# Noncommutative analytical assembly maps

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**Abstract**

As part of a project of defining geometric quantization of (symplectic) noncommutative manifolds, we define a noncommutative version of analytical assembly maps. As an example, we consider proper actions on groupoid $C^*$-algebra's.

## Introduction

The purpose of this note is part of a project to provide a noncommutative framework for geometric quantization. In particular, this would provide a setting for quantizing singular symplectic manifolds. Instead of such a manifold one would use a suitable algebra of functions on it. But the goal is more general: there might even not be a space lurking in the background at all.

The approach to geometric quantization that we propose is based on a suggestion by Landsman (cf. [10, 5]). Before explaining this approach step by step let's summarize it in one sentence: he suggests to view the geometric quantization of a Hamiltonian action as the image under the Baum-Connes analytical assembly map of the $K$-homology class of the Spin$^c$-Dirac operator on the underlying manifold coupled to the prequantization line bundle. This approach embeds the Guillemin-Sternberg conjecture within a more general program of proving the functoriality of geometric quantization in a certain sense (cf. [10]). In this program geometric quantization is to be viewed as a functor from classical dual pairs (cf. e.g. [9]) to Kasparov $KK$-theory (cf. [7], [3]).

This transition from the traditional discussion of geometric quantization to the approach mentioned above proceeds in a few steps. The usual approach to geometric quantization, in particular Kähler quantization, is to consider the holomorphic sections of the prequantization line bundle with respect to some equivariant complex (Kähler) structure on the symplectic manifold $M$. A well-known fact is that in favorable cases this space of holomorphic sections coincides with the index of the Dolbeault operator associated to the complex structure (Kodaira's vanishing Theorem). So a first generalizing step is to define the geometric quantization simply as the index of the Dolbeault operator. However, a manifold $M$ need not always possess an equivariant Kähler structure. A somewhat weaker condition is for $M$ to have an equivariant Spin$^c$-structure. Therefore, one weakens the definition of geometric quantization to being the index of a Spin$^c$-Dirac operator on $M$. This approach makes sense for compact groups, and the index naturally carries a (virtual) representation of the group.

To enlarge the scope of geometric quantization to non-compact groups (and non-compact spaces) Landsman suggested to consider proper cocompact actions. This makes available the machinery of the Baum-Connes conjecture (cf. [2]). The purpose of the Baum-Connes conjecture is to describe the $K$-theory of the $C^*$-algebra of a locally compact group by topological means. Indeed, the topological space under consideration is the classifying space $\underline{E}H$ of *proper* actions of a group $H$. One considers the $K$-homology $K^i(\underline{E}H)$ of this space and a map

$$\mu_{BC} : K_i^H(\underline{E}H) \to K_i(C^*(H)),$$

$i = 0, 1$, called the Baum-Connes analytical assembly map. The group $K_i(C^*(H))$ should be seen as a generalized representation ring $\mathcal{R}(H)$ of $H$. Indeed, for compact $H$ it equals $\mathcal{R}(H)$. The Baum-Connes conjecture states that this map $\mu_{BC}$ is an isomorphism. Morally, this means that any generalized representation is the index of a generalized equivariant Fredholm operator on $\underline{E}H$.

Landsman's suggestion is to use the map

$$\mu_{BC} : K_0^H(M) \to K_0(C^*(H)),$$

which is defined for any locally compact Hausdorff space $M$ on which $H$ acts properly and cocompactly, to define the geometric quantization of a proper, cocompact, Hamiltonian action on $M$. This will be the image of the class in $K_0^H(M)$ of the Spin$^c$-Dirac operator coupled to the prequantization line bundle under the Baum-Connes assembly map $\mu_{BC}$. This approach is indeed successful, cf. [5, 6].

Our plan is to extend the approach another step further. We would like to apply it to (Hamiltonian) actions on noncommutative (symplectic) spaces. We construct a Baum-Connes assembly map

$$\mu_{BC} : KK_i^H(A, B) \to K_i(H \ltimes B),$$

($i = 0, 1$) for $C^*$-algebras $A$ endowed with a proper, counital action of $H$. For this purpose we review the notion of proper actions on $C^*$-algebras by Rieffel (cf. [14]). We introduce the notion of counital action, generalizing cocompact actions to the noncommutative setting. Our main examples of proper, counital actions are certain actions on $C_0(M)$-algebras and reduced groupoid $C^*$-algebras.

The discussion of *symplectic* noncommutative manifolds and Hamiltonian actions should be within the setting of Hochschild/cyclic (co)homology (cf. [20]). We leave this to a later paper.

# 1 Proper actions

In this section we review proper actions of groups on $C^*$-algebras and introduce new classes of examples thereof.

