Foliated symplectic couplings and Hamiltonian Lie algebroid actions

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Abstract

In this note we explain the relation between foliated coupling forms on symplectic fibrations and Hamiltonian actions of Lie algebroids and Lie groupoids as introduced in [1].

Suppose $(B, \mathcal{F}_B)$ is a foliated smooth manifold and $p : M \to B$ is a smooth fiber bundle. Suppose $\omega : \mathrm{Ver} \wedge \mathrm{Ver} \to \mathbb{R}$ is a vertical symplectic form on M (where $\mathrm{Ver} = \ker(Tp)$). In [1] we assumed the existence of an extension of ω to a form $\tilde{\omega} : p^*\mathcal{F} \wedge p^*\mathcal{F} \to \mathbb{R}$, a so-called foliated coupling form, and discussed the notion of Hamiltonian Lie algebroid action in this setting. In this note we show how to construct such a foliated coupling form, and how this construction is entails a Hamiltonian action of a certain Lie algebroid (and groupoid).

1 foliated bundles and foliated connections

Suppose M and B are a smooth manifolds and $\mathcal{F}_B$ a regular foliation of B. Suppose $p : M \to B$ is a a fiber bundle. Endow M with the pullback foliation $\mathcal{F}^M := p^*\mathcal{F}^B$. Denote $\ker(Tp) \subset TM$ by Ver, $T\mathcal{F}_M$ by F^M and $T\mathcal{F}_B$ by F^B. Note that $\mathrm{Ver} \subset F^M$.

Definition 1.1. An **$\mathcal{F}$-partial (Ehresmann) connection** Γ on M is a $\mathcal{F}$-partial horizontal distribution Hor, i.e. for all $x \in B$

$$F^M_x = \mathrm{Hor}_x \oplus \mathrm{Ver}_x \,.$$

The $\mathcal{F}$-partial connection Γ is said to be **complete** if each it has the usual lifting property: for each $x \in B$, $m \in M_x$ and each curve $\gamma : [0,1] \to \mathcal{F}_B$, there exist a unique curve $\tilde{\gamma}_m : [0,1] \to \mathcal{F}_M$ such that $\gamma = p \circ \tilde{\gamma}$ and $d^{\mathcal{F}}\tilde{\gamma}([0,1]) \subset \mathrm{Hor}$.

If the fibers of $M \to B$ are compact, then any partial connection is complete. Denote the horizontal lifting map $TB \to \mathrm{Hor}$ by $v \mapsto v^\sharp$. The **curvature form**

$$\Gamma(F^B) \wedge \Gamma(F^B) \to \Gamma(\mathrm{Ver})$$

is denoted by

$$\mathrm{curv}(v,w) = \tau_{\mathrm{Ver}}([v^\sharp, w^\sharp]) = [v^\sharp, w^\sharp] - [v,w]^\sharp,$$

where $\tau_{\mathrm{Ver}} : F^M \to \mathrm{Ver}$ denotes the projection along Hor.

2 Fiber-compatible forms and connections

Now suppose $p : M \to B$ is endowed with a vertical symplectic form $\omega : \mathrm{Ver} \wedge \mathrm{Ver} \to \mathbb{R}$, i.e. $d^{\mathrm{Ver}}\omega = 0$ and ω is non-degenerate.

Definition 2.1. An $\mathcal{F}$-partial 2-form $\tilde{\omega} : F^M \wedge F^M \to \mathbb{R}$ of ω is called fiber-compatible if

$$\tilde{\omega}|_{\mathrm{Ver} \wedge \mathrm{Ver}} = \omega.$$

Remark 2.2. This is the situation encountered in [1]. There we simply *assumed* the existence of an extension of ω to a *closed* form $\tilde{\omega} : F^M \wedge F^M \to \mathbb{R}$. In this note we will show that there is a natural way to form such an extension, using the so-called coupling construction. Actually, a special case of this construction was already implicit in the example of the gauge groupoid in that paper.

