Continuous representations of groupoids

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Abstract

We introduce unitary representations of continuous groupoids on continuous fields of Hilbert spaces. We investigate some properties of these objects and using several examples. We present a palette of results, including, among others: a comparision of the different notions of continuity for representations, a description of the representations of families of groups, a version of the Peter-Weyl theorem for groupoids.

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1 Introduction

The purpose of this paper is to study some of the basic theory of continuous representations in the context of groupoids.

To some extent the study of representations of groupoids was initiated by Westman in [21, 22]. He studied representations of locally trivial groupoids on continuous vector bundles. We shall look at representations not only on continuous vector bundles, but on continuous fields of Hilbert spaces. Continuous fields of Hilbert spaces were introduced and studied by Dixmier and Douady [8]. They play an important rôle in noncommutative geometry, as they occur as Hilbert C^*-modules of commutative C^*-algebras (cf. Theorem 2.10).

The main reason of our interest in representations of groupoids on continuous fields of Hilbert spaces is because of its rôle in groupoid equivariant KK-theory. Groupoid equivariant KK-theory is needed in the groupoid version of the Baum-Connes conjecture [12]. Moreover, it can be used in a theoretical framework for index theorems on foliations (cf. [6]). A groupoid equivariant KK-cycle consists of a representation of a groupoid on a continuous field of Hilbert spaces endowed with other structure (which does not concern us here) (cf. [12]).

In [2, 3] we study geometric quantization of families with symmetry described by a groupoid action. As remarked in this paper, one obtains a groupoid equivariant KK-cycle. The geometric quantization one obtains *is* a continuous representation of a groupoid on a continuous field of Hilbert spaces.

Another reason why we not only consider representations on continuous vector bundles is the following. One should note that the regular representation of a groupoid $G \rightrightarrows M$ with Haar system is defined on a continuous field of L^2 functions on the target fibers. Even for very simple étale groupoids this is not a locally trivial field (consider e.g. the family of groups $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R})\backslash\{(-1,0)\} \to \mathbb{R}$).

Our study should be contrasted with the standard work of Renault [16] on measured representations of groupoids on measured fields of Hilbert spaces. His approach involves the construction of a quasi-invariant measure on the base space of the groupoid. Advantage of this approach is the bijective relation he obtains with the representations of the groupoid C^*-algebra. An advantage of this work is that we are not dependent on the existence of such a measure. In addition to this, we obtained a bijection with the representations of the continuous C^*-category associated to the groupoid in the continuous case (cf. [3]).

Section 3 introduces representations of groupoids on continuous fields of Hilbert spaces. We discuss several notions of continuity of representations and show how they relate. Then we treat two examples, namely the regular representation of a groupoid and representations of continuous families of groups. In the last part of this section we "embed" the theory of continuous groupoid representations in the theory of group representations. We discuss the topological group of global bisections of a groupoid and give a theorem that explains which representations of this group correspond to representations of the groupoid. Hence one could view the representation theory of groupoids as a way to understand some specific representations of certain 'infinite-dimensional' groups.

Section 4 treats harmonic analysis in the case of groupoids. We prove an analogue of Schur's Lemma and of the Peter-Weyl Theorem. As one will see, proofs of theorems in representation theory of groupoids heavily rely on the representation theory of groups. The differences mostly arise in dealing with the global topology of the groupoid and its orbit foliation.

2 Continuous fields of Banach and Hilbert spaces

2.1 Definition and known results

Suppose X is a locally compact Hausdorff space.

Definition 2.1. ([8]) A **continuous field of Banach spaces over** X is a family of Banach spaces $\{\mathcal{B}_x\}_{x\in X}$ and a space of sections $\Delta \subset \prod_{x\in X} \mathcal{B}_x$, such that

(i) the set $\{\xi(x) \mid \xi \in \Delta\}$ equals $\mathcal{B}_x$ for all $x \in X$.

(ii) For every $\xi \in \Delta$ the map $x \mapsto \|\xi(x)\|$ is in[1] $C_0(X)$.

(iii) Δ is locally uniformly closed, i.e. if $\xi \in \prod_{x\in X} \mathcal{B}_x$ and for each $\varepsilon > 0$ and each $x \in X$, there is an $\eta \in \Delta$ such that $\|\xi(y) - \eta(y)\| < \varepsilon$ on a neighborhood of x, then $\xi \in \Delta$.

[1] $f \in C_0(X)$ iff f is continuous and for every $\varepsilon > 0$ there exists a compact set K such that $f(x) < \varepsilon$ if $x \notin K$

Remark 2.2. By composing the map $x \mapsto \|\xi(x)\|_{\mathcal{B}_x}$ with the norm on $C_0(X)$ one obtains a norm

$$\|\xi\| = \sqrt{\sup_{x\in X} \|\xi(x)\|^2_{\mathcal{B}_x}}$$

on Δ. From (iii) it follows at once that Δ is complete in this norm.

There is a subclass of these continuous fields which has our special interest.

Definition 2.3. A **continuous field of Hilbert spaces over** X is a family of Hilbert spaces $\{\mathcal{H}_x\}_{x\in X}$ and a space of sections $\Delta \subset \prod_{x\in X} \mathcal{H}_x$ that form a continuous field of Banach spaces.

Remark 2.4. In the case of a continuous field of Hilbert spaces, the condition (ii) in Definition 2.1 can be replaced by the requirement that for any $\xi, \eta \in \Delta$ the map $x \mapsto \langle \xi(x), \eta(x)\rangle_{\mathcal{H}_x}$ is in $C_0(X)$. The field is called **upper (lower) semi-continuous** if $x \mapsto \|\xi(x)\|$ is just upper (lower) continuous for every $\xi \in \Delta$.

Lemma 2.5. *([8]) If $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ is a continuous field of Banach spaces, then there is a topology on the total space $\mathcal{B} := \coprod_{x\in X} \mathcal{B}_x$ such that Δ equals the set of continuous sections $\Gamma_0(\mathcal{B}) := \{\xi \in \Gamma(\mathcal{B}) \mid \|\xi\| \in C_0(X)\}$.*

The topology is defined as follows. For each $\varepsilon > 0$, $V \subset X$ open and $\xi \in \Delta$, we define

$$U(\varepsilon, \xi, V) := \{h \in \mathcal{B} \mid \|h - \xi(p(h))\| < \varepsilon \text{ and } p(h) \in V\},$$

where $p : \mathcal{B} \to X$ is the projection of the total space on the base. One easily sees that these sets form a basis for a topology on $\mathcal{B}$.

Remark 2.6. As a short notation we often denote a continuous field of Banach spaces $(\{\mathcal{B}_x\}_{x\in X}, \Delta_{\mathcal{B}})$ by $(\mathcal{B}, \Delta)$.

Lemma 2.7. *([3]) For any continuous field of Banach spaces $(\mathcal{B}, \Delta)$ the map $\|.\| : \mathcal{B} \to \mathbb{R}_{\geq 0}$ is continuous.*

Definition 2.8. ([8]) A **morphism** $\Psi : (\mathcal{B}^1, \Delta^1) \to (\mathcal{B}^2, \Delta^2)$ **of continuous fields of Banach spaces** is a family of bounded linear maps $\{\Psi_x : \mathcal{B}^1_x \to \mathcal{B}^2_x\}_{x\in X}$ such that the induced map $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ on the total spaces satisfies

$$\{\Psi \circ \xi \mid \xi \in \Delta^1\} \subset \Delta^2$$

and

$$x \mapsto \|\Psi_x\|$$

is a locally bounded map.

Here $\|\Psi_x\|$ is the operator norm of Ψ_x,

$$\|\Psi_x\| := \sup_{\|h\|_{\mathcal{B}^1_x}=1} \|\Psi_x(h)\|_{\mathcal{B}^2_x}.$$

The first condition has to be satisfied only on a dense subset of Δ^1 ([8], Proposition 5).

Lemma 2.9. *([8]) The map $\Psi : \mathcal{B}^1 \to \mathcal{B}^2$ is continuous iff Ψ is a morphism of continuous fields of Banach spaces.*

The map $\Psi : (\mathcal{B}^1, \Delta^1) \to (\mathcal{B}^2, \Delta^2)$ is an (isometric) isomorphism of continuous fields of Banach spaces if all the Ψ_x are (isometric) isomorphisms and $\Psi(\Delta^1) = \Delta^2$. In fact, one can replace the second condition by $\Psi(\Lambda) \subset \Delta^2$ for a dense subset $\Lambda \subset \Delta^1$ ([8], Proposition 6).

If $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ is a continuous field of Banach/Hilbert spaces, then Δ has the structure of Banach/Hilbert C^*-algebra (cf. [8] or [3]). One can also go the other way around.

Theorem 2.10. *[8] §4) There is an equivalence of categories of continuous fields of Banach (respectively Hilbert) spaces and left Banach (respectively Hilbert) $C_0(X)$-modules.*

Example 2.11. Suppose $\pi : Y \to X$ is a continuous surjection endowed with a continuous family of Radon measures $\{\nu_x\}_{x\in X}$ (cf. [16]). For any $p \in \mathbb{R}_{\geq 1}$ consider the norm on $C_c(Y)$ given by

$$\|f\|_p := \sup_{x\in X} \|f|_{Y_x}\|_{L^p(Y_x,\nu_x)}.$$

Define $\Delta^p_\pi(Y)$ to be the closure of $C_c(Y)$ with respect to this norm. One easily sees that this is a Banach $C_0(X)$-module with $C_0(X)$-valued norm given by

$$\|f\|(x) := \|f|_{Y_x}\|_{L^p(Y_x,\nu_x)} = \left(\int_{Y_x} |f(y)|^p \nu_x(dy)\right)^{1/p}.$$

The continuous field associated to this Banach $C_0(X)$-module is denoted by

$$(\hat{L}^p_\pi(Y), \Delta^p_\pi(Y))$$

. The fiber at $x \in X$ equals $L^p(Y_x, \nu_x)$.

If $p = 2$, one obtains a Hilbert $C_0(X)$-module and hence a continuous field of Hilbert spaces. The $C_0(X)$-valued inner product is given on $C_c(Y)$ by

$$\langle f, f'\rangle(x) := \langle f|_{Y_x}, f'|_{Y_x}\rangle_{L^2(Y_x,\nu_x)} = \int_{Y_x} \overline{f(y)} f'(y)\nu_x(dy).$$

2.2 Dimension and local pseudo-trivializations

This section contains a characterization of (uniformly) finite-dimensional continuous fields of Hilbert spaces.

The **dimension of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ over X is the supremum of the function

$$\dim : X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, x \mapsto \dim(\mathcal{H}_x).$$

A continuous field of Hilbert spaces is **uniformly finite-dimensional** if it has finite dimension. One should distinguish between uniformly finite-dimensional and **finite-dimensional** continuous fields, which means that each fiber is finite dimensional.

Example 2.12. Consider the field over $\mathbb{R}$ with $\mathcal{H}_x := \mathbb{C}^n$ if $x \in [-n, -n+1) \cup (n-1, n]$ for all $n \in \mathbb{N}$ and $\mathcal{H}_0 = 0$. The topology on the field comes from the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ on the first n coordinates. The inner product on each fiber is the standard Hermitian metric on $\mathbb{C}^n$. This field is finite-dimensional, but not uniformly finite-dimensional.

Definition 2.13. A continuous field $(\mathcal{H}, \Delta)$ is **locally trivial** if for every $x \in X$ there exist a neighborhood $U \ni x$, a Hilbert space $\mathcal{H}'$ and an isomorphism of continuous fields $\mathcal{H}|_U \to U \times \mathcal{H}'$.

Example 2.14. Locally trivial finite-dimensional continuous fields of Hilbert spaces are known as complex vector bundles with Hermitian metric.

Obviously a continuous field is not always locally trivial, see e.g. Example 2.12 or

Example 2.15 (Winding stair)**.** Consider the continuous field over $\mathbb{R}^2$ defined as follows. Fix any $d \in \mathbb{N}$ (the number of stairs). For $\vec{x} \in \mathbb{R}_{\geq 0} \times \{0\}$ let $\mathcal{H}_{\vec{x}} = 0$. For $\vec{x}$ in the sector between angles $i\, 2\pi/d$ and $(i+1)\, 2\pi/d$ (including the positive halfline at $(i+1)\, 2\pi/d$, unless $i = d-1$) take $\mathcal{H}_{\vec{x}} = \mathbb{C}^{i+1}$, where $i = 0, \ldots d-1$. Again, the topology on the field comes from the inclusion $\mathbb{C}^i \hookrightarrow \mathbb{C}^{i+1}$ on the first i coordinates. The inner product on each fiber is the standard Hermitian inner product on $\mathbb{C}^i$. Obviously, the field is not locally trivial at the origin.

Therefore, we introduce the notion of local pseudo-trivializations.

Definition 2.16. A **local pseudo-trivialization** of a continuous field $(\mathcal{H}, \Delta)$ on an open set $U \subset X$ is a family of open sets $\{U_i\}_{0 \geq i \geq \dim(\mathcal{H}|_U)}$, such that

$$\bigcup_{0 \leq i \leq \dim(\mathcal{H}|_U)} U_i = U$$

and

$$\mathcal{H}|_U \simeq \sum_{0 \leq i \leq \dim(\mathcal{H}|_U)} U_i \times \mathbb{C}^i,$$

where on intersections $U_i \cap U_j$ $(i < j)$, the topology is given by identifying along the inclusion $1_i \times 0_{j-i} : \mathbb{C}^i \hookrightarrow \mathbb{C}^j$.

Pay attention to the fact that this is not a direct sum. Such local pseudo-trivializations shall be useful in Section 3.4.

Lemma 2.17. *Suppose $(\mathcal{H}, \Delta)$ is a uniformly finite-dimensional continuous field of Hilbert spaces over X. Then for any $x \in X$ there exists a neighborhood U_x of x such that $(\mathcal{H}, \Delta)$ admits a local pseudo-trivialization on U_x.*

Proof. We prove by induction on the dimension d of $(\mathcal{H}, \Delta)$. Suppose $d = 1$. If $\mathcal{H}_x \neq 0$, then there exists a $\xi \in \Delta$ such that $\xi(x) \neq 0$. By continuity, $\xi(y) \neq 0$ for y in a neighborhood U of x. Obviously, the map $U \times \mathbb{C} \to \mathcal{H}|_U$ given by $(y, z) \mapsto z\, \xi(y)$ is a local (pseudo-)trivialization.