Suppose $G$ is a locally compact unimodular group. Let $\alpha : G \to \mathrm{Aut}(A)$ be a continuous left action of $G$ on a $C^*$-algebra $A$. For a subalgebra $A_0 \subset A$ let $M(A_0)$ denote the subalgebra of the multiplier algebra $M(A)$ of $A$ consisting of $m \in M(A)$ that satisfy $m\, A_0 \subset A_0$

**Definition 1.1.** (cf. [14]) The action $\alpha$ is **proper** with respect to a dense $G$-invariant $*$-subalgebra $A_0$ of $A$ if

(i) for all $a, b \in A_0$
$$x \mapsto \alpha_x(a)\, b$$
is in $L^1(G, A, \lambda)$ with respect to a Haar measure $\lambda$ on $G$ and

(ii) for all $a, b, c \in A_0$ there exists an element $d \in M(A_0)^G$ such that
$$\int_G \alpha_x(a\, b)\, c\, \lambda(dx) = d\, c.$$

**Remark 1.2.** In more recent work [15] Rieffel introduced a more intrinsic notion of proper action without reference to a subalgebra $A_0$. For our purposes it is convenient to stick to the old definition. A special rôle for a subalgebra of a $C^*$-algebra is certainly within the spirit of noncommutative geometry.

A large class of examples of proper actions on $C^*$-algebras is of the following kind. First we recall a definition by G. Kasparov (cf. [7]). Suppose $M$ is a locally compact space endowed with a continuous action of a locally compact group $G$.

**Definition 1.3.** A $C^*$-algebra $A$ is a $C_0(M)$**-algebra** if there exists a homomorphism
$$\pi : C_0(M) \to Z(M(A))$$
and an approximate unit $\{u_i\}$ in $C_c(M)$ such that
$$\lim_{i\to\infty} \|u_i a - a\| = 0.$$
A $G$-$C_0(M)$**-algebra** is a $C_0(M)$-algebra $A$ on which $G$ acts continuously such that
$$x \cdot (\pi(f)a) = \pi(x \cdot f)(x \cdot a). \tag{1.1}$$

One can prove that $G$-$C_0(M)$-algebras correspond to $G$-equivariant upper semi-continuous fields of $C^*$-algebras over $M$.

**Proposition 1.4.** *If $G$ acts properly on $M$ and $A$ is a $G$-$C_0(M)$-algebra then $G$ acts properly on $A$ with respect to $A_0 := C_c(M)A$.*

*Proof.* Equation (1.1) implies that $A_0$ is $G$-invariant. The existence of the approximate unit implies that $A_0$ is dense in $A$. The fact that $\pi$ maps $C_c(M)$ into the *center* of the multiplier algebra of $A$ implies that $A_0$ is a $*$-subalgebra of $A$. Moreover, for $f, f' \in C_c(M)$ and $a, a' \in A$
$$x \cdot (\pi(f)a)\pi(f')a' = \pi((x \cdot f)f')a\, a',$$
hence $x \mapsto x \cdot (\pi(f)a)\pi(f')a'$ has compact support, say $C_{f\, f'}$. One has
$$\int_G x \cdot (\pi(f)a)\pi(f')a'\, dx = \int_{C_{f\, f'}} \pi(x \cdot f)a\, dx\, \pi(f')a',$$
where $\int_{C_{f\, f'}} \pi(x \cdot f)a\, dx$ is in $A$ and hence a multiplier of $A_0$. □

**Example 1.5.** Consider a space $M$ on which $G$ acts properly and a $C^*$-algebra $B$. Let $A$ be the $C^*$ algebra $C_0(M, B)$ of continuous function with values in $B$ that vanish at infinity. Then $A$ is a $G$-$C_0(M)$-algebra, with

$$(\pi(f)\theta)(m) := f(m)\,\theta(m)$$

and

$$(x \cdot \theta)(m) := \theta(x^{-1} \cdot m)$$

for all $x \in G$, $f \in C_0(M)$ and $\theta \in A$. This is Example 2.6 in [14].

## 2 Example: proper actions on groupoid $C^*$-algebras

In this section we introduce another large class of examples. It comes from proper group actions on proper groupoids.

Suppose $H \rightrightarrows M$ is a proper groupoid and $G$ a locally compact unimodular group. Suppose $G$ **acts continuously on the groupoid** $H \rightrightarrows M$ (cf. e.g. [11]), i.e. $G$ acts continuously on $H$ and on $M$ in the usual sense such that the action behaves well with respect to the groupoid structure maps: for all $x \in G$, $m \in M$ and $h, h' \in H$ one has

$$\begin{aligned} s(x \cdot h) &= x \cdot s(h), \\ t(x \cdot h) &= x \cdot t(h), \\ x \cdot (h\,h') &= (x \cdot h)(x \cdot h'), \\ u(x \cdot m) &= x \cdot u(m), \\ (x \cdot h)^{-1} &= x \cdot h^{-1}. \end{aligned}$$

**Remark 2.1.** The quotient $H/G \rightrightarrows M/G$ of such an action of a group $G$ on a groupoid $H \rightrightarrows M$ inherits a groupoid structure from $H \rightrightarrows M$. An example of this is the gauge groupoid of a principal $G$-bundle $P \to M$. This is the quotient of the pair groupoid $P \times P \rightrightarrows P$ under the diagonal action of $H$.

**Remark 2.2.** Consider the group of continuous global bisections $\mathrm{Bis}(H)$ of $H \rightrightarrows M$ ([19]). This group gives rise to "inner" actions on $H \rightrightarrows M$. Indeed, one has the left action of $\mathrm{Bis}(H)$ on $H$ given by $l_\sigma h := \sigma(t(h))h$, the right action given by $r_\sigma h := h(\sigma(s(h)))^{-1}$ and hence conjugation $r \circ l = l \circ r$.