Proposition 2.3. *Given ω, there is a canonical bijection between fiber-compatible $\mathcal{F}$-partial forms $\tilde{\omega} : F^M \wedge F^M :\to \mathbb{R}$ extending ω and $\mathcal{F}$-partial connections.*

Proof. Given $\tilde{\omega}$, define Hor to be the annihilator of Ver with respect to $\tilde{\omega}$. Conversely, given Hor, define for all $x \in B$ and $v, w \in F^M_x$

$$\tilde{\omega}_x(v,w) := \omega_x(\tau_{\mathrm{Ver}}(v), \tau_{\mathrm{Ver}}(w)),$$

which finishes the proof. □

The form $\tilde{\omega}$ constructed here is obviously not necessarily closed.

Suppose $p : (M, \mathcal{F}_M) \to (B, \mathcal{F}_B)$ is endowed with an $\mathcal{F}$-partial connection Γ and a partial 2-form $\tilde{\omega} : F^M \wedge F^M \to \mathbb{R}$.

Definition 2.4. We say $\tilde{\omega}$ is Γ-compatible if Hor equals the annihilator of Ver with respect to $\tilde{\omega}$.

3 Connection compatible partial forms

Definition 3.1. A partial connection Γ is said to be **symplectic** with respect to a vertical symplectic form ω if the associated parallel transport preserves the symplectic form, i.e. the induced maps $M_{\gamma(0)} \to M_{\gamma(1)}, m \mapsto \tilde{\gamma}_m(1)$ are symplectomorphisms for all paths $\gamma : [0,1] \to B$.

The existence of an $\mathcal{F}$-partial connection hence implies that restricting to a leaf $L \in \mathcal{F}_B$ the bundle $M|_L \to L$ is a symplectic fiber bundle in the sense of [2].

Suppose $\tilde{\omega}$ is a fiber-compatible $\mathcal{F}$-partial forms. Infinitessimaly, the condition of a partial connection being symplectic is equivalent to the following.

Proposition 3.2. *The $\mathcal{F}$-partial connection Γ is symplectic iff $i_{v_1 \wedge v_2} d^{\mathcal{F}} \tilde{\omega} = 0$ for all $v_1, v_2 \in \Gamma(\mathrm{Ver})$.*

Proof. The connection is symplectic iff for all $v \in \Gamma(F^B)$

$$L_{v^\sharp}\omega = 0.$$

Since $\exp(tv^\sharp)$ preserves vertical fibers for all $t \geq 0$,

$$[\mathrm{Hor}, \mathrm{Ver}] \subset \mathrm{Ver}\,.$$

Hence for $v_1, v_2 \in \Gamma(\mathrm{Ver})$

$$\begin{aligned} i_{v_1\wedge v_2} L_{v^\sharp}\omega &= (-L_{v^\sharp} i_{v_1\wedge v_2} + i_{[v^\sharp, v_1\wedge v_2]})\omega \\ &= (-L_{v^\sharp} i_{v_1\wedge v_2} + i_{[v^\sharp, v_1\wedge v_2]})\tilde\omega \\ &= i_{v_1\wedge v_2} L_{v^\sharp}\tilde\omega = i_{v_1\wedge v_2} i_{v^\sharp} d^{\mathcal{F}}\tilde\omega \\ &= i_{v^\sharp}\, i_{v_1\wedge v_2} d^{\mathcal{F}}\tilde\omega \end{aligned}$$

since $i_{v^\sharp}\tilde\omega = 0$. □

Suppose $p : (M, \mathcal{F}_M) \to (B, \mathcal{F}_B)$ is a foliated symplectic fibration with vertical symplectic form ω.

