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Dirac's Theorem on Simplicial Matroids

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Abstract. We introduce the notion of k-hyperclique complexes, i.e., the largest simplicial complexes on the set $[n]$ with a fixed k-skeleton. These simplicial complexes are a higher-dimensional analogue of clique (or flag) complexes (case $k = 2$) and they are a rich new class of simplicial complexes. We show that Dirac's theorem on chordal graphs has a higher-dimensional analogue in which graphs and clique complexes get replaced, respectively, by simplicial matroids and k-hyperclique complexes. We prove also a higher-dimensional analogue of Stanley's reformulation of Dirac's theorem on chordal graphs.

Keywords: clique (flag) complexes, Dirac's theorem on chordal graphs, simplicial matroids, k-hyperclique complexes, Helly dual k-property, strong triangulable simplicial matroids

1. Introduction and Notations

Set $[n] := \{1, 2, \ldots, n\}$. The simplicial matroids $\mathrm{S}_k^n(E)$ on the ground set $E \subseteq \binom{[n]}{k}$ have been introduced by Crapo and Rota [6,7] as one of the six most important classes of matroids. These matroids generalize graphic matroids: More precisely $\mathrm{S}_2^n(E)$ is the cycle matroid (or graphic matroid) of the graph $([n], E)$. With the aid of Alexander's duality theorem for manifolds applied to simplices, they prove the following beautiful isomorphism

$$\left[\mathrm{S}_k^n\left(\binom{[n]}{k}\right)\right]^* \simeq \mathrm{S}_{n-k}^n\left(\binom{[n]}{n-k}\right), \quad X \mapsto [n] \setminus X, \tag{1.1}$$

where $\left[\mathrm{S}_k^n\left(\binom{[n]}{k}\right)\right]^*$ denotes the dual (or orthogonal) matroid of $\mathrm{S}_n^k\left(\binom{[n]}{k}\right)$, see [7, Theorem 11.4] or, for an elementary proof (depending only on matrix algebra), [4, Theorem 6.2.1]. In this paper, we use an equivalent definition of simplicial matroids, see Definition 1.4 below.

We introduce the notion of k-hyperclique complexes. These simplicial complexes are a natural higher-dimensional analogue of clique (or flag) complexes (case $k = 2$), see Definition 1.2 below. The k-hyperclique complexes are a rich new class of simplicial complexes of intrinsic interest. To get a better understanding of the structure of $\mathrm{S}_k^n(E)$ we attach to it the k-hyperclique complex on $[n]$ canonically determined by the family E.

In this paper, we introduce the notion of a strong triangulable simplicial matroid, a higher-dimensional generalization of the notion of a chordal graph. We prove an analogue of Dirac's theorem on chordal graphs (see Theorem 3.2) using a natural generalization of a perfect sequence of vertices of a chordal graph (see Theorem 5.2). We prove also a higher-dimensional generalization of Stanley's reformulation of Dirac's theorem on chordal graphs (see Theorem 4.3).

Let us set some preliminary notation.

Definition 1.1. *An* (abstract) oriented simplicial complex *on the set* $[n]$ *is a family* Δ *of linear ordered subsets of* $[n]$ (*called the* faces *of* Δ) *satisfying the following two conditions.* (*We identify the linear ordered set* $\{v_1, v_2, \ldots, v_m\}$, $v_1 < v_2 < \cdots < v_m$, *with the symbol* $v_1 v_2 \cdots v_m$.)

(1.1.1) *Every* $v \in [n]$ *is a face of* Δ.
(1.1.2) *If* F *is a face of* Δ *and* $F' \subset F$, *then* F' *is also a face of* Δ.

Given two faces F' *and* $F = i_1 i_2 \cdots i_m$, *the "incidence number"* $[F' \colon F]$ *is*

$$[F' \colon F] = \begin{cases} (-1)^j, & \text{if } F' = i_1 i_2 \cdots i_{j-1} i_{j+1} \cdots i_m; \\ 0, & \text{otherwise}. \end{cases}$$

Let $\mathcal{S}_d(\Delta)$ denote the *d-skeleton* of Δ, i.e., the family of faces of size d (or d-faces) of Δ. A *facet* is a face of Δ, maximal to inclusion. If nothing in contrary is indicated, we suppose that $\mathcal{S}_1(\Delta) = [n]$.

