

List of contributions

Rogier Bos

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Abstract

We construct Hermitian representations of Lie algebroids and associated unitary representations of Lie groupoids by a geometric quantization procedure. For this purpose we introduce a new notion of Hamiltonian Lie algebroid actions. The first step of our procedure consists of the construction of a prequantization line bundle. Next, we discuss a version of Kähler quantization suitable for this setting. This article is a summary of [1] and announces some of the results of that paper. It was originally a poster at the XV international workshop on geometry and physics.

Keywords: Lie algebroid, prequantization, geometric quantization.
2000 Mathematics Subject Classification: 22A22, 58H05, 53D50

1. Hamiltonian Lie algebroid actions

1.1. The symmetry of a map

The symmetries of an object S , within a certain category \mathcal{C} are given by its automorphisms $\phi \in \text{Aut}_{\mathcal{C}}(S)$. One studies the symmetry of the object through group homomorphisms

$$G \rightarrow \text{Aut}(S),$$

called **actions** of G , for groups G .

Consider the arrow category $\mathbf{Arrow}(\mathcal{C})$ of \mathcal{C} . This is the category whose objects are the arrows of \mathcal{C} and whose arrows

$$(J_1 : S_1 \rightarrow M_1) \rightarrow (J_2 : S_2 \rightarrow M_2)$$

are pairs of arrows $(\psi : S_1 \rightarrow S_2, \phi : M_1 \rightarrow M_2)$ in \mathcal{C} such that the following diagram commutes

$$\begin{array}{ccc} S_1 & \xrightarrow{J_1} & M_1 \\ \psi \downarrow & & \downarrow \phi \\ S_2 & \xrightarrow{J_2} & M_2 \end{array}$$

In particular one has automorphisms groups $\text{Aut}_{\mathbf{Arrow}(\mathcal{C})}(J)$ for every arrow J in \mathcal{C} .

Assume that $\mathcal{C} \subset \mathbf{Sets}$ and $J : S \rightarrow M$ an object in $\mathbf{Arrow}(\mathcal{C})$. One can decompose an automorphism (ψ, ϕ) , into isomorphisms $\psi_m : J^{-1}(m) \rightarrow J^{-1}(\phi(m))$ for $m \in M$. If $\phi(m) = m$, then ψ_m is called an **internal symmetry**, else it is called an **external symmetry**. The collection of all internal and external symmetries

$$J^{-1}(m) \xrightarrow{\cong} J^{-1}(m')$$

$(m, m' \in M)$ of J form a **groupoid**, denoted by $\text{Aut}_{\mathbf{Arrow}(\mathcal{C})}^M(J) \rightrightarrows M$. Recall that a groupoid is a small category in which all arrows are invertible. **Global bisections** of $\text{Aut}_{\mathbf{Arrow}(\mathcal{C})}^M(J) \rightrightarrows M$, i.e. subsets of $\text{Aut}_{\mathbf{Arrow}(\mathcal{C})}^M(J)$ such that the source and target map are bijections when restricted to them, correspond to automorphisms in $\text{Aut}_{\mathbf{Arrow}(\mathcal{C})}(J)$. Therefore, the right way to study the symmetry of a map $J : S \rightarrow M$ is via groupoid morphisms

$$G \rightarrow \text{Aut}_{\mathbf{Arrow}(\mathcal{C})}^M(J),$$

for groupoids $G \rightrightarrows M$, which are called **actions** of $G \rightrightarrows M$ on $J : S \rightarrow M$. Usually J is called the moment map of the action, but we will not use this terminology, because of the notion of momentum map introduced later.

1.2. Hamiltonian actions

We shall now focus on the situation where $\mathcal{C} = \mathbf{Man}$, the category of smooth manifolds and $J : S \rightarrow M$ is a fiber bundle. The symmetry of J can be

described by a **Lie groupoid** $G \rightrightarrows M$ and an action

$$\alpha : G \rightarrow \text{Aut}_{\mathbf{Arrow}(\mathbf{Man})}^M(J).$$

To a Lie groupoid is associated a Lie algebroid $(\mathcal{A}(G) \rightarrow M, [\cdot, \cdot], \rho)$ and the action of $G \rightrightarrows M$ induces a Lie algebroid action

$$\alpha : \Gamma(\mathcal{A}(G)) \rightarrow \mathfrak{X}(S)$$

As a references for Lie groupoids, Lie algebroids and their actions and representations the author suggests [3] and [2]. We shall from now on only study Lie algebroid actions (for more on the relation to Lie groupoids see [1]). Suppose $(\mathcal{A} \rightarrow M, [\cdot, \cdot], \rho)$ is a regular Lie algebroid and $\alpha : \Gamma(\mathcal{A}) \rightarrow \mathfrak{X}(S)$ an action of \mathcal{A} on $J : S \rightarrow M$. Note that such an action can be thought of as a map of Lie algebroids $\mathcal{A} \times J \rightarrow TS$. We shall fix some terminology.