Proposition 3.3. *There exists $\mathcal{F}$-partial symplectic connection Γ on $M \to B$ and a fiber-compatible, Γ-compatible 2-form $\tilde\omega : F^M \wedge F^M \to \mathbb{R}$.*

Proof. On a local trivialization $\phi_U : M \to F \times U \to U$ with $U \subset B$ one takes the canonical flat connection Γ_U and defines $\tilde\omega_U = pr_1^*\omega_F|_{F^M}$. Choose a cover $\mathcal{U}$ of B with a partition of unity $\{\rho_U\}_{U\in\mathcal{U}}$ subordinate to $\mathcal{U}$ and define

$$\Gamma = \sum_{U\in\mathcal{U}} (\rho_U \circ p)\phi_U^*\Gamma$$

and

$$\tilde\omega = \sum_{U\in\mathcal{U}} (\rho_U \circ p)\phi_U^*\tilde\omega_U.$$

For all $v \in \Gamma(F^B)$ the horizontal lift equals $\sum_{U\in\mathcal{U}}(\rho_U \circ p)(v|_U)^\sharp$, and hence

$$\begin{aligned} L_{v^\sharp}\omega &= L_{\sum_{U\in\mathcal{U}}(\rho_U\circ p)(v|_U)^\sharp}\omega \\ &= \sum_{U\in\mathcal{U}} \Big((\rho_U \circ p)\, L_{(v|_U)^\sharp}\omega + i_{(v|_U)^\sharp} d(\rho_U \circ p) \wedge \omega\Big) \\ &= \sum_{U\in\mathcal{U}} (\rho_U \circ p)\, L_{(v|_U)^\sharp}\omega = 0, \end{aligned}$$

hence the connection is symplectic. □

The only problem left is that $\tilde\omega$ is not $\mathcal{F}$-closed. This will be solved by the coupling construction.

4 Constructing the coupling form

Suppose $p : (M, \mathcal{F}_M) \to (B, \mathcal{F}_B)$ is a foliated symplectic fibration with vertical symplectic form ω and suppose Γ is a $\mathcal{F}$-partial symplectic connection on $M \to B$ and $\tilde{\omega} : F^M \wedge F^M \to \mathbb{R}$ a fiber-compatible, Γ-compatible 2-form. This set of data is called **geometric data** on $p : (M, \mathcal{F}_M) \to (B, \mathcal{F}_B)$.

Suppose $p : (M, \mathcal{F}_M) \to (B, \mathcal{F}_B)$ is a foliated symplectic fibration with vertical symplectic form ω and Γ an $\mathcal{F}$-partial symplectic connection.

Theorem 4.1. *If $H^1_{dR}(F) = 0$, then one can complete this to a set of geometric data with $d^{\mathcal{F}}\tilde{\omega} = 0$.*

Proof. Being fiber-compatible determines $\tilde{\omega}$ on $\mathrm{Ver} \wedge \mathrm{Ver}$ to equal ω. Being Γ-compatible determines $\tilde{\omega}$ to be zero on $\mathrm{Ver} \wedge \mathrm{Hor}$. The only freedom we have left is determining $\tilde{\omega}$ on $\mathrm{Hor} \wedge \mathrm{Hor}$. About the behaviour on $\mathrm{Hor} \wedge \mathrm{Hor}$ the following is known. Given the geometric data one can prove the following formula for $v, w \in F^B$

$$(v^\sharp \lrcorner (w^\sharp \lrcorner d^{\mathcal{F}}\tilde{\omega}))|_{\mathrm{Ver}} = ([v^\sharp, w^\sharp] \lrcorner \tilde{\omega})|_{\mathrm{Ver}} - (d^{\mathcal{F}}(\tilde{\omega}(v^\sharp, w^\sharp)))|_{\mathrm{Ver}}.$$

We already know that $(\mathrm{Hor} \wedge \mathrm{Ver}) \lrcorner d^{\mathcal{F}}\tilde{\omega} = 0$ and $(\mathrm{Ver} \wedge \mathrm{Ver}) \lrcorner d^{\mathcal{F}}\tilde{\omega} = 0$. The formula gives us a condition for the last ingredient, namely that $(\mathrm{Hor} \wedge \mathrm{Hor}) \lrcorner d^{\mathcal{F}}\tilde{\omega} = 0$. This is the case iff

$$[v^\sharp, w^\sharp] \lrcorner \tilde{\omega}|_{\mathrm{Ver}} = (d^{\mathcal{F}}(\tilde{\omega}(v^\sharp, w^\sharp)))|_{\mathrm{Ver}}$$