Definition 1.2. *Let* $\mathcal{S}_k := \{F_1, F_2, \ldots, F_m\}$ *be a family of* k*-subsets of* $[n]$. *Let* $\langle \mathcal{S}_k \rangle$ *be the simplicial complex such that* $F \subseteq [n]$ *is a face of* $\langle \mathcal{S}_k \rangle$ *provided that*:

(1.2.1) $|F| < k$; *or*
(1.2.2) *if* $|F| \geq k$, *every* k*-subset of* F *belongs to* $\{F_1, F_2, \ldots, F_m\}$.

We say that $\langle \mathcal{S}_k \rangle$ *is the* k-hyperclique complex *generated by the set*

$$\mathcal{S}_k = \{F_1, F_2, \ldots, F_m\}.$$

We see $\langle \mathcal{S}_k \rangle$ *as an oriented simplicial complex, with the natural orientation induced by* $[n]$.

Note that $\langle \mathcal{S}_k \rangle$ is the largest simplicial complex Δ on the set $[n]$ with the fixed k-skeleton $\mathcal{S}_k$. (In the hypergraph literature, a family of sets satisfying Property (1.2.2) is said to have the *Helly dual k-property*.)

Throughout this work $\mathcal{S}_k$ denotes a family of k-subsets of $[n]$ and let $\langle \mathcal{S}_k \rangle$ denote the corresponding k-hyperclique complex. $\big($So, we have $\mathcal{S}_k = \mathcal{S}_k(\langle \mathcal{S}_k \rangle)$.$\big)$ The paradigm examples of the k-hyperclique complexes are the clique (or flag) complexes (the 2-hyperclique complexes).

Example 1.3. Set $\mathcal{S}_k = \{F_1, F_2, \ldots, F_m\}$. If $\bigcap_{i=1}^m F_i \neq \emptyset$ then $F_1, \ldots, F_m$ are facets of $\langle \mathcal{S}_k \rangle$. The other facets of $\langle \mathcal{S}_k \rangle$ are the $(k-1)$-subsets of $[n]$ which are in no F_i. For every $k, n \geq k \geq 1$, $2^{[n]}$ is a (full) k-hyperclique complex.

Let $\mathcal{S}_\ell$ be a subset of $\binom{[n]}{\ell}$, where $n \geq \ell \geq 2$. Let $\langle \mathcal{S}_\ell \rangle$ be the oriented ℓ-hyperclique complex determined by $\mathcal{S}_\ell$. Let $\mathbb{F}$ be a field. Consider the two vector spaces $\mathbb{F}^{\mathcal{S}_\ell}$ and $\mathbb{F}^{\binom{[n]}{\ell-1}}$ over $\mathbb{F}$. Let us define the (boundary) map

$$\partial_\ell \colon \mathbb{F}^{\mathcal{S}_\ell} \longrightarrow \mathbb{F}^{\binom{[n]}{\ell-1}}$$

as the vector space map determined by linearity specifying its values in the basis elements:

$$\partial_\ell F = \sum_{F' \in \binom{[n]}{\ell-1}} \big[F' \colon F\big]\, F',$$

for every $F \in \mathcal{S}_\ell$. By duality let us define the (coboundary) map

$$\delta^{\ell-1} \colon \mathbb{F}^{\binom{[n]}{\ell-1}} \longrightarrow \mathbb{F}^{\mathcal{S}_\ell}$$

as the vector space map determined by linearity specifying its values in the basis elements:

$$\delta^{\ell-1} F' = \sum_{F \in E} \big[F' \colon F\big]\, F$$

for every $F' \in \binom{[n]}{\ell-1}$. $\big($The symbol $[F' \colon F]$ denotes the incidence number of the faces F' and F in the oriented simplicial complex $\langle \mathcal{S}_\ell \rangle$.$\big)$

Definition 1.4. [4] *The* simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$*, on the ground set* $\mathcal{S}_k$ *and over the field* $\mathbb{F}$*, is the matroid such that*

$$\{X_1, X_2, \ldots, X_m\} \subseteq \mathcal{S}_k$$

is an independent set *if and only if the vectors,*

$$\partial_k X_1, \partial_k X_2, \ldots, \partial_k X_m,$$

are linearly independent in the vector space $\mathbb{F}^{\binom{[n]}{k-1}}$.

