

Lecture notes on groupoid cohomology

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Abstract

These lecture notes discuss several approaches to groupoid cohomology found in the literature. The main purpose is to understand the relation between these approaches.

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1 Introduction

These lecture notes have been written for a series of lectures in November and December 2007 at the Groupoid Seminar of Rui Loja Fernandes and Pedro Resende at IST in Lisbon. Almost all results come from references, mostly papers of Moerdijk and Crainic, except the part on abstract pseudogroups.

The goal of the lectures was to discuss and compare several notions of cohomology associated with groupoids for people with diverse backgrounds. It has been tried to keep the presentation as accessible as possible. This means that a minimal knowledge of homological algebra and groupoids is assumed and the most important preliminary notions and techniques are explained in the appendices. This also means that the use of spectral sequences has been avoided in the main body of the text, although they sometimes occur in remarks, since in concrete computations they are very useful.

Groupoids play an important rôle in modeling noncommutative spaces ([4]). Groupoids are still nice geometric object, but the point is that the space represented has not such a nice structure. Therefore, groupoids still allow natural constructions of invariants, like cohomology, whereas the represented space often does not. For example, one should think of leaf spaces of foliations, which can be modeled by the holonomy groupoid of the foliation ([14]). Instead of defining a cohomology on the leaf space, the cohomology is approached through the groupoid cohomology of the holonomy groupoid.

To a groupoid can be associated many other objects. This can be helpful in the computations of the cohomology or the description of (characteristic) classes that live in the cohomology. We shall compare the cohomology of these associated objects to the cohomology of the original groupoid.

Two versions of groupoid cohomology will be discussed. The first is differentiable cohomology with coefficient in a representation of a Lie groupoid (based on [6, 18]). This shall turn out to vanish for proper groupoids. The other groupoid cohomology will be discussed in the second part. It is cohomology with coefficients in a sheaf. This is actually the topos cohomology (with coefficients) of the topos of G -sheaves, but we shall not discuss this here. Let us remark here that this cohomology does not vanish (in general) for proper groupoids. These two cohomology theories associated to groupoids will be our main interest and will be compared to the cohomology of the objects associated to the groupoids.

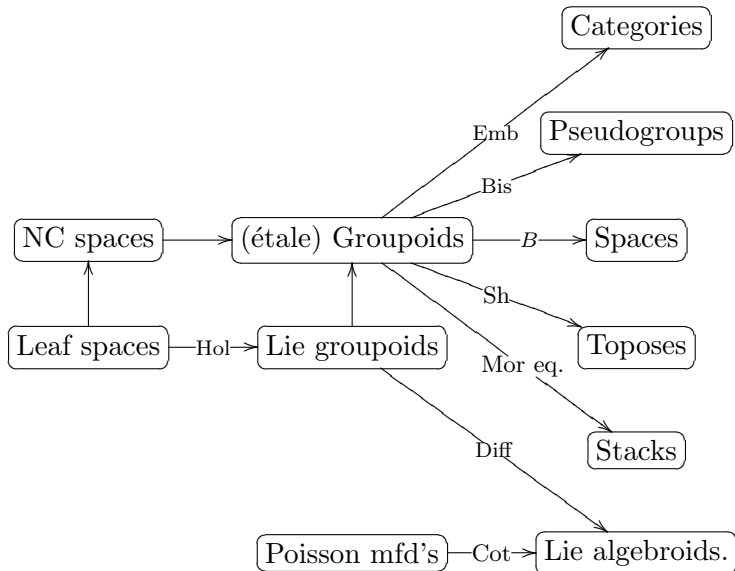


Figure 1: The relation between the objects whose cohomology we study. The arrows are explained in the introduction.

One of these associated objects is a (differentiable) stack. One point of view is that a stack is groupoid *up to Morita equivalence* ([1]), i.e. up to isomorphisms in a certain category of groupoids (cf. Appendix III). We show that the differentiable groupoid cohomology is invariant under Morita equivalence. Actually, one can show that the same is true for the cohomology with coefficients in a sheaf ([16]). Hence one can view the cohomology of the groupoid as the cohomology of the stack that is represented by the groupoid.

To a Lie groupoid is associated a Lie algebroid, loosely speaking by differentiation at the unit manifold. We discuss Lie algebroid cohomology with coefficients in a representation. Since there is still no good notion of Morita equivalence for Lie algebroids, there is obviously also no analogue of Morita invariance of the algebroid cohomology.

Through Lie algebroids one finds the main application of the differentiable cohomologies discussed in the first part. To any Poisson manifold P is associated a Lie algebroid structure on the cotangent space T^*P and the cohomology of this Lie algebroid equals the Poisson cohomology. A Poisson manifold is under some conditions integrable to a symplectic groupoid, which is exactly the Lie groupoid integrating this Lie algebroid. Extensions of Lie groupoids by bundles of Abelian groups are classified by the second cohomology group with coefficients in this bundle of groups of this Lie groupoid. These extensions play an important rôle in the prequantization of Poisson manifolds. As another application, note that the modular class of the Poisson manifold ([10]) lies in the Poisson cohomology.

Another cohomology theory related to groupoids is groupoid invariant cohomology for the action of a groupoid on a map. As in the previous cohomologies this notion generalizes the similar notion for Lie groups. It is the key ingredient for the comparison of Lie groupoid cohomology and Lie algebroid cohomology. The result proven in this section is a generalization of a classical result on Lie groups by Van

Est (cf. [9]). Using this result one finds that the cohomology of a Lie groupoid is isomorphic to the cohomology of the associated Lie algebroid if the source fibers are homologically n -connected. This is quite a strong condition, not satisfied in many examples. Therefore, the main conclusion should be that these cohomologies are *not* isomorphic in general! But, as a consolation, there still is a spectral sequence, that can be used to compare the cohomologies.

As announced, the second part discusses cohomology of étale groupoids with coefficients in a sheaf. In our exposition we follow [7]. The most important example is the constant sheaf \mathbb{R} . In this case one obtains a De Rham cohomology for groupoids and Section 5 can be seen as an intermediate step between part I and II, since this groupoid-De Rham cohomology can be defined for more general groupoids than étale.

The main application of this sheaf cohomology lies in foliation theory, since the holonomy and monodromy groupoid of a foliation are étale. Characteristic classes of foliations live in the cohomology that we describe.

One can associate cohomologies to G without reference to the specific “points” $g \in G$. Such an approach is called a point-free (or pointless) approach. Instead, the constructions use just the topology $\Omega(G)$ of G . We discuss two of such approaches, which turn out to be closely related.

One approach uses the inverse semi-group associated to a groupoid. This semi-group consists of the so-called local bisections of the groupoid. We discuss the notion of sheaves for inverse semigroups or, in fact, for a special kind of inverse semigroups: abstract pseudogroups. This allows us to define a cohomology with values in such sheaves. If the inverse semigroup comes from a groupoid it turns out that this cohomology is isomorphic to the groupoid cohomology.

The other approach was taken by Moerdijk (and Crainic) (cf. [8]) and associates to a groupoid an embedding category. One can describe the cohomology with values in a G -sheaf of such categories. They use the terminology of Čech-cohomology, which is not entirely justified, since the construction uses a basis instead of just any (good) cover. Nevertheless, the definition is obviously inspired by Čech cohomology and if G is a space one obtains the Čech cohomology with respect to the chosen basis. The main theorem is again that this cohomology is isomorphic to the groupoid cohomology.

We finish the main body of the lecture notes with some remarks on the classifying space BG of a groupoid G following [13]. We discuss a comparison of the cohomology of a groupoid G with values in a sheaf \mathcal{A} and the sheaf cohomology of the induced sheaf $\tilde{\mathcal{A}}$ on the classifying space BG .

Missing in this discussion are the Hochschild, cyclic and periodic cohomology of the convolution algebra of an étale groupoid. For this we refer the reader to [5]. It is important to stress again that the cohomology theories as discussed here come to life, if one discusses the characteristic classes and obstruction classes to some (integration) problems that live in them (cf. e.g. [6, 7, 8]). Alas, this has to wait to a future set of lecture notes!

The author would like to thank the participants of the seminar for the comments and discussions.

Part I

Differentiable cohomology

2 Differentiable Groupoid cohomology

2.1 Basic definition and examples

Suppose $G \rightrightarrows G^{(0)}$ is a smooth groupoid (although the results of this section still hold in the continuous case). We use the notation $G^{(n)}$ for the set of n -composable arrows

$$G^{(n)} = \{(g_1, \dots, g_n) \mid s(g_i) = t(g_{i+1}), 1 \leq i \leq n-1\}$$

It is a simplicial set, as it is a nerve, cf. Appendix I. There are maps

$$\delta_i^n : G^{(n)} \rightarrow G^{(0)}$$

defined by $\delta_i^n := t \circ pr_i$ for $i = 1, \dots, n$ and $\delta_{n+1}^n = s \circ pr_n$.

A smooth (locally trivial, complex) **representation** of $G \rightrightarrows G^{(0)}$ is a smooth complex vector bundle $\pi : E \rightarrow G^{(0)}$ and a smooth map

$$G \times_{s \times \pi} E \rightarrow E$$

preserving the linear structure on the fibers and satisfying the usual conditions of an action. Denote by $\text{Rep}(G)$ the semi-ring of isomorphism classes of such representation.

Let $\pi : E \rightarrow G^{(0)}$ be a representation of $G \rightrightarrows G^{(0)}$. For $n \in \mathbb{Z}_{\geq 0}$ a **smooth n -cochain** on G with values in E is a smooth section of the pullback vector bundle of $E \rightarrow G^{(0)}$ along $\delta_1^n : G^n \rightarrow G^{(0)}$. The set of smooth n -cochains with values in E is denoted by

$$C^n(G, E) := \Gamma^\infty(G^{(n)}, (\delta_1^n)^* E).$$

The family $C^*(G, E)$ can be turned into a cochain complex with (co)differentials

$$d^n : C^n(G, E) \rightarrow C^{n+1}(G, E)$$

defined by

$$\begin{aligned} (d^n c)(g_1, \dots, g_{n+1}) &:= g_1 \cdot c(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} c(g_1, \dots, g_n), \end{aligned}$$

and

$$(d^0 c)(g) = g \cdot c(s(g)) - c(t(g))$$

One easily checks that $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. The associated **differentiable groupoid cohomology** is

$$H_d^n(G, E) := \ker(d^n) / \text{im}(d^{n-1})$$

for $n \in \mathbb{N}$ and $H_d^0(G, E) = \ker(d^0)$. One sees at once that $H_d^0(G, E) = \Gamma^\infty(E)^G$.

Remark 2.1. Note that $H_d^n(G, E)$ is the simplicial cohomology adapted to the differentiable case with coefficients.

Example 2.2 (Baby example: trivial groupoid). Consider the trivial groupoid $M \rightrightarrows M$ for a smooth manifold M . Obviously $M^{(p)} \cong M$. Any smooth vector bundle $E \rightarrow M$ is a representation of M . Hence $C^n(G, E)$ can be identified with $\Gamma^\infty(E)$. One computes

$$d^i = \begin{cases} 0 & \text{if } i = \text{even} \\ id_E & \text{if } i = \text{odd} \end{cases}$$

Hence $H_d^0(M, E) = \Gamma^\infty(E)$ and $H_d^n(M, E) = 0$ for $n \in \mathbb{N}$.

Example 2.3. For Lie groups we recover the usual differential cohomology (cf. [9]).