Suppose $H \rightrightarrows M$ is endowed with a Haar system $\{\lambda^m\}_{m \in M}$ that is $G$-invariant, in the sense that

$$\int_{H^m} f(x \cdot h)\lambda^m(dh) = \int_{H^{x \cdot m}} f(h)\lambda^{x \cdot m}(dh)$$

for all $m \in M$ and $x \in G$. LConsider $C_r^*(H)$, the reduced $C^*$-algebra associated to $H \rightrightarrows M$ and $\{\lambda^m\}_{m \in M}$. Suppose that the action of $G$ is proper and that the groupoid $H \rightrightarrows M$ itself is proper. We shall show that there exists an induced action of $G$ on $C_r^*(H)$ which is proper.

Define $A := C_r^*(H)$ and $A_0 := C_c(H)$. The action of $G$ on $A_0$ is defined in the obvious way by

$$(x \cdot f)(h) := f(x^{-1} \cdot h),$$

where $x \in G$, $f \in A_0$ and $h \in H$. We can extend this action to $A$, since $\|x \cdot f\| = \|f\|$ in the reduced $C^*$-algebra norm.

**Lemma 2.3.** *The reduced $C^*$-algebra norm is $G$-invariant on $A_0$, i.e. for all $x \in G$ and $f \in A_0$*

$$\|x \cdot f\| = \|f\|$$

*Proof.* Because of $G$-invariance of the Haar system, one has

$$\begin{aligned} \|x \cdot f\| &= \sup_{\|\xi\|_{\hat{L}^2_t(H)}=1} \|(x \cdot f) * \xi\|_{\hat{L}^2_t(H)} \\ &= \sup_{\|\xi\|_{\hat{L}^2_t(H)}=1} \|f * (x^{-1} \cdot \xi)\|_{\hat{L}^2_t(H)}. \end{aligned}$$

But, since $\|x^{-1} \cdot \xi\|_{\hat{L}^2_t(H)} = \|\xi\|_{\hat{L}^2_t(H)}$, the above term equals $\|f\|$. □

**Lemma 2.4.** *The action of $G$ on $A$ is strongly continuous.*

*Proof.* It suffices to show continuity on $A_0$ at the unit in $G$. Suppose $f \in A_0$ and $x \in G$. Then,

$$\begin{aligned} \|x \cdot f - f\| &= \sup_{\xi \in \hat{L}^2_t(H), \|\xi\|=1} \|(x \cdot f - f) * \xi\| \\ &\leq \|x \cdot f - f\|_{\hat{L}^2_t(H)} \\ &= \sup_{m \in M} \sqrt{\int_{H^m} |f(x^{-1} \cdot h) - f(h)|^2 \lambda^m(dh)}. \end{aligned}$$

By continuity of $f$ one has for any $\varepsilon > 0$ that there is an open neighborhood $U$ of $e$ in $G$ such that $x \in U$ implies $|f(x^{-1} \cdot h) - f(h)| < \varepsilon / \sup_{m \in M} \lambda^m(\mathrm{supp}(f))$. From this the statement easily follows. □

**Proposition 2.5.** *If $G$ acts properly on a proper groupoid $H \rightrightarrows M$, then the induced action of $G$ on the reduced $C^*$-algebra $C^*_r(H)$ is proper with respect to $C_c(H)$.*

*Proof.* One easily checks that $x \cdot (f * f') = (x \cdot f) * (x \cdot f')$ for all $x \in G$ and $f \in A_0 = C_c(H)$. For any $f, f' \in A_0$ the support of $x \mapsto (x \cdot f) * f'$ is compact, since the $G$-action is proper and $H \rightrightarrows M$ is a proper groupoid.

Next, we want to prove property 1.1.ii. Suppose $f, f' \in A_0$ and consider

$$\begin{aligned} \int_G (x \cdot f) * f'(h')\, dx &= \int_G \int_{H^{t(h')}} f(x^{-1} \cdot h)\, f'(h^{-1}h')\, \lambda^{t(h')}(dh)\, dx \\ &= \int_{H^{t(h')}} \int_G f(x^{-1} \cdot h)\, dx\, f'(h^{-1}h')\, \lambda^{t(h')}(dh) \\ &= \tilde{f} * f'(h'), \end{aligned}$$

where

$$\tilde{f}(h) := \int_G f(x^{-1} \cdot h)\, dx.$$

Note that $\tilde{f} \in C_b(H)$. In [16] Proposition 4.3 it is proved that $A = C^*_r(H)$ can be identified with the compact operators on $\hat{L}^2_s(H)$ made $H$-equivariant by averaging. From Lemma 4.4 in the same paper one can conclude that the multiplier algebra $M(A)$ equals the $H$-equivariant (adjointable) bounded operators on $\hat{L}^2_s(H)$. If $\pi_l(f) \in C^*_r(H)$ is an $H$-equivariant and bounded operator then

$$\begin{aligned}\int_G \pi_l(x \cdot f)dx &= \pi_l(\int_G x \cdot f\, dx) \\ &= \pi_l(\tilde{f})\end{aligned}$$

is an $H$-equivariant bounded operator, hence an element of $M(A)$.