If $\mathcal{H}_x = 0$, then we proceed as follows. For every $y \in X$ for which with $\mathcal{H}_y \neq 0$ there exists a section ξ_y such that $\xi_y(y)$ spans $\mathcal{H}_y$. Since X is locally compact, there exists an open neighborhood U (with compact closure) and a subset $\{y_i\}_{i\in I} \subset X$ such that $\{\mathrm{supp}(\xi_{y_i})\}_{i\in I}$ forms a locally finite cover of $U \cup \mathrm{supp}(\mathcal{H})$. For any subsequence $\{y_j\}_{j\in J}$ converging to x, we add the $\lim_{j\to J} \xi_{y_j} \in \Delta$ to the set of sections indexed by I. By continuity the sum $\xi(y) := \sum_{i\in I} \xi_{y_i}(y)$ still spans $\mathcal{H}_y$ for y in a small enough open neighborhood U' of x. Again, the map $U' \times \mathbb{C} \to \mathcal{H}|_U$ given by $(y, z) \mapsto z\,\xi(y)$ is a local (pseudo-)trivialization.

Suppose $(\mathcal{H}, \Delta)$ has dimension d. If $\mathcal{H}_x \neq 0$, then there exists a $\xi \in \Delta$ such that $\xi(x) \neq 0$. Again, then $\xi(y) \neq 0$ for y in a neighborhood U of x. Hence $\mathcal{H}|_U \simeq \mathrm{span}\,\xi|_U \oplus \mathcal{H}'$ for some continuous field $(\mathcal{H}', \Delta')$ over U. The field $(\mathcal{H}', \Delta')$ has dimension $d-1$, so by the induction hypothesis there is an isomorphism

$$\phi : \sum_{i=0}^{d-1} U'_i \times \mathbb{C}^i \to \mathcal{H}'|_{U'}$$

on an open neighborhood U' of x. Hence, an isomorphism

$$\sum_{i=1}^{d} U'_{i-1} \times \mathbb{C}^i \to \mathcal{H}|_{U'}$$

is given by

$$(y, \vec{z}) \mapsto \phi(y, z_1, \ldots, z_{i-1}) + z_i\,\xi(y).$$

If $\mathcal{H}_x = 0$, then we construct a local section ξ on a neighborhood of x as in the case $d = 1$. Proceed as above. □

Corollary 2.18. *A continuous field $(\mathcal{H}, \Delta)$ over a compact space X is uniformly finite-dimensional iff Δ is finitely generated over $C_0(X)$.*

Lemma 2.19. *For a uniformly finite-dimensional continuous field of Hilbert spaces over X the dimension is a lower semi-continuous function*

$$\dim : X \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}.$$

That is, $\dim : X \to \mathbb{Z}_{\geq 0}$ *has a local minimum at every point.*

Proof. Suppose $(\mathcal{H}, \Delta)$ is such a continuous field of Hilbert spaces and $x \in X$. Choose sections $\xi^x_j \in \Delta$ for $j = 1, \ldots, \dim(\mathcal{H}_x)$, such that $\{\xi^x_j(x)\}_{j=1}^{\dim(\mathcal{H}_x)}$ forms a basis of $\mathcal{H}_x$. Let V_x be the set on which their images stay linearly independent and non-zero. This set is open, since, for a local pseudo-trivializations ϕ,

$$x \mapsto \det(\phi^*\xi^x_1 \mid \ldots \mid \phi^*\xi^x_{\dim \mathcal{H}_x}) = \det((\langle \phi^*\xi^x_k, \phi^*\xi^x_l \rangle)_{kl})$$

is continuous. Indeed, this last expression is a polynomial in $\langle \phi^*\xi^x_k, \phi^*\xi^x_l \rangle$ for $1 \leq k, l \leq j$ which are continuous . □

3 Continuous representations of groupoids

3.1 Continuous representations of groupoids

In this section we introduce continuous representations of groupoids on continuous fields of Hilbert spaces. As far as we know this notion as we define it does not appear anywhere in the literature. There is a preprint by Amini [1], which treats continuous representations on Hilbert bundles, which is rather different from the notion of continuous field of Hilbert spaces as we use it. It seems as though his article does not give full attention to the 'continuity-issues' involved.

As for representations of groups there are several forms of continuity for such representations. We consider "normal", weak and strong continuity and in Section 3.2 also continuity in the operator norm. All these forms of continuity can be compared, cf. Lemma 3.6, Lemma 3.7 and Lemma 3.14, generalizing similar results for groups (cf. e.g. [10]). In Definition 3.8 we introduce the notion of a morphism of representations and we show in Proposition 3.10 that any representation of a proper groupoid is isomorphic to a unitary representation, generalizing a similar result for compact groups.

For groupoids we use the terminology and notational conventions from [13] and [16]. Let M be a locally compact space and $G \rightrightarrows M$ a continuous groupoid.

Definition 3.1. A **bounded representation** of $G \rightrightarrows M$ on a continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m\in M}, \Delta)$ over M is a family of invertible bounded operators

$$\{\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}\}_{g\in G}$$

satisfying

(i) $\pi(1_m) = id_{\mathcal{H}_m}$ for all $m \in M$,

(ii) $\pi(gg') = \pi(g)\pi(g')$ for all $(g, g') \in G_2 = G\ {}_t\times_s G$,

(iii) $\pi(g^{-1}) = \pi(g)^{-1}$ for all $g \in G$ and

(iv) $g \mapsto \|\pi(g)\|$ is locally bounded.

We denote such a representation by a triple $(\mathcal{H}, \Delta, \pi)$. Recall from Lemma 2.5 that $\mathcal{H}$ can be endowed with a topology such that the sections Δ equals the set of continuous sections $\Gamma_0(\mathcal{H})$ of the projection $\mathcal{H} \to M$ onto the base space M.

Definition 3.2. A representation $(\mathcal{H}, \Delta, \pi)$ is **strongly continuous** if the map

$$g \mapsto \pi(g)\xi(s(g))$$

is continuous $G \to \mathcal{H}$ for all $\xi \in \Delta$. A representation is **weakly continuous** if the map

$$g \mapsto \langle \pi(g)\xi(s(g)), \eta(t(g))\rangle$$

is continuous $G \to \mathbb{C}$ for all $\xi, \eta \in \Delta$. A representation $(\pi, \mathcal{H}, \Delta)$ is **continuous** if

$$\Psi : (g, h) \mapsto \pi(g)h$$

is a continuous map $G\ {}_s\times_p \mathcal{H} \to \mathcal{H}$. The representation is **unitary** if the operators $\{\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}\}_{g\in G}$ are unitary.

For any $\xi, \eta \in \Delta^\pi$ we use the notation $\langle \xi, \pi\eta \rangle$ for the map $G \to \mathbb{C}$ given by

$$g \mapsto \langle \xi(t(g)), \pi(g)\eta(s(g)) \rangle \,,$$

which we call a **matrix coefficient**.

Condition (iv) of Definition 3.1 is perhaps somewhat strange at first sight. The following Example 3.3, Lemma 3.4 and Example 3.5 should clarify it. Moreover, recall that for morphism Ψ of continuous fields the map $m \mapsto \|\Psi_m\|$ has to be locally bounded too, cf. Definition 2.8.

Example 3.3. A simple example shows that $g \mapsto \|\pi(g)\|$ is not always continuous. Consider the groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$, with a continuous representation on a field given by the trivial representation on $\mathbb{C}$ at each $x \in \mathbb{R}$ except in 0, where it is the zero representation. In this case, the norm of π drops from 1 to 0 at 0.

Lemma 3.4. *For any continuous representation $(\mathcal{H}, \pi, \Delta)$ the map $g \mapsto \|\pi(g)\|$ is lower semi-continuous $G \to \mathbb{R}$.*

Proof. Using the above definition and Lemma 2.7 we know that the map $(g, h) \mapsto \|\pi(g)h\|$ is continuous $G_s \times_p \mathcal{H} \to \mathbb{R}_{\geq 0}$. For any $g \in G$, let $\varepsilon > 0$ be given. Let $h' \in \mathcal{H}_{s(g)}$ be such that

$$|\|\pi(g)h'\| - \|\pi(g)\|| < \varepsilon/2.$$

by continuity there exists an open neighborhood $U \subset G_s \times_p \mathcal{H}$ of (g, h') such that $(g'', h'') \in U$ implies

$$|\|\pi(g'')h''\| - \|\pi(g)h'\|| < \varepsilon/2.$$

Take $V := pr_1(U) \subset G$. Then $g'' \in V$ implies, for an $h'' \in pr_2(U)$ such that $(g'', h'') \in U$, one has

$$\|\pi(g'')\| \geq \|\pi(g'')h''\| > \|\pi(g)h'\| - \varepsilon/2 > \|\pi(g)\| - \varepsilon,$$

and we are done. □

The function $g \mapsto \|\pi(g)\|$ is locally bounded if, for example, $(\mathcal{H}, \Delta)$ is uniformly finite-dimensional.

Example 3.5. A counterexample of a continuous representation of a proper groupoid where $g \mapsto \|\pi(g)\|$ is not locally bounded $G \to \mathbb{R}$, even though the restriction to G_m is bounded for each m, is as follows.

Consider the trivial bundle of groups $[0,1] \times \mathbb{Z}/2\mathbb{Z} \rightrightarrows [0,1]$. Define a continuous field of Hilbert spaces over $[0,1]$ by $\mathcal{H}_0 := \mathbb{C}^2 =: \mathcal{H}_1$ and $\mathcal{H}_x := \mathbb{C}^{2\,n}$ if $x \in [\frac{1}{n+1}, \frac{1}{n})$ for all $n \in \mathbb{N}$. The topology on the field is obtained from the inclusions $\mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2(n+1)}$ given by $\vec{v} \mapsto (0, \vec{v}, 0)$. Define, for every $n \in \mathbb{N}$ and $x \in [\frac{1}{n+1}, \frac{1}{n})$,

$$\pi(x, -1) := \mathrm{diag}'(1/n, \ldots, 1/2, 1, 1, 2, \ldots n),$$

where diag′ denotes the matrix filled with zeros except the diagonal from the upper right corner to the lower left corner, where the above sequence is filled in. Furthermore, $\pi(0, -1) :=$ diag'$(1, 1)$. This representation is strongly continuous, but

$$\|\pi(x, -1)\| = n \text{ if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

Hence $g \mapsto \|\pi(g)\|$ is not locally bounded at $(0, -1)$.

Lemma 3.6. *If a representation $(\pi, \mathcal{H}, \Delta)$ is strongly continuous, then it is weakly continuous. The converse implication holds if the representation is unitary.*

Proof. Suppose $(\pi, \mathcal{H}, \Delta)$ is strongly continuous. Suppose $\xi, \eta \in \Delta$ and $g \in G$. Write $n = t(g)$. Let $\varepsilon > 0$ be given. Let $\xi' \in \Delta$ be a section satisfying $\xi'(n) = \pi(g)\xi(s(g))$. Choose a neighborhood $U \subset M$ of n such that $n' \in U$ implies $| \langle \eta(n'), \xi'(n') \rangle_{\mathcal{H}_{n'}} - \langle \eta(n), \xi'(n) \rangle_{\mathcal{H}_n} | < \varepsilon/2$. This is possible since $\langle \eta, \xi' \rangle$ is continuous on M. Since π is strongly continuous there exists an open set $V \subset G$ containing g such that for all $g' \in V$ one has $t(g') \in U$ and

$$\|\pi(g')\xi(s(g')) - \xi'(t(g'))\|_{\mathcal{H}_{t(g')}} < \varepsilon/(2 \sup_{n' \in U} \|\eta(n')\|).$$

Hence, for all $g' \in V$

$$\begin{aligned}
&| \langle \eta(t(g')), \pi(g')\xi(s(g')) \rangle_{\mathcal{H}_{t(g')}} - \langle \eta(n), \xi'(n) \rangle_{\mathcal{H}_n} | \\
&\quad \leq | \langle \eta(t(g')), \pi(g')\xi(s(g')) \rangle_{\mathcal{H}_{t(g')}} - \langle \eta(t(g')), \xi'(t(g')) \rangle_{\mathcal{H}_{t(g')}} | \\
&\qquad + | \langle \eta(t(g')), \xi'(t(g')) \rangle_{\mathcal{H}_{t(g')}} - \langle \eta(n), \xi'(n) \rangle_{\mathcal{H}_n} | \\
&\quad < \|\eta(t(g'))\| \varepsilon/(2 \sup_{n' \in U} \|\eta(n')\|) + \varepsilon/2 \leq \varepsilon.
\end{aligned}$$

The converse implication is proven as follows. Suppose $(\pi, \mathcal{H}, \Delta)$ is weakly continuous and unitary. Let $U(\varepsilon, \eta, V)$ be a neighborhood of $\pi(g)\xi(s(g))$ in $\mathcal{H}$ for a given $g \in G$ and $\xi \in \Delta$, where $\eta \in \Delta$ satisfies $\eta(t(g)) = \pi(g)\xi(t(g))$. We compute for any $g' \in G$,

$$\begin{aligned}
&\|\eta(t(g')) - \pi(g')\xi(s(g'))\|_{\mathcal{H}_{t(g')}} \qquad (3.1)\\
&\quad = | \langle \eta(t(g')), \eta(t(g')) \rangle - \langle \eta(t(g')), \pi(g')\xi(s(g')) \rangle \\
&\qquad - \langle \pi(g')\xi(s(g')), \eta(t(g')) \rangle + \langle \pi(g')\xi(s(g')), \pi(g')\xi(s(g')) \rangle |^{1/2} \\
&\quad \leq (| \langle \eta(t(g')), \eta(t(g')) \rangle - \langle \eta(t(g')), \pi(g')\xi(s(g')) \rangle | \\
&\qquad + | \langle \xi(s(g')), \xi(s(g')) \rangle - \langle \pi(g')\xi(s(g')), \eta(t(g')) \rangle |)^{1/2} \qquad (3.2)
\end{aligned}$$