Example 2.4 (Pair groupoid). Consider the pair groupoid $M \times M \rightrightarrows M$ for a smooth manifold M . Obviously $G^{(p)} \cong M^{p+1}$. A non-zero representation of the pair groupoid is necessarily on a trivial vector bundle $E = M \times V$ for some vector space V . Hence $C^n(G, E)$ can be identified with $C^\infty(M^n, V)$. Note that $(d^0 c)(m, n) = c(m) - c(n)$, hence the kernel of d^0 consists of constant functions

$$H_d^0(G, E) \cong V$$

The higher cohomology vanishes

$$H_d^n(G, E) = 0 \text{ for } n \in \mathbb{N},$$

which we show by constructing a cochain homotopy for $n \in \mathbb{N}$ (cf. Lemma 8.3)

$$H_d^n : C^n(G, E) \rightarrow C^{n-1}(G, E)$$

from the identity to zero on $C^n(G, E)$ for $n \in \mathbb{N}$. Indeed, choose any $m \in M$ and define

$$(h^n c)(m_1, \dots, m_n) = c(m, m_1, \dots, m_n).$$

This is indeed the desired homotopy:

$$\begin{aligned} d(h^{n-1}(c))(m_1, \dots, m_n) &= c(m, m_2, \dots, m_n) \\ &\quad + \sum_{i=2}^n c(m, m_1, \dots, \hat{m}_i, \dots, m_n) \\ &\quad + (-1)^n c(m, m_1, \dots, m_{n-1}) \\ &= -h^n(d(c))(m_1, \dots, m_n) + c(m_1, \dots, m_n). \end{aligned}$$

Actually, these two examples follow from a theorem.

Theorem 2.5. ([6]) *For any proper Lie groupoid $G \rightrightarrows G^{(0)}$ and $E \in \text{Rep}(G)$*

$$H_d^n(G, E) = 0 \text{ for } n \in \mathbb{N},$$

Recall that $H_d^0(G, E) \cong \Gamma^\infty(E)^G$.

Proof. The proof uses the same technique of cochain homotopy as the example. We need the notion of Haar system and cutoff function explained in Appendix II. Suppose $\{\lambda_x\}_{x \in G^{(0)}}$ is a Haar system for $G \rightrightarrows G^{(0)}$. Since $G \rightrightarrows G^{(0)}$ is proper, there exists a cutoff function $\mathbf{c} : G^{(0)} \rightarrow \mathbb{R}$. This time the homotopy between id and zero is given by

$$h(c)(g_1, \dots, g_n) = \int_{g \in G^x} g \cdot c(g^{-1}, g_1, \dots, g_n) \mathbf{c}(s(g)) \lambda^x(dg),$$

where $x = t(g_1)$. We compute

$$\begin{aligned} h(d(c))(g_1, \dots, g_n) &= \int_{g \in G^x} g \cdot (g^{-1} \cdot c(g_1, \dots, g_n)) \\ &\quad - \sum_{i=1}^{n-1} (-1)^i c(g^{-1}, g_1, \dots, g_i g_{i+1}, \dots, g_n) \\ &\quad + (-1)^{n+1} c(g^{-1}, g_1, \dots, g_{n-1}) \mathbf{c}(s(g)) \lambda^x(dg) \\ &= c(g_1, \dots, g_n) - \int_{G^{s(g_1)}} g_1 g \cdot c(g^{-1}, g_2, \dots, g_n) \mathbf{c}(s(g)) \lambda^{t(g_2)}(dg) \\ &\quad - \sum_{i=1}^{n-1} (-1)^i \int_{G^{t(g_1)}} g \cdot c(g^{-1}, g_1, \dots, g_i g_{i+1}, \dots, g_n) \mathbf{c}(s(g)) \lambda^{t(g_1)}(dg) \\ &\quad + (-1)^{n+1} \int_{G^{t(g_1)}} g \cdot c(g^{-1}, g^1, \dots, g_{n-1}) \mathbf{c}(s(g)) \lambda^{t(g_1)}(dg) \\ &= c(g_1, \dots, g_n) - d(h(c))(g_1, \dots, g_n), \end{aligned}$$

where the second equality follows from invariance of the Haar system and the properties of the cutoff. \square

Example 2.6. Suppose a Lie group H acts properly on a manifold M and $E \rightarrow M$ and H -equivariant vector bundle. Then, the action groupoid $H \ltimes M$ is proper and hence

$$H_d^n(H \ltimes M, E) = 0 \text{ for } n \in \mathbb{N},$$

and $H_d^0(H \ltimes M, E) = \Gamma^\infty(E)^H$.

Remark 2.7 (Čech double complex for Lie groupoid cohomology). Suppose $G \rightrightarrows G^{(0)}$ is a Lie groupoid and $\mathcal{U} = \{U_\alpha\}$ a good cover of $G^{(0)}$ by G -invariant (i.e. saturated) sets. Consider the double complex

$$C^{p,q} := \check{C}^p(\mathcal{U}, C^q(G)) := \prod_{\alpha_1, \dots, \alpha_p} C^\infty(G^{(q)}|_{U_{\alpha_1 \dots \alpha_p}}),$$

where

$$G^{(p)}|_U := \{(g_1, \dots, g_p) \in G^{(p)} \mid s(g_i), t(g_i) \in U \forall 1 \leq i \leq p\}$$

and with vertical differential $d_v : C^{p,q} \rightarrow C^{p,q+1}$ simply the restriction of the Lie groupoid differential and horizontal differential $d_h : C^{p,q} \rightarrow C^{p+1,q}$ the well-known Čech differential (cf. [3]):

$$(d_h c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

There is a coaugmentation

$$\eta : C^*(G) \rightarrow C^{0,*}$$

and, since the cohomology of the rows vanish (the cover is good),

$$H^*(G) \cong H_{tot}^*(C^{*,*}).$$

2.2 Functorial behavior for generalized morphisms

In this section we describe the functorial behavior of differentiable groupoid cohomology.

Proposition 2.8. *The application $G \rightarrow \text{Rep}(G)$ is a contravariant functor*

$$\text{GPD}_b \rightarrow \text{Semi-rings}.$$

Proof. (sketch) Given a generalized morphism of smooth groupoids $[P] : G \rightarrow H$ and a representation $E \rightarrow H^{(0)}$ of H . One can form the fibered product

$$(E \times_{H^0} P)/H,$$

quotiented by the diagonal H -action. The quotient is well-defined since the H -action is principal. It canonically induces a linear action of G from the action of G on P . \square

Theorem 2.9. ([6]) *The application $(G, E) \rightarrow H_d^*(G, E)$ is functorial*

$$\text{im}(\text{Rep}) \rightarrow \text{GradedRings}.$$

Proof. Suppose $G \rightrightarrows G^{(0)}$ and $H \rightrightarrows H^{(0)}$ are smooth groupoids and $[P]$ a generalized homomorphism $G \rightarrow H$. Consider the double complex

$$C^{p,q}(H, P, G) := C^\infty(H^{(p)} \times_{H_0} P \times_{G^{(0)}} G^{(q)}),$$

with differentials obtained by observing that the q -th column is the complex of the action groupoid $H \times (P \times_{G^{(0)}} G^{(q)})$ and the p -th row is the complex of the action groupoid $(H^{(p)} \times_{H_0} P) \times G$. The double complex comes with canonical coaugmentations

$$C^*(H) \xrightarrow{\eta} C^{*,\bullet}(H, P, G) \xleftarrow{\varepsilon} C^\bullet(G).$$

Since the action of H on P is principal, it is proper. Hence, the columns are exact by Theorem 2.5 and η induces an isomorphism on cohomology by Proposition 8.5. We obtain a homomorphism

$$(\eta^*)^{-1} \circ \varepsilon^* : H_d^*(G) \rightarrow H_d^*(H).$$

The proof that functor indeed preserves compositions uses a triple complex, cf. [6]. \square

Remark 2.10. More information on the relation of the cohomology of G and H can be obtained from the spectral sequence of the double complex $C^{*,*}$. The isomorphism is actually just a simple consequence, if it degenerates.

Example 2.11. Any transitive groupoid is the gauge groupoid $P \times_H P \rightrightarrows M$ for a principal H -bundle $P \rightarrow M$ and a Lie group H . Obviously, $[P]$ is a Morita equivalence $H \rightarrow P \times_H P$. Suppose V is a representation of H . Then, $P \times_H V \rightarrow M$ is the representation of $P \times_H P$ induced by Morita equivalence. Hence,

$$H_d^*(P \times_H P, P \times_H V) \simeq H_d^*(H, V).$$

Example 2.12. Suppose $G \rightrightarrows G^{(0)}$ is a smooth groupoid and $j : Y \rightarrow G^{(0)}$ an immersion whose image intersects at least once with each G -orbit. Then $j^!G$ and G are Morita equivalent and consequently

$$H_d^*(j^!G) \cong H_d^*(G).$$

This is useful for computing the cohomology of foliation groupoid, where Y is a transversal to the foliation. In that case, the pullback is étale, whereas G is not, in general.

The same can be done, if $j : Y \rightarrow G^{(0)}$ is a submersion.

Remark 2.13 (The product structure). There is a product on the complexes

$$C^p(G, E) \otimes C^q(G, E') \rightarrow C^{p+q}(G, E \otimes E')$$

given by the cup product

$$c_1 \cup c_2(g_1, \dots, g_{p+q}) := c_1(g_1, \dots, g_p) \otimes (g_1 \dots g_p) c_2(g_{p+1}, \dots, g_{p+q}).$$

Moreover, the double complex can also be endowed with a product, such that the coaugmentations are morphisms of differential algebras. As a consequence, the cup product is a natural transformation of the groupoid cohomology functor. This means that for $[P] : (G, E) \rightarrow (H, F)$ and $[P] : (G, E') \rightarrow (H, F')$ the diagram

$$\begin{array}{ccc} H_d^p(G, E) \otimes H_d^q(G, E') & \longrightarrow & H_d^{p+q}(G, E \otimes E') \\ [P]^* \uparrow & & \uparrow [P]^* \\ H_d^p(H, F) \otimes H_d^q(H, F') & \longrightarrow & H_d^{p+q}(H, F \otimes F') \end{array}$$

commutes.

3 Lie algebroid cohomology

Suppose $(\mathcal{A} \rightarrow M, \rho, [\cdot, \cdot])$ is a Lie algebroid (\mathcal{A} for short). An \mathcal{A} -connection on a smooth vector bundle $E \rightarrow M$ is

$$\pi : \Gamma^\infty(\mathcal{A}) \times \Gamma^\infty(E) \rightarrow \Gamma^\infty(E)$$

satisfying the usual axioms of a connection, where the Leibniz rule is substituted by

$$\pi(X)(f\xi) = f\pi(X)\xi + \rho(X)f\xi$$

for $f \in C^\infty(M)$, $\xi \in \Gamma^\infty(E)$. A **representation of \mathcal{A}** on a smooth vector bundle $E \rightarrow M$ is a flat \mathcal{A} -connection, i.e. $[\pi(X), \pi(Y)] = \pi([X, Y])$. The representations of \mathcal{A} form a semi-ring $\text{Rep}(\mathcal{A})$.

Definition 3.1. For $n \in \mathbb{Z}_{\geq 0}$, a **smooth n -cochain** on \mathcal{A} with values in a representation $E \rightarrow M$ is a smooth section of $\bigwedge^n \mathcal{A}^* \otimes E$. Denote the set of such smooth n -cochains by

$$C^n(\mathcal{A}, E) := \Gamma^\infty(\bigwedge^n \mathcal{A}^* \otimes E).$$

The family $C^*(\mathcal{A}, E)$ can be turned into a cochain complex with differential

$$\begin{aligned} (d\omega)(X_1, \dots, X_{n+1}) &:= \sum_{i < j} (-1)^{i+j-1} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}) \\ &\quad + \sum_{i=1}^{n+1} (-1)^i \pi(X_i) (\omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1})). \end{aligned}$$

and

$$(d^0)(\xi)(X) := \pi(X) \xi,$$

for $\xi \in C^0(\mathcal{A}, E) = \Gamma^\infty(E)$. Denote the associated **Lie algebroid cohomology** by $H^*(\mathcal{A}, E)$.

Note that

$$H^0(M, E) = \Gamma^\infty(E)^{\mathcal{A}} := \{\xi \in \Gamma^\infty(E) \mid \pi(X)\xi = 0 \text{ for all } X \in \Gamma^\infty(\mathcal{A})\}$$

This corresponds nicely to the cohomology of the cohomology of an integrating Lie groupoid G , if the source fibers of G are connected.

Example 3.2 (The baby revisited). Consider the zero Lie algebroid over M and $E \rightarrow M$ a vector bundle. Obviously, $H^k(M, E) = 0$ if $k \geq 1$.

Example 3.3. For Lie algebras we recover the usual Chevalley-Eilenberg cohomology.