Left to show is that $\tilde{f}$ is a left multiplier of $A_0$, in the sense that $\tilde{f}A_0 = A_0$. We first prove this for the case that $H \rightrightarrows M$ is the pair groupoid $M \times M \rightrightarrows M$ with the diagonal action of $G$. Note that, in this case, convolution of $\tilde{f}, f'$ with $f, f' \in C_c(M \times M)$ is given by

$$\tilde{f} * f'(m,p) \int_M \tilde{f}(m,n)f'(n,p)dn,$$

w.r.t. a Radon measure on $M$. Consider the map $M \times M \times M \to \mathbb{C}$ given by $(m,n,p) \mapsto \tilde{f}(m,n)f'(n,p)$. The support of this map is

$$G\ \mathrm{supp}(f) \times M \cap M \times \mathrm{supp}(f').$$

Since the $G$-action on $M$ is proper, this set is compact. Hence the support of $\tilde{f} * f'$ is compact.

The map $t \times s : G \to M \times M$ is a proper groupoid homomorphism. Hence for $f, f' \in C_c(G)$ the support of $\tilde{f} * f'$ is contained in

$$(t \times s)^{-1}\big((t \times s)(G\ \mathrm{supp}(f)) \times M \cap M \times (t \times s)(\mathrm{supp}(f'))\big),$$

which is compact. We conclude that the support of $\tilde{f} * f'$ is compact since it is a closed subset of a compact set. □

**Example 2.6.** As an example of Proposition 2.5, consider a proper action of $G$ on a space $M$ with a Radon measure $\mu$. Then the diagonal action of $G$ on the pair groupoid $M \times M \rightrightarrows M$ is proper. Hence the induced action of $G$ on $C^*(M \times M)$ is proper. Since $C^*(M \times M)$ is canonically isomorphic to the $C^*$-algebra of compact operator on $L^2(M,\mu)$ we have a proper action of $G$ on $\mathcal{K}(L^2(M,\mu))$. This generalizes Rieffel's Example 2.1 of [14].

**Example 2.7.** If $H$ is a compact group. Then $H$ acts on itself from the left, for example by left multiplication or by conjugation. Both actions induce a proper action of $H$ on $C^*_r(H)$ with $A_0 = C_c(H)$.

**Example 2.8.** Suppose $M$ is a space with commuting proper actions of locally compact groups $G$ and $H$. Consider the action groupoid $H \ltimes M \rightrightarrows M$. A proper action of $G$ on $H \ltimes M \rightrightarrows M$ is defined by

$$g \cdot (h,m) := (h, g \cdot m),$$

where $g \in G$, $h \in H$ and $m \in M$. By Proposition 2.5 the induced action of $G$ on

$$C^*(H \ltimes M) \cong H \ltimes C_0(M)$$

is proper. This is Example 2.5 in [14].

**Example 2.9.** Suppose $X$ is an orbifold and $H \rightrightarrows M$ a proper étale groupoid representing $X$, i.e. there exists a homeomorphism $M/H \to X$. A group $G$ acts properly on $X$ iff it corresponds to a proper action of $G$ on $H \rightrightarrows M$. Hence it induces a proper action of $G$ on the $C^*$-algebra $C^*(H)$ by Proposition 2.5.

**Definition 2.10.** We call the action of $G$ on $A$ **amenable** if the natural projection $G \ltimes A \to G \ltimes_r A$ is an isomorphism of $C^*$-algebras. The action is $K$**-amenable** if $K_i(G \ltimes A) \to K_i(G \ltimes_r A)$ is an isomorphism of groups for $i = 0, 1$.

Obviously amenability implies $K$-amenability. A proper action of $G$ on $M$ induces an amenable (and proper) action of $G$ on $C_0(M)$. Also, if $G$ is amenable, then any action of $G$ on a $C^*$-algebra is amenable. But not any proper action of $G$ on a $C^*$-algebra is amenable.

**Example 2.11.** In [14] Rieffel gives the example of $G$ acting on $A := \mathcal{K}(L^2(G))$ by conjugation with the regular representation, which is a special case of Proposition 2.5. In this case,

$$G \ltimes A \cong C^*(G) \otimes \mathcal{K}$$

and

$$G \ltimes_r A \cong C^*_r(G) \otimes \mathcal{K}.$$

Hence if $C^*(G) \ncong C^*_r(G)$, then the action is not amenable. An example of this is $G$ being the free group on two generators.

**Lemma 2.12.** *If $G$ acts properly on $M$ and $A$ is a $G$-$C_0(M)$-algebra, then the action of $G$ on $A$ is amenable.*

*Proof.* This is an immediate consequence of the fact that $G \ltimes M$ is proper groupoid, hence amenable and Theorem 3.4 in [1], which states (among other things) that, if the action of $G$ on $M$ is amenable, then the action of $G$ on $A$ is amenable. □

# 3 Hilbert $C^*$-modules associated to proper actions

The main ingredient in the definition of a noncommutative Baum-Connes analytical assembly map for a action of a group $G$ on a $C^*$-algebra $A$ is a certain Hilbert $G \ltimes_r A$-module associated to this action. For this to exist, the action has to be proper. In this section we shall construct this module, following Rieffel [14]. For the actual construction of the analytical assembly map we shall need an extra condition. This condition corresponds in the commutative case to the action being cocompact. Since compactness of a space $X$ corresponds to the commutative $C^*$-algebra $C_0(X)$ being unital, we shall name actions satisfying this condition counital.