Beause ω should be fiber-compatible and Γ-compatible this is equivalent to

$$\mathrm{curv}(v, w) \lrcorner \omega = d^{\mathrm{Ver}}(\tilde{\omega}(v^\sharp, w^\sharp)). \tag{4.1}$$

Thus, finding a suitable $\tilde{\omega}(v^\sharp, w^\sharp)$ is simply an integration problem. Note $d^{\mathrm{Ver}}(\mathrm{curv}(v, w) \lrcorner \omega) = 0$, hence the class

$$[\mathrm{curv}(v, w) \lrcorner \omega] \in H^{\mathrm{Ver},1}_{dR}(M)$$

is an obstruction to the problem. Since $p : M \to B$ is a fiber bundle with fiber F, $H^{\mathrm{Ver},1}_{dR}(M) = 0$ iff $H^1_{dR}(F) = 0$. In this case the obstruction vanishes. $\square$

5 Hamiltonian actions and coupling forms

Instead of posing the strong condition of $H^1_{dR}(F) = 0$ on F in the previous theorem, a more elegant way to look at the problem is as follows. One can prove that it is equivalent to the existence of a weak momentum map for the action of a certain family of Lie algebras.

Consider the holonomy group $\mathrm{Hol}^\Gamma_x(p) \subset \mathrm{Diff}(M_x)$ of Γ at $x \in B$. These groups form a bundle of Fréchêt-Lie groups $I(\mathrm{Hol}^\Gamma(p)) := \bigcup_{x \in B} \mathrm{Hol}^\Gamma_x(p) \to B$, with associated bundle of Lie algebras $\mathcal{A}(I(\mathrm{Hol}^\Gamma(p))) \to B$. These notations are explained in the next section. By the Holonomy Theorem (or Ambrose-Singer Theorem) the Lie algebra $\mathcal{A}(I(\mathrm{Hol}^\Gamma(p)))_x$ at $x \in B$ equals the set

$$\{\mathrm{curv}(v, w) \mid v, w \in \Gamma(F^B)\} \subset \mathfrak{X}(M_x).$$

Theorem 5.1. *For a foliated symplectic fibration $p : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B)$ with vertical symplectic form ω and an $\mathcal{F}$-partial symplectic connection Γ, one can form an associated coupling form $\tilde{\omega}$, if there exists a weak momentum map for the canonical action of $\mathcal{A}(I(\mathrm{Hol}^\Gamma(p))) \to B$ on $(p : M \to B, \omega)$ (i.e. if this action is weakly Hamiltonian).*

Proof. Suppose μ is a momentum map for the action. Define

$$\tilde{\omega}(v^\sharp, w^\sharp) := \langle \mu, \mathrm{curv}(v, w) \rangle \,,$$

for all $v, w \in \mathfrak{X}(B)$, then $\tilde{\omega}$ is closed (cf. Equation 4.1), by the defining relation of internal momentum maps. □

Remark 5.2. The notion of foliation-coupling form is different from foliated coupling forms. In the first case, which was studied by Vaisman, a partial form on $\mathcal{F}$ is extended to the whole space M. In our case we just have a foliated version of the classical coupling construction for fiber bundles.

6 Hamiltonian Lie algebroid actions and coupling forms

In [1] we introduce the notion of Hamiltonian Lie algebroid action. Let us recapitulate this definition, slightly adapting the terminology. Suppose $(\mathcal{A} \to B, \rho)$ is a Lie algebroid and $(p : M \to B, \omega)$ a symplectic fibration, with $F^B = \rho(\mathcal{A})$. Suppose $\mathcal{A}$ acts symplectically on $p : M \to B$. Denote the action by $\alpha : \mathcal{A} \to TM$. Suppose $\alpha(\mathcal{A}) \subset F^M$. Suppose $\tilde{\omega} : \mathcal{F}^M \wedge \mathcal{F}^M \to \mathbb{R}$ is a closed fiber-compatible $\mathcal{F}$-partial 2-form.