Example 3.4. Any Poisson structure on a smooth manifold P gives rise to a Lie algebroid structure on the cotangent bundle T^*P . The associated Lie algebroid cohomology is equal to the Poisson cohomology.

Example 3.5. For a smooth manifold M consider the tangent algebroid $TM \rightarrow M$. The cohomology (with trivial coefficients) is the De Rham-cohomology of M . Note the difference here with the cohomology of the pair groupoid, whose cohomology vanishes for $n \in \mathbb{N}$. This is obviously not the case for the De Rham cohomology in general!

Example 3.6. Suppose G is a Lie group with Lie algebra \mathfrak{g} and $P \rightarrow B$ is a principal G bundle. Consider the bundle of Lie algebras $P \times_G \mathfrak{g} \rightarrow B$ obtained by associating \mathfrak{g} to P using the adjoint action of G on \mathfrak{g} .

Since $\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra automorphism, G acts on the cohomology space $H^*(\mathfrak{g})$ of \mathfrak{g} .

Proposition 3.7. *There is a canonical isomorphism*

$$H^*(P \times_G \mathfrak{g}) \cong \Gamma^\infty(P \times_G H^*(\mathfrak{g})).$$

Proof. Note that

$$C^k(P \times_G \mathfrak{g}) \cong \Gamma^\infty(P \times_G \bigwedge^k(\mathfrak{g}^*)),$$

which in turn equals $\Gamma^\infty(P \times_G C^k(\mathfrak{g}))$. \square

Example 3.8. Suppose \mathcal{A} is a Lie algebroid over M and $\pi : P \rightarrow M$ a submersion with connected fibers. One can form the pullback Lie algebroid over P defined by

$$\pi^!\mathcal{A} = \{(X_p, v_m) \in \mathcal{A} \times TP \mid \rho(X_p) = T\pi(v_m)\}.$$

The anchor is defined by the canonical projection on TP . Note that $\Gamma^\infty(\pi^!\mathcal{A})$ is generated over $C^\infty(P)$ by $\pi^*(\Gamma^\infty(\mathcal{A}))$. Hence we can define the Lie bracket by extension of

$$[(\pi^*X, v), (\pi^*Y, w)] = (\pi^*[X, Y], [v, w])$$

using the Leibniz identity.

Consider the double complex

$$C^{p,q} := C^p(F(\pi), \wedge^q \mathcal{A}^*)$$

with the horizontal differential being the foliated De Rham differential of $C^*(F(\pi))$ and vertical differential being the differential of the Lie algebroid cohomology of \mathcal{A} . One easily checks that the total complex C_{tot}^* of the double complex is isomorphic to the complex $C^*(\pi^!\mathcal{A})$ of the pullback Lie algebroid.

The zeroth column of the double complex equals the complex of \mathcal{A} . Hence, if the cohomology of the rows vanish, i.e. if $H^k(F(\pi)) = 0$ for $1 \leq k \leq n$, then

$$H^n(\pi^!\mathcal{A}, \pi^*E) \cong H^n(\mathcal{A}, E),$$

using Proposition 8.5. This is a part of Theorem 2 in [6].

Remark 3.9. Suppose \mathcal{A} is a Lie algebroid with an algebroid ideal \mathcal{B} . There is a finite filtration on $C^*(\mathcal{A})$ given by

$$F^p(C^{p+q}(\mathcal{A})) := \{\omega \in C^{p+q}(\mathcal{A}) \mid X_1 \wedge \dots \wedge X^{p+1} \lrcorner \omega = 0 \text{ if } X_1, \dots, X^p \in \mathcal{B}\}.$$

Since the filtration is finite, the associated spectral sequence degenerates.

Hence, in the above situation more information on the relation of the cohomology of \mathcal{A} and $\pi^!\mathcal{A}$ is stored in the spectral sequence associated $F(\pi) < \pi^!\mathcal{A}$.

The other part of Theorem 2 in [6] states:

Lemma 3.10. *For any $n \in \mathbb{N}$, if $H_{DR}^k(\pi^{-1}(m)) = 0$ for all $1 \leq k \leq n$ and $m \in M$, then $H^k(F(\pi)) = H_{DR}^k(\mathcal{F}(\pi)) = 0$ for all $1 \leq k \leq n$.*

This follows from very general theory of equivariant sheaves (cf. [2]). If this is the case for some $n \in \mathbb{N}$, we say the leaves of the foliation are homologically n -connected. By $H_{DR}^k(\mathcal{F}(\pi))$ we denote the foliated De Rham cohomology.

There exists a wedge (or shuffle) product $C^*(\mathcal{A}, E) \otimes C^*(\mathcal{A}, F) \rightarrow C^*(\mathcal{A}, E \otimes F)$ defined by

$$\omega_1 \wedge \omega_2(X_1, \dots, X_{p+q}) = \sum_{\sigma \in S(p,q)} |\sigma| \omega_1(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \otimes \omega_2(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}),$$

where $S(p, q)$ is the set of (p, q) -shuffles and $|\sigma| = \text{sign}(\sigma)$ the sign of σ :

$$S(p, q) := \{\sigma \in S(p+q) \mid \sigma(1) < \dots < \sigma(p) \text{ and } \sigma(p+1) < \dots < \sigma(p+q)\}.$$

The product is compatible with the codifferential and hence induces product on cohomology.

Remark 3.11 (Čech double complex for Lie algebroid cohomology). Suppose \mathcal{A} is a Lie algebroid over M and $\mathcal{U} = \{U_\alpha\}$ a good cover of M . Consider the double complex

$$C^{p,q} := \check{C}^p(\mathcal{U}, C^q(\mathcal{A})) := \prod_{\alpha_1, \dots, \alpha_p} C^q(\mathcal{A}|_{U_{\alpha_1 \dots \alpha_p}}),$$

with vertical differential $d_v : C^{p,q} \rightarrow C^{p,q+1}$ simply the restriction of the Lie algebroid differential and horizontal differential $d_h : C^{p,q} \rightarrow C^{p+1,q}$ the well-known Čech differential (cf. [3]):

$$(d_h \omega)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

There is a coaugmentation

$$\eta : C^*(\mathcal{A}) \rightarrow C^{0,*}$$

and, since the cohomology of the rows vanish (the cover is good),

$$H^*(\mathcal{A}) \cong H^*(C_{tot}).$$

This double complex is useful in a definition of characteristic classes for Lie algebroid representations, cf. [6].

4 Groupoid invariant cohomology

4.1 The Van Est-Crainic Theorem

Suppose a Lie groupoid $G \rightrightarrows G^{(0)}$ acts on a submersion $\pi : P \rightarrow G^{(0)}$. Then it induces an action on the tangent space $F(\pi) \rightarrow G^{(0)}$ to the fibers $(p, v) \cdot g = (p \cdot g, Tg^{-1}(v))$, where we see g as a diffeomorphism $\pi^{-1}(s(g)) \rightarrow \pi^{-1}(t(g))$.

Let $E \rightarrow G^{(0)}$ be a representation of G . We can consider foliated forms on $\mathcal{F}(\pi)$ with values in $\pi^*E \rightarrow P$. Denote the subset of equivariant foliated forms by $\Omega^*(\mathcal{F}(\pi), E)^G$. The differential of an equivariant form is again equivariant and hence there is an associated cohomology $H_{G\text{-inv}}^*(\mathcal{F}(\pi), E)$ called the **G -invariant cohomology of $\pi : P \rightarrow G^{(0)}$** with values in E .

Remark 4.1. In the case of groups one has the notion of equivariant cohomology. I do not know if this has been studied for groupoid actions. The problem is that the adjoint representation of groupoid is only a representation up to homotopy.

Example 4.2 (Baby examples). Suppose M is a smooth manifold and $E \rightarrow M$ a vector bundle over M . The trivial groupoid $M \rightrightarrows M$ acts on $\pi : P \rightarrow M$. The invariant cohomology with values in E is simply the foliated De Rham cohomology of $\mathcal{F}(\pi)$ with values in E .

The pair groupoid $M \times M \rightrightarrows M$ acts on $\pi = id : M \rightarrow M$. A representation of $M \times M$ is trivial on a trivial bundle $M \times V \rightarrow M$, for a vector space V . One easily sees that the invariant cohomology vanishes except in degree zero, where it is V .

If $M = pt$ and hence G is a Lie group acting on a manifold P , we recover the usual group invariant cohomology $H_{G\text{-inv}}^*(P)$.

Theorem 4.3. (Van Est-Crainic Theorem, [6]) *If $G \rightrightarrows G^{(0)}$ acts properly on a submersion $P \rightarrow G^{(0)}$ with connected fibers, then for any $E \in \text{Rep}(G)$*

(i) *there is a map*

$$\Phi_P : H_d^*(G, E) \rightarrow H_{G\text{-inv}}^*(\mathcal{F}(\pi), E)$$

compatible the product structures.

(ii) *If the leaves of $\mathcal{F}(\pi)$ are homologically n -connected, then Φ_P is an isomorphism in degrees $\leq n$ and injective in degree $n + 1$.*

(iii) *If the leaves of $\mathcal{F}(\pi)$ are contractible, then Φ_P is an isomorphism.*

Proof. For $n \in \mathbb{Z}_{\geq 0}$ define the foliation $\mathcal{F}(n)$ of $P_{\pi \times_{\delta_1^n} G^{(n)}}$ by to be the fibers of the projection $pr_2^n : P_{\pi \times_{\delta_1^n} G^{(n)}} \rightarrow G^{(n)}$. Denote the associated distribution by $F(n)$. Note that as a Lie algebroid $F(n) := (pr_1^n)^!(F(\pi))$, where $pr_1^n : P_{\pi \times_{\delta_1^n} G^{(n)}} \rightarrow P$ is the canonical projection.

We can form the double cocomplex

$$C^{p,q} := C^\infty(P_{\pi \times_{\delta_1^p} G^{(p)}}; \bigwedge^q F(p)^*)$$

The differential in column p is the partial De Rham differential on $\mathcal{F}(p)$. The differential in row q is the one from the groupoid cohomology of the action groupoid $P \rtimes G$ with coefficients in $\bigwedge^q F(\pi)^*$.

Vertically there is a coaugmentation

$$\varepsilon : C^p(G) \rightarrow C^{p,0}$$

and horizontally there is a coaugmentation

$$\eta : \Omega^q(\mathcal{F}(\pi))^G \rightarrow C^{0,q}.$$

Since the action of G on P is proper $P \rtimes G$ is a proper groupoid, the cohomology of the rows vanishes. Hence by Proposition 8.5 η induces an isomorphism to the total cohomology and we can define

$$\Phi_P := (\eta^*)^{-1} \circ \varepsilon^*.$$

Claim (ii) follows at one from the observation that $C^{p,*}$ is the cocomplex computing the Lie algebroid cohomology of $(pr_1^n)^!(F(\pi))$ tensored by $C^\infty(P \times_{\pi \times \delta_1^p} G^{(p)})$ and using Example 3.8 and Proposition 8.5. \square

Example 4.4. Suppose $G \rightrightarrows G^{(0)}$ is a transitive proper Lie groupoid. It canonically acts transitively and properly on the base $G^{(0)}$. Hence, by the Van Est-Crainic Theorem

$$H_d^*(G, E) = H_{G\text{-inv}}^*(G^{(0)}, E).$$

But since G acts transitively, the invariant forms are constant. Hence, the comology vanishes, except in degree zero. This we of course already knew by Theorem 2.5.

Remark 4.5. Suppose $G \rightrightarrows G^{(0)}$ is an étale groupoid acting on a smooth map $\pi : P \rightarrow G^{(0)}$. In this case, there is an induced action of G not only on the vertical tangent space $F(\pi)$, but on the whole tangent space TP .

Indeed, for $g \in G$, let $\sigma_g : U \rightarrow G$ be a local bisection such that $\sigma(s(g)) = g$. Then σ induces a local diffeomorphism

$$\tilde{\sigma} : P|_U \rightarrow P|_{t(\sigma(U))}$$

by

$$\tilde{\sigma}(p) := \sigma(\pi(p)) \cdot p.$$

Define the action of G on TP by

$$g \cdot v := T_p \tilde{\sigma}_g(v)$$

for $v \in T_p P$ and $g \in G_{\pi(p)}$. This definition does not depend on the choice of bisection, since G is étale.