Remark 1.5. [3, 4] Let $(s_{p,q})$ be the matrix whose rows and columns are labeled by the sets of $\binom{[n]}{k-1}$ and $\mathcal{S}_k$ respectively, with $s_{p,q} = 0$ if $p \not\subseteq q$ and $s_{p,q} = (-1)^j$ if $q - p = i_j$, $q = \{i_1, \ldots, i_j, \ldots i_k\}$. The simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ (over the field $\mathbb{F}$) is

the independent matroid of the columns of the $\{-1, 1, 0\}$ matrix $(s_{p,q})$, over the field $\mathbb{F}$. If the matrix $(s_{p,q})$ is not totally unimodular, the simplicial matroid depends of the field $\mathbb{F}$. Since the time of Henri Poincaré, it is known that if $k = 2$, the matrix $(s_{p,q})$ is totally unimodular. The 2-hyperclique complex $\langle \mathcal{S}_2 \rangle$ is the clique complex of the simple graph $([n], \mathcal{S}_2)$ and $\mathrm{S}_2^n(\mathcal{S}_2)$ is its corresponding cycle matroid. So $\mathrm{S}_2^n(\mathcal{S}_2)$ is a regular (or unimodular) matroid, i.e., it is irrespective of the field $\mathbb{F}$.

If nothing in contrary is said, the simplicial matroids here considered are over the field $\mathbb{F}$.

For background, motivation, and matroid terminology left undefined here, see any of the standard references [7, 11, 13, 14] or the encyclopedic survey [15–17]. For a description of the developments on simplicial matroids before 1986, see [4]. See also [2] for an interesting application. For a topological approach to combinatorics, see [1]. Dirac characterization of chordal graphs (see [8]) is treated extensively in [9, Chapter 4]. For an algebraic proof of Dirac's theorem, see [10].

2. Simplicial Matroids

The following two propositions are folklore and they are included for completeness. For every vector v of $\mathbb{F}^E$, where

$$v = a_1 e_1 + a_2 e_2 + \cdots + a_m e_m \ (e_i \in E, a_i \in \mathbb{F}^*),$$

let $\underline{v} := \{e_1, e_2, \ldots, e_m\}$ denote the *support* of v.

Proposition 2.1. *Let $\mathcal{S}_k$ be a subset of $\binom{[n]}{k}$ and $\mathrm{S}_k^n(\mathcal{S}_k)$ be the corresponding simplicial matroid* (*over the field* $\mathbb{F}$). *Consider the linear map $\partial_k \colon \mathbb{F}^{\mathcal{S}_k} \to \mathbb{F}^{\binom{[n]}{k-1}}$. Then*:

(2.1.1) *Each circuit of $\mathrm{S}_k^n(\mathcal{S}_k)$ has at least $k+1$ elements.*
(2.1.2) *For every $(k+1)$-face X of $\langle \mathcal{S}_k \rangle$, $\underline{\partial_{k+1} X}$ is a circuit of $\mathrm{S}_k^n(\mathcal{S}_k)$. Each circuit with exactly $k+1$ elements is of this type.*

For each $X \in \mathcal{S}_{k+1}(\langle \mathcal{S}_k \rangle)$ we say that $\underline{\partial_{k+1} X}$ is a *small circuit* of $\mathrm{S}_k^n(\mathcal{S}_k)$.

Proposition 2.2. *Let $\mathcal{S}_k$ be a subset of $\binom{[n]}{k}$, $k \geq 2$, and $\mathrm{S}_k^n(\mathcal{S}_k)$ be the corresponding simplicial matroid* (*over the field* $\mathbb{F}$). *Consider the linear map $\delta^{k-1} \colon \mathbb{F}^{\binom{[n]}{k-1}} \to \mathbb{F}^{\mathcal{S}_k}$. Then*:

(2.2.1) *The cocircuit space of $\mathrm{S}_k^n(\mathcal{S}_k)$ is generated by the set of vectors $\left\{\delta^{k-1} V \neq 0 \colon V \in \binom{[n]}{k-1}\right\}$.*
(2.2.2) *If non-empty, the set $\underline{\delta^{k-1} V}$, $V \in \binom{[n]}{k-1}$, is a union of cocircuits of $\mathrm{S}_k^n(\mathcal{S}_k)$.*

Proof. The oriented simplicial complex $\left\langle \binom{[n]}{k} \right\rangle = 2^{[n]}$ is the oriented full k-hyperclique complex. The matroid $\mathrm{S}_k^n\left(\binom{[n]}{k}\right)$ is the full simplicial matroid on the ground set $\binom{[n]}{k}$. Consider the linear map

$$\delta^{k-1} \colon \mathbb{F}^{\binom{[n]}{k-1}} \to \mathbb{F}^{\binom{[n]}{k}}. \tag{2.1}$$

From Isomorphism (1.1), we know that

$$\mathcal{C}^* := \left\{ \delta^{k-1} V : V \in \binom{[n]}{k-1} \right\}$$

is a generating set of the cocircuit space of $\mathrm{S}_k^n\left(\mathcal{S}_k'\right)$. The linear map

$$\delta^{k-1} : \mathbb{F}^{\binom{[n]}{k-1}} \longrightarrow \mathbb{F}^{\mathcal{S}_k}$$

is the composition of the map (2.1) and the natural projection

$$\iota : \mathbb{F}^{\binom{[n]}{k}} \longrightarrow \mathbb{F}^{\mathcal{S}_k}.$$

