A momentum map for symplectic fibrations

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- Use geometric quantization to construct representations.
- Is there an orbit method/correspondence (Kirillov)?
- Does quantization commute with reduction in this setting?

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- Prequantization and quantization (if time allows).

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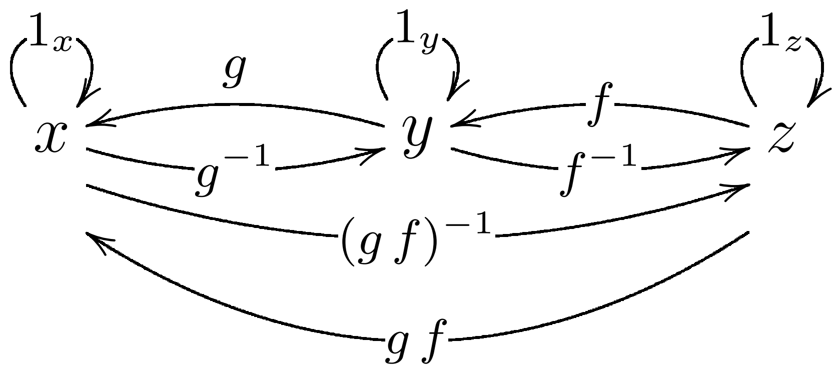
Example. *A (transitive) groupoid with 3 objects* x, y, z*:*



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- a set of objects G_0, a set of arrows G_1;
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- an associative partial multiplication $m : G_2 := G_1 \,{}_s\!\times_t G_1 \to G_1$ satisfying $s(h\,g) = s(g)$ and $t(h\,g) = t(h)$ for all $(h, g) \in G_2$;

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- a unit map $u : G_0 \to G_1$ such that $u(x) =: 1_x$ is a left unit for $G^x := t^{-1}(x)$ and a right unit for $G_x := s^{-1}(x)$ for all $x \in G_0$;

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- an inverse map $i : G_1 \to G_1$ such that $g^{-1} := i(g)$ is a 2-sided inverse of g for all $g \in G$.

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- For an action of a group G on a set B, the action groupoid $G \ltimes B \rightrightarrows B$;
- For a submersion $p : M \to B$, the groupoid of diffeomorphisms between the fibers $\mathrm{Diff}(p) \rightrightarrows B$.

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- A groupoid is *transitive* if it has just one orbit $t(s^{-1}(x))$

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Example: iff $G = B \times B \rightrightarrows B$, then $\mathrm{Bis}^\infty(G) = \mathrm{Diff}^\infty(B)$.

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such that the following Leibniz identity is satisfied

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Proposition. *A Lie groupoid $G \rightrightarrows B$ gives rise to a Lie algebroid structure on $\mathcal{A}(G) := u^* \ker(Ts) \to B$.*

Actions of Lie groupoids/algebroids

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Indeed,

$$m \cdot \sigma = \sigma(\pi(m))^{-1} \cdot m,$$

for $m \in M$ and $\sigma \in \mathrm{Bis}^\infty(G)$.

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Example. *Suppose $p : P \longrightarrow B$ is a principal G-bundle and G acts symplectically on (F, ω). Then $P \times_G F \longrightarrow B$ is a symplectic fibration.*

Coadjoint orbits of groupoids

A groupoid $G \rightrightarrows B$ acts on itself from the left and from the right by multiplication. The action by conjugation $c_g(h) = g\,h\,g^{-1}$ is only defined on the isotropy groupoid (family of groups)

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Proposition. *The orbits of the coadjoint action are symplectic fibrations over the leaves of $\mathcal{F}^B$.*

Foliated Ehresmann connections

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A *foliated Ehresmann connection* on p is a distribution $\mathrm{Hor} \subset T\mathcal{F}^M$ such that for all $x \in B$

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- *symplectic* if the parallel transport preserves the symplectic form.

Foliated coupling forms

Using a complete symplectic foliated Ehresmann connection, we want to extend the form $\omega : \mathrm{Ver} \wedge \mathrm{Ver} \longrightarrow \mathbb{R}$ to a *foliated coupling form*

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In particular, if $p : M \to B$ is a fiber bundle with fiber F, satisfying $H^1(F) = 0$, then this condition is satisfied.