Consequently, we can consider the invariant cohomology of the whole of P :

$$H_{G\text{-inv}}^*(P, E)$$

for a representation $E \rightarrow G^{(0)}$ of G . We shall come back to this in Section 5.

4.2 The relation of groupoid and algebroid cohomology

Suppose $G \rightrightarrows G^{(0)}$ is a source-connected Lie groupoid and $E \rightarrow G^{(0)}$ a representation of G . Then there is a representation of the associated Lie algebroid $\mathcal{A}(G)$ on $E \rightarrow G^{(0)}$. Consider the action of G on $s : G \rightarrow G^{(0)}$ by multiplication from the right. Note that $H_{G\text{-inv}}(\mathcal{F}(s), E) = H(\mathcal{A}(G), E)$, by definition of the Lie algebroid.

Theorem 4.3 implies that

Theorem 4.6. ([6]) *there are homomorphisms*

$$\Phi_G^k : H_d^k(G, E) \rightarrow H^k(\mathcal{A}(G), E)$$

for all $k \geq 0$. If the source fibers are homologically n -connected, then Φ_G^k is an isomorphism for $k \geq n$ and an injective for $k = n + 1$.

Proof. This follows immediately from Theorem 4.3. The double complex in the proof of Theorem 4.3 in this case takes the form

$$C^{p,q} = C^\infty(G^{(p+1)}; \bigwedge^q t^* \mathcal{A}^*) \cong C^\infty(G^{(p+1)}) \otimes_{C^\infty(G^{(0)})} \Gamma^\infty(\bigwedge^q T^{s,*}(G)).$$

□

A concrete homomorphism, generalizing Van Est original map, was given by Xu and Weinstein, cf. [18].

$$\Phi^k(c)(X_1, \dots, X_k) = \sum_{\sigma \in S(k)} |\sigma| X_{\sigma(k)} \dots X_{\sigma(1)} c,$$

where $X c$ is the vector field $\tilde{X}|_{G_x}$ on G_x induced by X using left translation applied to

$$g_1 \mapsto g_1^{-1} \cdot c(g_1, g_2, \dots, g_k), G_x \rightarrow E_x,$$

and then evaluated in 1_x , where $x = t(g_2)$. Xu and Weinstein show that it is a cochain map.

Crainic gives a concrete *left* inverse of the coaugmentation η , which can be defined on simple tensors by

$$\eta^{-1}(c \otimes \omega) := \Phi(c) \wedge \omega,$$

where \wedge is the wedge product on the Lie algebroid $T^s G \rightarrow G$. The fact that this is a cochain map now simply follows from the fact that Φ is and that the product is. One easily checks that indeed $(\eta^*)^{-1} \circ \varepsilon^* = \Phi$.

Example 4.7. A Poisson manifold (P, Π) is integrable if there exists a symplectic groupoid $(G \rightrightarrows P, \omega)$ such that the induced Poisson structure on the base equals Π . As a consequence of Theorem 4.6, the cohomology of G is isomorphic to the Poisson cohomology of P up to degree n if the source fibers are homologically n -connected. Suppose this is true, then, if $G \rightrightarrows P$ is proper, the Poisson cohomology vanishes up to degree n .

Morita equivalence of Poisson manifolds P_1 and P_2 corresponds to Morita equivalence of the integrating symplectic groupoids $G(P_1)$ and $G(P_2)$. Suppose the source fibers of $G(P_1)$ and $G(P_2)$ are homologically n -connected, then we conclude that the Poisson cohomology of P_1 is isomorphic to that of P_2 .

Example 4.8 (Action Lie algebroid). Suppose G is a Lie group and $\alpha : G \times M \rightarrow M$ a smooth action on a smooth manifold M . There is an induced action $\alpha : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of the Lie algebra \mathfrak{g} of G on M . Suppose $E \rightarrow M$ is an G -equivariant vector bundle.

Proposition 4.9. • *There is a canonical isomorphism*

$$H^*(\mathfrak{g} \times M, E) \cong \bigoplus_{k+l=n} H^k(G \times M, E) \otimes H_{dR}^l(G).$$

• *If the action is proper, then there is a canonical isomorphism*

$$H^*(\mathfrak{g} \times M, E) \cong \Gamma^\infty(E)^G \otimes H_{dR}^*(G).$$

- If G is compact, then

$$H^*(\mathfrak{g} \times M, E) \cong \Gamma^\infty(E)^G \otimes H^*(\mathfrak{g}).$$

where $H^*(\mathfrak{g})$ denotes the Chevalley-Eilenberg cohomology of \mathfrak{g} .

Proof. Consider the tensor double complex $C^{k,l}$ ($k, l \geq 0$) of the proof of Theorem 4.3. It equals

$$\begin{aligned} C^{k,l} &= C^\infty((G \times M)^{(k)}) \otimes_{C^\infty(M)} \Gamma^\infty(\wedge^l T^*G \times M) \\ &= C^\infty((G \times M)^{(k)}) \otimes \Omega^l(G). \end{aligned}$$

From the previous Theorem we know

$$H^*(\mathfrak{g} \times M) \cong H_{tot}^*(C^{*,*}).$$

The Künneth formula for the total cohomology implies

$$H_{tot}^n(C^{*,*}) \cong \bigoplus_{k+l=n} H^k(G \times M) \otimes H_{dR}^l(G)$$

The second result follows from Theorem 2.5 and the third from the well-known result that $H_{dR}^*(G) \cong H^*(\mathfrak{g})$, if G is compact. \square

5 Lie Groupoid-De Rham cohomology

Suppose $G \rightrightarrows G^{(0)}$ is a Lie groupoid and $E \rightarrow G^{(0)}$ a representation of G . Consider the following double complex of smooth p -forms on $G^{(q)}$ with values in E

$$C^{p,q} := \Omega^p(G^{(q)}, (\delta_1^n)^* E)$$

with horizontal differential $d_h^{p,q} : C^{p,q} \rightarrow C^{p+1,q}$ is the De Rham differential on $G^{(q)}$ and vertical differential $d_v^{p,q} : C^{p,q} \rightarrow C^{p,q+1}$

$$\begin{aligned} (d_v^{p,q}\omega)(g_1, \dots, g_{q+1}) &:= g_1 \cdot \omega(g_2, \dots, g_{q+1}) \\ &\quad + \sum_{i=1}^n (-1)^i \omega(g_1, \dots, g_i g_{i+1}, \dots, g_{q+1}) \\ &\quad + (-1)^{q+1} \omega(g_1, \dots, g_q). \end{aligned}$$

Obviously, the zeroth column equals the groupoid cohomology of G . The cohomology of total complex of the double complex is called the (differentiable) Lie groupoid-De Rham cohomology of G ,

$$H_{DR}^*(G, E) := H^*(C_{tot}).$$

Proposition 5.1. [17] *If $G \rightrightarrows G^{(0)}$ is a proper étale groupoid and $E \rightarrow G^{(0)}$ a representation of G , then*

$$H_{DR}^*(G, E) \cong H_G^* \text{-inv}(G^{(0)}, E).$$

Proof. The double complex has a coaugmentation by the complex of equivariant forms

$$\eta : \Omega^*(G^{(0)}, E)^G$$

Since G is étale, we can identify $TG^{(q)}$ equivariantly with $(\delta_1^q)^*TG^{(0)}$. This induces an isomorphism

$$C^{p,q} \cong C^p(G, (\delta_1^q)^*TG^{(0)})$$

compatible with the horizontal differentials. Since G is proper, we can conclude that the cohomology of the rows vanishes. Using Proposition 8.5 again, the statement follows. \square

Remark 5.2. This notion of Lie groupoid-De Rham cohomology is more or less what Haefliger calls differentiable cohomology, cf. [11] p.36. In fact, he only discusses two examples: the first is the action groupoid $H \ltimes M$ of a smooth action of a Lie group on a manifold M . He endows H with the discrete topology to obtain an étale groupoid. The second is the case of the groupoid of local diffeomorphisms on a manifold, cf. [11] p. 39.

For étale groupoids the Lie groupoid-De Rham cohomology equals the cohomology of the groupoid with values in the constant sheaf \mathbb{R} , as we shall see in the next Part.

Part II

Sheaves and cohomology

6 Cohomology of étale groupoids with coefficients in a sheaf

We now change our focus from Lie groupoids to étale groupoids. Whereas applications of the material of the previous sections should be sought in Poisson geometry, applications of the forthcoming sections lie in the theory of foliations, as discussed in the introduction.

Suppose $G \rightrightarrows G^{(0)}$ is an étale groupoid. We shall assume that it is Hausdorff, although constructions are available in the non-Hausdorff case as well.

Definition 6.1. A G -sheaf is a sheaf \mathcal{S} over $G^{(0)}$ on which G acts continuously from the right.

After étalification this corresponds to an étale space $E \rightarrow G^{(0)}$ with right G -action. We shall be interested in G -sheaves of Abelian groups.

Example 6.2. For any Abelian group A , one can consider the constant sheaf $\mathcal{A} = A$ with the trivial G -action. The associated étale space is $A \times G^{(0)} \rightarrow G^{(0)}$ with sheaf topology, i.e. the continuous local sections are constant.

Example 6.3. The sheaf of germs of continuous functions C^0 on $G^{(0)}$ has a canonical right G -action. If G is smooth, then the same holds for the sheaf smooth functions C^∞ (and C^n for any $n \in \mathbb{N}$).

Example 6.4. If G is smooth, then there is a sheaf of germs of smooth n -forms Ω^n on $G^{(0)}$ with a canonical G -action.

Example 6.5. Suppose $E \rightarrow G^{(0)}$ is a continuous vector bundle with a representation of $G \rightrightarrows G^{(0)}$. The sheaf \mathcal{E} of germs of continuous sections of E has a canonical G -action.

Definition 6.6. (i) A G -sheaf \mathcal{I} is **injective** if for any injective map of G -sheaves $i : \mathcal{A} \rightarrow \mathcal{B}$, every morphism $f : \mathcal{A} \rightarrow \mathcal{I}$ has a lift $\tilde{f} : \mathcal{B} \rightarrow \mathcal{I}$, i.e. $\tilde{f} \circ i = f$.

(ii) An **injective resolution** of a G -sheaf of Abelian groups \mathcal{A} is an exact sequence of G -sheaves

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots,$$

where \mathcal{I}^n is an injective G -sheaf of Abelian groups for all $n \in \mathbb{Z}_{\geq 0}$.

(iii) For any G -sheaf \mathcal{A} of Abelian groups, with an injective resolution as above the **groupoid cohomology** $H^*(G, \mathcal{A})$ of G with coefficients in a sheaf \mathcal{A} is the cohomology of the cocomplex

$$\Gamma_{G\text{-inv}}(\mathcal{I}^0) \rightarrow \Gamma_{G\text{-inv}}(\mathcal{I}^1) \rightarrow \Gamma_{G\text{-inv}}(\mathcal{I}^2) \rightarrow \dots,$$

where $\Gamma_{G\text{-inv}}$ denotes G -invariant sections.

Remark 6.7. It is well-known (claimed in [7]) that the category of G -sheaves has enough injectives, so that each Abelian G -sheaf indeed has an injective resolution.

Example 6.8. Suppose A is the constant sheaf for some injective Abelian group, say $A = \mathbb{Q}$, with trivial G -action. Then the sheaf is injective and hence we can consider the trivial injective resolution $0 \rightarrow A \rightarrow A \rightarrow 0$. As a consequence $H^i(G, A)$ is zero for $i > 0$ and $H^0(G, A) = \Gamma_{G\text{-inv}}(G^{(0)}, \mathbb{Q})$.

Remark 6.9. One can show that the category of G -sheaves of Abelian groups has enough injective, so an injective resolution as above always exists.

Remark 6.10. One can show using a spectral sequence (cf. [7]) that any resolution of \mathcal{A} by sheaves

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{A}_0 \rightarrow \mathcal{A}_1 \rightarrow \dots$$

such that $(\delta_1^p)^* \mathcal{A}_q$ is acyclic for all $p, q \in \mathbb{Z}_{\geq 0}$ gives rise to a double complex

$$C(G^{(p)}, (\delta_1^p)^* \mathcal{A}_q)$$

whose total cohomology is isomorphic to $H^*(G, \mathcal{A})$. The argument is similar to the proof of the Van Est-Crainic theorem. This is the approach used as a definition in [11] (see p. 29, 31).