So, Assertion (2.2.1) holds. We know that C^* is a cocircuit of $\mathrm{S}_k^n\left(\binom{[n]}{k}\right)$ if and only if C^* is the support of a non-null vector of $\mathcal{C}^*$, minimal for inclusion. Note that $\mathrm{S}_k^n\left(\mathcal{S}_k\right)^*$ is obtained from $\mathrm{S}_k^n\left(\binom{[n]}{k}\right)^*$ by contracting the set $\binom{[n]}{k} \setminus \mathcal{S}_k$. So, Assertion (2.2.2) holds. ∎

Throughout this work $V, V', V_1, V_2, \ldots$ denote $(k-1)$-subsets of $[n]$. So, they are $(k-1)$-face of $\langle \mathcal{S}_k \rangle$. Let $\langle \mathcal{S}_k \rangle \backslash\backslash V$ denote the k-hyperclique complex $\langle \mathcal{S}_k \setminus \underline{\delta^{k-1} V} \rangle$, i.e., the k-hyperclique complex determined by the set $\mathcal{S}_k \setminus \underline{\delta^{k-1} V}$. Note that, for every pair of $(k-1)$-faces V and V', we have:

$$(\langle \mathcal{S}_k \rangle \backslash\backslash V) \backslash\backslash V' = \left(\langle \mathcal{S}_k \rangle \backslash\backslash V'\right) \backslash\backslash V = \left\langle \mathcal{S}_k \setminus \left(\underline{\delta^{k-1} V} \cup \underline{\delta^{k-1} V'} \right) \right\rangle.$$

Definition 2.3. *Let $\Delta_0 = \langle \mathcal{S}_k \rangle$ be a k-hyperclique complex such that $\mathrm{S}_k^n\left(\mathcal{S}_k\right)$ has rank r. A sequence $V_1, V_2, \ldots, V_r$ of $(k-1)$-faces of Δ_0 is said to be* basic linear sequence *when*

$$C_j^* := \underline{\delta^{k-1} V_j} \setminus \bigcup_{i=1}^{j-1} \underline{\delta^{k-1} V_i}$$

is a cocircuit of $\mathrm{S}_k^n\left(\mathcal{S}_k\left(\Delta_{j-1}\right)\right)$, for $j \in \{1, 2, \ldots, r\}$, where

$$\Delta_{j-1} := \Delta_{j-2} \backslash\backslash V_{j-1}, \quad j \in \{2, \ldots, r\}.$$

The following result is a corollary of Proposition 2.2.

Corollary 2.4. *Let $\langle \mathcal{S}_k \rangle$ be a k-hyperclique complex such that $\mathrm{S}_k^n(\mathcal{S}_k)$ has rank r. If $\mathcal{V} = (V_1, V_2, \ldots, V_r)$ is a basic linear sequence of $(k-1)$-faces of $\langle \mathcal{S}_k \rangle$, then*

$$\beta = \left\{ \delta^{k-1} V_1, \delta^{k-1} V_2, \ldots, \delta^{k-1} V_r \right\}$$

is a basis of the cocircuit space of $\mathrm{S}_k^n(\mathcal{S}_k)$.

Proof. Suppose that β is a dependent set. Choose a dependent subset of β

$$\left\{ \delta^{k-1} V_{i_1}, \delta^{k-1} V_{i_2}, \ldots, \delta^{k-1} V_{i_s} \right\},$$

such that $i_1 < i_2 < \cdots < i_s$ and s is minimum. Therefore

$$\underline{\delta^{k-1} V_{i_s}} \subseteq \bigcup_{j=1}^{s-1} \underline{\delta^{k-1} V_{i_j}}$$

and $V_{i_s} \notin \mathcal{V}$, a contradiction. As the cocircuit space of $\mathrm{S}_k^n(\mathcal{S}_k)$ has dimension r the result follows. ∎

3. D-Perfect k-Hyperclique Complexes

In this section we extend to k-hyperclique complexes the notions of "simplicial vertex" and "perfect sequence of vertices", introduced in the Dirac characterization of the clique complexes of chordal graphs, see [8, 9].