By weak continuity we can choose a neighborhood $W_g \subset G$ of g such that $g' \in W_g$ implies

$$| \langle \eta(t(g')), \pi(g')\xi(s(g')) \rangle - \langle \eta(t(g)), \pi(g)\xi(s(g)) \rangle | < \varepsilon.$$

Since t is open and $\eta \in \Delta$, we can choose an open neighbor hood $W'_g \subset W_g$ of g such that

$$| \langle \eta(t(g')), \eta(t(g')) \rangle - \langle \eta(t(g)), \eta(t(g)) \rangle | < \varepsilon$$

Hence the first two terms of Equation (3.2) are smaller than 2ε. Analogously, the last two terms of Equation (3.2) are also smaller than 2ε, which finishes the proof. □

Lemma 3.7. *If a representation $(\pi, \mathcal{H}, \Delta)$ is continuous, then it is strongly continuous. The converse holds if π is unitary.*

Proof. Suppose $(\pi, \mathcal{H}, \Delta)$ is continuous. Suppose $g \in G$ and $\xi \in \Delta$. There exists an open neighborhood $U(\varepsilon, \eta, V) \subset \mathcal{H}$ of $\pi(g)\xi(s(g))$ such that $\eta(t(g) = \pi(g)\xi(s(g))$. Then, by continuity of π there exists a neighborhood $W_g \subset G_s \times_p \mathcal{H}$ of g such that $g' \in W_g$ implies $\pi(W_g) \subset U(\varepsilon, \eta, V)$. Now, define a subset of G

$$W_G := \{g' \in G \mid (g', \xi(s(g'))) \in W_g\}.$$

This set is open since it equals $s^{-1}\xi^{-1}p_2(W_g) \cap p_1(W_g)$. If $g' \in W_G$, then

$$\|\eta(t(g')) - \pi(g')\xi(s(g'))\| < \varepsilon.$$

Conversely, suppose $(\pi, \mathcal{H}, \Delta)$ is strongly continuous and unitary. Suppose $(g, h) \in G_s \times_p \mathcal{H}$. Let $U(\varepsilon, \eta, V)$ be an open neighborhood of $\pi(g)h$ with $\eta(t(g)) = \pi(g)h$ as usual. Let ξ be any section in Δ such that $\xi(s(g)) = h$. Then by strong continuity there exists an open set $V_g \subset G$ such that $g' \in V_g$ implies $\|\eta(t(g')) - \pi(g')\xi(s(g'))\| < \varepsilon$. Define the set

$$W_{g,h} := \{(g', h') \in G_s \times_p \mathcal{H} \mid \|h' - \xi(s(g'))\| < \varepsilon, g' \in V_g\}.$$

It is easily seen to be open and $(g', h') \in W_{g,h}$ implies

$$\begin{aligned} \|\eta(t(g')) - \pi(g')h'\| &\leq \|\eta(t(g')) - \pi(g')\xi(s(g'))\| + \|\pi(g')\xi(s(g')) - \pi(g')h'\| \\ &< \varepsilon + \|\pi(g')\|\|\xi(s(g')) - h'\| < 2\varepsilon, \end{aligned}$$

which finishes the proof. □

Definition 3.8. A **morphism of continuous (unitary) representations**

$$(\mathcal{H}^1, \Delta^1, \pi_1) \to (\mathcal{H}^2, \Delta^2, \pi_2)$$

of a groupoid is a morphism $\Psi : (\mathcal{H}^1, \Delta^1) \to (\mathcal{H}^2, \Delta^2)$ of continuous fields of Hilbert spaces (cf. Definition 2.8) that intertwines the groupoid representations

$$\begin{array}{ccc} \mathcal{H}^1_{s(g)} & \xrightarrow{\pi_1(g)} & \mathcal{H}^1_{t(g)} \\ \downarrow{\scriptstyle \Psi_{s(g)}} & & \downarrow{\scriptstyle \Psi_{t(g)}} \\ \mathcal{H}^2_{s(g)} & \xrightarrow[\pi_2(g)]{} & \mathcal{H}^2_{t(g)}. \end{array}$$

Example 3.9. The trivial representation of a groupoid $G \rightrightarrows M$ is given by the continuous field $(\mathcal{H}, \Delta)$ that has fiber $\mathbb{C}$ over each $m \in M$ and a map $\pi : G \to U(M \times \mathbb{C}) \cong M \times U(\mathbb{C}) \times M$,

$$g \mapsto (t(g), 1, s(g)).$$

We give another example of a continuous unitary representation of a groupoid. For any continuous function $f : G \to \mathbb{R}$ we can construct the representation

$$\pi_f : g \mapsto (t(g), e^{2\pi i(f(t(g)) - f(s(g)))}, s(g)).$$

These representation are all isomorphic. Indeed, for $f, g : G \to \mathbb{R}$,

$$m \mapsto e^{2\pi i(f(m)-g(m))}$$

is an isomorphism $(\mathcal{H}, \Delta, \pi_g) \to (\mathcal{H}, \Delta, \pi_f)$. In particular all these representations are isomorphic to π_0, which is the trivial representation.

Proposition 3.10. *If $G \rightrightarrows M$ is a proper groupoid endowed with a Haar system $\{\lambda_m\}_{m\in M}$ (cf. [16]), then any continuous representation $(\mathcal{H}, \Delta, \pi)$ is isomorphic to a unitary representation.*

Proof. Suppose $(\mathcal{H}, \Delta, \pi)$ is a non-zero continuous representation of G. Let $c : M \to \mathbb{R}_{>0}$ be a cutoff function (cf. [18])). This exists, since $G \rightrightarrows M$ is proper. Define an inner product $\langle ., . \rangle^{new}$ on $\mathcal{H}$ by the following description: for all $m \in M$ and $h, h' \in \mathcal{H}_m$,

$$\langle h, h' \rangle^{new} (m) := \int_{G_m} \langle \pi(g)h, \pi(g)h' \rangle \, c(t(g))\lambda_m(dg).$$

This inner product is G-invariant, since the Haar system and t are right invariant. It gives rise to a new topology on $\mathcal{H}$. The isomorphism is the identity on $\mathcal{H}$, which is easily seen to be continuous. Indeed, let $h \in \mathcal{H}$ and let $U(\varepsilon, \xi, V) \ni h$ be an open set in $\mathcal{H}$ with respect to the old norm. Then there exists a an open set V' such that $V' \subset V$, $p(h) \in V'$ and $g \mapsto \|\pi(g)\|$ is bounded on $t^{-1}V' \cap \mathrm{supp}(c \circ t)$. Since $c \circ t$ has compact support on each s-fiber, the function

$$m' \mapsto \int_{g\in G_{m'}} \|\pi(g)\| c(t(g))\lambda_{m'}(dg)$$

is bounded on V'. Hence we can set

$$\delta := \frac{\varepsilon}{\sup_{m\in V'} \int_{g\in G_{m'}} \|\pi(g)\| c(t(g))\lambda_{m'}(dg)}.$$

Then $h' \in U(\delta, \xi, V')$ (in the old topology) implies

$$\begin{aligned}
\|h' - \xi(m')\|^{new}_{m'} &= \int_{G_{m'}} \|\pi(g)(h' - \xi(m'))\| c(t(g))\lambda_{m'}(dg) \\
&\leq \int_{G_{m'}} \|\pi(g)\| c(t(g))\lambda_{m'}(dg) \|h' - \xi(m')\| \\
&\leq \varepsilon,
\end{aligned}$$

which proves the continuity of the identity map.

The proof that the inverse (also the identity) is continuous proceeds similarly. One uses that

$$\begin{aligned}
\|h' - \xi(m')\| &= \int_{G_{m'}} \|h' - \xi(m')\| c(t(g)))\lambda_{m'}(dg) \\
&= \int_{G_{m'}} \|\pi(g^{-1})\pi(g)(h' - \xi(m'))\| c(t(g))\lambda_{m'}(dg) \\
&= \sup_{g\in G_{m'}} \|\pi(g)\| \int_{G_{m'}} \|\pi(g)(h' - \xi(m'))\| c(t(g))\lambda_{m'}(dg)
\end{aligned}$$

and local boundedness of $g \mapsto \|\pi(g)\|$. This finishes the proof. □

A representation $(\mathcal{H}, \Delta, \pi)$ is **locally trivial** if the continuous field $(\mathcal{H}, \Delta)$ is locally trivial. In [19] locally trivial representations of a groupoid $G \rightrightarrows M$ are called G-vector bundles. Representations of transitive groupoids are locally trivial.

3.2 Continuity of representations in the operator norm

In this section we go through quite some effort to define a suitable topology on the set of bounded linear operators $\{P : \mathcal{H}_m \to \mathcal{H}_n\}_{n,m \in M}$ for a continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m\in M}, \Delta_{\mathcal{H}})$. This is done to be able to consider representations which are continuous in the operator topology. At first reading one could consider skipping the proofs.

Let $(\{\mathcal{H}_m\}_{m\in M}, \Delta_{\mathcal{H}})$ be a continuous field of Hilbert spaces over M. Consider the continuous field of Banach spaces over $M \times M$ whose fiber at (n, m) is given by the bounded linear operators $\mathcal{H}_m \to \mathcal{H}_n$, i.e. $\mathcal{B}(\mathcal{H}, \mathcal{H})_{(n,m)} := \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$. This is indeed a Banach space with the norm

$$\|P\| = \sup_{h\in\mathcal{H}_m, \|h\|_{\mathcal{H}_m}=1} \|P(h)\|_{\mathcal{H}_n}.$$

We define a space of sections $\Delta_{\mathcal{B}}$ of the field to consist of those maps $(n, m) \mapsto P(n, m)$ in $\prod_{(n,m)\in M\times M} \mathcal{B}(\mathcal{H}, \mathcal{H})$ such that

(i) for every $m \in M$ and $h \in \mathcal{H}_m$

$$n \mapsto P(n, m)h$$

is in $\Delta_{\mathcal{H}}$,

(ii) for every $n \in M$ and $\xi \in \Delta_{\mathcal{H}}$ the map

$$m \mapsto P(n, m)\xi(m)$$

is continuous $M \to \mathcal{H}_n$,

(iii) The map $(n, m) \mapsto \|P(n, m)\|$ locally bounded, and

(iv) P is adjointable, which means that there exists a $P^* : R \to \mathcal{B}(\mathcal{H}, \mathcal{H})$, satisfying (i), (ii) and (iii), such that for all $\xi, \eta \in \Delta_{\mathcal{H}}$ one has $(\eta, P\xi) = (P^*\eta, \xi)$, more concretely: for all $(n, m) \in R$

$$\langle \eta(n), P(n, m)\xi(m)\rangle_{\mathcal{H}_n} = \langle P^*(m, n)\eta(n), \xi(m)\rangle_{\mathcal{H}_m}.$$

Lemma 3.11. *The pair $(\{\mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)\}_{(n,m)\in M\times M}, \Delta_{\mathcal{B}})$ is a lower semi-continuous field of Banach spaces.*

Proof. First, we prove lower semi-continuity of the norm of a section $P \in \Delta_{\mathcal{B}}$. This follows from the fact that the map

$$(n, m, h) \mapsto \|P(n, m)h\|_{\mathcal{H}_n}$$

is a continuous map $M \times M \times_p \mathcal{H} \to \mathbb{R}$, analogously to the proof of Lemma 3.4. This last statement is proven as follows. Let $\varepsilon > 0$ be given. Suppose $(n, m, h) \in M \times M \times_p \mathcal{H}$. There exists a $\xi \in \Delta_{\mathcal{H}}$ such that $\xi(m) = h$. Then by condition (i), (ii), (iii) and continuity of $\|\xi\|$, there exists a neighborhood $W \in M \times M \times_p \mathcal{H}$ such that for any $(n', m', h') \in W$ the map $\|P\|$ is bounded on W and we have

$$\begin{aligned}
&|\|P(n', m')h'\| - \|P(n, m)h\|| \\
&\quad \leq |\, \|P(n', m')h'\| - \|P(n, m')h'\|\,| + |\, \|P(n, m')h'\| - \|P(n, m')\xi(m')\|\,| \\
&\quad\quad + |\, \|P(n, m')\xi(m')\| - \|P(n, m)\xi(m)\|\,| \\
&\quad \leq \varepsilon + \|P(n, m')\|\varepsilon + \varepsilon.
\end{aligned}$$

Next, we prove that for every $P \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ and every $\varepsilon > 0$ there exist a $Q \in \Delta_{\mathcal{B}}$ such that $\|Q(n, m) - P\| < \varepsilon$. Suppose $P \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ and let $\varepsilon > 0$ be given. Let $\xi_1, \ldots, \xi_k \in \Delta_{\mathcal{H}}$ be such that for any $h \in \mathcal{H}_m$

$$\|h - \sum_{i=1}^{k} \langle \xi_i(m), h \rangle \, \xi_i(m)\| < \varepsilon.$$

Let $\eta_1, \ldots, \eta_l \in \Delta_{\mathcal{H}}$ be such that for any $h \in \mathcal{H}_n$

$$\|h - \sum_{i=1}^{l} \langle \eta_i(n), h \rangle \, \xi_i(n)\| < \varepsilon.$$

Define, for $(n', m') \in M \times M$,

$$Q(n', m')h := \sum_{i=1}^{k} \sum_{j=1}^{l} \langle \xi_i(m'), h \rangle \, \langle \eta_j(n), P\xi_i(m) \rangle \, \eta_j(n')$$

One easily checks that $Q \in \Delta_{\mathcal{B}}$. Furthermore,

$$\begin{aligned}
&\|Ph - Q(n, m)h\| \\
&\quad \leq \|Ph - \textstyle\sum_{i=1}^{k} \langle \xi_i(m), h \rangle \, P\xi_i(m)\| + \| \textstyle\sum_{i=1}^{k} \langle \xi_i(m), h \rangle \, P\xi_i(m) \\
&\quad\quad - \textstyle\sum_{i=1}^{k} \sum_{j=1}^{l} \langle \xi_i(m'), h \rangle \, \langle \eta_j(n), P\xi_i(m) \rangle \, \eta_j(n')\| \\
&\quad < \|P\|\varepsilon + \varepsilon.
\end{aligned}$$

The last step is to show that $\Delta_{\mathcal{B}}$ is locally uniformly closed. Suppose

$$Q \in \prod_{(n,m) in M \times M} \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m).$$

Suppose that for all $\varepsilon > 0$ and all $(n, m) \in M \times M$ there is a $Q' \in \Delta_{\mathcal{B}}$ such that

$$\|Q(n', m') - Q'(n', m')\| < \varepsilon$$

on a neighborhood V of (n, m). We shall now show that this implies $Q \in \Delta_{\mathcal{B}}$. Indeed, let $\varepsilon > 0$ be given and suppose $n \in M$. Then there exist Q' and V as above. Define $U := p_1(V)$. Then $n' \in U$ implies, for any $h \in \mathcal{H}_m$, that

$$\|Q(n', m)h - Q'(n', m)h\| \leq \|Q(n', m) - Q'(n', m)\|\|h\| < \varepsilon\|h\|.$$

Hence $n \mapsto \|Q(n, m)h\|$ is continuous. In a similar way one proves condition (ii) for Q which finishes the proof. □

Actually, $\mathcal{B}(\mathcal{H}, \mathcal{H})$ is a so-called lower semi-continuous Fell bundle over $M \times M \rightrightarrows M$ and therefore a (full) lower semi-continuous C^*-category over M, but we shall not discuss this here (cf. [3]). The collection of sets

$$\{U(\varepsilon, \xi, V) \mid \xi \in \Delta_{\mathcal{B}}, \varepsilon > 0, V \subset M \times M \text{ open}\},$$

as defined in Lemma 2.5 for a continuous field of Banach spaces, is in general a subbasis for the topology on $\coprod_{(n,m)\in M\times M} \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$, instead of a basis. Since the field is not continuous in general, we do not have $\Delta = \Gamma_0(M \times M, \mathcal{B}(\mathcal{H}, \mathcal{H}))$. Consider the restriction of the total space $\mathcal{B}(\mathcal{H}, \mathcal{H})$ to the unitary operators, i.e.

$$U^{op}(\mathcal{H}) := \coprod_{(n,m)\in M\times M} U(\mathcal{H}_m, \mathcal{H}_n),$$

endowed with the subspace topology.