Suppose $G$ is a locally compact unimodular group acting on a $C^*$-algebra $A$. Consider the convolution $*$-algebra $C_c(G, A)$ with convolution defined by ($f, g \in C_c(G, A)$, $x \in G$)

$$f * g(x) := \int_G \alpha_y(f(y^{-1}x))\, g(y)\, \lambda(dy)$$

and involution defined by

$$f^*(x) = \alpha_x(f(x^{-1}))^*,$$

where $\lambda$ is a Haar measure on $G$.

Consider the right representation $C_c(G, A) \to \mathcal{B}(L^2(G, A, \lambda))$ of $C_c(G, A)$ defined by

$$g \mapsto . * g$$

This is just the integration $\int \pi_L(x)\, g(x)\, \lambda(dx)$ of the left regular (covariant) representation $\pi_L$ of $G$ on $L^2(G, A, \lambda)$:

$$\pi_L(y) f(x) := \alpha_y(f(y^{-1}\, x)).$$

The closure of the image of $C_c(G, A)$ under this map is called the **reduced crossed product**, denoted by $G \ltimes_r A$. Obviously, if $A = \mathbb{C}$ we get the reduced group $C^*$-algebra.

Suppose a locally compact group $G$ acts properly on a $C^*$-algebra $A$ with respect to $A_0$. We shall now construct the needed Hilbert $G \ltimes_r A$-module using an appropriate closure of $A_0$. Define a sesquilinear form on $A_0$ with values in $G \ltimes_r A$ by $(a, b \in A_0, x \in G)$

$$\langle a, b \rangle\, (x) := \alpha_x(a^*)\, b.$$

Denote the space of finite linear combinations of elements $\langle a, b \rangle \in G \ltimes_r A$ by $E_0$. Then Rieffel proves the following in [14]:

(i) The space $E_0$ is a $^*$-subalgebra of $G \ltimes_r A$.

(ii) A right action of $E_0$ on $A_0$ is given by $(a \in A_0,\ f \in E_0)$

$$a \cdot f := \int_G \alpha_x(a)\ f(x)\, \lambda(dx).$$

(iii) The pair $(A_0, \langle ., . \rangle)$ forms a full pre-Hilbert $E_0$-module (or rigged $E_0$-space).

(iv) Let $E$ denote the closure of $E_0$ in $G \ltimes_r A$. The algebra $E$ is an ideal in $G \ltimes_r A$.

Let $\bar{A}_0$ denote the closure of $A_0$ with respect to the norm $\|a\| := \sqrt{\|\, \langle a, a \rangle\, \|_{G \ltimes_r A}}$. The action of $E_0$ on $A_0$ extends to an action of $E$ on $\bar{A}_0$ by continuity. The right action of $E$ on $\bar{A}_0$ extends uniquely to a right action of $G \ltimes_r A$, since the action of $E$ on $\bar{A}_0$ is non-degenerate (i.e. $E\, \bar{A}_0$ is dense in $\bar{A}_0$). Indeed, suppose $f \in G \ltimes_r A$ and $a \in \bar{A}_0$. There exists a sequence $\{e_i\, a_i \in E\, \bar{A}_0\}_{i \in \mathbb{N}}$ that converges to $a$. Define

$$a \cdot f := \lim_{i \to \infty} a_i\, (e_i \cdot f).$$

**Proposition 3.1.** *The pair* $(\bar{A}_0, \langle ., . \rangle)$ *forms a Hilbert* $G \ltimes_r A$*-module.*

**Definition 3.2.** We call an action of $G$ on $A$ which is proper with respect to $A_0$ **quantizable** if the associated Hilbert $G \ltimes_r A$-module $(\bar{A}_0, \langle ., . \rangle)$ is rank one and projective.

If this is the case, then a proper, quantizable action of $G$ on $A$ determines a class

$$[(\bar{A}_0, \langle .,. \rangle)] \in K_0(G \ltimes_r A).$$

If the action is K-amenable then $[(\bar{A}_0, \langle .,. \rangle)]$ induces a class in $K_0(G \ltimes A)$. The following Lemma and two Propositions treat the classes of examples that we have been considering before.

**Definition 3.3.** Suppose $G$ is a locally compact group with Haar measure $\lambda$. We say an action of $G$ on a $C^*$-algebra $A$ is **counital** with respect to $\lambda$ and a dense subalgebra $A_0$, if $M(A_0)^G$ is unital and if there exists a positive element $k \in A_0$ such that

$$\int_G \alpha_x(k) a \, \lambda(dx) = a$$

for all $a \in A_0$.