- \mathcal{F}_ρ is the foliation of M associated to $\text{im}(\rho)$ and $\tilde{J} := p \circ J$, where $p : M \rightarrow M/\mathcal{F}_\rho$ is the canonical projection on the space of leaves. The \tilde{J} -foliation of S is regular, since \mathcal{A} is regular. $T^J S := \ker T J$ and $T^{\tilde{J}} S := \ker T \tilde{J}$, which is well-defined as one can check. From this one obtains notions like $T^{*,J} S = (T^J S)^*$, $\Omega^J(S) := \Gamma(T^{*,J} S)$, etc.
- A **J -presymplectic form** is a closed 2-form $\tilde{\omega} \in \Omega^{*,\tilde{J}}(S)$, which is non-degenerate when restricted to $T^J S$.
- The action α is said to be **J -presymplectic** if $(\mathcal{L}_{\alpha(X)} \tilde{\omega})|_{T^J S} = 0$ for all $X \in \Gamma(\mathcal{A})$.
- The **action Lie algebroid** associated to α is denoted by $\mathcal{A} \times J \rightarrow S$.
- If the action is J -presymplectic, then $d^{\mathcal{A} \times J} \alpha^* \tilde{\omega} = 0$ (where $d^{\mathcal{A} \times J}$ is the differential in **Lie algebroid-de Rham cohomology** of $\mathcal{A} \times J$) and $d^J(\alpha(X) \lrcorner \tilde{\omega})|_{T^J S} = 0$ (where d^J is the J -foliated de Rham differential). A J -presymplectic action is **Hamiltonian** if there exists a section $\tilde{\mu} \in \Gamma((\mathcal{A} \times J)^*)$, that satisfies

$$d^{\mathcal{A} \times J} \tilde{\mu} = -\alpha^* \tilde{\omega} \tag{1.1}$$

and

$$d^J \langle \tilde{\mu}, X \rangle = -(\alpha(X) \lrcorner \tilde{\omega})|_{T^J S} \tag{1.2}$$

for all $X \in \Gamma(\mathcal{A} \times J)$. Hence obstructions to a J -presymplectic action being Hamiltonian lie in $H^2(\mathcal{A} \times J)$ and $H_{dR}^{2,J}(S)$. The map $\tilde{\mu} : S \rightarrow J^* \mathcal{A}^*$ is called a **momentum map** for the action.

For more background on the classical notion of momentum map one could consult the recent overview [4] and references therein.

Example 1.1 Consider the the tangent bundle $TM \rightarrow M$ of a manifold M as a Lie algebroid. It trivially acts on $J := id : M \rightarrow M$. Let $\tilde{\omega} \in \Omega^2(M)$ be any closed form. It is trivially J -presymplectic. Moreover, the action is Hamiltonian if $\tilde{\omega}$ is exact, i.e. $\tilde{\omega} = -d\tilde{\mu}$ for a 1-form $\tilde{\mu} \in \Omega^1(M)$ which is the momentum map in this case.

Example 1.2 Suppose H is a Lie group which acts in a Hamiltonian fashion on a symplectic manifold (S, ω) with momentum map $\mu : S \rightarrow \mathfrak{h}^*$, $P \rightarrow M$ a principal H -bundle and $\tau \in \Gamma(T^*P \otimes \mathfrak{h})$ a Lie algebra-valued connection 1-form on P . Denote the action of H on P by α and the action of H on S by β . One can form the associated bundle $J : P \times_H S \rightarrow M$ and consider the canonical action of the gauge Lie algebroid TP/H on $P \times_H S \rightarrow M$. The tangent space $T^{\tilde{J}}(P \times_H S) = T(P \times_H S)$ can be identified with $(TP \times_H TS)/\sim$, where $(\alpha(X), 0) \sim (0, \beta(X))$. Define a J -presymplectic form on $P \times_H S$ by

$$\begin{aligned} \tilde{\omega}_{[p, \sigma]}([w_1, v_1], [w_2, v_2]) &:= \omega_{\sigma}^S((v_1 - \beta(\tau(w_1))), v_2 - \beta(\tau(w_2))) \\ &\quad - \langle \mu_{\sigma}, F_p(w_1, w_2) \rangle, \end{aligned}$$

where F is the \mathfrak{h} -valued curvature 2-form on P associated to τ . One can show that the action of TP/H on $P \times_H S$ is Hamiltonian with momentum map

$$\tilde{\mu} := \langle \mu, \tau \rangle.$$

2. Geometric quantization

2.1. Prequantization

Suppose $(p : \mathcal{A} \rightarrow M, \rho)$ is a Lie algebroid and $\alpha : \Gamma^{\infty}(\mathcal{A}) \rightarrow \mathfrak{X}^{\infty}(S)$ is an action of \mathcal{A} on $J : S \rightarrow M$. Suppose $E \rightarrow S$ is a smooth vector bundle over S . We introduce some terminology.