Example 6.11. Suppose $G \rightrightarrows G^{(0)}$ is a smooth étale groupoid. Then

$$0 \rightarrow \mathbb{R} \rightarrow \Omega_{G^{(0)}}^0 \rightarrow \Omega_{G^{(0)}}^1 \rightarrow \dots$$

is a resolution by acyclic G -sheaves. Hence,

$$H^*(G, \mathbb{R}) \cong H_{tot}^*(C(G^{(p)}, (\delta_1^p)^* \Omega_{G^{(0)}}^q)).$$

This is the Groupoid-De Rham cohomology of Section 5. Recall Proposition 5.1, that stated that for proper étale Lie groupoids

$$H_{G\text{-inv}}^*(\Omega_{G^{(0)}}) \cong H_{tot}^*(C(G^{(p)}), (\delta_1^p)^* \Omega_{G^{(0)}}^q).$$

Remark 6.12. If $E \rightarrow G^{(0)}$ is a smooth vector bundle endowed with a representation of G . Then the sheaf of sections \mathcal{E} has a resolution by acyclic sheaves

$$0 \rightarrow \mathcal{E} \rightarrow \Omega_{G^{(0)}}^0 \otimes \mathcal{E} \rightarrow \Omega_{G^{(0)}}^1 \otimes \mathcal{E} \rightarrow \dots$$

Using the above reasoning we find

$$H^*(G, \mathcal{E}) \cong H_{tot}^*(C(G^{(p)}), (\delta_1^p)^* \Omega_{G^{(0)}}^q \otimes \mathcal{E}).$$

7 Point-free approaches

In this section we discuss two approaches to the cohomology of an étale groupoid that are point-free. This means that the model that is used makes no reference to the “points”, i.e. elements, in G . Instead, it just uses the open sets of G .

7.1 Pseudogroups and cohomology

Definition 7.1. An **inverse semigroup** is a set S endowed with an associative multiplication $S \times S \rightarrow S, (\sigma, \tau) \mapsto \sigma \tau$ and an involution $S \rightarrow S, \sigma \rightarrow \sigma^{-1}$ such that

- (i) $\sigma \sigma^{-1} \sigma = \sigma$ for all $\sigma \in S$,
- (ii) the set of idempotents $\Omega(S) := \{p \in S \mid p^2 = p\}$ is a commutative.

An inverse semigroup S has associated source map $s(\sigma) := \sigma^{-1} \sigma$ and target map $t(\sigma) = \sigma \sigma^{-1}$.

Example 7.2. Our one and only main example in these notes is the inverse semigroup associated to an étale groupoid $G \rightrightarrows G^{(0)}$. The set of bisections of G is

$$S(G) := \{U \subset G \text{ open} \mid U \stackrel{s}{\cong} s(U), U \stackrel{t}{\cong} t(U)\}$$

The multiplication is defined as follows

$$U \cdot V := \{gh \mid g \in U, h \in V, s(g) = t(h)\}$$

and the inverse is defined by

$$U^{-1} := \{g^{-1} \mid g \in U\}.$$

The set of idempotents $\Omega(S)$ is isomorphic to the set of opens $\Omega(G_0)$ of G_0 via the unit map $G^{(0)} \rightarrow G$. The source and target map of inverse semigroup correspond to the source and target map of the groupoid also via the unit map.

Equivalently, one may think of local bisections as local sections of the source map such that composition with the target map is a homeomorphism. Also equivalent: as local section of target map such that composition with the source map is homeomorphism. These last two definitions are also valid for general continuous groupoids.

To develop a cohomology theory with coefficients in a sheaf, we need to have a good notion of sheaves on $\Omega(S)$. To this purpose we restrict ourselves to inverse semigroups S that are **abstract pseudogroups**, i.e. those inverse semigroups for which $\Omega(S)$ is a locale with partial order $p \leq q$ iff there is a $p' \in \Omega(S)$ such that $p = p'q$.

Definition 7.3. A **locale** is a sup-lattice Ω (with partial order denoted by \geq , meets denoted by \wedge and joins denoted by \vee), closed under infinite joins and satisfying the following distributive law

$$p \wedge \bigvee_{i \in I} q_i = \bigvee_{i \in I} (p \wedge q_i)$$

for all $p, q_i \in \Omega, i \in I$ and index families I .

One should think of a locale as a generalized space because of the following example.

Example 7.4. For any topological space X . The set of opens $\Omega(X)$ is a locale with $\leq = \subset, \wedge = \cap$ and $\vee = \cup$.

Another important feature of locales is that it has exactly those ingredients needed to define the notion of a sheaf. Indeed, the definition of a sheaf on a space makes no mention of the points of that space, but just of the (locale of) open sets.

One can consider Ω as a category, with objects elements of Ω and an arrow $p \rightarrow q$ iff $p \leq q$.

Definition 7.5. Suppose \mathcal{C} is a category. A **sheaf on a locale** Ω is a contravariant functor $\mathcal{S} : \Omega \rightarrow \mathcal{C}$ such that for all families $\{p_i\}_{i \in I}$ the following diagram has an equalizer e

$$\mathcal{S}(\bigvee_{i \in I} p_i) \xrightarrow{e} \prod_{i \in I} \mathcal{S}(p_i) \begin{array}{c} \xrightarrow{\mathcal{S}(\leq_j)} \\ \xrightarrow{\mathcal{S}(\leq_i)} \end{array} \prod_{i, j \in I} \mathcal{S}(p_i \wedge p_j).$$

Example 7.6. For the locale of open sets $\Omega(X)$ of a space X , a sheaf is a sheaf in the usual sense.

Definition 7.7. Suppose S is an abstract pseudogroup. An **S -sheaf** is a sheaf \mathcal{S} over the locale $\Omega(S)$ with a right S action, i.e. to every $\sigma \in S$ is associated an isomorphism

$$\sigma : \mathcal{S}(t(\sigma)) \rightarrow \mathcal{S}(s(\sigma)).$$

Lemma 7.8. *Suppose $G \rightrightarrows G^{(0)}$ is a étale groupoid. One can identify G -sheaves and $S(G)$ -sheaves. Moreover, there is an equivalence of of categories (toposes) of G -sheaves and $S(G)$ -sheaves*

Proof. One might as well think of bisections as local sections of the source map, such that composition with the target is a local homeomorphism of $G^{(0)}$. If we think of a G -sheaf \mathcal{S} as the sheaf of local sections of the associated étale space, then given $\sigma \in S(G)$ and $\xi \in \mathcal{S}(t(\sigma))$ we define the S -action by

$$(\xi \cdot \sigma)(x) := \xi(t(\sigma(x))) \cdot \sigma(x)$$

for $x \in s(\sigma)$.

Conversely, given an $S(G)$ -sheaf \mathcal{S} , for $g \in G$ and y in the stalk $\in \mathcal{S}_{t(g)}$ at $t(g)$, pick a $\xi \in \mathcal{S}(V)$ such that $y = \lim_{\substack{\longrightarrow \\ t(g) \in \tilde{U} \subset V}} \xi(U)$ and a $\sigma \in S(G)$ such that $g \in \sigma$ and define

$$y \cdot g := \lim_{\substack{\longrightarrow \\ t(g) \in \tilde{U} \subset V}} r_U^V \xi \cdot \sigma.$$

Since G is étale this does not depend on the choices of σ and ξ . \square

Analogously to Definition 6.6 one has the following definition.

Definition 7.9. Suppose S is an abstract pseudogroup.

- (i) An S -sheaf \mathcal{I} is **injective** if for any injective map of S -sheaves $i : \mathcal{A} \rightarrow \mathcal{B}$, every morphism $f : \mathcal{A} \rightarrow \mathcal{I}$ has a lift $\tilde{f} : \mathcal{B} \rightarrow \mathcal{I}$, i.e. $\tilde{f} \circ i = f$.
- (ii) An **injective resolution** of an S -sheaf of Abelian groups \mathcal{A} is an exact sequence of S -sheaves

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots,$$

where \mathcal{I}^n is an injective S -sheaf of Abelian groups for all $n \in \mathbb{Z}_{\geq 0}$.

- (iii) For any S -sheaf \mathcal{A} of Abelian groups, with an injective resolution as above the **abstract pseudogroup cohomology $H^*(S, \mathcal{A})$ of S with coefficients in a sheaf \mathcal{A}** is the cohomology of the cocomplex

$$\Gamma_{S\text{-inv}}(\mathcal{I}^0) \rightarrow \Gamma_{S\text{-inv}}(\mathcal{I}^1) \rightarrow \Gamma_{S\text{-inv}}(\mathcal{I}^2) \rightarrow \dots,$$

where $\Gamma_{S\text{-inv}}$ denotes S -invariant sections.

Remark 7.10. We state without proof that for any abstract pseudogroup there are enough injective S -sheaves of Abelian groups.

As an immediate consequence of Lemma 7.8 the following theorem holds.

Theorem 7.11. *If $G \rightrightarrows G^{(0)}$ is an étale groupoid and \mathcal{A} a G -sheaf of Abelian groups. Then*

$$H^*(G, \mathcal{A}) \cong H^*(S(G), \mathcal{A}).$$

Remark 7.12. This should come as no surprise from the point of view of topos theory. Equivalent toposes have isomorphic associated topos cohomologies.

Remark 7.13. Suppose $S'(G)$ is a basis for the topology on G and a sub-pseudogroup of $S(G)$. By inspection of the proof of Lemma 7.8 one easily sees that $S'(G)$ -sheaves extend to $S(G)$ -sheaves and hence

$$H^*(S'(G), \mathcal{A}) \cong H^*(S(G), \mathcal{A}).$$

7.2 The embedding category of a groupoid

In [8] the authors associate a category to an étale groupoid. The construction is similar to the construction of inverse semigroup or pseudogroup associated to a groupoid.

Suppose $G \rightrightarrows G^{(0)}$ is a étale groupoid. The **embedding category** $\text{Emb}(G)$ of G has as objects the (locale of) open sets $\text{Emb}(G)^{(0)} = \Omega(G^{(0)})$ of $G^{(0)}$. The morphisms are pairs (U, V) , where V is a local bisection of G and $U \in \Omega(G^{(0)})$ such that $t(V) \subset U$, i.e.

$$\text{Emb}(G)^{(1)} := \{(U, V) \in \Omega(G^{(0)}) \times S(G) \mid t(V) \subset U\}$$

One defines $s(U, V) := s(V)$, $t(U, V) = U$ and composition $(U, V) \circ (U', V') = (U, VV')$ if $s(V) = U'$. We sometimes simply write $e \in \text{Emb}_G$ instead of pairs $(U, V) = e$.

An $\text{Emb}(G)$ -sheaf is a sheaf on $G^{(0)}$ (or $\Omega(G^{(0)})$) with an $\text{Emb}(G)$ -action.

Lemma 7.14. *Suppose $G \rightrightarrows G^{(0)}$ is a étale groupoid. One can identify G -sheaves and $\text{Emb}(G)$ -sheaves.*

Remark 7.15. Using a basis \mathcal{U} for the topology on $G^{(0)}$ one can consider a subcategory of $\text{Emb}(G)$ denoted by $\text{Emb}_{\mathcal{U}}(G)$ with

$$\text{Emb}_{\mathcal{U}}(G)^{(0)} := \mathcal{U}$$

and

$$\text{Emb}_{\mathcal{U}}(G)^{(1)} := \{(U, V) \in \text{Emb}(G) \mid s(V), U \in \mathcal{U}\},$$

that is, $\text{Emb}_{\mathcal{U}}(G)$ is the pullback category of $\text{Emb}(G)$ along the inclusion

$$\mathcal{U} \hookrightarrow \Omega(G^{(0)}).$$

One easily sees that one can identify $\text{Emb}(G)$ -sheaves and $\text{Emb}_{\mathcal{U}}(G)$ -sheaves.