Lemma 3.12. *The total space $U^{op}(\mathcal{H})$ is a continuous groupoid over M.*

Proof. We show that the composition $\mathcal{B}(\mathcal{H}, \mathcal{H})^{(2)} \to \mathcal{B}(\mathcal{H}, \mathcal{H})$ is a continuous map. First note that for every $(P, Q) \in \mathcal{B}(\mathcal{H}, \mathcal{H})^{(2)}$ the inequality $\|PQh\| \leq \|P\|\|Qh\|$ implies

$$\|PQ\| \leq \|P\|\|Q\|.$$

Suppose that $m, n, p \in M$, $P_2 \in \mathcal{B}(\mathcal{H}_p, \mathcal{H}_n)$, $P_1 \in \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$ and $U(\varepsilon, Q, V)$ is an open neighborhood of P_2P_1 such that $Q(p, m) = P_2P_1$. There are $Q_1, Q_2 \in \Delta_{\mathcal{B}}$ such that $Q_1(n, m) = P_1$ and $Q_2(p, n) = P_2$. Choose $\varepsilon_i > 0$ and an open subset $V_i \subset M$ such that $P_i' \in U(\varepsilon_i, Q_i, V_i)$ implies $\|P_i'\|\varepsilon_i < \varepsilon/3$ for $i = 1, 2$. Furthermore, note that by condition (i), for each $m' \in M$ and $h \in \mathcal{H}_{m'}$ the map $n' \mapsto Q_1(n', m')h$ is in $\Delta_{\mathcal{B}}$. Hence by condition (ii) the map for each $p', m' \in M$ the map $n' \mapsto Q_2(p', n')Q_1(n', m')$ is continuous. The map $(p', m') \mapsto Q_2(p', n')Q_1(n', m')$ is easily seen to be continuous too. Hence we can shrink V_1 and V_2 such that $(p', n', m') \in V_2 \times_M V_2$ implies

$$|\,\|Q_2(p', n)Q_1(n, m') - Q_2(p', n')Q_1(n', m')\|\,| < \varepsilon/3.$$

Define $Q \in \Delta_{\mathcal{B}}$ by $Q(p', m') := Q_1(p', n)Q_2(n, m')$ Suppose

$$(P_2', P_1') \in U(\varepsilon_2, Q_2, V_2)_s \times_t U(\varepsilon_1, Q_1, V_1),$$

then

$$\begin{aligned}
\|P_2'P_1' - Q(p',m')\| &= \|P_2'P_1' - Q_2(p',n)Q_1(n,m')\| \\
&\le \|P_2'P_1' - Q_2(p',n')P_1'\| + \|Q_2(p',n')P_1' - Q_2(p',n')Q_1(n',m')\| \\
&\quad + \|Q_2(p',n')Q_1(n',m') - Q_2(p',n)Q_1(n,m')\| \\
&< \|P_2' - Q_2(p',n')\|\,\|P_1'\| + \|Q_2(p',n')\|\,\|P_1' - Q_1(n',m')\| + \varepsilon/3 \\
&< \varepsilon_2\|P_1'\| + \|Q_2(p',n')\|\varepsilon_2 + \varepsilon/3 < \varepsilon.
\end{aligned}$$

Proving that the other structure maps are continuous is similar, but easier. □

Definition 3.13. A representation $(\pi, \mathcal{H}, \Delta)$ is **continuous in the operator norm** if the map

$$G \to \mathcal{B}(\mathcal{H},\mathcal{H}), g \mapsto \pi(g)$$

is continuous. If G is unitary, then the representation is continuous if

$$G \to U^{op}(\mathcal{H}), g \mapsto \pi(g)$$

is a continuous map of groupoids.

Lemma 3.14. *A representation is continuous if it is continuous in the operator norm. The converse implication is true if the representation Δ^π is finitely generated over $C_0(M)$ and unitary.*

Proof. Suppose $(g,h) \in G_s \times_p \mathcal{H}$ and let $n = t(g)$ and $m = s(g)$. Suppose $U(\varepsilon, V, \xi)$ is a neighborhood of $\pi(g)h$, with $\xi(n) = \pi(g)h$. Let $Q \in \Delta_{\mathcal{B}}$ be any section with $Q(n,m) = \pi(g)$, which exists since $(\mathcal{B}(\mathcal{H},\mathcal{H}), \Delta_{\mathcal{B}})$ is a lower semi-continuous field of Banach spaces. Let $\eta \in \Delta_{\mathcal{H}}$ be a section such that $\eta(m) = h$. By the conditions (i), (ii) and (iii) above there exists a neighborhood $S \subset M \times M$ of (n,m) such that for all $(n',m') \in S$

$$\|\xi(n') - Q(n',m)h\| < \varepsilon/4,$$

the function $\|Q\|$ is bounded on S and

$$\|Q(n',m)\eta(m) - Q(n',m')\eta(m')\| < \varepsilon/4.$$

Define

$$\begin{aligned}
\delta &:= \tfrac{\varepsilon}{4\sup_{(n',m)\in S}\|Q(n',m')\|}, \\
W' &:= U(\delta, \eta, p_2(S)), \\
K &:= \sup_{h'\in W'}\|h'\|,
\end{aligned}$$

and

$$W := \pi^{-1}(U(Q, \frac{\varepsilon}{4K}, S)),$$

where $p_2 : M \times M \to M$ is the projection on the second entry. We claim that $(g',h') \in W_s \times_p W'$ implies $\pi(g')h' \in U(\varepsilon, V, \xi)$. Indeed, suppose $(g',h') \in W_s \times_p W'$ and $m' =$

$s(g')$, $n' = t(g')$, then

$$\begin{aligned} \|\xi(n') - \pi(g')h'\| &\leq \|\xi(n') - Q(n',m)h\| + \|Q(n',m)\eta(m) - Q(n',m')\eta(m')\| \\ &+ \|Q(n',m')\eta(m') - Q(n',m')h'\| + \|Q(n',m')h' - \pi(g')h'\| \\ &< \varepsilon/4 + \varepsilon/4 + \|Q(n',m')\|\delta + \|h'\|\tfrac{\varepsilon}{4K} < \varepsilon. \end{aligned}$$

We shall now prove the converse implication. Suppose $(\mathcal{H}^\pi, \Delta^\pi, \pi)$ is a strongly continuous unitary representation on a continuous field of Hilbert spaces with Δ^π finitely generated. There exist a finite set $\{\xi_i\}_{i\in I}$ of sections in Δ^π such that for each $m' \in M$ the set $\{\xi_i(m')\}_{i\in I}$ contains a (normalized) basis for $\mathcal{H}_{m'}$. Suppose $U(\varepsilon, Q, V)$ is a neighborhood of $\pi(g)$, $s(g) = m$, $t(g) = n$ and $Q(n,m) = \pi(g)$. Note that by condition (i) $n' \mapsto Q(n',m)\xi_i(m)$ is in Δ^π, so by strong continuity of π there exists an open set $U_i \subset G$ such that $g' \in U_i$ implies

$$\|\pi(g')\xi'(s(g')) - Q(t(g'),m)\xi_i(m)\| < \varepsilon/(2|I|).$$

Moreover, by condition (ii) we can shrink U_i such that $g' \in U_i$ implies that

$$\|Q(t(g'),m)\xi_i(m) - Q(t(g'),s(g'))\xi_i(s(g')\| < \varepsilon/(2|I|).$$

Hence

$$\|\pi(g')\xi'(s(g')) - Q(t(g'),s(g'))\xi_i(s(g') < \varepsilon/|I|$$

for $g' \in U_i$. Define $U := \bigcap_{i\in I} U_i$, then $g' \in U$ implies

$$\begin{aligned} &\|\pi(g') - Q(t(g'),s(g'))\| \\ &\quad = \sup_{h'\in\mathcal{H}_{s(g')},\|h'\|=1} \|\pi(g')h' - Q(t(g'),s(g'))h'\|_{\mathcal{H}_{t(g')}} \\ &\quad < \textstyle\sum_{i\in I} \|\pi(g')\xi_i(s(g')) - Q(t(g'),s(g'))\xi_i(s(g')\|_{\mathcal{H}_{t(g')}} \\ &\quad < \textstyle\sum_{i\in I} \varepsilon/|I| = \varepsilon, \end{aligned}$$

which finishes the proof. □

From these comparison lemmas (Lemma 3.6, Lemma 3.7 and Lemma 3.14) we can conclude that for unitary representations any of these topologies are equivalent. Hence, from now on, we shall not specify which notion we mean, but only say that a unitary representation is continuous (if it is).

3.3 Example: the regular representations of a groupoid

The following example considers the regular representation. In a different form it was studied by Renault (cf. [16]), but he considered $L^2(G)$ as a measurable field of Hilbert spaces. We are interested in representations on continuous fields of Hilbert spaces. Therefore, the statement of Proposition 3.15 is actually new. It generalizes the analogous statement for groups.

Suppose a continuous groupoid $G \rightrightarrows M$ is endowed with a left Haar system.

Proposition 3.15. *The* ***left regular representation*** *of a continuous groupoid $G \rightrightarrows M$ on $(\hat{L_t^2}(G), \Delta_t^2(G))$ (cf. Example 2.11) defined by (continuous extension of)*

$$(\pi_L(g)f)(g') = f(g^{-1}g'),$$

for $g \in G$, $f \in C_c(G^{s(g)})$ and $g' \in G^{t(g)}$, is a continuous unitary representation.

Proof. Unitarity is immediate from the G-invariance of the Haar system.

We have to check that for all $\xi \in \Delta_t^2(G)$ the map $g \mapsto \pi_L(g)\xi(s(g))$ is continuous $G \to \hat{L_t^2}(G)$. Let $g \in G$. Suppose a neighborhood $U(\varepsilon, \eta, V) \subset \hat{L_t^2}(G)$ of $\pi_L(g)\xi(s(g))$ is given, where $\varepsilon > 0$, V an open set in M and $\eta \in \Delta_t(G)$ is a section satisfying $\pi_L(g)\xi(s(g)) = \eta(s(g))$. There exist $\xi', \eta' \in C_c(G)$ such that $\|\eta - \eta'\|_{\hat{L}^2} < \varepsilon/3$, $\|\xi - \xi'\|_{\hat{L}^2} < \varepsilon/3$ and $\pi_L(g)\xi'(s(g)) = \eta'(s(g))$. To continue we first need the following lemma:

Lemma 3.16. *(A. Connes [5]) If f is a compactly supported continuous function on $G^{(2)}$, then the map*

$$g \mapsto \int_{h \in G^{s(g)}} f(g,h)\lambda^{s(g)}(dh)$$

is continuous on G.

Now, apply this lemma to the map

$$f(g', h') := |\xi'((g')^{-1}h') - \eta'(h')|^2.$$

As a result,

$$F(g') := \sqrt{\int_{h' \in G^{t(g')}} |\xi'((g')^{-1}h') - \eta'(h')|^2 \lambda^{t(g')}(dh')}$$

depends continuously on g'. Note that $F(g) = 0$, so that we can choose a neighborhood $U \subset G$ of g such that $F(g') < \varepsilon/3$ whenever $g' \in U$. Finally, intersect U with $t^{-1}(V)$ to obtain the required open set in G whose image is a subset of $U(\varepsilon, \eta, V)$. ☐

In the same way one proves that the **right regular representation** of G on $(\hat{L_s^2}(G), \Delta_s^2(G))$ given by

$$\pi_L(g)h(g') := h(g'g)$$

(where $h \in C_c(G_{s(g)})$ and $g' \in G_{t(g)}$) is strongly continuous and unitary.

Let's consider two very simple examples. Other examples will occur at other places of the text.

Example 3.17. Note that the left (and right) regular representation of the groupoid $M \rightrightarrows M$ is $m \mapsto 1_m : M \to M \times U(1)$. The left regular representation of the pair groupoid $M \times M \rightrightarrows M$

$$\pi_L : M \times M \to U(\hat{L}^2(M \times M)) \cong M \times U(L^2(M, \nu)) \times M$$

is given by

$$(m, n) \mapsto (m, 1_{L^2(M,\nu)}, n),$$

for a given Radon measure ν on M.