**Lemma 3.4.** *If an action of $G$ on a $C^*$-algebra $A$ is proper and counital with respect to $\lambda$ and a dense subalgebra $A_0$, then the action of $G$ on $A$ is quantizable.*

*Proof.* Suppose $k = (k')^* k'$ for some $k' \in A_0$ The Hilbert $G \ltimes_r A$-module $(\bar{A}_0, \langle .,. \rangle)$ is rank one projective iff there exists an element $k' \in \bar{A}_0$ such that

$$1_{\bar{A}_0} = |k'\rangle\langle k'|.$$

So, in particular, the fact that for all $a \in A_0$ one has

$$|k'\rangle\langle k'|a = \int_G \alpha_x((k')^* k') \, a \, \lambda(dx) = a$$

implies that $(\bar{A}_0, \langle .,. \rangle)$ is rank one projective. □

**Proposition 3.5.** *Suppose the action of $G$ on a proper groupoid $H \rightrightarrows M$ is proper and cocompact, then the action of $G$ on $A := C^*_r(H)$ is proper and counital with respect to $A_0 := C_c(H)$.*

*Proof.* The first part is Proposition 2.5. As for the second part, let $l \in C_c(H)$ be a positive function with support on each $G$-orbit. Such a function exists since the $G$-action is cocompact. Then, the function $k \in C_c(H)$ defined by

$$k(h) := \frac{l(h)}{\int_G l(x^{-1} \cdot h) dx},$$

for all $h \in H$, has the desired property. □

The following definition is a variation on the notion of a quasi-local algebra (cf. e.g. [12]) in the case of a $C_0(M)$-algebra.

**Definition 3.6.** A $C_0(M)$-algebra $A$ is said to have **local units** if for every $f \in C_c(M)$ there exists $u_f \in A$ such that for all $a \in A$

$$\pi(f) \, u_f \, a = \pi(f) \, a.$$

The element $u_f$ is called a local unit for $f$.

**Example 3.7.** Suppose $B$ is a $C^*$-algebra. The $C_0(M)$-algebra $A := C_0(M, B)$ has local units iff $B$ is unital.

**Remark 3.8.** Interpreting $A$ as the space of continuous sections

$$A \cong \Gamma_0(M, \coprod_{m \in M} A_m)$$

of an upper semi-continuous field of $C^*$-algebras $\{A_m\}_{m \in M}$, the above notion of having local units can be stated as follows. For any compact set $C \subset M$ there exists a continuous section $u_C \in A$ such that $u_C(m)$ equals the unit of the fiber $A_m$ at $m$ for all $m \in C$. So, in particular, $A_m$ has to be unital for all $m \in M$.

**Definition 3.9.** A $C_0(M)$-algebra $A$ is said to have **$G$-equivariant local units** if it has local units and if for every $f \in C_c(M)$ and $x \in G$ the element $x \cdot u_f$ is a local unit for $x \cdot f$.

In terms of continuous fields, as in the above remark, this means that $x$ applied to the unit of $A_m$ should equal the unit of $A_{x \cdot m}$.

**Proposition 3.10.** *Suppose $G$ acts properly and cocompactly on $M$. If $A$ is a $G$-$C_0(M)$-algebra that has $G$-equivariant local units, then the $G$ action on $A$ is proper and counital with respect to $A_0 := C_c(M)A$.*

*Proof.* The first part is proven in Proposition 1.4. As for the second part, choose any $f' \in C_c(M)$ with support on all $G$-orbits. Let $u_{f'}$ be a local unit for $f'$. Define $f \in C_c(M)$ by

$$f(m) := \frac{f'(m)}{\int_G f'(x^{-1} \cdot m) dx}$$

and

$$k := \pi(f)\, u_{f'} \in A_0.$$

Then, for any $\pi(g)a \in A_0$ ($g \in C_c(M)$ and $a \in A$), one has

$$\begin{aligned} \int_G (x \cdot k)\pi(g)a\, dx &= \int_G \pi(x \cdot f)(x \cdot u_{f'})\pi(g)a\, dx \\ &= \int_G \pi(x \cdot f)\, dx\, \pi(g)a \\ &= \pi(g)a, \end{aligned}$$

which finishes the proof. □

# 4 The noncommutative analytical assembly map

We are now ready to define the noncommutative analytical assembly map. We shall use equivariant $KK$-theory extensively in what follows. References for this are e.g. [3] or [7].

Suppose a locally compact group $G$ acts on a $C^*$-algebra $A$ in a proper and counital (or, more generally, quantizable) way with respect to $A_0$. Denote the isomorphism class of the associated projective $(G \ltimes_r A)$-module by

$$[\bar{A}_0] \in K_0(G \ltimes_r A).$$

**Theorem 4.1.** *Suppose $G$ acts properly and counitally on $A$ with respect to $A_0$ (or, more generally, the action is quantizable) and $G$ acts on $B$. Then, there exists a canonical (relative to $A_0$) group homomorphism (the **noncommutative analytical assembly map**)*

$$\mu_{BC} : KK_i^G(A, B) \rightarrow\longrightarrow K_i(G \ltimes_r B)$$

*Proof.* The map $\mu_{BC}$ is defined as follows; for any $[\mathcal{H}, \pi, F] \in KK_i^G(A, B)$, it is the 'index' isomorphism (cf. [3])

$$KK_i(\mathbb{C}, G \ltimes_r B) \xrightarrow{\cong} K_i(G \ltimes_r B)$$

applied to the Kasparov product of $[\bar{A}_0] \in KK_0(\mathbb{C}, G \ltimes_r A)$ and $j_r([\mathcal{H}, \pi, F]) \in KK_i(G \ltimes_r A, G \ltimes_r B)$, that is

$$\mu_{BC}([\mathcal{H}, \pi, F]) := \operatorname{index}([\bar{A}_0]\hat{\otimes}_{G\ltimes_r A} j_r([\mathcal{H}, \pi, F])) \in K_i(G \ltimes_r B). \tag{4.2}$$