Remark 7.16. One could proceed along the lines of Definition 6.6 and Definition 7.9 by defining the cohomology of an $\text{Emb}_{\mathcal{U}}(G)$ -sheaf using injective resolutions. The isomorphism with groupoid cohomology would readily follow. But we shall follow [8] and discuss the cohomology directly in terms of the bar complex.

A sheaf \mathcal{A} is called \mathcal{U} -acyclic if \mathcal{A} is acyclic when restricted to any $U \in \mathcal{U}$ i.e. $H^i(U, \mathcal{A}) = 0$. Suppose

$$\mathcal{A} \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \rightarrow \dots$$

is a resolution of \mathcal{A} by \mathcal{U} -acyclic G -sheaves. Then one can form the double complex

$$C^{p,q}(\text{Emb}_{\mathcal{U}}(G), \mathcal{A}) := \prod_{(e_0, \dots, e_p) \in \text{Emb}_{\mathcal{U}}(G)^{(p)}} \mathcal{A}^q(s(e_p))$$

endowed with a horizontal differentials defined by the usual formula familiar from the differentiable groupoid cohomology and vertical differentials obtained from the resolution. The **cohomology of the embedding category $\text{Emb}_{\mathcal{U}}(G)$ with coefficients in the sheaf \mathcal{A}** is defined to be the total cohomology of this double complex:

$$H(\text{Emb}_{\mathcal{U}}(G), \mathcal{A}) := H_{tot}^*(C^{*,*}(\text{Emb}_{\mathcal{U}}(G), \mathcal{A})).$$

Remark 7.17. If \mathcal{A} is a constant sheaf of Abelian group \mathcal{A} and all sets $U \in \mathcal{U}$ are contractible, then this equals the simplicial cohomology of the nerve of $\text{Emb}_{\mathcal{U}}(G)$ with values in \mathcal{A} as defined in the Appendix.

Theorem 7.18 ([8]). *For an étale groupoid $G \rightrightarrows G^{(0)}$, a basis \mathcal{U} of $G^{(0)}$ and any \mathcal{U} -acyclic G -sheaf \mathcal{A}*

$$H^*(G, \mathcal{A}) \cong H^*(\text{Emb}_{\mathcal{U}}(G), \mathcal{A}).$$

Proof. Consider the double graded space

$$S_{p,q} := \{(e_1, \dots, e_p, g, g_1, \dots, g_q) \in \text{Emb}_{\mathcal{U}}(G)^{(q)} \times G \times G^{(q)} \mid s(g) = t(g_1), t(g) \in s(e_p)\}.$$

We shall assume that \mathcal{A} is injective, so the cohomology is computed without a resolution. The sheaf \mathcal{A} pulls back to a sheaf $\mathcal{A}_{p,q}$ on $S_{p,q}$. Hence we can consider the double complex

$$C^{p,q} := \Gamma(S_{p,q}, \mathcal{A}_{p,q})$$

with differentials coming from the complexes of $\text{Emb}_{\mathcal{U}}(G)$ and G .

There are obvious horizontal and vertical coaugmentations by the complexes computing the groupoid cohomology and the cohomology of the embedding category. So we want to prove that the rows and columns are acyclic.

For fixed p (the p -th column) we get a product

$$S_{p,*} = \prod_{\text{Emb}_{e \in \mathcal{U}}(G)^{(p)}} G^{s(e_p)} \times_t G^{(*)}$$

that are isomorphic to the complex of the comma groupoid s_{e_p}/G . Indeed, the isomorphisms are given by

$$(g, g_1, \dots, g_q) \mapsto (g_1, \dots, g_q),$$

where the g_i 's on the left are seen as a map $g g_1, \dots, g_{i-1} \rightarrow g g_1, \dots, g_i$. Each comma groupoid s_{e_p}/G is Morita equivalent to $s(e_p) \rightrightarrows s(e_p)$ (cf. Remark 8.24). Using a sheaf-version of Theorem 2.9 and the fact that the sheaf is \mathcal{U} -acyclic, we conclude that the cohomology of the columns vanish.

Let's now focus on the rows. Consider the sheaf over G^0 defined by

$$\mathcal{E}(U) := \prod_{\{(e_1, \dots, e_p, e) \in \text{Emb}_{\mathcal{U}}(G)^{(p+1)} \mid s(e) = U\}} \mathcal{A}(U).$$

We can pullback this sheaf to $G^{(q)}$ and the stalk is precisely the complex $C^{*,q}$.

For fixed U , $\mathcal{E}(U)$ is isomorphic to the complex computing the cohomology of the comma category $\text{Emb}_{\mathcal{U}}(G) \setminus U$ with values in the Abelian group $\mathcal{A}(U)$. By Example 8.23 this cohomology vanishes. As a consequence the cohomology of the rows vanish, which finishes the proof. \square

8 Some remarks on the classifying space of a groupoid

For discrete groupoids $\mathbb{G} \rightrightarrows \mathbb{G}^{(0)}$ one can consider then nerve $N\mathbb{G}$ and the classifying space $B\mathbb{G}$. It follows immediately from Theorem 8.19 that for any Abelian group A the simplicial cohomology of \mathbb{G} , i.e. the simplicial cohomology of $N\mathbb{G}$, is isomorphic to the singular cohomology of $B\mathbb{G}$:

$$H^*(\mathbb{G}, A) \cong H^*(B\mathbb{G}, A).$$

For étale groupoids $G \rightrightarrows G^{(0)}$ this statement can be improved in two ways. The material that follows can be found in [13]. A related approach, from a topos point-of-view, is found in [12]. On the one hand, the topology of G can be taken into account in the construction of the classifying space. Indeed, the nerve $N(G)^n = G^{(n)}$ inherits a topology from G . This in turn gives a product topology on

$$\coprod_{n \in \mathbb{Z}_{\geq 0}} G^n \times \Delta^n,$$

and hence a quotient topology on BG .

On the other hand coefficient in G -sheaves should be allowed.

Lemma 8.1. *A G -sheaf canonically induces a sheaf on BG .*

Proof. Let \mathcal{A} be a G -sheaf. For each n pulling back \mathcal{A} along the maps $G^{(n)} \times \Delta^n \rightarrow G^{(0)}$, $(g_1, \dots, g_n, \vec{x}) \mapsto s(g_n)$ for $n \in \mathbb{Z}_{\geq 0}$ gives a sheaf $\bar{\mathcal{A}}$ on

$$\coprod_{n \in \mathbb{Z}_{\geq 0}} G^n \times \Delta^n.$$

Now apply the quotient with respect to the equivalence relation, generated by the following relation given on the stalks on for all $n \in \mathbb{Z}_{\geq 0}$ by $((g_1, \dots, g_n, x) \in G^{(n)} \times \delta\Delta^n)$

$$\mathcal{A}_{(g_1, \dots, g_n, \delta_i(\vec{x}))} \sim \mathcal{A}_{(\delta_i(g_1, \dots, g_n), \vec{x})}$$

if $i < n$ and

$$\mathcal{A}_{(g_1, \dots, g_n, \delta_n(\vec{x}))} \sim \mathcal{A}_{(\delta_n(g_1, \dots, g_n), \vec{x})} \cdot g_n.$$

One easily sees this gives a well defined sheaf $\tilde{\mathcal{A}} = \bar{\mathcal{A}} / \sim$ on BG . □

The improvement that we mentioned, that started as a conjecture by A. Haefliger and was proven by I. Moerdijk is the following.

Theorem 8.2. ([13]) *For any étale groupoid $G \rightrightarrows G^{(0)}$ and Abelian G -sheaf \mathcal{A} , there is an isomorphism*

$$H^*(G, \mathcal{A}) \xrightarrow{\cong} H^*(BG, \tilde{\mathcal{A}}),$$

natural in G and \mathcal{A} .

For a proof we refer the reader to the original source [13].

Appendix I: homological algebra

For a good introduction to homological algebra I suggest the books by C. A. Weibel [19] and R. Bott and L. Tu [3]. We shall state results in this appendix in terms of cocomplexes, because the notes deal with cohomology.

8.1 Chain homotopy

Suppose (C^*, d_C) and (D^*, d_D) are cochain complexes over some ring and $f, g : C^* \rightarrow D^*$ are morphism of cochain complexes. Then, f and g are said to be **homotopic** if there exists a **cochain homotopy**, i.e. a family of maps

$$h : C^* \rightarrow D^{*-1}$$

satisfying

$$f - g = d_D h + h d_C.$$

This notion generalizes the notion of homotopy for continuous maps and its main use is the following Lemma analogous to the topological case.

Lemma 8.3. *If $f, g : C^* \rightarrow D^*$ are cochain homotopic, then they induce the same map on cohomology*

$$f^* = g^* : H^*(C) \rightarrow H^*(D).$$

Proof. It is enough to prove that if f is homotopic to zero then $f^* = 0$. Suppose $[x] \in H^*(C)$, i.e. $d(x) = 0$. Then $f(x) = d(h(x))$ and $[f(x)] = 0$. \square

Example 8.4. Suppose $f, g : X \rightarrow Y$ are continuous maps of topological spaces X, Y and H a homotopy $f \rightarrow g$. Then H induces a cochain homotopy h of the induced maps $f^*, g^* : C^*(X) \rightarrow C^*(Y)$ on the singular cochain cocomplex.

8.2 Double cocomplexes and coaugmentations

Suppose $C^{*,*}$ is a doubly graded cocomplex with commuting codifferentials $d_h : C^{*,*} \rightarrow C^{*+1,*}$ and $d_v : C^{*,*} \rightarrow C^{*,*+1}$, i.e. a cocomplex in the category of cocomplexes (in two ways!). One can form the total complex

$$C_{tot}^n = \bigoplus_{p+q=n} C^{p,q}$$

with codifferential

$$D = d_h + (-1)^p d_v.$$

Suppose the double cocomplex is $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ -graded, (D^*, d_D) a cocomplex, and $\eta^* : D^* \rightarrow C^{0,*}$ is a coaugmentation of the rows, i.e. η satisfies $d_h \circ \eta = 0$ and $d_v \eta = d_D \eta$. As a consequence η commutes with the total codifferential D .

Proposition 8.5. *If the first n rows are exact cocomplexes (i.e. the cohomology vanishes), then the map η^k induces an isomorphism on cohomology*

$$\eta^* : H^k(D) \xrightarrow{\cong} H^k(C_{tot}),$$

for all $k \leq n$ and an injection if $k = n + 1$.

Proof. First we show that a cocycle in $c \in C_{tot}^n$ is cohomologous to a cocycle which is nonzero only in $C^{0,n}$. Indeed, suppose q is the smallest for which the component c^q of c in $C^{n-q,q}$ is non-zero. Then $d_h c^q = 0$. Since the rows are exact, $c^q = d_h(b^q)$ for some $b^q \in C^{n-q-1,q}$. Hence c is cohomologous to $c - b^q$. By induction, c is cohomologous to a cochain $c' \in C^{0,n}$.

Surjective: Suppose $[c] \in H^n(C_{tot})$. By the previous, we can assume $c \in C^{0,n}$. Since $d_h c = 0$ and the rows are exact, there exists a $d \in D^n$ such that $\eta(d) = c$. Since η is injective and $d_v(c) = 0$, we have $d_D c' = 0$.

Injective: Suppose $\eta([d]) = D([c])$. Again, we can assume $c \in C^{0,n-1}$. Since $d_h(c) = 0$, there is a $d' \in D^{n-1}$ such that $\eta(d') = c$. Since $\eta(d_D(d')) = d_v \eta(d') = \eta(d)$ and η^* is injective, one has $d_D(d') = d$. \square

The same argument holds for the case of a coaugmentation of the columns and exact columns.

Example 8.6. Suppose M is a smooth manifold and \mathcal{U} a cover of M . The well-known Čech-De Rham double cocomplex $C^*(\mathcal{U}, \Omega^*)$ has a horizontal coaugmentation by de De Rham cocomplex $\Omega^*(M)$ and a vertical coaugmentation by the Čech cocomplex. Moreover, the rows are exact and the columns are exact if the cover is “good”. Hence, by the previous proposition we get the isomorphisms

$$H_{DR}(M) \cong H_{tot}(C(\mathcal{U}, \Omega)) \cong \check{H}(M, \mathbb{R}).$$

8.3 Simplicial objects

This will be an unpretentious short introduction to simplicial objects, addressing the absolute minimum needed for the main body of the text.