Definition 3.1. Let $\Delta_0 = \langle \mathcal{S}_k \rangle$ be a k-hyperclique complex and suppose that the simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ has rank r. We say that a $(k-1)$-face V is *simplicial* in Δ_0, if there is exactly one facet X of Δ_0 such that $V \subsetneq X$. We say that Δ_0 is D-*perfect* if there is a basic linear sequence of $(k-1)$-faces, $\mathcal{V} = (V_1, V_2, \ldots, V_r)$, such that every $V_i \in \mathcal{V}$ is simplicial in the k-hyperclique complex Δ_{i-1} where

$$\Delta_{i-1} := \Delta_{i-2} \backslash\backslash V_{i-1}, \quad i \in \{2, \ldots, r\}.$$

We will call $\mathcal{V}$ a D-*perfect sequence of* Δ_0.

Chordal graphs are an important class of graphs. The following theorem is one of their fundamental characterizations, reformulated in our language.

Theorem 3.2. (Dirac's theorem on chordal graphs [8,9]) *Let $G = ([n], \mathcal{S}_2)$, $\mathcal{S}_2 \subseteq \binom{[n]}{2}$ be a graph and $\langle \mathcal{S}_2 \rangle$ be its clique complex. Then G is chordal if and only if $\langle \mathcal{S}_2 \rangle$ is* D-*perfect.*

Proposition 3.3. *Let V be a $(k-1)$-subset of $[n]$. If V is simplicial in the k-hyperclique complex $\langle \mathcal{S}_k \rangle$ then $\underline{\delta^{k-1} V}$ is a cocircuit of $\mathrm{S}_k^n(\mathcal{S}_k)$.*

Proof. From Proposition 2.2 we know that $\underline{\delta^{k-1} V}$ is a union of cocircuits of $\mathrm{S}_k^n(\mathcal{S}_k)$. Suppose for a contradiction that there are two different cocircuits C_1^* and C_2^* contained in $\underline{\delta^{k-1} V}$. Choose elements $F_1 \in C_1^* \setminus C_2^*$ and $F_2 \in C_2^* \setminus C_1^*$. As V is simplicial it follows that $C = \binom{F_1 \cup F_2}{k}$ is a circuit of $\mathrm{S}_k^n(\mathcal{S}_k)$ and $C \cap C_1^* = \{F_1\}$, a contradiction to orthogonality. ∎

The reader can easily see that the converse of Proposition 3.3 is not true.

Example 3.4. Set

$$\mathcal{S}_3 = \{123, 124, 125, 145, 245, 136, 137, 167, 367, 238, 239, 289, 389\}.$$

Consider the 3-hyperclique complex $\langle \mathcal{S}_3 \rangle$ on the set $[9]$. From Property (1.2.2) we know that $\mathcal{S}_4(\langle \mathcal{S}_3 \rangle) = \{1245, 1367, 2389\}$ and $\mathcal{S}_5(\langle \mathcal{S}_3 \rangle) = \emptyset$. From Property (1.2.1)

we can see that the sets of 2-faces and 1-faces of $\langle \mathcal{S}_3 \rangle$ are respectively $\mathcal{S}_2(\langle \mathcal{S}_3 \rangle) = \binom{[9]}{2}$ and $\mathcal{S}_1(\langle \mathcal{S}_3 \rangle) = \binom{[9]}{1}$. We can see that the set of facets of $\langle \mathcal{S}_3 \rangle$ is

$$\{18, 19, 26, 27, 34, 35, 46, 47, 48, 49, 56, 57,\\ 58, 59, 68, 69, 78, 79, 123, 1245, 1367, 2389\}.$$

Note that $\mathrm{S}_3^9(\mathcal{S}_3)$ has rank 10 and $\langle \mathcal{S}_3 \rangle$ is D-perfect with the D-perfect sequence: $45, 67, 89, 15, 14, 16, 17, 28, 29, 12$.

Proposition 3.5. *Let V be a $(k-1)$-subset of $[n]$. Suppose that V is not a facet of the k-hyperclique complex $\langle \mathcal{S}_k \rangle = \langle F_1, F_2, \ldots, F_m \rangle$. Then the following two assertions are equivalent*:

(3.5.1) *V is simplicial in $\langle \mathcal{S}_k \rangle$.*
(3.5.2) *The set $X = \bigcup_{F_i \in \underline{\delta^{k-1} V}} F_i$ is the unique facet of $\langle \mathcal{S}_k \rangle$ containing V.*

Proof. The implication (3.5.2) $\Rightarrow$ (3.5.1) is clear.