The fact that it is a group homomorphism follows immediately from the properties of the Kasparov product. □

By the distributivity of the Kasparov product and the fact that $j_r$ is a homomorphism of Abelian groups, this map $\mu_{BC}$ is a homomorphisms of Abelian groups. If the action is K-amenable, then we can use the class in $K_0(\mathbb{C}, G \ltimes A)$ determined by $[\bar{A}_0]$ and the unreduced version of the descent map $j$ to define an analytical assembly map

$$\mu_{BC} : KK_i^G(A, B) \rightarrow K_i(G \ltimes B)$$

analogously to Formula 4.2.

**Example 4.2.** Suppose $H$ is a locally compact group and $K$ a compact subgroup. Consider the action groupoid $G := K \ltimes H \rightrightarrows H$ of the left action of $K$ on $H$ by left multiplication. There exists a proper, cocompact action of $H$ on $G \rightrightarrows H$ given by $h' \cdot (k, h) := (k, h(h')^{-1})$ for $h' \in H$ and $(k, h) \in G = K \ltimes H$. By Proposition 3.5 this induces a quantizable action of $H$ on $C_r^*(G)$. Hence one can construct an analytical assembly map

$$\mu_{BC} : KK_0^H(C_r^*(K \ltimes H), \mathbb{C}) \rightarrow K_0(C_r^*(H)). \tag{4.3}$$

We shall construct $H$-Kasparov $(C_r^*(K \ltimes H), \mathbb{C})$-modules that represent classes of $KK_0^H(C_r^*(K \ltimes H), \mathbb{C})$. Suppose $(\pi, V)$ is a representation of $K$. This induces a representation $\bar{\pi}$ of $K \ltimes H \rightrightarrows M$ on the trivial vector bundle $V \times H \rightarrow H$ defined by

$$\bar{\pi}(k, h)(v, h) := (\pi(k)v, k \cdot h),$$

for $k \in K$, $v \in V$ and $h \in H$.

Suppose $\lambda$ is a Haar measure on $H$. As explained in [13], the measurable representation $\bar{\pi}$ of $K \ltimes H$ on $V \times H$ gives rise to representation $\tilde{\pi}$ of $C_r^*(K \ltimes H)$ on $L^2(H, V, \lambda)$. Obviously, there exists a representation of $H$ on $L^2(H, V \times H, \lambda)$. One easily checks that the triple $(L^2(H, V, \lambda), \tilde{\pi}, 0)$ is an $H$-Kasparov $(C_r^*(K \ltimes H), \mathbb{C})$-module and hence represents a class

$$[L^2(H, V, \lambda), \tilde{\pi}, 0] \in KK_0^H(C_r^*(K \ltimes H, \mathbb{C})$$

Strictly speaking we should have started with an element in the representation ring of $K$, and the $KK$-cycle would consist of the direct sum corresponding to two representations representing the formal difference.

The $K$-theory of the $C^*$-algebra of a compact group $K$ is isomorphic to its representation ring,

$$K_0(C^*_r(K)) \cong \mathcal{R}(K).$$

In particular, an isomorphism is given as follows. A representation $(\pi, V)$ of $K$ corresponds to the projection in $C^*_r(K)$ induced by the function on $K$ given by

$$p_\pi : k \mapsto \dim(V) \langle v, \pi(k)v \rangle ,$$

where $v \in V$ with $\|v\| = 1$. The function $p_\pi \in C(K)$ is interpreted as projection operator in $C^*_r(H)$ by the formula

$$\bar{p}_\pi(f)(h) := \int_K p_\pi(k)\, f(k^{-1}h)\, dk,$$

where $f \in C_c(H)$.

**Lemma 4.3.** *The image of* $[L^2(H, V, \lambda), \tilde{\pi}, 0] \in KK_0^H(C^*_r(K \ltimes H), \mathbb{C})$ *under the assembly map* $\mu$ *(Equation 4.3) is* $[\bar{p}_\pi]$.

Hence the composition is the induction map

$$K_0(C^*_r(K)) \to KK_0^H(C^*_r(K \ltimes H), \mathbb{C}) \to K_0(C^*_r(H))$$

given by $[p_\pi] \mapsto [\bar{p}_\pi]$. This map is well-known, but is usually obtained via $KK_0^H(C_0(H/K), \mathbb{C})$ instead of $KK_0^H(C^*_r(K \ltimes H)$. But $C_0(H/K)$ is Morita equivalent to $C^*_r(K \ltimes H)$, since the groupoids $H/K \rightrightarrows H/K$ and $K \ltimes H \rightrightarrows H$ are Morita equivalent,

$$\begin{array}{ccccc} & & KK_0^H(C^*_r(K \ltimes H), \mathbb{C}) & & \\ & \nearrow & & \searrow & \\ K_0(C^*_r(K)) & & \downarrow \cong & & K_0(C^*_r(H)). \\ & \searrow & & \nearrow & \\ & & KK_0^H(C_0(H/K), \mathbb{C}) & & \end{array}$$

Using the commutativity of this diagram the proof of the Lemma follows from the commutative case (cf. [18]).