Semi-simplicial objects are tools that represent some of the combinatorics in algebraic topology. The simplicial objects that we will introduce later are even better for this purpose. Not only cohomology can be discussed in a combinatorial way using these as we will discuss, but even homotopy theory.

Suppose \mathcal{C} is a category.

Definition 8.7. A **semi-simplicial object** in \mathcal{C} is a family of objects $\{S_n\}_{n \in \mathbb{Z}_{\geq 0}}$ and morphisms

$$\delta_i^n : S_n \rightarrow S_{n-1}$$

for $n \in \mathbb{N}$, $0 \leq i \leq n$, called faces, satisfying

$$\delta_i^{n-1} \delta_j^n = \delta_{j-1}^{n-1} \delta_i^n$$

if $i < j$.

Example 8.8. For $n \in \mathbb{Z}_{\geq 0}$ the set

$$\Delta^n := \{(x_0, \dots, x_n) \in [0, 1]^n \mid \sum_{i=0}^n x_i = 1\}$$

is called the geometric n -simplex. This family is endowed with face maps $\delta_i^n : \Delta^{n-1} \rightarrow \Delta^n$ for $0 \leq i \leq n$ defined by

$$(x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}),$$

which is the inclusion into the face opposite of the vertex on the $(i + 1)$ -th axis.

For any topological space one can consider the sets

$$S_n(X) := C(\Delta_n, X),$$

for $n \in \mathbb{Z}_{\geq 0}$. Together with the induced maps $(\delta_i^n)^*$, these form a semi-simplicial object $S(X)$ in the category of sets; a semi-simplicial set for short. The application $X \mapsto S(X)$ obviously extends to a contravariant functor.

Example 8.9. For any category \mathcal{D} one can construct the **nerve**, which is a family of sets for $n \in \mathbb{N}$

$$N_n(\mathcal{D}) := \mathcal{D}^{(n)} = \mathcal{D}^{(1)} \times_{\mathcal{D}^{(0)}} \dots \times_{\mathcal{D}^{(0)}} \mathcal{D}^{(1)},$$

where the fibered product (pullback) contains n copies of $\mathcal{D}^{(1)}$ and $N_0(\mathcal{D}) = \mathcal{D}^{(0)}$. This family forms a simplicial set with faces

$$\delta_i^n(d_1, \dots, d_n) = (d_1, \dots, d_i \circ d_{i+1}, \dots, d_n).$$

For continuous (say étale) groupoids one takes the nerve within the category of topological spaces. Hence one obtains a simplicial space.

Definition 8.10. The **simplicial cohomology** with values in an Abelian group A of a semi-simplicial set S is the cohomology $H^*(S, A)$ of the cochain complex

$$C^n(S, A) := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[S^n], A), \quad n \in \mathbb{Z}_{\geq 0},$$

with codifferential

$$d^n := \sum_{i=0}^n (-1)^i \delta_i^n.$$

Example 8.11. In the case of Example 8.8 we get the usual singular cohomology of the space X . The complex $C(S(X), A)$ is called the singular cochain complex with values in A of X .

Example 8.12. Consider the situation of Example 8.9 with \mathcal{D} . We can conclude that we have obtained a cohomology associated to a category:

$$H^*(\mathcal{D}, A) := H^*(N(\mathcal{D}), A).$$

In particular, if the category is actually a groupoid, then we get a groupoid cohomology. If the groupoid is actually étale and one wants a cohomology that takes into account the topology and takes values in a sheaf, then one needs to adapt the adapt the cohomology as is done in the main body of the text.

We now turn our attentions to a richer combinatorial notion, extending the notion of semi-simplicial objects.

Definition 8.13. A **simplicial object** is a semi-simplicial object that is also endowed with maps

$$\sigma_i^n : S_n \rightarrow S_{n+1}$$

for $n \in \mathbb{Z}_{\geq 0}$, $0 \leq i \leq n$, called degeneracies, satisfying

$$\sigma_i^{n+1} \sigma_j^n = \sigma_{j+1}^{n+1} \sigma_i^n$$

if $i \leq j$, and moreover the interplay between faces and degeneracies is as follows

$$\delta_i^{n+1} \sigma_j^n = \begin{cases} \sigma_{j-1}^{n-1} \delta_i^n & \text{if } i < j \\ id & \text{if } i = j \text{ or } i = j + 1 \\ \sigma_j^{n-1} \delta_{i-1}^n & \text{if } i > j + 1. \end{cases}$$

Example 8.14. Consider the geometric n -simplices Δ_n again. There are also degeneracy maps $\sigma_i^n : \Delta^{n+1} \rightarrow \Delta^n$ for $0 \leq i \leq n$ defined by

$$(x_0, \dots, x_n) \mapsto (x_0, \dots, x_i + x_{i+1}, \dots, x_n),$$

that can be thought of as projections onto n of the $n + 1$ faces of Δ^n . These maps induce degeneracies on $S_n(X)$ for a topological space X .

Example 8.15. On the nerve $N(\mathcal{D})$ of a category \mathcal{D} one can define degeneracies by

$$\sigma_i^n(d_1, \dots, d_n) := (d_1, \dots, d_i, 1_{s(d_i)}, d_{i+1}, \dots, d_n).$$

Definition 8.16. The **geometric realization** $|S|$ of a simplicial space (or set) S is

$$\left(\coprod_{n \in \mathbb{Z}_{\geq 0}} S_n \times \Delta_n \right) / \sim,$$

where

$$(\delta_i^n(s), x) \sim (s, \delta_i^n(x)) \text{ and } (\sigma_i^n(s), x) \sim (s, \sigma_i^n(x)).$$

Any morphism of simplicial spaces $S \rightarrow T$ (i.e. a family of maps $f_n : S_n \rightarrow T_n$ that commute with the faces and degeneracies) canonically induces a continuous map $|S| \rightarrow |T|$. Hence, geometric realization is a covariant functor.

Example 8.17. Let us continue the first of our previous two examples. One might think that the geometric realization $|S(X)|$ of the simplicial set $S(X)$ of a topological space X is homeomorphic to X and $S(|T|)$ is isomorphic to T for a simplicial set T , but this is not the case. In fact the following is the case:

Proposition 8.18.

- (i) The functor $|-|$ is left adjoint of S ;
- (ii) $H^*(|S(X)|, A) \cong H^*(X, A)$ for topological spaces X

The prove of (i) is straight-forward. Item (ii) follows from a more general Theorem

Theorem 8.19. For simplicial sets T

$$H^*(|T|, A) \cong H^*(T, A).$$

If one takes $T = S(X)$ one gets Proposition 8.18.ii.

8.4 Categories and cohomology

For a category \mathcal{D} the geometric realization $|N(\mathcal{D})|$ of the nerve $N(\mathcal{D})$ is called the classifying space of \mathcal{D} and is denoted by

$$B\mathcal{D} := |N(\mathcal{D})|.$$

As a corollary of Theorem 8.19

$$H^*(B\mathcal{D}, A) \cong H^*(\mathcal{D}, A).$$

Example 8.20. In the case of a group G , the geometric realization of the nerve equals the well-known classifying space BG that classifies principal G -bundles in the sense that there is a bijection between homotopy classes of maps $X \rightarrow BG$ and principle G -bundles. Moreover, it is a so-called Eilenberg-MacLane space $K(G, 1)$ in the sense that all fundamental groups vanish except the first, which equals G .

We now discuss a simple result needed in the main body of the text. We use that B is a functor from categories to topological spaces.

Lemma 8.21. *Suppose $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are functors and $\alpha : F \rightarrow G$ a natural equivalence, then $BF, BG : B\mathcal{C} \rightarrow B\mathcal{D}$ are homotopic.*

Proof. Let \mathcal{I} be the category with one non-identity arrow $j : 0 \rightarrow 1$. We define a bifunctor $\mathcal{I} \times \mathcal{C} \rightarrow \mathcal{D}$ as follows. For any $f : x \rightarrow y \in \mathcal{C}$, $H(id_0, f) := F(f)$, $H(j, f) := G(f)\alpha_x = \alpha_y F(f)$ and $H(id_1, f) = G(f)$. One easily sees that $B\mathcal{I}$ is homeomorphic to the interval $[0, 1]$. Hence,

$$BH : [0, 1] \times B\mathcal{C} \rightarrow B\mathcal{D}.$$

Moreover, close inspection shows that $BH(0, \cdot) = BF$ and $BH(1, \cdot) = BG$. Thus BH is the desired homotopy. \square

Corollary 8.22. *If \mathcal{C} and \mathcal{D} are equivalent categories, then $B\mathcal{C}$ is homotopy equivalent to $B\mathcal{D}$. Therefore, for any Abelian group A*

$$H^*(B\mathcal{C}, A) \cong H^*(B\mathcal{D}, A),$$

and hence, by Proposition 8.18,

$$H^*(\mathcal{C}, A) \cong H^*(\mathcal{D}, A).$$

Example 8.23. Suppose \mathcal{C} is a category and x is an object in \mathcal{C} . The **comma category** x/\mathcal{C} is by definition the category whose objects are arrows $f : y \rightarrow x$ and whose morphisms $f \rightarrow f'$ are commuting triangles

$$\begin{array}{ccc} y & \xrightarrow{g} & y' \\ & \searrow f & \swarrow f' \\ & x & \end{array}$$

of arrows in \mathcal{C} .

The comma category x/\mathcal{C} is equivalent to the trivial category on x (i.e. contractible). Indeed, let $F : x/\mathcal{C} \rightarrow x$ be the canonical projection and $G : x \rightarrow x/\mathcal{C}$ the canonical inclusion, then a natural transformation $\alpha : GF \rightarrow id_{\mathcal{C}}$ is given by $\alpha_f = f$. As a consequence

$$H^i(x/\mathcal{C}, A) = 0$$

for $i > 0$.

Remark 8.24. For a continuous groupoid $G \rightrightarrows G^{(0)}$, and an open set $U \subset G^{(0)}$, there is a notion of comma groupoid U/G extending the previous notion of comma category pointwise. The comma groupoid $U/G \rightrightarrows G_U$ is Morita equivalent to $U \rightrightarrows U$. Indeed, the Morita bibundle is G_U and one easily sees that $U/G \cong G_U \times G_U$.

Appendix II: Haar systems and cut-off functions

In this appendix we give a short introduction to Haar systems on groupoids and cutoff functions for proper groupoids. We focus on examples.

Suppose X and Y are locally compact spaces and $p : Y \rightarrow X$ is a continuous surjection. A **continuous family of Radon measures on $p : Y \rightarrow X$** is a family of Radon measures $\{\nu_x\}_{x \in X}$ on Y such that

- (i) the support of ν_x is a subset of $p^{-1}(x) =: Y_x$ and
- (ii) for every function $f \in C_c(Y)$ the function

$$x \mapsto \int_{y \in Y_x} f(y) \nu_x(dy)$$

is continuous $X \rightarrow \mathbb{C}$.

Suppose $G \rightrightarrows X$ is locally compact, second countable continuous groupoid.

Definition 8.25. A **left Haar system on $G \rightrightarrows X$** is a continuous family of Radon measures $\{\lambda^x\}_{x \in X}$ on $t : G \rightarrow X$ that is left-invariant, i.e. for all $x, y \in X$, $h \in G_x^y$, and $f \in C_c(G)$,

$$\int_{g \in G^x} f(hg) \lambda^x(dg) = \int_{g \in G^y} f(g) \lambda^y(dg).$$

There is an analogous notion of right Haar system. We shall sometimes be sloppy and not distinguish between a measure λ_x on G_x and its pushforward along the inclusion $G_x \hookrightarrow G$.

Example 8.26. Suppose X is a locally compact space. The Dirac measures $\{\lambda^x\}_{x \in X}$ form a Haar system on the trivial groupoid $X \rightrightarrows X$. If ν is a Radon measure on X , then $\{\nu^x := \nu\}_{x \in X}$ is a Haar system on the pair groupoid $X \times X \rightrightarrows X$.

