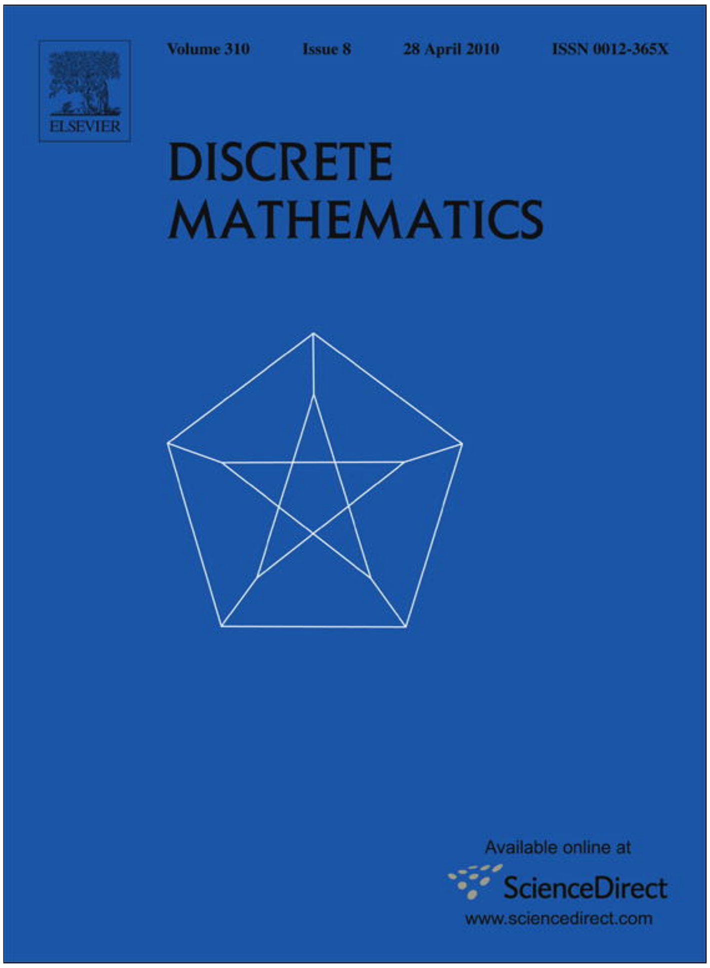
Volume 310
Issue 8
28 April 2010
ISSN 0012-365X
ELSEVIER
DISCRETE
MATHEMATICS
Available online at
ScienceDirect
www.sciencedirect.com

Discrete Mathematics 310 (2010) 1354–1365

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The 3-connected matroids with circumference 6

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ARTICLE INFO

Article history:
Received 15 January 2008
Received in revised form 1 October 2009
Accepted 7 January 2010
Available online 20 January 2010

Keywords:
Matroid
Circumference
3-connected

ABSTRACT

We construct all 3-connected matroids with circumference equal to 6 having rank at least 8. A matroid belongs to this family if and only if it is a generalized parallel connection of a set of planes along a common line (which may have some virtual points).

1. Introduction

We assume familiarity with matroid theory. The notation and terminology used in this article follow Oxley [10]. In recent years, the circumference of a matroid has appeared in some bounds, for example, in an upper bound for the size of a minimally n-connected matroid and in a lower bound for the size of a n-connected matroid having a circuit whose deletion is also n-connected, for $n \in \{2, 3\}$ (see [6–8]). Using these bounds and results about matroids with small circumference, it is possible to improve some bounds found in the literature.

In this paper, we construct all 3-connected matroids with circumference 6 and large rank. This is the first interesting case because there is no relevant family of 3-connected matroids with circumference smaller than 6, since Lemos and Oxley [8] proved the following:

Theorem 1.1. *Suppose that M is a* 3*-connected matroid. If $r(M) \geq 6$, then $\mathrm{circ}(M) \geq 6$.*

By this result, every 3-connected matroid with circumference at most 5 has rank at most 5. Junior and Lemos [4] proved that a 3-connected matroid having a rank at most 5 is Hamiltonian, unless it is isomorphic to $U_{1,1}$, F_7^*, $\mathrm{AG}(3, 2)$, J_9, or J_{10}, where J_{10} is the matroid whose representation over $GF(2)$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and J_9 is the matroid obtained from J_{10} by deleting the last column.