3.4 Example: continuous families of groups

The following example can give the reader a feeling for the issues on the global topology with continuous groupoid representations. We express the set of finite-dimensional continuous representations of a family of groups on a given continuous field of Hilbert spaces in terms of continuous sections of the family $\mathrm{Rep}^{\mathcal{H}}(G) \to M$ of the sets of finite-dimensional continuous representations of each of the groups.

Suppose H is a locally compact group. Let $\mathrm{Rep}(H)$ denote the set of non-zero continuous unitary representations of H. This set can be endowed with a topology. Indeed, one uses the Jacobson topology on the primitive spectrum of the C^*-algebra $C^*(H)$. We shall not go into the details, since there is an easier description of the case that has our interest. For $n \geq 1$, denote by $\mathrm{Rep}^n(H)$ the subspace of continuous non-zero unitary representations on $\mathbb{C}^n$ with standard inner product $\langle z, z' \rangle = \bar{z}z'$. Let $\mathrm{Rep}^0(H)$ be the set consisting of just the zero representation. Let $S(\mathbb{C}^n)$ denote the unit sphere in $\mathbb{C}^n$.

Lemma 3.18. *([7], 18.1.9) For every integer $n \geq 0$ a subbasis for the topology on $\mathrm{Rep}^n(H)$ is given by the sets*

$$U(\pi, \varepsilon, K) := \{\pi' \in \mathrm{Rep}^n(H) \mid \max_{g \in K} |\langle h', \pi(g)\, h \rangle - \langle h', \pi'(g)\, h \rangle| < \varepsilon, \forall h, h' \in S(\mathbb{C}^n)\},$$

for compact sets $K \subset H$, representations $\pi \in \mathrm{Rep}^n(H)$ and $\varepsilon > 0$.

We need the following technical notion. Suppose $p : N \to M$ is a continuous map.

Definition 3.19. We say a set $K \subset N$ is **p-open-compact** if the restriction $K \cap p^{-1}(m)$ is compact for all $m \in M$ and the image $p(K) \subset M$ is open. We say that $p : N \to M$ is **locally open-compact** if every $n \in N$ has a p-open-compact neighborhood.

Example 3.20. If $p : N \to M$ is a fiber bundle with locally compact fiber, then it is easy to show that p is locally open-compact.

Suppose $s : G \to M$ is a continuous family of groups, i.e. a continuous groupoid with $s = t$. *Fix* a uniformly finite-dimensional continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$. We proceed in a few steps to define the surjection $\mathrm{Rep}^{\mathcal{H}}(G) \to M$ and endow it with a suitable topology.

(1) Choose for each $m \in M$ a group $H_m \cong s^{-1}(m)$ and an isomorphism $\psi_m : s^{-1}(m) \to H_m$, fixing the group structure at each fiber. Endow $\coprod_{m \in M} H_m$ with the topology such that

$$\coprod_{m \in M} \psi_m : G \to \coprod_{m \in M} H_m$$

is a homeomorphism. Denote the canonical projection $\coprod_{m \in M} H_m \to M$ by s'.

(2) Suppose

$$\left\{ \phi_i : \mathcal{H}|_{U_i} \hookrightarrow U_i \times \sum_{n=0}^{\dim(\mathcal{H}_{U_i})} \mathbb{C}^n \right\}_{i \in I}$$

is a local pseudo-trivialization of $(\mathcal{H},\Delta)$ (cf. Definition 2.16). Define for all $i,j\in I$ the homeomorphism

$$\gamma_{ij}:=\phi_j(\phi_i)^{-1}:\mathrm{im}(\phi_i)|_{U_i\cap U_j}\to\mathrm{im}(\phi_j)|_{U_i\cap U_j}.$$

(3) For any $i\in I$, define

$$\mathrm{Rep}^{\mathcal{H}}(G|_{U_i}):=\coprod_{m\in U_i}\mathrm{Rep}^{\dim(\mathcal{H}_m)}(H_m)$$

and denote the canonical projection by

$$p_i:\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})\to U_i.$$

(4) For each $i\in I$ the following sets form a subbasis of a topology on $\mathrm{Rep}^{\mathcal{H}}(G)|_{U_i}$: For any $\xi,\eta\in\Delta$, $V\subset\mathbb{C}$ open and $K\subset\coprod_{m\in U_i}H_m$ s'-open-compact,

$$U(\xi,\eta,K,V):=\left\{\pi\in\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})\mid\langle\xi,\pi\eta\rangle\left(K\cap H_{p_i(\pi)}\right)\subset V\right\}.$$

(5) Define

$$\mathrm{Rep}^{\mathcal{H}}(G):=(\coprod_{i\in I}\mathrm{Rep}^{\mathcal{H}}(G|_{U_i}))/\sim,$$

where $\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})|_{U_i\cap U_j}\ni\pi_i\sim\pi_j\in\mathrm{Rep}^{\mathcal{H}}(G|_{U_j})|_{U_i\cap U_j}$ iff $\pi_j=\gamma_{ij}\pi_i\gamma_{ij}^{-1}$.

The space $\mathrm{Rep}^{\mathcal{H}}(G)$ is uniquely determined up to homeomorphism by the chosen local pseudo-trivialization of $(\mathcal{H},\Delta)$ and isomorphisms $\{\psi_m:s^{-1}(m)\to H_m\}_{m\in M}$. One can see that $s:G\to M$ being locally open-compact implies that the topology of $\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})$ restricted to each fiber is equivalent to the topology of Lemma 3.18.

Proposition 3.21. *Suppose that $s:G\to M$ is locally open-compact family of groups. Then there is a one-to-one correspondence between continuous representations of $s:G\to M$ on $(\mathcal{H},\Delta)$ and continuous sections of $\mathrm{Rep}^{\mathcal{H}}(G)\to M$.*

Proof. A continuous unitary representation π of G on $(\mathcal{H},\Delta)$ corresponds to a continuous section of $\mathrm{Rep}^{\mathcal{H}}(G)$, i.e. to a family of sections $\tilde{\pi}_i:U_i\to\mathrm{Rep}^{\mathcal{H}}(G)|_{U_i}$ given by

$$\tilde{\pi}_i(m)=\phi_i\circ\pi\circ(\psi_m^{-1}\times\phi_i^{-1}).$$

These are easily seen to be compatible, i.e. $\tilde{\pi}_j=\gamma_{ij}\tilde{\pi}_i\gamma_{ij}^{-1}$. It remains to show that each $\tilde{\pi}_i$ is continuous. Consider an open set $U(\xi,\eta,K,V)$ as above. Note that

$$\begin{aligned}\tilde{\pi}^{-1}(U(\xi,\eta,K,V)) &= \{m\in U_i\mid\langle\xi,\pi\eta\rangle|_{K\cap H_m}\subset V\}\\ &= s'(K\cap\{g\in\coprod_{m\in U_i}H_m\mid\langle\xi,\pi\eta\rangle(g)\subset V\}),\end{aligned}$$

which is open since K is s'-compact and π is continuous.

A continuous section $\tilde{\pi}$ of $\mathrm{Rep}^{\mathcal{H}}(G)$ determines a continuous unitary representation by

$$\pi(g) := \phi_i^{-1} \circ \tilde{\pi}_i \circ (\psi_{s(g)}(g) \times \phi_i) \in U(\mathcal{H}_{s(g)}),$$

where $i \in I$ such that $s(g) \in U_i$. We only need to show that $\pi|_{G_{U_i}} \psi^{-1}$ is continuous. Suppose $\xi, \eta \in \Gamma_0(\mathrm{im}(\phi_i))$. Given $g \in \coprod_{m \in U_i} H_m$ and $V \subset \mathbb{C}$, let K be an s'-compact neighborhood of g and $W \subset K$ an open neighborhood of g. Consider $U(\xi, \eta, K, V)$. Define $W' := W \cap s^{-1}\tilde{\pi}^{-1}(U(\xi, \eta, K, U))$, which is open since s and $\tilde{\pi}$ are continuous. Then $g' \in W'$ implies

$$\langle \xi(s(g')), \pi(g')\eta(s(g'))\rangle = \langle \xi(s(g')), \tilde{\pi}(s(g'))(g')\eta(s(g'))\rangle \in U.$$

Moreover π is locally bounded, since $\mathcal{H}$ is uniformly finite-dimensional. This finishes the proof. □

Example 3.22. Consider a locally compact group H and a continuous principal H-bundle $\tau : P \to M$. From this we can construct a continuous bundle of groups $P \times_H H \to M$, where the action of H on H is given by conjugation. Consider a local trivialization $\{\chi_i : P|_{U_i} \to U_i \times H\}_{i \in I}$ of $P \to M$. Suppose $I = \mathbb{N}$. One can fix the group structure at each fiber of $P \times_H H \to M$ as follows: for every $m \in M$ choose the smallest $i \in I$ such that $m \in U_i$ and define

$$\psi_m : (P \times_H H)_m \to H, [p, h] \mapsto \chi_i(p)\, h\, \chi_i(p)^{-1}.$$

Given a representation $(\pi, \mathbb{C}^n) \in \mathrm{Rep}^n(H)$, one can construct a vector bundle $\mathcal{H} := P \times_\pi \mathbb{C}^n \to M$. Obviously, the trivialization of $P \to M$ gives rise to a trivialization $\{\phi_i : \mathcal{H}|_{U_i} \to U_i \times \mathbb{C}^n\}_{i \in I}$ of $\mathcal{H} \to M$, by $\phi_i([p, z]) = (\tau(p), \pi(\chi_i(p))z)$. Using these data one can form the bundle $\mathrm{Rep}^{\mathcal{H}}(P \times_H H) \to M$ and a topology on it. A continuous section of this bundle is given by

$$\tilde{\pi}_i(m) = (h \mapsto \pi(\gamma_{ij}^{-1} h \gamma_{ij}),$$

for all $i \in \mathbb{N}$, $m \in U_i$, $h \in H$ and the smallest $j \in \mathbb{N}$ such that $m \in U_j$. This section corresponds to the representation of $P \times_H H \to M$ on $(\mathcal{H}, \Delta)$ given by $\tilde{\pi}([p, h])[p, z] = [p, \pi(h)z]$.

Remark 3.23. One can "twist" $\mathcal{H} := P \times_\pi \mathbb{C}^n$ by another continuous field $(\mathcal{H}', \Delta')$, carrying the trivial representation of $P \times_H H \to M$, to obtain a representation on $\mathcal{H} \otimes \mathcal{H}'$. In general, the representation ring of G is a module over the representation ring of M, cf. [3].

3.5 Representations of the global bisections group

For the reader who prefers representation theory of groups and wonders why one should be interested in representations of groupoids at all, the next section will be of particular interest. Namely, to any continuous groupoid is associated a topological group: the group of global bisections. For a large class of continuous groupoids (the ones we call locally bisectional) we establish a bijection between the continuous representations of the groupoid on continuous fields of Hilbert spaces and a specific type of continuous representations of the group of global bisections on Banach spaces. Hence the representation theory of such groupoids can be “embedded” in the representation theory of groups. From this point of view, the groupoid offers a way to study some representations of these groups of bisections.

Suppose $G \rightrightarrows M$ is a continuous groupoid. A continuous **global bisection** is a continuous map $\sigma : M \to G$ such that $t \circ \sigma = id_M$ and $\tilde{\sigma} := s \circ \sigma : M \to M$ is a homeomorphism. Denote the set of global bisections of $G \rightrightarrows M$ by $\mathrm{Bis}(G)$. This set has a group structure, cf. [20]. Moreover, it is even a topological group.

Lemma 3.24. $\mathrm{Bis}(G)$ *has the structure of a topological group in the compact-open topology.*

Proof. The multiplication is given by

$$(\sigma_1 \cdot \sigma_2)(m) := \sigma_1(m)\sigma_2(\tilde{\sigma_1}(m)).$$

The unit is given by the unit section $u : M \to G$ and the inverse is defined by

$$\sigma^{-1}(m) := (\sigma(\tilde{\sigma}^{-1}(m)))^{-1}.$$

The group laws are easily checked, for example

$$\begin{aligned}(\sigma \cdot \sigma^{-1})(m) &= \sigma(m)\sigma^{-1}(\tilde{\sigma}(m)) \\ &= \sigma(m)(\sigma(\tilde{\sigma}^{-1}\tilde{\sigma}(m)))^{-1} \\ &= 1_m.\end{aligned}$$

We prove that multiplication is continuous $\mathrm{Bis}(G)\times\mathrm{Bis}(G) \to \mathrm{Bis}(G)$. Suppose $\sigma_1\cdot\sigma_2 \in U(C,V)$, where C is a compact set in M, V open in G and $U(C,V)$ the set of maps $\tau : M \to G$ that satisfy $\tau(C) \subset V$, i.e. $U(C,V)$ is in the standard subbasis of the topology on $\mathrm{Bis}(G)$. For each $m \in C$, let V_m be a neighborhood of $(\sigma_1 \cdot \sigma_2)(m) = \sigma_1(m)\sigma_2(\tilde{\sigma_1}(m))$. These V_m cover $\sigma_1 \cdot \sigma_2(C)$ which is compact by continuity of the multiplication in G and σ_1, σ_2. Let $\{V_i\}_{i\in I}$ be a finite subcover. The inverse image $m^{-1}(V_i)$ is open and contains a Cartesian product $W_i^1 \times W_i^2$ of open sets W_i^1, W_i^2 for each $i \in I$. Then $\sigma_1' \in U(C, \bigcup_{i\in I} W_i^1)$ and $\sigma_2' \in U(\tilde{\sigma_1}(C), \bigcup_{i\in I} W_i^2)$ implies $\sigma_1' \cdot \sigma_2' \in U(C,V)$. □

Example 3.25. The global bisection group of the pair groupoid $M \times M$ is the group of homeomorphisms of M.

Example 3.26. For the trivial group bundle $G \times M \to M$ (with fiber G) the group of global bisections is just the group of sections with the pointwise multiplication. In particular, if M is the circle $\mathbb{S}^1$ and G a Lie group then the group of global bisections is the loop group $C(\mathbb{S}^1, G)$ with its usual topology (cf. [14]).

Lemma 3.27. *A continuous unitary representation* $(\pi, \mathcal{H}, \Delta)$ *of a groupoid* $G \rightrightarrows M$ *canonically induces a continuous isometric representation of* $\mathrm{Bis}(G)$ *on* Δ.