Definition 6.1. An action α of a Lie algebroid $(\mathcal{A} \to B, \rho)$ on $(\pi : M \to B, \tilde{\omega})$ is **weakly Hamiltonian** if there exists a smooth section $\tilde{\mu} : M \to (\mathcal{A} \ltimes p)^*$, satisfying

$$d^{\mathrm{Ver}} \langle \tilde{\mu}, p^* X \rangle = - \left. (\alpha(X) \lrcorner \tilde{\omega}) \right|_{\mathrm{Ver}} \text{ for all } X \in \Gamma^\infty(\mathcal{A}), \tag{6.2}$$

$$\tag{6.3}$$

$\tilde{\mu}$ is called a **weak momentum map** for the action.

Obviously, one can simply view $\tilde{\mu}$ as a map

$$\tilde{\mu} : M \to \mathcal{A}^*,$$

(with certain properties) clarifying the analogy with the case of Lie algebra actions.

Consider the situation of the above Theorem 5.1. The bundle $I(\mathrm{Hol}^\Gamma(p)) \to B$ of holonomy groups is the isotropy groupoid of a certain groupoid. Let $\mathrm{Diff}(p) \rightrightarrows B$ denote the Fréchèt-Lie groupoid of diffeomorphisms $p^{-1}(x) \to p^{-1}(y)$ for all $x, y \in B$. Any path $\gamma : [0, 1] \to \mathcal{F}^B$ gives rise to diffeomorphism $T_\gamma : M_{\gamma(0)} \to M_{\gamma(1)}$, i.e. an element $T_\gamma \in \mathrm{Diff}(p)$.

Definition 6.2. The holonomy groupoid of the foliated bundle $p : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B)$ with respect to Γ is the wide subgroupoid $\mathrm{Hol}^\Gamma(p) \rightrightarrows B$ of $\mathrm{Diff}(p) \rightrightarrows B$ defined by

$$\mathrm{Hol}^\Gamma(p) := \{T_\gamma \mid \gamma : [0, 1] \to \mathcal{F}^B\} \subset \mathrm{Diff}(p).$$

Note that $I(\mathrm{Hol}^\Gamma(p))$ is the isotropy subgroupoid of $\mathrm{Hol}^\Gamma(p)$,

$$I(\mathrm{Hol}^\Gamma(p)) \subset \mathrm{Hol}^\Gamma(p),$$

as was suggested by the notation.

Remark 6.3. If the connection Γ is flat, then the parallel transport T_γ along a path γ only depends on the homotopy class $[\gamma] \in \mathrm{Mon}(B, \mathcal{F}^B)$ of γ. In this case, the map of parallel transport

$$T : \mathrm{Mon}^\Gamma(B, \mathcal{F}^B) \twoheadrightarrow \mathrm{Hol}^\Gamma(p)$$

is a surjective map of groupoids. The quotient of $\mathrm{Mon}^\Gamma(B, \mathcal{F}^B)$ by the kernel of T is denoted by $\mathrm{Hol}^\Gamma(B, \mathcal{F}^B)$ and hence

$$\mathrm{Hol}^\Gamma(B, \mathcal{F}^B) \cong \mathrm{Hol}^\Gamma(p).$$

Example 6.4. For any foliated manifold $(B, \mathcal{F}^B)$, the normal bundle $p : TB/F^B \to B$ comes equipped with a canonical flat connection called the Bott connection. In this case the associated holonomy groupoid is denoted by by $\mathrm{Hol}(B, \mathcal{F}^B) \rightrightarrows B$. This groupoid is known as the holonomy groupoid of $(B, \mathcal{F}^B)$.

If $p : TB/F^B \to B$ is a symplectic fibration, and the Bott connection is symplectic, then the foliation is called transversely symplectic. The existence of a coupling form follows from the fact that the typical fiber of $p : TB/F^B \to B$ is a vector space whose first cohomology hence vanishes. The coupling is horizontaly trivial though, since the curvature of the Bott connection is zero.

