. 1, 157–218.

[8] S. A. Gaal, *Linear analysis and representation theory*, Die Grundlehren der mathematischen Wissenschaften, Band 198. Springer-Verlag, New York-Heidelberg, 1973.

[9] G. J. Heckman, *Orbits and multiplicities for compact groups*, Invent. math. **67** (1982).

[10] M. Hilsum and G. Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture de Kasparov)*, Ann. Sci. Ens. **20** (1987), 325-390.

[11] A. Kumjian, *Fell bundles over groupoids*, Proc. Amer. Math. Soc. **126** (1998), no. 4, 1115–1125.

[12] N. P. Landsman, *The Muhly-Renault-Williams theorem for Lie groupoids and its classical counterpart*, Lett. Math. Phys. **54** (2000), no. 1, 43-59.

[13] N. P. Landsman, *Mathematical Topics Between Classical and Quantum Mechanics*, Springer Mathematical Monographs, Springer, New York, 1998.

[14] P.-Y. Le Gall, *Théorie de Kasparov équivariante et groupoïdes I*, K-Theory 16 (1999), no. 4, 361–390.

[15] J. Mrčun, *Stability and Invariants of Hilsum-Skandalis maps*, PhD Thesis, University of Utrecht, 1996.

[16] J. Mrčun and I. Moerdijk, *Introduction to Foliations and Lie Groupoids*, Cambridge University Press, Cambridge, 2003.

[17] P. Muhly, *Bundles over groupoids*, Groupoids in analysis, geometry, and physics (Boulder, CO, 1999), 67–82, Contemp. Math. **282**, Amer. Math. Soc. Providence, RI, 2001.

[18] P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987), no. 1, 3–22.

[19] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.

[20] D. Pronk, *Groupoid representations for sheaves on orbifolds*, PhD-thesis, University of Utrecht, 1995.

[21] J. Renault, *A groupoid approach to C^*-algebras*, Lecture notes in mathematics **793**, Springer Verlag, 1980.

[22] J. Renault, *Représentations des produits croisés d'algèbres de groupoïdes*, J. Operator Theory **18** (1987), 67-97.

[23] J.-L. Tu, *La conjecture de Novikov pour les feuilletages hyperboliques*, K-theory **16** (1999), 129–184.

[24] J.-L. Tu, P. Xu and C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Sci. Ens **37** (2004), no. 6, 841–910.

[25] A. Weinstein, *Groupoids: unifying internal and external symmetry*, Notices Amer. Math. Soc. **43** (1996), 744–752.

[26] A. Weinstein and A. Cannas da Silva, *Geometric models for noncommutative algebras*, Berkeley Mathematics Lecture Notes **10**, American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[27] J. J. Westman, *Locally trivial C^r-groupoids and their representations*, Pacific J. Math. **20** (1967), 339–349.

[28] J. J. Westman, *Harmonic analysis on groupoids*, Pacific J. Math. **27** (1968), no. 3, 621-632.

[29] S. Yamagami, *On primitive ideal spaces of C^*-algebras over certain locally compact groupoids*, Mappings of operator algebras (Philadelphia, PA, 1988), 199–204, Progr. Math. **84**, Birkhäuser Boston, Boston, MA, 1990.