(3.5.1) $\Rightarrow$ (3.5.2) Let X' be the unique facet of $\langle \mathcal{S}_k \rangle$ containing V. Then it is clear that $F_i \subseteq X'$ for each F_i containing V. We conclude that $X \subseteq X'$ and so X is a face of $\langle \mathcal{S}_k \rangle$. Suppose, for a contradiction, that X is not a facet of $\langle \mathcal{S}_k \rangle$. Then there is an $F \in \mathcal{S}_k$ such that $F \not\subset X$ but $F \subset X'$. So, for every $x \in F \setminus X$, we know that $V \cup x \in \mathcal{S}_k$ and so $V \cup x \in \underline{\delta^{k-1} V}$. We have the contradiction $F \subset X$. Therefore $X = X'$. ∎

4. Superdense Simplicial Matroids

A matroid M on the ground set $[n]$ and of rank r is called *supersolvable* if it admits a maximal chain of modular flats

$$\mathrm{cl}(\emptyset) = X_0 \subsetneq X_1 \subset \cdots \subsetneq X_{r-1} \subsetneq X_r = [n]. \tag{4.1}$$

The notion of "supersolvable lattices" was introduced and studied by Stanley in [12]. For a recent study of supersolvability for chordal binary matroids, see [5].

Proposition 4.1. *Let $\mathrm{S}_k^n(\mathcal{S}_k)$, $k > 2$, be a simplicial matroid. The matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ is supersolvable if and only if it does not have circuits.*

Proof. All the circuits of $\mathrm{S}_k^n(\mathcal{S}_k)$ have at least $k+1$ elements. So a hyperplane H is modular if and only if $|\mathcal{S}_k \setminus H| = 1$. Indeed, if $F, F' \in \mathcal{S}_k \setminus H$, then the line $\mathrm{cl}(\{F, F'\})$ cannot intersect the hyperplane H. From (4.1) we conclude that if $\mathrm{S}_k^n(\mathcal{S}_k)$ is supersolvable then it cannot have circuits. The converse is clear. ∎

So, the notion of supersolvability is not interesting for the class of non-graphic simplicial matroids. The following definition gives the "right" extension of the notion of supersolvable.

Definition 4.2. *Suppose that $\mathrm{S}_k^n(\mathcal{S}_k)$ has rank r. A hyperplane H of $\mathrm{S}_k^n(\mathcal{S}_k)$ is said to be* dense *if there is a simplicial $(k-1)$-face, V, of $\langle \mathcal{S}_k \rangle$ such that*

$$H = \mathcal{S}_k \setminus \underline{\delta^{k-1} V}.$$

We say that the simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ *is* superdense *if it admits a maximal chain of "relatively dense" flats*

$$\emptyset = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{r-1} \subsetneq X_r = \mathcal{S}_k,$$

i.e., such that X_i *is a dense hyperplane of* $\mathrm{S}_k^n(X_{i+1})$, $i \in \{0, 1, \ldots, r-1\}$.

A hyperplane H of $\mathrm{S}_2^n(\mathcal{S}_2)$ is dense if and only if H is modular. Then $\mathrm{S}_2^n(\mathcal{S}_2)$ is superdense if and only if it is supersolvable. So, Theorem 4.3 below can be seen as higher-dimensional generalization of Stanley's reformulation of Dirac's theorem on chordal graphs, see [12].

Theorem 4.3. *Let* $\Delta_0 = \langle \mathcal{S}_k \rangle$ *be a k-hyperclique complex. Then the following two assertions are equivalent*:

(4.3.1) Δ_0 *is* D-*perfect;*
(4.3.2) $\mathrm{S}_k^n(\mathcal{S}_k)$ *is superdense.*

Proof. (4.3.1) $\Rightarrow$ (4.3.2) Let $\mathcal{V} = (V_1, V_2, \ldots, V_r)$ be a D-perfect sequence of Δ_0. From Proposition 3.3 we know that the sets

$$C_j^* := \underline{\delta^{k-1} V_j} \setminus \bigcup_{i=1}^{j-1} \underline{\delta^{k-1} V_i}, \quad j \in \{1, 2, \ldots, r\},$$

are cocircuits of $\mathrm{S}_k^n(\mathcal{S}_k(\Delta_{j-1}))$ where

$$\Delta_{j-1} := \Delta_{j-2} \setminus\setminus V_{j-1}, \quad j \in \{2, 3, \ldots, r\}.$$

So, $\mathcal{V}$ determines a maximal chain of flats of $\mathrm{S}_k^n(\mathcal{S}_k)$:

$$\emptyset = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_{r-1} \subsetneq X_r = \mathcal{S}_k,$$

where

$$X_{r-j} = \mathcal{S}_k(\Delta_{j-1}) \setminus C_j^*, \quad j = 1, \ldots, r.$$