Junior [3] constructed all matroids with circumference at most 5. With the knowledge of all matroids with circumference c, for example, one can calculate all the Ramsey numbers $n(c + 1, y)$ for matroids, for every value of y (for a

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.01.001

definition of $n(x, y)$ see Reid [13]). These numbers were completely determinate by Lemos and Oxley [9] using a sharp bound for the number of elements of a connected matroid as a function of its circumference and cocircumference.

The definitions of paddle and k-separating set, for a positive integer k, can be found in [11,12]. For an integer n exceeding 2, a family $C_1^*, \ldots, C_n^*$ of pairwise disjoint cocircuits of a 3-connected matroid M is said to be a *book* of M provided:

$$r(C_I^*) = |I| + 2, \tag{1.1}$$

for every non-empty subset I of $\{1, \ldots, n\}$, where $C_I^* = \bigcup_{i \in I} C_i^*$. For $i \in \{1, \ldots, n\}$, choose $a_i \in C_i^*$. Observe that $\{a_i : i \in I\}$ is a set of coloops of $M \setminus (C_I^* - \{a_i : i \in I\})$. In particular,

$$r(E(M) - C_I^*) = r(M \setminus (C_I^* - \{a_i : i \in I\})) - |\{a_i : i \in I\}| \leq r(M) - |I|. \tag{1.2}$$

By (1.1) and (1.2), C_I^* is a 3-separating set of M unless $I = \emptyset$ or $I = \{1, \ldots, n\}$ and $|E(M) - C_{\{1,\ldots,n\}}^*| \leq 2$. By Lemmas 2.1 and 4.7 of [11], $\{C_1^*, \ldots, C_n^*, E(M) - C_{\{1,\ldots,n\}}^*\}$ is a paddle of M provided $|E(M) - C_{\{1,\ldots,n\}}^*| \geq 3$. If $L = \mathrm{cl}_M(C_1^*) \cap \mathrm{cl}_M(C_2^*)$, then $\mathrm{cl}_M(C_I^*) = C_I^* \cup L$, for every $\emptyset \neq I \subseteq \{1, \ldots, n\}$. We say that L is the *back* of this book. Our main result is the following:

Theorem 1.2. *If M is a 3-connected matroid such that $r(M) \geq 8$, then the following statements are equivalent:*

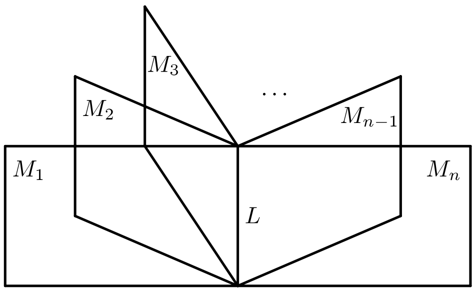
(i) $\mathrm{circ}(M) = 6$.
(ii) *There is a book $C_1^*, \ldots, C_n^*$ of M having back L such that $\{C_1^*, \ldots, C_n^*, L\}$ is a partition of $E(M)$.*

In the previous theorem, it is easy to see that (ii) implies (i). Therefore we only need to establish (ii) assuming (i). This shall be done in Section 5. Now, we sketch its proof. We fix a circuit C of M such that $|C| = 6$. In Section 2, we establish that each connected component of M/C has rank 0 or 1. In particular, $E(M) - \mathrm{cl}_M(C)$ is the disjoint union of cocircuits $C_1^*, C_2^*, \ldots, C_{n-3}^*$ of M. It is not difficult to prove the existence of a 3-subset Z_i of C_i^* which is independent in M. In Section 3, we describe the possibilities for $M|(C \cup Z_i)$. In Section 4, we show that $A_i \cup A_j \cup A_k$ is a dependent set of M, where i, j and k are pairwise different and A_l is any 2-subset of C_l^*. This piece of information is essential to establish, in Section 5, that $C_1^*, C_2^*, \ldots, C_{n-3}^*$ is a book of M. Moreover, this book can be increased with 3 more cocircuits, say C_{n-2}^*, C_{n-1}^* and C_n^*, such that this new book satisfies (ii) of Theorem 1.2.