- Let $\mathcal{D}(E)$ denote the Lie algebroid whose sections are derivations D on $E \rightarrow S$. The anchor is the symbol map $\Theta : \mathcal{D}(E) \rightarrow T_{\tilde{J}}S$, i.e. for all $f \in C_{\tilde{J}}^{\infty}(S)$ and $\sigma \in \Gamma^{\infty}(E)$ one has

$$D(f\sigma) = fD\sigma + \Theta(D)f\sigma.$$

- A **\mathcal{A} -connection on a complex vector bundle** $p : E \rightarrow S$, is a map of vector bundles $\pi : \mathcal{A} \times J \rightarrow \mathcal{D}(E)$, such that $\alpha = \Theta \circ \pi$.

- If π preserves the Lie bracket, then it is called a **representation** or **flat \mathcal{A} -connection**; in this case $\pi : \mathcal{A} \times J \rightarrow \mathcal{D}(E)$ is a morphism of Lie algebroids.
- A representation $\pi : \mathcal{A} \times J \rightarrow \mathcal{D}(E)$ is **Hermitian** with respect to a Hermitian metric g on $E \rightarrow M$ if for all $X \in \Gamma(\mathcal{A})$, $\sigma, \tau \in \Gamma(E)$

$$g(\pi(X)\sigma, \tau) + g(\sigma, \pi(X)\tau) = \alpha(X)g(\sigma, \tau).$$

- The set of Hermitian rank one representations of \mathcal{A} over $J : S \rightarrow M$ is denoted by $\text{Pic}_{\mathcal{A}}(J)$. It can be given a group structure.

Proposition 2.1 *Any representation $(\pi, L) \in \text{Pic}_{\mathcal{A}}(J)$ is of the form*

$$\pi(X) := \nabla_{\alpha(X)} - 2\pi i \langle \tilde{\mu}, J^* X \rangle$$

where α denotes the action of \mathcal{A} on $J : S \rightarrow M$, $\tilde{\mu} \in \Gamma^\infty((\mathcal{A} \times J)^*)$ and ∇ a Hermitian $T^{\tilde{J}}S$ -connection on S .

Proposition 2.2 *There is an exact sequence of groups*

$$0 \rightarrow H^1(\mathcal{A} \times J) \rightarrow \text{Pic}_{\mathcal{A}}(J) \rightarrow \text{Pic}(S) \xrightarrow{c_1^{\mathcal{A} \times J}} H^2(\mathcal{A} \times J),$$

where $\text{Pic}(S) = \text{Pic}_{TS}(id_S)$. The first map send $[\mu]$ to the representation $X \mapsto \langle \mu, X \rangle$ on a trivial line bundle, the second map forgets the representation of \mathcal{A} .

The last arrow denotes the first $\mathcal{A} \times J$ -Chern class. One can define a notion of **longitudinal Čech cohomology** $\check{H}_{\tilde{J}}^*(S, \mathbb{R})$ of S along the orbits of the action \mathcal{A} -action and show that the cohomology class $[\tilde{\omega}] \in H_{\tilde{J}, dR}^2(S)$ of a J -presymplectic form $\tilde{\omega}$ determines a unique class in $\check{H}_{\tilde{J}}^2(S, \mathbb{R})$. One can show that

$$\text{Pic}(S) \simeq \check{H}^2(S, \mathbb{Z}) \simeq \check{H}_{\tilde{J}}^2(S, \mathbb{Z}) \hookrightarrow \check{H}_{\tilde{J}}^2(S, \mathbb{R}) \simeq H_{\tilde{J}, dR}^2(S).$$

Theorem 2.3 *A \tilde{J} -closed form $\tilde{\omega} \in \Omega_{\tilde{J}}^2(S)$ is the curvature 2-form of a \tilde{J} -partial Hermitian connection ∇ on a complex line bundle $L \rightarrow S$ iff $[\tilde{\omega}] \in H_{\tilde{J}, dR}^2(S)$ is integral, i.e. in the image of the above inclusion.*

Theorem 2.4 *If the action of \mathcal{A} on $J : S \rightarrow M$ is Hamiltonian, or, more weakly, Equation 1.1 is satisfied (which is equivalent to $c_1^{\mathcal{A} \times J}([\tilde{\omega}]) = 0$) and $[\tilde{\omega}]$ is integral, then there exists a Hermitian rank one representation of the Lie algebroid \mathcal{A} on $(L \rightarrow M, h)$ given by*

$$\pi(X) := \nabla_{\alpha(X)} - 2\pi i \langle \tilde{\mu}, J^* X \rangle,$$

(which is no surprise, see Proposition 2.1).