As V_j is simplicial in Δ_{j-1}, we know that X_{r-j} is dense in $\mathrm{S}_k^n(\mathcal{S}_k(\Delta_{j-1}))$. So, $\mathrm{S}_k^n(\mathcal{S}_k)$ is superdense. The proof of the converse part is similar. ∎

5. Triangulable Simplicial Matroids

Now we introduce a generalization of the notion of "triangulable" for the classes of simplicial matroids. Given a union of circuits D of $\mathrm{S}_k^n(\mathcal{S}_k)$, let $\overrightarrow{D}$ denote a vector of $\mathbb{F}^{\mathcal{S}_k}$ whose support is D. Set $\underline{\overrightarrow{D}} = D$.

Definition 5.1. *Let* $\langle \mathcal{S}_k \rangle = \langle F_1, F_2, \ldots, F_m \rangle$ *be a k-hyperclique complex. We say that* $\mathrm{S}_k^n(\mathcal{S}_k)$ (*over the field* $\mathbb{F}$) *is* triangulable *provided that the vector family*

$$\{\partial_{k+1} X : X \in \mathcal{S}_{k+1}(\langle \mathcal{S}_k \rangle)\}$$

spans the circuit space.

Moreover, when generators $\partial_{k+1}X_1, \partial_{k+1}X_2, \ldots, \partial_{k+1}X_{m'}$ *can be chosen such that, for every circuit C, there are non-null scalars* $a_j \in \mathbb{F}^*$ *such that*

$$\overrightarrow{C} = \sum_{j=1}^{s} a_j \partial_{k+1} X_{i_j} \quad \textit{and} \quad \bigcup_{F_\ell \in \underline{C}} F_\ell = \bigcup_{i=1}^{s} X_{i_j} \quad \textit{where} \quad X_{i_j} \in \{X_1, \ldots, X_{m'}\},$$

we say that $\mathrm{S}_k^n(\mathcal{S}_k)$ *is* strongly triangulable.

Note that we can replace in Definition 5.1 the circuit C by a union of circuits D. It is clear that a simple graph $([n], \mathcal{S}_2)$ is chordal if and only if $\mathrm{S}_2^n(\mathcal{S}_2)$ is strongly triangulable. Theorem 5.2 is the possible generalization of Dirac's theorem on chordal graphs (see Theorem 3.2 above). Indeed, if $k > 2$, the converse of Theorem 5.2 is not true, see the remarks following the theorem.

Theorem 5.2. *Let* $\Delta_0 = \langle \mathcal{S}_k \rangle = \langle F_1, F_2, \ldots, F_m \rangle$ *be a k-hyperclique complex. If* Δ_0 *is* D*-perfect, then* $\mathrm{S}_k^n(\mathcal{S}_k)$ *is strongly triangulable.*

Proof. The proof is algorithmitic. Let $\mathcal{V} = (V_1, \ldots, V_r)$ be a D-perfect sequence. Let D be a union of circuits of $\mathrm{S}_k^n(\mathcal{S}_k)$. Let V_i be the first $(k-1)$-face of $\mathcal{V}$ contained in an element of D. From the definitions we know that V_i is a simplicial $(k-1)$-face of Δ_{i-1} and D is a union of circuits of $\mathrm{S}_k^n(\mathcal{S}_k(\Delta_{i-1}))$, where

$$\Delta_{i-1} = \Delta_{i-2} \setminus\setminus V_{i-1}, \quad i \in \{2, \ldots, r\}.$$

From Proposition 3.3 we know that

$$C_j^* := \underline{\delta^{k-1} V_j} \setminus \bigcup_{i=1}^{j-1} \underline{\delta^{k-1} V_i}$$

is a cocircuit of $\mathrm{S}_k^n(\mathcal{S}_k(\Delta_{i-1}))$. Set $D \cap C_j^* = \left\{F_{i_1}, F_{i_2}, \ldots, F_{i_h}\right\}$ and consider the family of vectors of $\mathbb{F}^{\mathcal{S}_k}$

$$\left\{\overrightarrow{C_s} = \partial_{k+1}\left(F_{i_1} \cup F_{i_s}\right), \quad s = 2, \ldots, h\right\}.$$