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$$(v^\sharp \lrcorner (w^\sharp \lrcorner d^{\mathcal{F}} \tilde{\omega}))|_{\mathrm{Ver}} = ([v^\sharp, w^\sharp] \lrcorner \tilde{\omega})|_{\mathrm{Ver}} - (d^{\mathcal{F}}(\tilde{\omega}(v^\sharp, w^\sharp)))|_{\mathrm{Ver}}.$$

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Hence $(\mathrm{Hor} \oplus \mathrm{Hor}) \lrcorner d^{\mathcal{F}} \tilde{\omega} = 0$ iff

$$[v^\sharp, w^\sharp] \lrcorner \tilde{\omega}|_{\mathrm{Ver}} = (d^{\mathcal{F}}(\tilde{\omega}(v^\sharp, w^\sharp)))|_{\mathrm{Ver}}$$

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Note $d^{\mathrm{Ver}}(\mathrm{curv}(v, w) \lrcorner\, \omega) = 0$, hence the class

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If $H^1(\mathrm{Ver}) = 0$, it has a global solution.

Weakly Hamiltonian actions

A Lie algebra action α of $\mathfrak{g}$ on a symplectic manifold (M, ω) is *weakly Hamiltonian* if there exists a map $\mu : M \to \mathfrak{g}^*$ (*momentum map*) satisfying for all $X \in \mathfrak{g}$

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Note that the existence of $\langle \mu, X \rangle$ is an integration problem with obstruction class $[(\alpha(X) \lrcorner \tilde{\omega})|_{\mathrm{Ver}}] \in H^1(\mathrm{Ver})$

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The converse does not hold in general. One could speak of a *p-weakly Hamiltonian action* of a Lie group (in this case $\mathrm{Bis}^{\infty}(G)$) on M.

Important example: parallel transport

The *holonomy groupoid* of the connection Hor on $p : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B)$ is the groupoid of parallel transports along all curves on $\mathcal{F}^B$,

$$\mathrm{Hol}^{\mathrm{Hor}}(p) := \{T_\gamma : M_{\gamma(0)} \to M_{\gamma(1)} \mid \gamma : [0,1] \to \mathcal{F}^B\}.$$

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Theorem. *A coupling form* $\tilde{\omega}$ *exists if the canonical action of* $\mathcal{A}(\mathrm{Hol}^{\mathrm{Hor}}(p))$ *on* $p:M\to B$ *is weakly Hamiltonian.*

Proof of the Theorem

Ingredients:

- By the Holonomy Theorem (or Ambrose-Singer Theorem) the Lie algebra $\mathcal{A}(I(\mathrm{Hol}^{\mathrm{Hor}}(p)))_x$ at $x \in B$ equals the set

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The converse

The converse (if a coupling form $\tilde{\omega}$ exists, then the canonical action of $\mathcal{A}(\mathrm{Hol}^{\mathrm{Hor}}(p))$ on $p : M \to B$ is weakly Hamiltonian) holds, if one can choose a smooth bundle map

$$v_1 \oplus v_2 : \mathcal{A}(I(\mathrm{Hol}^{\mathrm{Hor}}(p))) \longrightarrow TB \oplus TB$$

satisfying $\mathrm{curv} \circ (v_1 \oplus v_2) = id$.

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- Extend μ, using the connection, which gives a splitting

$$0 \to \mathcal{A}(I(\mathrm{Hol}^\Gamma(p))) \to \mathcal{A}(\mathrm{Hol}^\Gamma(p)) \leftrightarrows F^B \to 0.$$

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or equivalently,

- Poisson, where $\mathfrak{g}^*$ is endowed with its linear Lie-Poisson structure

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A solution lies in using Lie algebroid cohomology!

Lie algebroid cohomology

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Encompasses De Rham ($\mathcal{A} = TM$), Chevalley-Eilenberg ($\mathcal{A} = \mathfrak{g}$) and Poisson ($\mathcal{A} = T^*P$) cohomology.

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Hamiltonian actions of Lie algebroids

Suppose α is a weakly Hamiltonian action of a Lie algebroid $\mathcal{A}$ on a foliated symplectic fibration $(p : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B), \omega)$, with an $\mathcal{F}$-closed 2-form $\tilde{\omega}$ extending ω, and momentum map $\mu : M \to \mathcal{A}^*$.

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We define: the action is (strongly) *Hamiltonian* if

$$d^{\mathcal{A} \ltimes p} \mu = -\alpha^* \tilde{\omega}.$$

2 remarks on Hamiltonian actions

- The momentum map being Hamiltonian is (again) an integration problem to which the obstruction class is

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- Lemma: If the action is Hamiltonian, then

$$i^* \circ \mu : M \to \mathcal{A}^* \to \ker(\rho)^*$$

is equivariant for the coadjoint action on $\ker(\rho)^*$, where $i : \ker(\rho) \hookrightarrow \mathcal{A}$.