Proof. Define the representation $\tilde{\pi}$ of $\mathrm{Bis}(G)$ by

$$(\tilde{\pi}(\sigma)\xi)(m) := \pi(\sigma(m))\xi(\tilde{\sigma}(m)),$$

where $\xi \in \Delta$, $m \in M$ and $\sigma \in \mathrm{Bis}(G)$. This representation is isometric, since π is unitary:

$$\|\tilde{\pi}(\sigma)\xi\| = \sup_{m\in M} \|\pi(\sigma(m))\xi(\tilde{\sigma}(m))\|_{\mathcal{H}_m} = \|\xi\|.$$

Continuity is proven as follows. Suppose $\varepsilon > 0$ and $\xi \in \Delta$ are given. There exists a compactly supported section $\xi' \in \Delta_c := C_c(M)\Delta$ such that $\|\xi - \xi'\| < \varepsilon/6$. Denote the support of ξ' by K. Moreover, since π is continuous and unitary it is norm continuous and hence there exists an open set $V \subset G$ such that $g, g' \in V$ implies $\|\pi(g)\xi'(s(g)) - \pi(g')\xi'(s(g'))\| < \varepsilon/3$. Now, suppose that $\sigma, \sigma' \in U(K, V)$ and $\eta \in B(\xi, \varepsilon/6)$, then

$$\sup_{m\in M} \|\pi(\sigma(m))\eta(\tilde{\sigma}(m)) - \pi(\sigma'(m))\eta(\tilde{\sigma}'(m))\| < \varepsilon,$$

which finishes the proof. □

The obtained representation of $\mathrm{Bis}(G)$ is actually $C_0(M)$**-unitary** (or C^***-unitary** with respect to $C_0(M)$), in the sense that

$$\langle \tilde{\pi}(\sigma)\xi, \tilde{\pi}(\sigma)\eta\rangle = \langle \xi, \eta\rangle$$

for all $\sigma \in \mathrm{Bis}(G)$ and $\xi, \eta \in \Delta$.

For the following result we need a technical condition on groupoids. We call a continuous groupoid $G \rightrightarrows M$ **bisectional** if

(i) every $g \in G$ is in the image of a continuous global bisection;

(ii) for all compact sets $K \subset M$ and open sets $V \subset G$, the set $\bigcup_{\sigma\in U(K,V)} \mathrm{im}(\sigma) \subset G$ is open.

Item (i) was recently addressed for Lie groupoids in [4]. The author believes the second condition can also be proved for Lie groupoids.

Theorem 3.28. *Suppose $G \rightrightarrows M$ is bisectional. Then there is a bijective correspondence between continuous unitary representations of G and continuous $C_0(M)$-unitary representations of* $\mathrm{Bis}(G)$ *on a Hilbert $C_0(M)$-module satisfying*

(i) $C_0(M)$-linearity, i.e.

$$\tilde{\pi}(\sigma)(f\,\xi) = \tilde{\sigma}^*(f)\,\tilde{\pi}(\sigma)(\xi)$$

for all $\sigma \in \mathrm{Bis}(G)$, $\xi \in \Delta$ and $f \in C_0(M)$ and

(ii) locality, i.e. if $\sigma(m) = 1_m$ for some $m \in M$, then $\|\tilde{\pi}(\sigma)\xi - \xi\|(m) = 0$

Proof. Given a representation $(\tilde{\pi}, \Delta)$ of $\mathrm{Bis}(G)$ as above, define a representation $\pi : G \to U(\mathcal{H})$ as follows. Form the continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m\in M}, \Delta)$ associated to Δ (cf. Theorem 2.10). For any $g \in G$ and $h \in \mathcal{H}_{s(g)}$, define

$$\pi(g)h := (\tilde{\pi}(\sigma)\xi)(t(g)),$$

for any $\xi \in \Delta$ such that $\xi(s(g)) = h$ and $\sigma \in \mathrm{Bis}(G)$ such that $\sigma(t(g)) = g$, which exist by assumption. We now show that this definition does not depend on the choice of σ and ξ. Suppose ξ, ξ' satisfy $\xi(m) = h = \xi'(m)$. Let $\{U_i\}_{i\in\mathbb{N}}$ be a family of sets such that $\bigcap_{i\in\mathbb{N}} U_i = \{s(g)\}$ and $\{\chi_i : U_i \to [0,1]\}$ a family of functions such that $\chi_i(s(g)) = 0$ and $\chi_i(n) = 1$ for all $n \in M\backslash U_i$. Then

$$\begin{aligned}(\tilde{\pi}(\sigma)\xi)(t(g)) - (\tilde{\pi}(\sigma)\xi')(t(g)) &= \lim_{i\to\infty}(\tilde{\pi}(\sigma)\chi_i(\xi - \xi'))(t(g)) \\ &= \lim_{i\to\infty}\chi_i(\tilde{\sigma}(t(g)))(\tilde{\pi}(\sigma)(\xi - \xi'))(t(g)) \\ &= 0,\end{aligned}$$

since $\tilde{\pi}$ is $C_0(M)$-linear and $\tilde{\sigma}(t(g)) = s(g)$.

Suppose $\sigma(m) = \sigma'(m)$ for $\sigma, \sigma' \in \mathrm{Bis}(G)$ and $m \in M$. Then, by locality, for all $\xi \in \Delta$

$$\|\tilde{\pi}(\sigma^{-1}\,\sigma')\xi - \xi\|(m) = 0,$$

and hence $(\tilde{\pi}(\sigma)\xi)(m) = (\tilde{\pi}(\sigma')\xi)(m)$.

Unitarity of π follows at once from $C_0(M)$-unitarity of $\tilde{\pi}$.

Next, we prove continuity of π. Suppose $(g, h) \in G\,{}_s{\times}_p\mathcal{H}$ and $U(\varepsilon, \eta, V)$ open neighborhood of $\pi(g)h$, where $\eta(t(g)) = \pi(g)h$. We need to construct an open neighborhood of (g, h), which maps to $U(\varepsilon, \eta, V)$. Consider

$$B(\eta, \varepsilon) := \{\xi \in \Delta \mid \|\eta - \xi\| < \varepsilon\}.$$

Let $\sigma \in \mathrm{Bis}(G)$ be such that $\sigma(t(g)) = g$, which exists since G is bisectional. Define $\xi := \bar{\pi}(\sigma)^{-1}\eta$. By continuity of $\bar{\pi}$ there exists an open neighborhood $B(\xi, \delta)$ of ξ and an open neighborhood $U(K, W)$ of σ such that $\bar{\pi}(U(K, W) \times B(\xi, \delta)) \subset B(\eta, \varepsilon)$. Since $G \rightrightarrows M$ is bisectional, there exists an open neighborhood W' of g in $\bigcup_{\sigma\in U(K,W)} \mathrm{im}(\sigma)$.

Suppose that $(g', h') \in W'\,{}_s{\times}_p\, U(\xi, \delta, \tilde{\sigma}^{-1}(V))$, then

$$\pi(g')h' = (\bar{\pi}(\sigma')\xi')(t(g')) \in U(\varepsilon, \eta, V),$$

for some $\sigma' \in U(K, W)$ and $\xi' \in B(\xi, \delta)$.

One easily sees that the constructions given in this proof to obtain representations of G from representations of $\mathrm{Bis}(G)$ and vice versa in the proof of the above lemma are inverses of each other. □

Remark 3.29. As an intermediate step, one can also relate the representations of $G \rightrightarrows M$ to the representation of the inverse semi-group of continuous **local bisections** of $G \rightrightarrows M$. These are continuous maps $\sigma : U \to G$ for open $U \subset M$ such that $t\circ\sigma = id_U$ and $\tilde{\sigma} := s \circ \sigma : U \to U$ is a homeomorphism.

4 Groupoid representation theory

Is there a Schur's Lemma for groupoids? Is there a Peter-Weyl theorem for groupoids? In this chapter we give answers to these questions. We discuss a way to generalize these statements, that are well-known for groups, to groupoids. It turns out that you need extra conditions on the groupoid for the statements to be true (unlike what is suggested in [1]). A crucial rôle is played by the functors that restrict representations of a groupoid to representations of its isotropy groups. This chapter shows that representation theory of groupoids is quite different from representation theory for groups, but many results can be carried over using some caution.

4.1 Decomposability and reducibility

Definition 4.1. (i) The **direct sum of a countable family of continuous fields of Hilbert spaces** $\{(\mathcal{H}^i, \Delta_i\}_{i\in I}$ is the continuous field of Hilbert spaces $(\mathcal{H}^\oplus, \Delta^\oplus)$ whose fibers are given by $\mathcal{H}^\oplus_m := \bigoplus_{i\in I} \mathcal{H}^i_m$ and whose space of continuous sections $\Delta^\oplus$ is the closure of the pre-Hilbert $C_0(M)$-module of finite sums of sections $\sum_{j\in J} \xi_j$, where $J \subset I$ is a finite index set and $\xi_j \in \Delta_j$ for all $j \in J$.

(ii) The **direct sum of a countable family of continuous representations** $\{(\mathcal{H}^i, \Delta_i, \pi_i)\}_{i\in I}$ of a groupoid $G \rightrightarrows M$ is the representation of $G \rightrightarrows M$ on the direct sum of continuous fields of Hilbert spaces $(\mathcal{H}^\oplus, \Delta^\oplus)$, given by the continuous extension of the map $\bigoplus_{j\in J} \pi_j : g \mapsto \sum_{j\in J} \pi_j(g)$ on finite sums by continuity.

(iii) We say that that a continuous unitary representation $(\mathcal{H}, \pi)$ of a groupoid G is **decomposable** if it is equivariantly isomorphic to a direct sum of representations of $(\mathcal{H}^1, \pi_1)$ and $(\mathcal{H}^2, \pi_2)$

$$(\mathcal{H}, \Delta) \cong \mathcal{H}^1 \oplus \mathcal{H}^2.$$

and **indecomposable** if this is not possible.

(iv) A **continuous subfield** of a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$ is a continuous field of Hilbert spaces $(\mathcal{H}', \Delta')$, such that $\mathcal{H}'_m \subset \mathcal{H}_m$ is a closed linear subspace with the induced inner product for all $m \in M$ and $\Delta' \subset \Delta$ a Hilbert $C_0(M)$-submodule.

(v) A **continuous subrepresentation** of a continuous unitary representation $(\mathcal{H}, \pi)$ of a groupoid G is a continuous subfield of $(\mathcal{H}, \Delta)$ stable under π.

(vi) A continuous unitary representation is **reducible** if it has a proper continuous subrepresentation. It is **irreducible** if it is not reducible.

Proposition 4.2. *If $(\mathcal{H}, \Delta, \pi)$ is a continuous locally trivial unitary representation and $(\mathcal{H}', \Delta', \pi')$ a locally trivial subrepresentation $(\mathcal{H}, \Delta, \pi)$, then $(\mathcal{H}, \Delta, \pi)$ decomposes as a direct sum of $(\mathcal{H}', \Delta', \pi')$ and another locally trivial subrepresentation.*

Proof. For each $m \in M$ let $\mathcal{H}''_m$ be the orthogonal complement with respect to the inner product. The family $\{\mathcal{H}''_m\}_{m\in M}$ forms a continuous field, with

$$\Delta'' := \{\xi \in \Delta \mid \xi(m) \in \mathcal{H}''_m \text{ for all } m \in M\},$$

since $\mathcal{H}$ is locally trivial. Moreover, $(\mathcal{H}'', \Delta'')$ is locally trivial too. Since π is unitary, this complement is G-invariant. □

Decomposability implies reducibility (irreducible implies indecomposable), but not vice versa. Indeed, a representation can contain a subrepresentation without being decomposable.

Example 4.3. Consider the trivial representation of $\mathbb{R} \rightrightarrows \mathbb{R}$ on $(\mathbb{R} \times \mathbb{C}, C_0(\mathbb{R}))$. It has a subrepresentation given by the continuous field of Hilbert spaces which is 0 at 0 and $\mathbb{C}$ elsewhere, with space of sections

$$C_0^0(M) := \{f \in C_0(M) \mid f(0) = 0\}.$$

This subrepresentation has no complement, since this would be a field that is $\mathbb{C}$ at 0 and zero elsewhere, whose only continuous section could be the zero section. Hence it would not satisfy condition (i) of Definition 2.1. Note that $\mathbb{R} \rightrightarrows \mathbb{R}$ is an example of a groupoid that has no continuous irreducible representations.

Definition 4.4. Define the **support of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta_{\mathcal{H}})$ by

$$\operatorname{supp}(\mathcal{H}, \Delta) := \{m \in M \mid \mathcal{H}_m \neq 0\}.$$

This last set equals

$$\{m \in M \mid \xi(m) \neq 0 \text{ for some } \xi \in \Delta_{\mathcal{H}}\}.$$

One easily sees that for all continuous fields of Hilbert spaces $(\mathcal{H}, \Delta_{\mathcal{H}})$ the support $\operatorname{supp}(\mathcal{H}, \Delta_{\mathcal{H}})$ is open in M.

Lemma 4.5. *(i) If the support of a continuous representation $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ of a groupoid $G \rightrightarrows M$ properly contains a closed union of G-orbits, then it is reducible.*

(ii) If the support of a continuous representation $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ of a groupoid $G \rightrightarrows M$ properly contains a clopen set of G-orbits, then it is decomposable.

Proof. Let $(\mathcal{H}, \Delta, \pi)$ be a continuous representation of $G \rightrightarrows M$. Suppose $U \subset M$ is a closed union of orbits. Define a new continuous field of Hilbert spaces by

$$\mathcal{H}'_m := \begin{cases} \mathcal{H}_m & \text{if } m \notin U \\ 0 & \text{if } m \in U \end{cases}$$

and

$$\Delta_{\mathcal{H}'} := \{\xi \in \Delta \mid \xi|_U = 0\},$$

The groupoid $G \rightrightarrows M$ represents on $(\mathcal{H}', \Delta')$ by

$$\pi'(g) := \begin{cases} \pi(g) & \text{if } s(g) \notin U \\ id_0 & \text{if } m \in U \end{cases}$$

One easily sees that $(\mathcal{H}', \Delta_{\mathcal{H}'}, \pi')$ is a continuous subrepresentation of $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$. The second statement is proved analogously. □

The representation $(\mathcal{H}', \Delta_{\mathcal{H}'}, \pi')$ is called the **restriction of** $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ **to** U^c.