Express a vector $\overrightarrow{D}$ of support D in the canonical basis, say,

$$\overrightarrow{D} = a_{i_1} F_{i_1} + a_{i_2} F_{i_1} + \cdots + a_{i_h} F_{i_h} + a_{i_{h+1}} F_j + \cdots + a_{i_m} F_{i_m},$$

where $a_{i_\ell} \in \mathbb{F}^*, \ell = 1, \ldots, h$, $a_{i_\ell} \in \mathbb{F}$, $\ell = h+1, \ldots, m$, and $\{F_{i_1}, \ldots, F_{i_m}\} = \mathcal{S}_k$. For every $s \in \{2, 3, \ldots, h\}$, it is possible to choose $b_s \in \mathbb{F}^*$ such that F_{i_s} does not belong to the support of $b_s \overrightarrow{C_s} + \overrightarrow{D}$. As $(C_s \cap D) \cap C_j^* = \{F_{i_1}, F_{i_s}\}$ it follows that $F_{i_2}, F_{i_3}, \ldots, F_{i_h}$ does not belong to the support of

$$\overrightarrow{D'} := \overrightarrow{D} + b_2 \overrightarrow{C_2} + b_3 \overrightarrow{C_3} + \cdots + b_h \overrightarrow{C_h}.$$

The dependent set D' is a union of circuits and $D' \cap C_j^* \subseteq \{F_{i_1}\}$. So, by orthogonality we have $D' \cap C_j^* = \emptyset$. Note that

(i) for every $V_j \in \mathcal{V}$, $1 \leq j \leq i$, no element of D' contains V_j;

(ii)
$$\bigcup_{F_\ell \in D} F_\ell = \bigcup_{F_{\ell'} \in \bigcup_{s=2}^{h} C_s \cup D'} F_{\ell'}.$$

Replace D by the set D' and apply the same arguments. From (i) we know that the algorithm finish. It finishes only if D' is a small circuit. So the theorem follows. ∎

If $k > 2$, the converse of Theorem 5.2 is not true. Indeed, consider the triangulation of a projective plane

$$F_1 = 124, F_2 = 126, F_3 = 134, F_4 = 135, F_5 = 165,$$

$$F_6 = 235, F_7 = 236, F_8 = 245, F_9 = 346, F_{10} = 456.$$

Consider the 3-hyperclique complex $\langle \mathcal{S}_k \rangle = \langle F_1, F_2, \ldots, F_{10} \rangle$ on the set $[6]$. The simplicial matroid $\mathrm{S}_3^n(\mathcal{S}_k)$, over a field $\mathbb{F}$ of characteristic different of 2, does not have circuits and then it is (trivially) strongly triangulable. Every 2-face of a F_i is contained in another F_j, $j \in \{1, \ldots, 10\}$, $j \neq i$. The facets of $\langle \mathcal{S}_k \rangle$ are the sets $F_1, F_2, \ldots, F_{10}$ and all the 2-faces of $\langle \mathcal{S}_k \rangle$ are not contained in an F_i. Then $\langle \mathcal{S}_k \rangle$ does not contain simplicial 2-faces and it is not D-perfect.

We remark that the cycle matroid of a non-chordal graph can be triangulable. More generally, we have

Proposition 5.3. *For any $n, k, n-3 \geq k \geq 2$, there is a k-hyperclique complex $\langle \mathcal{S}_k \rangle$ such that*

(5.3.1) *the simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ is triangulable but not strongly triangulable;*
(5.3.2) *$\langle \mathcal{S}_k \rangle$ does not contain a simplicial $(k-1)$-face.*

Proof. Let $\langle \mathcal{S}_k \rangle$ be the k-hyperclique complex where

$$\mathcal{S}_k = \binom{12\cdots(k+1)}{k} \bigcup \binom{23\cdots(k+2)}{k} \setminus 23\cdots(k+1) \bigcup$$
$$\bigcup_{i=1}^{k+1} \binom{12\cdots\widehat{i}\cdots(k+1)n}{k} \bigcup \bigcup_{j=2}^{k+2} \binom{23\cdots\widehat{j}\cdots(k+2)n}{k}.$$

The simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ has $2k$ small circuits,

$$C_i := \binom{12\cdots\widehat{i}\cdots(k+1)n}{k}, \quad i \in \{1, 2, \ldots, k+1\},$$

$$C_j := \binom{23\cdots\widehat{j}\cdots(k+2)n}{k}, \quad j \in \{2, 3, \ldots, k+2\}.$$

The set

$$C := \binom{12\cdots(k+1)}{k} \bigcup \binom{23\cdots(k+2)}{k} \setminus 23\cdots(k+1)$$

is a circuit, symmetric difference of all the $2k$ small circuits. So, the simplicial matroid $\mathrm{S}_k^n(\mathcal{S}_k)$ over a field of characteristic 2 is triangulable but not strongly triangulable. The reader can check that there do not exist simplicial $(k-1)$-faces in $\langle \mathcal{S}_k \rangle$. ∎

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