For an integer n exceeding 2, a family of 3-connected matroids $M_1, \ldots, M_n$ is said to be a *tangerine* having *stem* L provided:

(i) $r(M_i) = 3$, for every $i \in \{1, \ldots, n\}$; and
(ii) L is a modular line of M_i, for every $i \in \{1, \ldots, n\}$; and
(iii) $E(M_i) \cap E(M_j) = L$, for every 2-subset $\{i, j\}$ of $\{1, \ldots, n\}$.

A geometric representation of a tangerine is given in the next figure.



The main result of this paper can be restated as:

Theorem 1.3. *If M is a 3-connected matroid such that $r(M) \geq 8$, then the following statements are equivalent:*

(i) $\mathrm{circ}(M) = 6$.
(ii) *There is a tangerine $M_1, \ldots, M_n$ having L as stem such that, for some $X \subseteq L$, $M = P_L(M_1, \ldots, M_n) \backslash X$, where $P_L(M_1, \ldots, M_n)$ denotes the generalized parallel connection of $M_1, \ldots, M_n$.*

Note that these results generalize the construction given by Cordovil, Junior and Lemos [2] of the 3-connected binary matroids having circumference 6 and large rank. We tried but we could not construct the 3-connected matroids with circumference 7 and a large rank, that is, we could not generalize the other main result of Cordovil, Junior and Lemos [2]. In the future, we hope to construct all the 3-connected matroids with circumference 6 and small rank. One needs this result, for example, to obtain all the matroids with circumference 6.

2. Contracting a maximum size circuit

In this section, we accomplish a major step necessary to establish the main result of this paper. We completely describe the result of a maximum size circuit contraction in a 3-connected matroid having circumference 6. The first difficulty in dealing with 3-connected matroids with circumference 7 is to show a similar result.

The proof of the next proposition is close to the proof of Proposition 2.1 of [2]. Dealing only with matroids having circumference 6 simplifies the proof, since we can use symmetry. New subcases emerge because we are considering non-binary matroids. Our approach to the construction of the non-binary matroids with circumference 6 became distinct from the binary case after the proof of Proposition 2.1, otherwise the number of subcases that we need to consider will become enormous. The other approach was appropriate to deal with the binary matroids with both circumference 6 and 7. We remind the reader that the lower bound on the rank is essential in Proposition 2.1.

In this paper, we use Tutte's geometry. It is natural to use Tutte's geometry because we are dealing with circuits of a matroid. A *T-flat* of a matroid M is a union of circuits of M. The *dimension* of a T-flat F of M is defined as $\dim(F) = r^*(F) - 1$. Note that a T-flat F of M has dimension 0 if and only if F is a circuit of M. A T-flat having dimension 1 or 2 in M is called respectively a *T-line* or a *T-plane* of M. For a T-flat F of a matroid M such that $\dim(F) \geq 1$, the set

$$\pi(F) = \{X \subseteq F : F - X \text{ is a T-flat of } M \text{ so that } \dim(F) = \dim(F - X) + 1\}$$

is a partition of F called *the canonical partition of* F. When F is a T-line of M, X belongs to $\pi(F)$ if and only if $F - X$ is a circuit of $M|F$. Note that $\pi(F)$ is the set of series classes of $M|F$. In particular,

(T1) If C is a circuit of M contained in a T-flat F of M, then

$$\begin{aligned} C &= \cup\{X : X \in \pi(F) \text{ and } X \subseteq C\} \\ &= \cup\{F - F' : F' \text{ is a T-flat of } M, C \not\subseteq F' \subseteq F \text{ and } \dim(F') = \dim(F) - 1\}. \end{aligned}$$

A T-flat F of M is said to be *connected* provided $M|F$ is connected. We also need the following results.