Example 8.27. If H is a locally compact group and κ a left Haar measure on H . Then κ is a Haar system on $H \rightrightarrows pt$. Suppose H acts on a locally compact space X . Then $\{\lambda^x := \kappa\}_{x \in X}$ forms a left Haar system on the action groupoid $H \ltimes X \rightrightarrows X$.

Suppose $p : P \rightarrow X$ is a left principal H -bundle. Suppose $x \in X$ and $\phi : P|_U \rightarrow U \times H$ is a local trivialization of $P \rightarrow X$ on a neighborhood U of x . The obvious Haar

system on $U \times H \rightarrow U$ can be pushed forward to $P|_U$, that is $\kappa^x := (\phi^{-1})_* \kappa$. Since κ is left H -invariant this unambiguously defines a continuous family of H -invariant Radon measures on $p : P \rightarrow X$. Suppose ν is a Radon measure on X . We define a continuous family of Radon measures on $p \circ pr_2 : P \times P \rightarrow X$ by

$$\tilde{\lambda}^x := \int_{y \in X} \kappa^y \times \kappa^x \nu(dy),$$

which is H -invariant under the diagonal action of H and hence descends to a left Haar system $\{\lambda^x\}_{x \in X}$ on the gauge groupoid $P \times_H P \rightrightarrows X$.

Example 8.28. Suppose $p : G \rightarrow X$ is a locally compact continuous family of groups. By a classical result there exists a left Haar measure on each group $G_x := p^{-1}(x)$, unique up to multiplication by a positive constant. Renault proves that there is a specific choice of measures λ^x on G_x for $x \in X$ such that they form a Haar system if and only if p is open. Indeed, one should construct a continuous function $F : G \rightarrow \mathbb{R}$ that is compactly supported on the fibers and that satisfies $0 \leq F \leq 1$ and $F \circ u = 1$. Then the measures λ^x should be chosen such that $\int_{G_x} F \lambda^x = 1$ for every $x \in X$.

For example, consider a group bundle $p : G \rightarrow X$ on a space X with fibers isomorphic to a fixed compact group K . We can take $F = 1$. Then by the above procedure the measure λ^x has to come from the normalized Haar measure on K for each $x \in X$.

Example 8.29. Suppose $G \rightrightarrows M$ is a *Lie* groupoid. There exists a Haar system on $G \rightrightarrows M$. Indeed, one easily sees that there exists a strictly positive smooth density ρ on the manifold $\mathcal{A}(G) = u^*(T^t G)$. This can be extended to a G -invariant density $\tilde{\rho}$ on $T^t G$. Then we define a Haar system on $G \rightrightarrows M$ by

$$\lambda^x(f) := \int_{G^x} f \tilde{\rho},$$

for all $f \in C_c(G)$.

Even for a proper groupoid $G \rightrightarrows M$, one cannot just integrate any function $f \in C(G)$ along a target fiber G^x . Consider, for example, the pair groupoid $M \times M \rightrightarrows M$ and any constant non-zero function on $M \times M$. A useful device for repairing this problem is the notion of a cutoff function for groupoids. Cutoff functions are useful in for example averaging processes. It turns out that a cutoff function for $G \rightrightarrows X$ exists iff $G \rightrightarrows X$ is proper (cf. [15]).

Definition 8.30. Suppose $G \rightrightarrows X$ is a groupoid endowed with a Haar system $\{\lambda_x\}_{x \in X}$. A **cutoff function** for $G \rightrightarrows X$ is a function $X \rightarrow \mathbb{R}_{\geq 0}$ such that

- the support of $(c \circ s)|_{t^{-1}K}$ is compact for all compact sets $K \subset M$;
- for all $x \in X$, $\int_{G^x} c(s(g)) \lambda^x(dg) = 1$.

Example 8.31. If G^x is compact for all $x \in X$, then one can simply take $c(x) = 1/\lambda^x(G^x)$.

Example 8.32. Consider the pair groupoid $X \times X \rightrightarrows X$. Suppose the Haar system is constructed using a measure μ on X (cf. Example 8.26). Any compactly supported function c on X satisfying $\int_X c \mu = 1$ is a cutoff function.

Appendix III: Morphisms of groupoids

In this appendix we recall the definition of actions of groupoids and generalized morphisms. Again, for the rest, our main purpose is to supply in some examples.

Definition 8.33. A morphism of continuous groupoids

$$(G \rightrightarrows G_0) \rightarrow (H \rightrightarrows H_0)$$

is a pair of continuous maps $\phi_1 : G \rightarrow H$ and $\phi_0 : G_0 \rightarrow H_0$ that commutes with the structure maps, i.e. $s \circ \phi_1 = \phi_0 \circ s$, $\phi_1(g \cdot g') = \phi_1(g) \cdot \phi_1(g')$, etcetera.

We denote the category of continuous groupoids with these morphisms by GPD.

Definition 8.34. The **pullback groupoid** $f^!G \rightrightarrows Y$ of a continuous groupoid $G \rightrightarrows X$ along a continuous map $f : Y \rightarrow X$ is defined as follows. The space of arrows is $f^!G := Y \times_{f \times t} G \times_{s \times f} Y$, the source map is projection on the third factor, the target map is projection on the first factor, composition is defined by $(y, g, y') \cdot (y', g', y'') := (y, g \cdot g', y'')$, the unit map is $u(y) := (y, u(f(y)), y)$ and inversion is defined by $(y, g, y')^{-1} := (y', g^{-1}, y)$.

Example 8.35. Suppose $G \rightrightarrows X$ is a continuous groupoid and $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X . Consider the canonical continuous map $j : \coprod_{i \in I} U_i \rightarrow X$. The pullback groupoid along this map is denoted by $G[\mathcal{U}] := j^!G \rightrightarrows \coprod_{i \in I} U_i$.

The cover groupoid of an open cover \mathcal{U} of a space X is an example of this construction. It equals $X[\mathcal{U}]$.

Suppose $G \rightrightarrows X$ is a continuous groupoid, Y a space and $J : Y \rightarrow X$ a continuous map.

Definition 8.36. A **continuous left action of $G \rightrightarrows X$ on $J : Y \rightarrow X$** is a continuous map

$$\alpha : G \times_{s \times J} Y \rightarrow Y$$

satisfying

- (i) $J(g \cdot y) = t(g)$ for all $(g, y) \in G \times_{s \times J} Y$,
- (ii) $1_{J(y)} \cdot y = y$ for all $y \in Y$,
- (iii) $g \cdot (g' \cdot y) = (gg') \cdot y$ for all $(g, g') \in G^{(2)}$ and $y \in J^{-1}(s(g'))$,

using the notation $g \cdot y := \alpha(g, y)$.

We shall also use the notation $\alpha(g) := \alpha(g, \cdot)$. There exists an analogous notion of a right action.

Example 8.37. Suppose $G \rightrightarrows X$ is a continuous groupoid. It acts from the left on $t : G \rightarrow X$ by left multiplication $l : G \times_t G \rightarrow G$, denoted by $l_g g' := g \cdot g'$. Analogously, $G \rightrightarrows X$ acts from the right on $s : G \rightarrow X$.

Suppose $G \rightrightarrows G_0$ is a continuous groupoid. Suppose G acts continuously from the left on a map $J : Y \rightarrow G_0$. The action is called **left principal** if the map

$$(g, y) \mapsto (g \cdot y, y)$$

is a homeomorphism

$$G \times_s Y \rightarrow Y \times_p Y,$$

where $G \backslash Y$ is endowed with the quotient topology, and $p : Y \rightarrow G \backslash Y$ is the projection on the orbit space. There is an analogous notion of right principal action. Suppose $H \rightrightarrows H_0$ is another continuous groupoid. A space Y is a (G, H) -**bibundle** if there is a left G -action on a map $J_G : Y \rightarrow G_0$ and a right H -action on a map $J_H : Y \rightarrow H_0$ that commute, i.e. $(g \cdot y) \cdot h = g \cdot (y \cdot h)$, $J_H(g \cdot y) = J_H(y)$ and $J_G(y \cdot h) = J_G(y)$. A **morphism of (G, H) -bibundles** Y, Y' is a (G, H) -equivariant continuous map $Y \rightarrow Y'$. An isomorphism class of a right principal (G, H) -bibundle can be interpreted as an arrow $G \rightarrow H$ in a category of groupoids. These arrows are called **Hilsum-Skandalis maps** or **generalized morphisms**. The category of continuous groupoids and generalized morphisms is denoted by \cdot . Composition of morphisms represented by a (G, H) -bibundle P and a (H, K) -bibundle Q is given by the fibered product $[P] \circ [Q] := [P \times_H Q]$. The unit morphism $U(G)$ at G is the class $[G]$ of G itself seen as a (G, G) -bibundle, with left and right multiplication as actions. One can show that a morphism given by a class of bibundles is an isomorphism if the representing bundles have principal left and right actions. In that case, one easily sees that $Y/H \cong G_0$ and $G \backslash Y \cong H_0$. Groupoids that are isomorphic in this category are called **Morita equivalent**. One can prove that a (G, H) -bibundle Y represents a Morita equivalence if it is left and right principal and $Y/H \cong G_0$ and $G \backslash Y \cong H_0$.

Example 8.38. Suppose H is a group and $P \rightarrow X$ is a continuous left principal H -bundle. The group H is Morita equivalent to the gauge groupoid $P \times_H P \rightrightarrows X$. Indeed, $[P]$ is an invertible generalized morphism $H \rightarrow P \times_H P$. Indeed, by definition, H acts from the left on P . The right action of $P \times_H P \rightrightarrows X$ on $P \rightarrow X$ is defined by $p \cdot [p, q] = q$.

Example 8.39. Suppose $(\phi_1, \phi_0) : (G \rightrightarrows G_0) \rightarrow (H \rightrightarrows H_0)$ is a continuous morphism of groupoids. This gives rise to a generalized morphism of groupoids $[G_0 \xrightarrow{\phi_0} \times_t H] : (G \rightrightarrows G_0) \rightarrow (H \rightrightarrows H_0)$, where we view $G_0 \xrightarrow{\phi_0} \times_t H$ as a (G, H) -bimodule as follows. The left action of $G \rightrightarrows G_0$ on the map $pr_1 : G_0 \xrightarrow{\phi_0} \times_t H \rightarrow G_0$ is given by $g \cdot (s(g), h) := (t(g), \phi_1(g)h)$ for all $g \in G$, $h \in H^{\phi_0(s(g))}$. The right action of $H \rightrightarrows H_0$ on $s \circ pr_2 : G_0 \xrightarrow{\phi_0} \times_t H \rightarrow H_0$ is given by right multiplication: $(x, h) \cdot h' := (x, h \cdot h')$. One easily sees that the action of $H \rightrightarrows H_0$ is right principal. This gives an inclusion functor

$$\text{GPD} \rightarrow \text{GPD}_b.$$

Example 8.40. Suppose $G \rightrightarrows X$ is a continuous groupoid and $j : Y \rightarrow X$ a continuous map. There is a canonical map $j^!G \rightarrow G$, which induces a generalized morphism $[Y \times_j G] : (j^!G \rightrightarrows Y) \rightarrow (G \rightrightarrows X)$. If $j(Y)$ intersect each G -orbit at least once, then it is actually a Morita equivalence with inverse $[G \times_s Y]$, endowed with the obvious actions.

In particular, we can conclude that for any continuous groupoid $G \rightrightarrows X$ and open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X , the groupoids $G[\mathcal{U}] \rightrightarrows \coprod_{i \in I} U_i$ and $G \rightrightarrows X$ are Morita equivalent.

Example 8.41. Suppose M is a smooth manifold and \mathcal{F} a regular foliation of M . Suppose $i_T : T \hookrightarrow M$ is a full transversal in the sense that it intersects each leaf at least once (transversally). By the previous example the holonomy groupoid $\text{Hol}(M, \mathcal{F}) \rightrightarrows M$ is Morita equivalent to the pullback groupoid $i_T^* \text{Hol}(M, \mathcal{F}) \rightrightarrows T$ obtained by restriction to the transversal T . Note that $i_T^* \text{Hol}(M, \mathcal{F}) \rightrightarrows T$ is étale. In general, one can show that any foliation groupoid is Morita equivalent to an étale groupoid.

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