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- The action is trivially weakly Hamiltonian for any momentum map $\mu : B \to T^*\mathcal{F}^B$.
- The action is Hamiltonian if $\tilde{\omega}$ is $\mathcal{F}^B$-exact: $\tilde{\omega} = -d^{\mathcal{F}}\mu$ for a momentum map $\mu \in \Omega^1(\mathcal{F}^B)$.

Example: gauge groupoids

Suppose G is a Lie group, with Lie algebra $\mathfrak{g}$ acting in Hamiltonian fashion on a symplectic manifold (F, ω^F) with momentum map $\mu^F : F \to \mathfrak{g}^*$. We denote the action by β.

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The associated bundle $p : P \times_G F \to B$ is a symplectic fibration, with $\mathrm{Ver} \cong P \times_G TF$, and using this isomorphism we can define

$$\omega([p, v], [p, w]) := \omega^F(v, w),$$

for $p \in P$ and $v, w \in T_f F$, $f \in F$.

Example: gauge groupoids cont.

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Using this, define a coupling form by

$$\tilde{\omega}_{[p,f]}\big([w_1, v_1], [w_2, v_2]\big) := \omega^F_\sigma\big((v_1 - \beta(\tau(w_1)), v_2 - \beta(\tau(w_2))\big) \\ - \langle \mu_\sigma, \mathrm{curv}_p(w_1, w_2)\rangle \,,$$

where curv is the $\mathfrak{g}$-valued curvature 2-form on P.

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The gauge groupoid $P \times_G P \rightrightarrows B$ acts on $p : P \times_G F \to B$ by

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Proposition. *The Lie algebroid action associated to* α *is Hamiltonian with momentum map* $\mu : P \times_G F \to T^*P/G$ *defined by*

$$\mu = \left\langle \mu^F, \tau \right\rangle.$$

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Construct a K-invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$. Denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ by $\mathfrak{p}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

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A semisimple Lie group G, maximal compact subgroup $K \subset G$.

The projection $p : G \to G/K$ is a principal K-bundle.

Denote the Lie algebra of G by $\mathfrak{g}$, of K by $\mathfrak{k}$.

Construct a K-invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$. Denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ by $\mathfrak{p}$: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Lemma. *The decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *determines a connection* $\tau \in \Gamma(T^*G \otimes \mathfrak{k})$ *on* $p : G \to G/K$ *with curvature* $\mathrm{curv} \in \Gamma(\bigwedge^2 T^*G \otimes \mathfrak{k})$ *satisfying*

$$\mathrm{curv}(X, Y) = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] \in \mathfrak{k},$$

for all $X, Y \in \mathfrak{g} = T_e G$.

Relation to Peter's Ch.12

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Proposition. *(cf. Formula 12.2 in Hochs) The coupling form on $M \to B$ is given by extending G-equivariantly after defining at the unit e, for all $n \in N$, $v, w \in T_nN$, $X, Y \in \mathfrak{p}$*

$$\tilde{\omega}_{[e,n]}(v + X, w + Y) := \omega^N_n(v,w) - \left\langle \mu^N_n, [X,Y] \right\rangle$$

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Proposition. *The canonical action of G on $G \times_K N$ factors through the action of the group of bisections $\mathrm{Bis}^\infty(G \times_K G)$ of the gauge groupoid $G \times_K G \rightrightarrows G/K$.*

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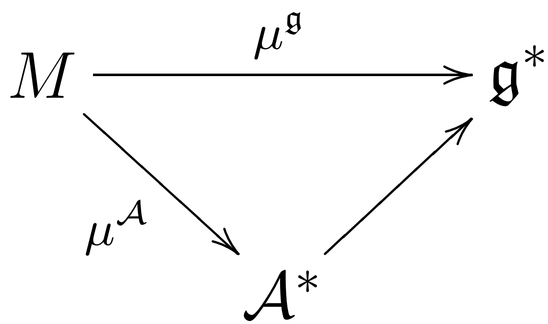

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Examples: Lie algebra representations $(\mathcal{A} = \mathfrak{g})$; flat connections $(\mathcal{A} = TB)$; foliated flat connections $(\mathcal{A} = T\mathcal{F}^B)$.