Example 4.6. If a groupoid $G \rightrightarrows M$ is proper and M, then the orbits are closed. Hence an irreducible representation must consist of one orbit that is clopen, since it is the support of a continuous field and the orbit of a proper groupoid. Therefore, a space $M \rightrightarrows M$ has an irreducible representation iff it has a discrete point $m \in M$.

4.2 Schur's lemma

In the previous section we have seen that in many cases of interest the irreducible representations do exist. Therefore, we introduce the weaker notion of internal irreducibility.

Definition 4.7. A continuous representation $(\pi, \mathcal{H}, \Delta)$ of a groupoid $G \rightrightarrows M$ is called **internally irreducible**, if the restriction of π to each of the isotropy groups is an irreducible representation.

Obviously, if a representation is irreducible, then it is internally irreducible. The converse does not hold as we have seen in Example 4.3.

Example 4.8. Suppose H is a topological group, $P \to M$ a continuous principal H-bundle and (π, V) an irreducible representation of H. Then, $P \times_H V \to M$ carries a canonical internally irreducible (but reducible, if $M \neq pt$ and Hausdorff) representation of the bundle of groups $P \times_H H \to M$ (cf. Section 3.4).

Example 4.9. If M is a topological space with a non-trivial rank 2 vector bundle $E \to M$. Then $E \to M$ is not internally irreducible as a representation of $M \rightrightarrows M$, even though it might be indecomposable.

Example 4.10. A morphism of internally irreducible continuous representations is not necessarily an isomorphism or the zero map. A counterexample is given by the following: let G be the constant bundle of groups $\mathbb{R} \times U(1) \rightrightarrows \mathbb{R}$. It represents internally irreducibly on the trivial rank one vector bundle $\mathcal{H} := \mathbb{R} \times \mathbb{C}$ over $\mathbb{R}$ by scalar multiplication. The map $\Psi : (x, z) \mapsto (x, x \cdot z)$ is an equivariant adjointable map $\mathcal{H} \to \mathcal{H}$, not equal to a scalar times the identity.

Remark 4.11. What one does see in this example is that Ψ is a function times the identity on $\mathcal{H}$, namely the function $\lambda : \mathbb{R} \to \mathbb{C}, x \mapsto x$, i.e. $\psi = \lambda 1_{\mathcal{H}}$. An alternative formulation of Schur's lemma for groupoids would be that an endomorphism of an internally irreducible representation $(\mathcal{H}, \pi)$ is a function $\lambda \in C(M)$ times the identity on $\mathcal{H}$. This we shall proof under some conditions in Lemma 4.13.

Notation 4.12. For a continuous groupoid $G \rightrightarrows M$ denote

(i) the set of isomorphism classes of continuous unitary representations by $\mathrm{Rep}(G)$;

(ii) the subset of isomorphism classes of indecomposable unitary representation by $\mathrm{IdRep}(G)$;

(iii) the subset of isomorphism classes of irreducible unitary representations by $\mathrm{IrRep}(G)$. For groups H this set is known as the unitary dual and denoted by $\hat{H}$;

(iv) the set of isomorphism classes of internally irreducible unitary representations by $\mathrm{IrRep}^i(G)$.

Lemma 4.13 (Schur's Lemma for groupoids)**.** *Suppose* $(\pi_i, \mathcal{H}^i, \Delta^i)$ *is an internally irreducible representation for* $i = 1, 2$.

(i) every equivariant endomorphism $\Psi : \mathcal{H}^1 \to \mathcal{H}^1$ *is equal to a continuous function* $\lambda \in C(M)$ *times the identity on* E*, i.e.* $\psi = \lambda\, 1_{\mathcal{H}^1}$.

(ii) If $\Phi : \mathcal{H}^1 \to \mathcal{H}^2$ *is a morphism of representations then* Φ_m *is either an isomorphism or the zero map* $\mathcal{H}^1_m \to \mathcal{H}^2_m$ *for all* $m \in M$.

(iii) If, furthermore, $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$ *is injective for every* $m \in M$*, then*

$$\mathrm{Hom}_G(\mathcal{H}^1, \mathcal{H}^2) = \begin{cases} \textit{line bundle} & \textit{if } (\pi_1, \mathcal{H}^1, \Delta^1) \cong (\pi_2, \mathcal{H}^2, \Delta^2); \\ 0 & \textit{if } (\pi_1, \mathcal{H}^1, \Delta^1) \ncong (\pi_2, \mathcal{H}^2, \Delta^2). \end{cases}$$

The proof follows easily from the analogous statement for groups.

Example 4.14. Suppose $P \to M$ is a principal H-bundle for a group H. If $G \rightrightarrows M$ is the gauge groupoid $P \times_H P \rightrightarrows M$, then every irreducible representation is internally irreducible. Moreover, $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$ is injective for all $m \in M$. Hence Schur's Lemma holds for all representations of these groupoids. Moreover, for two representations $E_i = P \times_H V_i \to M$ of $G \rightrightarrows M$ $(i = 1, 2)$, with V_1 and V_2 isomorphic representations of H.

$$\begin{aligned} \mathrm{Hom}_G(E_1, E_2) &= \mathrm{Hom}_G(P \times_H V_1, P \times_H V_2) \\ &\cong P \times_H \mathrm{Hom}_H(V_1, V_2) \\ &\cong P \times_H \mathbb{C} \end{aligned}$$

where we used Schur's Lemma for groups in the third equation. The group H acts on $\mathrm{Hom}_H(V_1, V_2)$ by $(h \cdot \phi)v_1 = h^{-1}\,\phi(h \cdot v)$.

Example 4.15. Consider the two-sphere as a groupoid $S^2 \rightrightarrows S^2$. It is proper and all indecomposable vector bundles over S^2 have rank one. These are internally irreducible representations, but obviously $\mathrm{Res}_m : \mathrm{IrRep}^i(S^2) \to \mathrm{IrRep}(\{m\})$ is not injective for any $m \in M$. Moreover, for non-isomorphic line bundles $L_1 \to S^2$ and $L_2 \to S^2$, one has

$$\mathrm{Hom}_M(L_1, L_2) \cong L_1^* \otimes L_2 \ncong 0.$$

4.3 Square-integrable representations

In this section we define the notion of square-integrability for continuous groupoid representations. In the end, we prove that for proper groupoids, with M/G compact, unitary representations are square-integrable, generalizing an analogous result for compact groups.

Suppose $G \rightrightarrows M$ is a locally compact groupoid endowed with a Haar system $\{\lambda_m\}_{m\in M}$, which desintegrates as $\lambda_m = \int_{n\in t(G_m)} \lambda_m^n \mu_m(dn)$, for a Haar system $\{\mu_m\}_{m\in M}$ on $R_G \rightrightarrows M$ and a continuous family of measures $\{\lambda_m^n\}_{(n,m)\in R_G}$ on $t \times s : G \to M \times M$ (cf. [17]).

Using the family $\{\lambda_m^n\}_{(n,m)\in R_G}$ one can construct the continuous field of Hilbert spaces

$$(\hat{L}^2(G), \Delta^2(G)) := (\hat{L}^2_{t\times s}(G), \Delta^2_{t\times s}(G)),$$

over R_G, cf. Example 2.11.

Example 4.16. A simple example of this is the following. If M is a space and μ a Radon measure on M and H a Lie group with Haar measure λ. Then the trivial transitive groupoid $M \times H \times M \rightrightarrows M$ with isotropy groups H has a Haar system $\{\lambda_m = \mu \times \lambda\}_{m\in M}$. Obviously, this decomposes as $\lambda_m = \int_{n\in M} \lambda\mu(dn)$, hence

$$(\hat{L}^2(G), \Delta^2(G)) = \left(L^2(G,\lambda) \times (M \times M), C_0(M \times M, L^2(G,\lambda))\right).$$

Definition 4.17. A map $f : G \to \mathbb{C}$ is called **$\hat{L}^2(G)$-square integrable** if the induced map

$$(m, n) \mapsto (g \mapsto f(g), G_m^n \to \mathbb{C})$$

is in $\Delta^2(G)$.

Definition 4.18. (i) The **conjugate $(\bar{\mathcal{H}}, \bar{\Delta})$ of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ is the family of Hilbert spaces is given by $\bar{\mathcal{H}}_m = \mathcal{H}_m$ as Abelian groups, but with conjugate complex scalar multiplication and the space of sections $\bar{\Delta} = \Delta$, but with conjugate $C_0(M)$-action.

(ii) The **conjugate representation** $(\bar{\mathcal{H}}, \bar{\Delta}, \bar{\pi})$ of a representation $(\mathcal{H}, \Delta, \pi)$ of $G \rightrightarrows M$ is the representation on the conjugate continuous field of Hilbert spaces $(\bar{\mathcal{H}}, \bar{\Delta})$ is given by $\bar{\pi}(g)h = \pi(g)h$, where $g \in G$ and $h \in \bar{\mathcal{H}}_{s(g)}$.

(iii) The **tensor product** $(\mathcal{H}^1 \otimes \mathcal{H}^2, \Delta^\otimes, \pi_1 \otimes \pi_2)$ **of two continuous continuous fields of Hilbert spaces** is the family of Hilbert spaces is given by $\mathcal{H}_m := \mathcal{H}^1_m \otimes \mathcal{H}^2_m$. The space $\Delta^\otimes$ is the closure of the pre-Hilbert $C_0(M)$-module of all finite sums of sections $\sum_{j\in J} \xi_j \otimes \eta_j$ of $\xi_j \in \Delta^1$ and $\eta_j \in \Delta^2$.

(iv) The **tensor product of two representations** $(\mathcal{H}^1, \Delta^1, \pi_1)$ and $(\mathcal{H}^2, \Delta^2, \pi_2)$ of a groupoid $G \rightrightarrows M$ is the representation of $G \rightrightarrows M$ on $(\mathcal{H}^\otimes, \Delta^\otimes)$ given by linearly extending the map $(\pi_1 \otimes \pi_2)(g)(h \otimes h') = \pi(g)h \otimes \pi(g)h'$ and then extending it continuously to the closure $(\mathcal{H}^\otimes, \Delta^\otimes)$.

Definition 4.19. A continuous representation $(\pi, \mathcal{H}, \Delta)$ is **square-integrable** if the map

$$(\bar{\mathcal{H}} \otimes \mathcal{H}, \Delta^{\otimes}) \to (\hat{L}^2(G), \Delta^2(G))$$

given by

$$h_2 \otimes h_1 \mapsto (g \mapsto (h_2, \pi(g)h_1)_{\mathcal{H}_{t(g)}}$$

is a map of continuous fields of Hilbert spaces.

This means that the matrix coefficients $\langle \xi, \pi\eta \rangle$, defined by

$$(n, m) \mapsto (g \mapsto \langle \xi(n), \pi(g)\eta(m) \rangle)$$

for $\xi, \eta \in \Delta$ are $\hat{L}^2(G)$-square-integrable maps.

Example 4.20. For example, consider a topological space M. A (finite-dimensional) vector bundle $E \to M$ is a square-integrable representation of $M \rightrightarrows M$.

Example 4.21. Consider the family of continuous groups $G := (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}) \backslash (0, -1) \rightrightarrows \mathbb{R}$. One easily sees that the trivial representation $g \mapsto id_{\mathbb{C}}$ on $(\mathbb{R} \times \mathbb{C}, C_0(\mathbb{R}))$ is not square-integrable. But, note that G is not proper (although for every $m \in M$ the set $s^{-1}(m) = t^{-1}(m)$ is compact).

Proposition 4.22. *If $G \rightrightarrows M$ is proper and M/G compact, then every unitary representation is square-integrable.*

Proof. Suppose $(\mathcal{H}, \Delta, \pi)$ is a unitary representation and $\xi, \eta \in \Delta$. Given $\varepsilon > 0$, choose $\xi', \eta' \in C_c(M)\Delta$ such that $\|\xi - \xi'\| < \varepsilon'$ and $\|\eta - \eta'\| < \varepsilon'$, where

$$\varepsilon' = \frac{\min\{\varepsilon, 1\}}{3M \max\{\|\xi\|, \|\eta\|\}}$$

and

$$M = \max_{(n,m) \in R_G} \lambda_m^n(G_m^n),$$

which exists since M/G is compact. First note that $\langle \xi', \pi\eta' \rangle$ has compact support, since $G \rightrightarrows M$ is proper. Moreover,

$$\begin{aligned}
\| \langle \xi, \pi\, \eta \rangle - \langle \xi', \pi\eta' \rangle \|_{\hat{L}^2} &\leq \| \langle (\xi - \xi'), \pi\, \eta \rangle \| + \| \langle \xi', \pi\, (\eta - \eta') \rangle \| \\
&\leq \max_{(n,m) \in R_G} \lambda_m^n(G_m^n)(\|\xi - \xi'\|\|\eta\| + \|\xi'\|\|\eta - \eta'\|) \\
&\leq \varepsilon'\|\eta\| + (\|\xi\| + \varepsilon')\varepsilon' \leq \varepsilon,
\end{aligned}$$

which finishes the proof. □

Proposition 4.23. *If a continuous groupoid G has the property that for all $m \in M$ the restriction map*

$$\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G_m^m)$$

is injective, then for any two non-isomorphic internally irreducible unitary square-integrable representations $(\mathcal{H}, \Delta, \pi)$, $(\mathcal{H}', \Delta', \pi')$ and $\xi, \eta \in \Delta$, $\xi', \eta' \in \Delta'$,

$$\langle \langle \xi, \pi\eta \rangle, \langle \xi', \pi'\eta' \rangle \rangle_{\hat{L}^2(G)} = 0$$

Proof. This easily follows from the version of this statement for compact groups and the invariance of the Haar system. □

4.4 The Peter-Weyl theorem

Suppose $G \rightrightarrows M$ is a continuous groupoid endowed with a Haar system $\{\lambda_m\}_{m\in M}$, which decomposes using a continuous family of measure $\{\lambda_m^n\}_{(n,m)\in R_G}$ as in Section 4.3. Let $\mathcal{E}(G) \subset \Delta^2(G)$ denote the $C_0(R_G)$-submodule spanned by the matrix coefficients (cf. Section 4.3) of all finite-dimensional representations of $G \rightrightarrows M$.