**Example 4.4.** Suppose $X$ is an orbifold, represented by a proper étale Lie groupoid $H \rightrightarrows M$, i.e. $X \cong M/H$. Suppose a Lie group $G$ acts properly and cocompactly on $X$. Then $G$ acts properly and cocompactly on $H \rightrightarrows M$. Hence, by Proposition 3.5, $G$ acts properly on $A := C^*_r(H)$ with respect to $A_0 := C_c(G)$ and we have constructed an analytical assembly map

$$\mu_{BC} : KK_0^G(C^*_r(H), \mathbb{C}) \to K_0(C^*_r(G)).$$

**Example 4.5.** We shall use the notation of [12]. Consider a locally finite, directed graph $E = (E_0, E_1)$ and an action of a locally compact group $G$ on $E$. Let $E^*$ denote the set of finite paths in $E$, $s(\mu)$ the start vertex of path $\mu$, $t(\mu)$ the end vertex of $\mu$ and $|\mu|$ the length of $\mu$. Let $A := C^*(E)$ denote the $C^*$-algebra associated to $E$.

The action of $G$ on $E$ induces an action on $C^*(E)$ by

$$x \cdot S_e = S_{x \cdot e}, x \cdot p_v = p_{x \cdot v},$$

for all $x \in G$, $e \in E_1$ and $v \in E_0$.

There is another action on $C^*(E)$, namely the so-called gauge action of the circle (or 1-torus) $\mathbb{T}$ defined by

$$\alpha \cdot S_e = \exp(2\pi\, i\, \alpha)\, S_e, \alpha \cdot p_v = p_v$$

for all $e \in E_1$, $v \in E_0$ and $\alpha \in \mathbb{T}$.

In [12] a class $[V, F] \in KK_1(C^*(E), C^*(E)^{\mathbb{T}})$ is constructed. The Hilbert $C^*(E)^{\mathbb{T}}$-module $V$ is the closure of $A_0$ under the norm obtained from the $C^*(E)^{\mathbb{T}}$-valued inner product given by

$$\langle a, b \rangle := \int_{\mathbb{T}} \alpha \cdot (a^* b)\, d\alpha.$$

One can show that any element in $A_0$ can be written as a finite sum of elements of the form $S_\mu\, S_\nu^*$, where $\mu = e_1 \ldots e_{|\mu|}$ and

$$S_\mu := S_{e_1} \ldots S_{e_{|\mu|}}$$

and the same for $\nu$. The operator $F$ is defined by

$$F := \frac{D}{\sqrt{1 + D^2}},$$

where $D$ is given by the simple formula

$$D(S_\mu\, S_\nu^*) := (|\mu| - |\nu|) S_\mu\, S_\nu^*,$$

for all $\mu, \nu \in E^*$. From this formula one sees at once that $D$ is $G$-equivariant, hence it determines a class in $KK_1^G(C^*(E), C^*(E)^{\mathbb{T}})$.

Suppose the action of $G$ on $C^*(E)$ is proper and counital with respect to a subalgebra $A_0$. Then, we can consider its image under the noncommutative assembly map with respect to $A_0$

$$KK_1^G(C^*(E), C^*(E)^{\mathbb{T}}) \to K_1(G \ltimes_r C^*(E)^{\mathbb{T}}).$$

To what extent can we expect this map to be non-trivial?

**Lemma 4.6.** *If the action of $G$ on $E$ is free and every vertex receives an edge, then*

$$K_1(G \ltimes_r C^*(E)^{\mathbb{T}}) = 0.$$

*Proof.* Firstly, $C^*(E)^{\mathbb{T}}$ is Morita equivalent to $C^*(Z \times E)$, where $Z$ is the graph with edges and vertices indexed by $\mathbb{Z}$ and

$$s(e_k) = v_k, t(e_k) = v_{k+1}.$$

The product graph is defined by $(Z \times E)_i := Z_i \times E_i$ for $i = 0, 1$ and

$$t(z, e) = (t(z), t(e)), s(z, e) = (s(z), s(e)).$$

There is an obvious action of $G$ on $Z \times E$ and $G \ltimes_r C^*(E)^{\mathbb{T}}$ is Morita equivalent to $G \ltimes_r C^*(Z \times E)$. Since the action is free, $G \ltimes_r C^*(Z \times E)$ is isomorphic to

$$C^*((Z \times E)/G) \otimes \mathcal{K}(l^2(G)),$$

(cf. [8]) which equals $C^*(Z \times (E/G)) \otimes \mathcal{K}(l^2(G))$. Hence, since $K_1$ is invariant under Morita equivalence and stable,

$$K_1(G \ltimes C^*(E)^{\mathbb{T}}) \cong K_1(C^*(Z \times (E/G))).$$

But $K_1(C^*(Z \times (E/G))) = \mathbb{Z}^{(\text{number of loops in } Z \times (E/G))}$ (cf. [12]) and one easily sees that there are no loops in $Z \times (E/G)$. □

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