Foliated Čech cohomology

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We say that $\tilde{\omega} \in H^2_{DR}(\mathcal{F}^B)$ is *integral* if its class is the image of

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It is wellknown that

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Proposition. *There is an exact sequence*

$$0 \to H^1(\mathcal{A}) \to \mathrm{Pic}(\mathcal{A}) \to \mathrm{Pic}(B) \to H^2(\mathcal{A}).$$

Proof of the Proposition

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If $c^1_{\mathcal{A}}(L) = [d^{\mathcal{A}}\mu]$ for some $\mu \in C^1(\mathcal{A})$, then

$$X \mapsto \nabla_{\rho(X)} + 2\pi\, i\, \langle \mu, X \rangle$$

is a representation of $\mathcal{A}$ on $L \to B$ (Kostant-Souriau formula).

Prequantization

Theorem. *If* $\mathcal{A}$ *acts on* $(p : (M, \mathcal{F}^M) \to (B, \mathcal{F}^B), \tilde{\omega})$ *in a Hamiltonian fashion and* $\tilde{\omega}$ *is integral, then there exists representation of* $\mathcal{A} \ltimes p$ *in* $\mathrm{Pic}(\mathcal{A} \ltimes p)$ *on a line bundle with first Chern class* $[\tilde{\omega}]$*.*

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By exactness of the previous sequence, we get a representation of $\mathcal{A} \ltimes p$ on $L \to B$.

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- If ∇ is a $\mathcal{F}^B$-connection on a line bundle with curvature $\tilde{\omega}$, then $\nabla - \mu$ is flat.

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Recall the setup of this previous example.

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$$\nabla_{[w,v]}[\theta_1, \theta_2] = \left[(\theta_1, \left(\nabla^F_{v-\beta(\tau(w))} - (w - \gamma(\tau(w)))\right)\theta_2\right],$$

where γ is the action of $\mathfrak{g}$ on P, $w \in \mathfrak{X}(P)$, $v \in \mathfrak{X}(F)$, $\theta_1 : P \times F \to P$ and $\theta_2 : P \times F \to L$ all with the “right behaviour” with respect to the actions of G and $\mathfrak{g}$ and the fiber structures.

Gauge groupoid example cont.

Hence the prequantization representation

$$\Gamma(TP/H) \times \Gamma(P \times_G L) \to \Gamma(P \times_G L)$$

is given by

$$([w], [\theta_1, \theta_2]) \mapsto \left(\nabla_{[w,0]} - 2\,\pi\,i\,\left\langle \mu^F, \tau(w) \right\rangle\right)(\theta_1, \theta_2).$$

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Remark. *Integrating this representation, one gets the representations of* $P \times_G P \rightrightarrows B$ *on* $P \times_G L \to B$ *given by*

$$\pi([p, p'])[p', l] = [p', l],$$

where $p, p' \in P$ *and* $l \in L$*.*

Equivariant almost complex structures

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The *polarization* of $p : M \to B$ w.r.t. J is

$$\mathcal{P} := \{v \in \mathrm{Ver} \otimes \mathbb{C} \mid J(v) = -i\, v\}.$$

Geometric quantization

Suppose $\mathcal{A}$ acts on $(p : (M, \mathcal{F}^M \to (B, \mathcal{F}^B), \tilde{\omega})$ in a Hamiltonian fashion, $L \in \mathrm{Pic}(\mathcal{A})$ is a prequantization representation and J is an almost complex structure on p.

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For each $x \in B$ consider the vector space

$$Q_x := \{\xi|_{p^{-1}(x)} \mid \xi \in \Gamma(L), \nabla_v \xi = 0 \text{ for all } v \in \Gamma(\mathcal{P})\}.$$

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This representation is called the *geometric quantization representation*.

Example: the gauge groupoid cont.

Given an equivariant almost complex structure J^F on F, one defines an almost complex structure $J : P \times_G TF \to P \times_G TF$ on $p : P \times_G F \to B$ by

$$J[p, v] := [p, J^F v],$$

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for $p \in P$ and $v \in TF$.

Proposition. *If Q^F is the geometric quantization representation of (F, ω^F, J^F), then the geometric quantization Q of $(P \times_G F, \tilde{\omega}, J)$ satisfies*

$$Q \cong P \times_G Q^F,$$

with integrating $(P \times_G P)$-representation $\pi([p, p'])[p', q] = [p, q]$ for $p, p' \in P$ and $q \in Q^F$.

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- What in the case of a Poisson fibration, instead of a symplectic fibration?