A generalization of the Peter-Weyl theorem as we are going to prove (cf. Theorem 4.29) appears not to be true for all continuous groupoids. Therefore, we introduce an extra condition:

Definition 4.24. For a continuous groupoid $G \rightrightarrows M$ the restriction map

$$\mathrm{Res}_m : \mathrm{Rep}(G) \to \mathrm{Rep}(G_m^m)$$

is **dominant** if for every $m \in M$ and every continuous unitary representation (π, V) of G_m^m there exists a continuous unitary representation $(\pi', \mathcal{H}, \Delta)$ of $G \rightrightarrows M$ such that (π, V) is isomorphic to a subrepresentation of $(\pi'|_{G_m^m}, \mathcal{H}_m)$.

Example 4.25. Suppose H is a group and $P \to M$ a principal H-bundle. Since $(P \times_H P)_m^m \cong H$ and $P \times_H P \rightrightarrows M$ are Morita equivalent, $\mathrm{Res}_m : \mathrm{Rep}(P \times_H P) \to \mathrm{Rep}((P \times_H P)_m^m)$ is dominant for all $m \in M$.

Example 4.26. Suppose H is a compact connected Lie group that acts on manifold M. Consider the action groupoid $G := H \ltimes M \rightrightarrows M$.

Proposition 4.27. *The restriction map* $\mathrm{Res}_m : \mathrm{Rep}(H \ltimes M) \to \mathrm{Rep}((H \ltimes M)_m^m)$ *is dominant for all* $m \in M$.

Proof. First we note that from every representation $(\pi, V) \in \mathrm{Rep}(H)$ we can construct a representation $\tilde{\pi} : H \ltimes M \to U(M \times V)$ of $H \ltimes M \rightrightarrows M$ on $M \times V \to M$ by $\tilde{\pi}(h, m) : (m, v) \mapsto (h\cdot m, \pi(h)v)$. Note that the isotropy groups of $H \ltimes M \rightrightarrows M$ coincide with the isotropy groups of the action. These are subgroups of H, hence the question is whether every representation of a subgroup of H occurs as the subrepresentation of the restriction of a representation of H.

Suppose K is a compact Lie subgroup of H. Fix a maximal tori $T_K \subset K$ and $T_H \subset H$ such that $T_K \subset T_H$, with Lie algebras $\mathfrak{t}_K$ and $\mathfrak{t}_H$. Note that $T_K \cong \mathfrak{t}_K/\Lambda_K$ and $T_H \cong \mathfrak{t}_H/\Lambda_H$ for lattices $\Lambda_K \subset \mathfrak{t}_K$ and $\Lambda_H \subset \mathfrak{t}_H$. There is an injective linear map $M : \mathfrak{t}_K \to \mathfrak{t}_H$ that induces the inclusion $\mathfrak{t}_K/\Lambda_K \hookrightarrow \mathfrak{t}_H/\Lambda_H$. Let P_K denote the integral weight lattice of T_K and P_H the integral weight lattice of T_H. Hence $q := M^T : \mathfrak{t}_H^* \to \mathfrak{t}_K^*$ is surjective map, mapping P_H onto P_K. Hence restriction of representations $\mathrm{Rep}(T_H) \to \mathrm{Rep}(T_K)$ is surjective too, since for tori irreducible representations correspond to integral weights.

The following argument is valid if one fixes positive root systems R_K^+, R_H^+ and hence fundamental Weyl chambers C_K^+, C_H^+ in a way specified in [11]. Suppose (π_λ, V) is an irreducible representation of K corresponding to the dominant weight $\lambda \in P_K \cap C_K^+$. One can choose any integral weight $\Lambda \in q^{-1}(\lambda) \cap P_H \cap C_H^+$; this set is non-empty, since q is surjective and the positive root systems have been fixed appropriately. Let π_Λ denote the irreducible representation of H associated to Λ. Then the multiplicity of π_λ in $\pi_\Lambda|_K$ is a positive integer (not necessarily 1), as follows from the Multiplicity Formula (3.5) in [11]. This finishes the proof. □

Example 4.28. A simple, but non-Hausdorff example of a proper groupoid which has a non-dominant restriction map is defined as follows. Consider $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z} \rightrightarrows \mathbb{R}$ and identify $(x, 0)$ with $(x, 1)$ for all $x \neq 0$. Endow the obtained family of groups $(\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})/\sim \rightrightarrows \mathbb{R}$, with the quotient topology. The non-trivial irreducible representation of $\mathbb{Z}/2\mathbb{Z}$ is not in the image of $\mathrm{Res}_0 : \mathrm{Rep}(G) :\longrightarrow \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$.

We now prove a generalization of the Peter-Weyl theorem for groupoids. Consider the continuous field of Hilbert spaces $(\hat{L}^2(G), \Delta^2(G))$ associated to a groupoid $G \rightrightarrows M$. Let $\overline{\mathcal{E}(G)}$ denote the closure of $\mathcal{E}(G)$ to a Hilbert $C_0(R_G)$-module.

Theorem 4.29 (Peter-Weyl for groupoids I)**.** *If $G \rightrightarrows M$ is a proper groupoid, M/G is compact and* Res_m *is dominant for all $m \in M$, then*

$$\overline{\mathcal{E}(G)} = \Delta^2(G).$$

Proof. Note that G_m^m is compact so Peter-Weyl for compact groups applies. Using the dominance property

$$\overline{\{\Theta(m, m) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^m, \lambda_m^m),$$

since $(\mathcal{H}, \Delta, \pi) < (\mathcal{H}', \Delta', \pi')$, implies $\langle \xi, \pi' \eta \rangle = \langle \xi, \pi \eta \rangle$ for $\xi, \eta \in \Delta$.

Note that $l_g^* : L^2(G_m^m, \lambda_m^m) \to L^2(G_m^n, \lambda_m^n)$ is an isometry for a chosen $g \in G_n^m$. Thus $\overline{\{l_g^*(\Theta(m, m)) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^n, \lambda_m^n)$. But, for all $h \in G_m^n$ and every continuous unitary finite-dimensional representation $(\mathcal{H}, \Delta, \pi)$

$$\begin{aligned} l_g^* \langle \xi, \pi \eta \rangle (h) &= \langle \xi(t(g)), \pi(gh) \eta(s(h)) \rangle_{\mathcal{H}^\pi_{t(g)}} \\ &= \textstyle\sum_{k=1}^{\dim(\mathcal{H}_n)} \langle \xi(m), \pi(g) e_k(n) \rangle_{E_n} \langle e_k(n), \pi(h) \eta(m) \rangle_{E_m}, \end{aligned}$$

where $e_1, \ldots, e_{\dim(\mathcal{H}_n)} \in \Delta$ are sections that form a basis of $\mathcal{H}$ at n. Thus $l_g^* \langle \xi, \pi \eta \rangle$ is a linear combination of matrix coefficients $\langle e_k, \pi \eta \rangle$ restricted to G_m^n, which implies $\overline{\{\Theta(n, m) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^n)$.

Let $f \in \Delta^2(G)$ and $\varepsilon > 0$ be given, then there exists a section $\tilde{f} \in \Delta^2(G)$ with compact support K such that $\|f - \tilde{f}\| < \varepsilon/2$, where the norm is the one associated to the $C_0(M)$-valued inner product. Moreover, for all $(m, n) \in R$ there are representations $(\mathcal{H}_{m,n}, \Delta_{m,n}, \pi_{m,n})$ and sections $u_{m,n}, v_{m,n} \in \Delta_{m,n}$, such that

$$\|\tilde{f} - (u_{m,n}, \pi_{m,n} v_{m,n})\|_{L^2(G_n^m)} < \varepsilon/2.$$

Since $\pi_{m,n}, u_{m,n}$ and $v_{m,n}$ are continuous we can find an open neighborhood $S_{m,n} \subset R$, such that still

$$\|\tilde{f} - (u_{m,n}, \pi_{m,n} v_{m,n})\|_{\hat{L}^2(G)|_{S_{m,n}}} < \varepsilon/2,$$

for all $(m, n) \in R$. These $S_{m,n}$ cover K, thus there is a finite subcover, which we denote by $\{S_i\}_{i \in I}$ to reduce the indices. Denote the corresponding representations by π_i and sections by u_i and v_i for $i \in I$. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\{S_i\}$. Define $\tilde{u}_i = \sqrt{\lambda_i} u_i$ and $\tilde{v}_i = \sqrt{\lambda_i} v_i$, then

$$\phi = \sum_{i \in I} (\tilde{u}_i, \pi_i \tilde{v}_i)$$

is a finite sum of matrix coefficients and

$$\begin{aligned}
\|f-g\| &\leq \|f-\tilde{f}\|+\|\tilde{f}-\phi\| \\
&\leq \varepsilon/2+\sup_{(m,n)\in R}\|\tilde{f}-\textstyle\sum_{i\in I}(\tilde{u}_i,\pi_i\tilde{v}_i)\|_{L^2(G_n^m)} \\
&= \varepsilon/2+\sup_{(m,n)\in R}\|\textstyle\sum_{i\in I}\lambda_i\tilde{f}-\sum_{i\in I}(\sqrt{\lambda_i}u_i,\pi_i\sqrt{\lambda_i}v_i)\|_{L^2(G_n^m)} \\
&\leq \varepsilon/2+\textstyle\sum_{i\in I}\lambda_i\sup_{(m,n)\in R}\|\tilde{f}-\sum_{i\in I}(u_i,\pi_i v_i)\|_{L^2(G_n^m)} \\
&\leq \varepsilon/2+\textstyle\sum_{i\in I}\lambda_i\varepsilon/2=\varepsilon,
\end{aligned}$$

which finishes the proof. □

Example 4.30. For a space M, $\overline{\mathcal{E}(M\rightrightarrows M)}=C_0(M)$ and $\overline{\mathcal{E}(M\times M\rightrightarrows M)}=C_0(M\times M)$ as Theorem 4.29 asserts.

Example 4.31. If H is a compact group and $P\to M$ an H-principal bundle. Then, for the bundle of groups $P\times_H H\to M$ one finds (cf. Example 4.8),

$$\begin{aligned}
\overline{\mathcal{E}(P\times_H H\rightrightarrows M)} &\cong \Gamma_0(P\times_H\overline{\mathcal{E}(H)}) \\
&\cong \Gamma_0(P\times_H L^2(H)) \\
&\cong \Delta^2(P\times_H H),
\end{aligned}$$

where in the second line we used the Peter-Weyl theorem for the group H.

Remark 4.32. Suppose $G\rightrightarrows M$ is a proper groupoid. In [19] the K-theory of $C^*_r(G)$ is shown to be generated by the locally trivial irreducible unitary representations of G, if $G\rightrightarrows M$ is approximately finitely generated projective (AFGP). In [9] it is proved that $G\rightrightarrows M$ is AFGP if the restriction functors are dominant as above. In the case of compact groups the fact that the K-theory of the C^*-algebra is generated by the irreducible unitary representations can be derived directly from the Peter-Weyl Theorem. It would be interesting to be able to do the same for proper groupoids.

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References

[1] M. Amini, *Tannaka-Krein duality for compact groupoids I, II and III*, preprint, ArXiv:math.OA/0308259, math.OA/0308260, math.OA/0308261.

[2] R. Bos, *Geometric quantization of Hamiltonian actions of Lie algebroids and Lie groupoids*, Int. J. Geom. Methods Mod. Phys. **4** (2007), no. 3, 389–436.

[3] R. Bos, *Groupoids in geometric quantization*, PhD-thesis, Radboud University Nijmegen, 2007.

[4] Z. Chen, Z.-J. Liu, D.-S. Zhong, *On the existence of global bisections of Lie groupoids*, preprint ArXiv:0710.3909.

[5] A. Connes, *Sur la théorie non commutative de l'intégration*, Springer Lecture Notes in Math. **725** (1979), 19–143.

[6] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[7] J. Dixmier, *les C*-algebras et leurs representations*, Deuxième édition, Paris, Gauthier-Villars, 1969.

[8] J. Dixmier and A. Douady, *Champs continues d'espaces hilbertiens et de C^*-algèbres*, Bul. de la S.M.F. **19** (1963), 227–284.

[9] H. Emerson and R. Meyer, *Equivariant representable K-theory*, preprint, ArXiv:0710.1410.

[10] S. A. Gaal, *Linear analysis and representation theory*, Die Grundlehren der mathematischen Wissenschaften, Band 198. Springer-Verlag, New York-Heidelberg, 1973.

[11] G. J. Heckman, *Orbits and multiplicities for compact groups*, Invent. math. **67** (1982).

[12] P.-Y. Le Gall, *Théorie de Kasparov équivariante et groupoïdes I*, K-Theory **16** (1999), no. 4, 361–390.

[13] J. Mrčun and I. Moerdijk, *Introduction to Foliations and Lie Groupoids*, Cambridge University Press, Cambridge, 2003.

[14] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.

[15] D. Pronk, *Groupoid representations for sheaves on orbifolds*, PhD-thesis, University of Utrecht, 1995.

[16] J. Renault, *A groupoid approach to C^*-algebras*, Lecture notes in mathematics **793**, Springer Verlag, 1980.

[17] J. Renault, *Représentations des produits croisés d'algèbres de groupoïdes*, J. Operator Theory **18** (1987), 67-97.

[18] J.-L. Tu, *La conjecture de Novikov pour les feuilletages hyperboliques*, K-theory **16** (1999), 129–184.

[19] J.-L. Tu, P. Xu and C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Sci. Ens **37** (2004), no. 6, 841–910.

[20] A. Weinstein and A. Cannas da Silva, *Geometric models for noncommutative algebras*, Berkeley Mathematics Lecture Notes **10**, American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[21] J. J. Westman, *Locally trivial C^r-groupoids and their representations*, Pacific J. Math. **20** (1967), 339–349.

[22] J. J. Westman, *Harmonic analysis on groupoids*, Pacific J. Math. **27** (1968), no. 3, 621-632.

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