(T2) If L_1 and L_2 are different T-lines contained in a T-plane of M, then $L_1 \cap L_2$ contains exactly one circuit of M (4.171 of [15]).

(T3) If L is a connected T-line of M, then $|\pi(L)| \geq 3$ (4.23 of [15]).

(T4) If C is a circuit of $M|P$, where P is a connected T-plane of M, then there are at least two connected T-lines of $M|P$ containing C (4.26 of [15]).

Now, assume that L is a T-line of M having canonical partition $\{L_1, L_2, \ldots, L_n\}$, for some $n \geq 2$. Note that C is a circuit of $M|L$ if and only if $C = L - L_i$, for some $i \in \{1, 2, \ldots, n\}$. If C_1 and C_2 are different circuits of M contained in L, then $L - C_1$ and $L - C_2$ belong to the canonical partition of L, say $L_1 = L - C_1$ and $L_2 = L - C_2$. Therefore $L = C_1 \cup C_2$. Moreover, when $n \geq 3$, $L - L_3$ is a circuit of M that contains $C_1 \triangle C_2 = L_1 \cup L_2$. We resume these observations in the next two properties:

(T5) The union of two different circuits contained in a T-line L of M is equal to L.

(T6) The symmetric difference of two different circuits contained in a connected T-line L of M is contained in another circuit C of $M|L$. Moreover, C may be chosen avoiding any element belonging to the intersection of these two circuits.

A result of Seymour [14] that gives conditions to extend a k-separation of a restriction to the whole matroid will be fundamental in the proof of the next proposition. To apply this result, we need to give more definitions. Let M be a matroid. For $F \subseteq E(M)$, an *F-arc* (see Section 3 of [14]) is a minimal non-empty subset A of $E(M) - F$ such that there exists a circuit C of M with $C - F = A$ and $C \cap F \neq \emptyset$. Such a circuit C is called an *F-fundamental for A*. Let A be a F-arc and $P \subseteq F$. Then $A \to P$ if there is a F-fundamental for A contained in $A \cup P$. Thus $A \not\to P$ denotes that there is no such F-fundamental. Note that, when F is a connected T-flat of M, A is an F-arc if and only if $F \cup A$ is a connected T-flat of M such that $\dim(F \cup A) = \dim F + 1$. We use also the next observation:

(T7) Let L be a T-line of M and let A be an L-arc. If C is an L-fundamental for A and D is a circuit of $M|L$ such that $L - D \not\subseteq C$, then $C \cup D$ is T-line of M. Moreover, $C \cup D$ is connected provided $A \not\to L - D$.

Note that (T7) holds because $C \cup D$ is a T-flat of M satisfying $C \subsetneq C \cup D \subsetneq L \cup A$; C and $L \cup A$ are T-flats of M having respectively dimensions 0 and 2; and the dimension is an increasing function.

Proposition 2.1. *Suppose that M is a 3-connected matroid such that $r(M) \geq 8$. If $\operatorname{circ}(M) = 6$, then the rank of every connected component of M/C is at most one, for every maximum size circuit C of M.*

Proof. Fix a maximum size circuit C of M. It is enough to show that $\operatorname{circ}(M/C) \leq 2$ because a connected matroid with a circumference of 1 or 2 is isomorphic to $U_{0,1}$ or $U_{1,n}$, for some $n \geq 2$, respectively. Suppose that $\operatorname{circ}(M/C) \geq 3$. By Lemma 2.2 of [2], there is a circuit A of M/C such that $|A| \geq 3$ and A is an C-arc. Hence $L = C \cup A$ is a connected T-line of M. If the canonical partition of L is equal to $\{X_1, X_2, \ldots, X_m\}$, for some m exceeding two, then $A = X_i$, for some $i \in \{1, 2, \ldots, m\}$, say $A = X_1$. As $C = L - A$ is a circuit of M having maximum size, it follows that $3 \leq |A| \leq |X_i|$, for every $i \in \{1, 2, \ldots, m\}$. Thus $m = 3$, $|A| = 3$ and $|X_2| = |X_3| = 3$ because

$$6 = |C| = |L - A| = |X_2| + |X_3| + \cdots + |X_m| \geq |A|(m - 1) \geq \max\{3(m - 1), 2|A|\} \geq 6.$$