Denote the Lie algebroid associated to $\mathrm{Hol}^\Gamma(p, \mathcal{F}^M)$ by $\mathcal{A}(\mathrm{Hol}^\Gamma(p, \mathcal{F}^M))$. Note that

$$\mathcal{A}(\mathrm{Hol}^\Gamma(p)) \subset \mathcal{A}(\mathrm{Diff}(p)),$$

where $\mathcal{A}(\mathrm{Diff}(p))$ denotes the Lie algebroid of $\mathrm{Diff}(p)$. One easily sees that

$$\Gamma(\mathcal{A}(\mathrm{Diff}(p))) = \{v \in \mathfrak{X}(M) \mid (Tp \circ v)|_{M_x} \text{ is constant for all } x \in B\}.$$

There is a short exact sequence of Lie algebroids over B

$$0 \to \mathcal{A}(I(\mathrm{Hol}^\Gamma(p))) \to \mathcal{A}(\mathrm{Hol}^\Gamma(p)) \to F^B \to 0.$$

Using the connection there is a splitting of this sequence. This induces a vector bundle isomorphism

$$\Phi : \mathcal{A}(I(\mathrm{Hol}^\Gamma(p))) \oplus F^B \xrightarrow{\cong} \mathcal{A}(\mathrm{Hol}^\Gamma(p)).$$

Indeed, for every $x \in B$ a vector field $v \in \mathfrak{X}(M_x)$ and a vector $w \in F^B_x$ map to a "vector field" $m \mapsto v(m) + w^\sharp_m, M_x \to TM$.

For vector fields $u, v, w \in \mathfrak{X}(B)$, we denote by $\Psi(u, v, w)$ the following section of $\mathcal{A}(\mathrm{Hol}^\Gamma(p, \mathcal{F}^M))$

$$\Psi(u, v, w) = \Phi(\mathrm{curv}(u, v) \oplus w).$$

The canonical action of $\mathrm{Hol}^\Gamma(p) \rightrightarrows B$ on $p : M \to B$ corresponds to the canonical action of $\mathcal{A}(\mathrm{Hol}^\Gamma(p))$ on $p : M \to B$.

Theorem 6.5. *In the set-up of Theorem 5.1, the map* $\tilde{\mu} : M \to \mathcal{A}^*(\mathrm{Hol}^{\Gamma}(p))$ *defined by*

$$\langle \tilde{\mu}, \Psi(u, v, w) \rangle := \langle \mu, \tau_{\mathrm{Ver}}(\Psi(u, v, w)) \rangle$$

is a weak momentum map for the action of $\mathcal{A}(\mathrm{Hol}^{\Gamma}(p))$ *on* $(p, \tilde{\omega})$*.*

Thus, the construction of a coupling form on $\pi : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B)$ entails a weakly Hamiltonian action of $\mathcal{A}(\mathrm{Hol}^{\Gamma}(p))$ on $p : M \to B$.

Remark 6.6. In [1] we introduce Hamiltonian actions of Lie algebroids. For completeness, let us shortly recall this notion.

Definition 6.7. A weakly Hamiltonian action $\alpha : \mathcal{A} \to TM$ of a Lie algebroid $\mathcal{A}$ on $p : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B)$ with foliated coupling form $\tilde{\omega} : \mathcal{F}^M \times \mathcal{F}^M \to \mathbb{R}$ is (strongly) Hamiltonian if the momentum map $\tilde{\mu} \in \Gamma((\mathcal{A} \ltimes p)^*)$ satisfies

$$d^{\mathcal{A} \ltimes p} \tilde{\mu} = \alpha^* \tilde{\omega}.$$

Note that

$$d^{\mathcal{A} \ltimes p} \alpha^* \tilde{\omega} = \alpha^* d^{\mathcal{F}^M} \tilde{\omega},$$

hence $\tilde{\mu}$ is the solution to another integration problem. If $H^2(\mathcal{A} \ltimes p) = 0$, then every weakly Hamiltonian action is Hamiltonian.

References

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