Example 2.5 We proceed with Example 1.1. In this case, the first Theorem expresses that an integral 2-form $\tilde{\omega} \in \Omega^2(M)$ is the curvature of a Hermitian connection a line bundle, which is well-known (from “classical” prequantization theory/Chern class theory). The second theorem, applied to this simple example, states that if the curvature 2-form is exact, say $d\tilde{\mu}$ (which corresponds to the action of TM on $M \rightarrow M$ being Hamiltonian) then one can add $\tilde{\mu}$ to the connection to obtain a flat connection.

Example 2.6 We proceed with Example 1.2. Suppose $(L \rightarrow S, h, \nabla)$ is a prequantization of the action of H on (S, ω) . Then one can form the line bundle $P \times_H L \rightarrow P \times_H S$. On this line bundle it easy to define an Hermitian inner product, given h . A Hermitian connection is given as follows: for each $\theta = (\theta_1, \theta_2) \in \Gamma^\infty(P \times_H L)$ and $[w, v] \in \mathfrak{X}^\infty(P \times_H S)$, define

$$\nabla_{[w,v]}\theta := (\theta_1, \nabla_{v-\beta(\tau(w))}^L \theta_2 - \tau_h(w) \cdot \theta_2).$$

One can show that these form a prequantization of the action of TP/H on $P \times_H S$.

2.2. Quantization

Suppose $J : S \rightarrow M$ is proper and \mathcal{A} acts on $J : S \rightarrow M$ in a Hamiltonian fashion, with momentum map $\tilde{\mu} : S \rightarrow J^*\mathcal{A}^*$. We introduce some more terminology.

- An **almost complex structure on $J : S \rightarrow M$** is a smooth map $j : T^J S \rightarrow T^J S$ that satisfies $j \circ j = -id$.
- An almost complex structure $j : T^J S \rightarrow T^J S$ is **\mathcal{A} -equivariant** if

$$[\alpha(X), j(v)]_{T^J S} = j[\alpha(X), v],$$

for all $X \in \Gamma(\mathcal{A})$ and $v \in \mathfrak{X}_J S$.

- The **Kähler polarization** $\mathcal{P}(S, j)$ of $(J : S \rightarrow M, \omega)$ is defined by

$$\mathcal{P}(S, j) := \{v \in T^J S \otimes \mathbb{C} \mid j_{\mathbb{C}}(v) = -iv\},$$

where $j_{\mathbb{C}}$ denotes the extension of j to $T^J S \otimes \mathbb{C}$.

- The **geometric quantization** of a prequantization line bundle $(L \rightarrow S, \nabla, h)$ for the Hamiltonian action of \mathcal{A} on $(J : S \rightarrow M, \tilde{\omega})$ is given by the vector space

$$\Delta(J, \tilde{\omega}, \alpha, \tilde{\mu}, j) := \{\sigma \in \Gamma(L) \mid \nabla_v \sigma = 0 \text{ for all } v \in \Gamma(\mathcal{P}(S, j))\}.$$

This space, shortly denoted by Δ , is a Hilbert $C_0(M)$ -module with $C_0(M)$ -valued inner product given by

$$\langle \theta, \theta' \rangle (m) := \int_{S_m} h(\theta, \theta') \Omega_m,$$

where $\{\Omega_m\}_{m \in M}$ is a smooth family of measures on $J : S \rightarrow M$ defined by

$$\Omega_m := \omega_m^d / (d!);$$

d is half the dimension of the fibers of J . Hence it gives rise to a finite-dimensional continuous field of Hilbert spaces. We shall *assume* here that it actually is a smooth vector bundle, denoted by $Q \rightarrow M$.

Theorem 2.7 *The geometric quantization vector bundle $(Q \rightarrow M, \langle \cdot, \cdot \rangle)$ carries a Hermitian representation of \mathcal{A} .*

From here one can proceed to define a generalized Marsden-Weinstein quotient (this explains the condition that $\tilde{\omega}$ is non-degenerate) and quantum reduction. Then it is possible to formulate and prove a “quantization commutes with reduction” theorem.

Acknowledgements

This work is part of the research programme of the ‘Stichting voor Fundamenteel Onderzoek der Materie (FOM)’, which is financially supported by the ‘Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)’. The author would like to thank the organizers for the very enjoyable conference in Tenerife. He would also like to thank Lgia Abrunheiro, Rui Loja Fernandes, Raquel Caseiro, Jesus Clemente, Eva Miranda, Miguel Olmos and Patrcia Santos.

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