In resume, the canonical partition of L is $\{X_1, X_2, X_3\}$ and so $X_1 \cup X_2$, $X_1 \cup X_3$ and $X_2 \cup X_3$ are the circuits of M contained in L. Moreover, each of them has 6 elements. In particular, there is a symmetry between X_1, X_2 and X_3. This symmetry shall be used in this proof.

Let $\mathcal{A}$ be the set of L-arcs. For a 2-subset $\{i, j\}$ of $\{1, 2, 3\}$, we define $\mathcal{A}_i = \{A' \in \mathcal{A} : A' \to X_i\}$, $\mathcal{A}_{ij} = \{A' \in \mathcal{A} : A' \to X_i \cup X_j\}$, $\mathcal{A}' = \mathcal{A} - (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$ and $\mathcal{A}'' = \mathcal{A}_{12} \cap \mathcal{A}_{13} \cap \mathcal{A}_{23}$. We divide the proof into some lemmas.

Lemma 2.1. *If $A' \in \mathcal{A}'$, then $|A'| = 1$. Moreover,*

(i) *$A' \in \mathcal{A}''$; or*
(ii) *there is a circuit $C_{A'}$ of M such that $A' = C_{A'} - L$ and $(|C_{A'} \cap X_1|, |C_{A'} \cap X_2|, |C_{A'} \cap X_3|) = \gamma$, for some $\gamma \in \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$.*

Proof. We argue by contradiction. Assume that $|A'| \geq 2$ or, when $|A'| = 1$, $A' \notin \mathcal{A}''$ and $C_{A'}$ does not exist. Let D be a circuit of $M|(L \cup A')$ such that $A' = D - L$. Remember that $L \cup A' = L \cup D$ is a connected T-plane of M. Assume that

$$|D \cap X_1| \leq |D \cap X_2| \leq |D \cap X_3|.$$

As $A' \notin \mathcal{A}_3$, it follows that $|D \cap X_2| \geq 1$. Observe that

$$|D \cap X_1| \leq 1 \quad \text{and} \quad |D \cap X_2| \leq 2, \tag{2.1}$$

since $|D| = |A'| + |D \cap X_1| + |D \cap X_2| + |D \cap X_3| \leq \text{circ}(M) \leq 6$.

Next, we establish that

$$X_3 \subseteq D. \tag{2.2}$$

If $|D \cap X_3| < |X_3|$, then, by (T7), $(X_1 \cup X_2) \cup D$ is a connected T-line of M. So, by (T6), there is a circuit D_1 of M such that $(X_1 \cup X_2) \triangle D \subseteq D_1$. Hence

$$|A'| + |X_1 - D| + |X_2 - D| + |D \cap X_3| \leq |D_1| \leq 6. \tag{2.3}$$

Observe that

$$3 = |X_2| \leq |X_2| + (|D \cap X_3| - |D \cap X_2|) = |X_2 - D| + |D \cap X_3|. \tag{2.4}$$

By (2.3) and (2.4),

$$|A'| + |X_1 - D| \leq 3. \tag{2.5}$$

By (2.1) and (2.5), $|A'| = 1$, $|X_1 - D| = 2$ and $|D \cap X_1| = 1$. Moreover, we have equality in (2.3)–(2.5). In particular, $|D \cap X_2| = |D \cap X_3|$. If $|D \cap X_2| = |D \cap X_3| = 1$, then $(|D_1 \cap X_1|, |D_1 \cap X_2|, |D_1 \cap X_3|) = (2, 2, 1)$; a contradiction. If $|D \cap X_2| = |D \cap X_3| = 2$, then $(|D \cap X_1|, |D \cap X_2|, |D \cap X_3|) = (1, 2, 2)$; a contradiction. Therefore (2.2) follows.

By (2.2), $X_3 \subseteq D$. Now, we establish that

$$D \cap X_1 = \emptyset. \tag{2.6}$$

If (2.6) does not hold, then $|D \cap X_1| = |D \cap X_2| = 1$. By (T7), $(X_2 \cup X_3) \cup D$ is a connected T-line of M. As $|D \cap X_2| = 1$, it follows, by (T6), that there is a circuit D_2 of M such that

$$(X_2 \cup X_3) \triangle D \subseteq D_2 \subseteq [(X_2 \cup X_3) \cup D] - (D \cap X_2). \tag{2.7}$$

As $X_1 \not\subseteq D_2$ and $X_2 \not\subseteq D_2$, it follows, by (2.2) applied to D_2, that $X_3 \subseteq D_2$ because $D_2 - L = A'$. But

$$7 \leq |A'| + |D \cap X_1| + |X_2 - D| + |X_3| \leq |D_2|;$$

a contradiction and (2.6) follows. By (2.6), $A' \in \mathcal{A}_{23}$. As $A' \notin \mathcal{A}''$, it follows that

$$A' \notin \mathcal{A}_{1i}, \quad \text{for some } i \in \{2, 3\}. \tag{2.8}$$

In the next paragraph, we construct a circuit D_3 such that $X_1 \cap D_3 = \emptyset$, $X_2 \subseteq D_3$ and $A' = D_3 - L$. Replacing D by D_3, when necessary, we may assume that $i = 2$ in (2.8).

Remember that $(X_2 \cup X_3) \cup D$ is a connected T-line of M. By (T7), for $e \in X_3$, there is a circuit D_3 of M such that

$$(X_2 \cup X_3) \triangle D \subseteq D_3 \subseteq [(X_2 \cup X_3) \cup D] - e. \tag{2.9}$$

As $X_1 \not\subseteq D_3$ and $X_3 \not\subseteq D_3$, it follows, by (2.2) applied to D_3, that $X_2 \subseteq D_3$. Thus we construct the circuit D_3 as described in the penultimate line of the previous paragraph.

Observe that $D \cup (X_1 \cup X_3)$ is a T-line of M. Let D_4 be the circuit of M contained in $[D \cup (X_1 \cup X_3)] - f$, for some $f \in X_3$. Note that

$$A' \cup (D \cap X_2) = D - (X_1 \cup X_3) \subseteq D_4$$

and $D_4 - L = A'$. By (2.2) applied to D_4, $X_1 \subseteq D_4$. In particular, D_4 meets both X_1 and X_2. By (2.6) applied to D_4, $D_4 \cap X_3 = \emptyset$ and so $A' \to (X_1 \cup X_2)$; a contradiction to (2.8) (remember that $i = 2$). $\square$

Lemma 2.2. $\mathcal{A}' - \mathcal{A}'' \neq \emptyset$.

Proof. Assume that $\mathcal{A}' - \mathcal{A}'' = \emptyset$. Hence $A' \to X_1$ or $A' \to (X_2 \cup X_3)$, for every L-arc A'. As $\{X_1, X_2 \cup X_3\}$ is a 2-separation of $M|L$, it follows, by (3.8) of [14], that there is a 2-separation $\{X, Y\}$ of M such that $X_1 \subseteq X$ and $X_2 \cup X_3 \subseteq Y$; a contradiction. Therefore Lemma 2.2 follows. □

Lemma 2.3. *$\mathcal{A}_i = \emptyset$, for each $i \in \{1, 2, 3\}$.*

Proof. Suppose that $\mathcal{A}_i \neq \emptyset$, say $i = 1$. By Lemma 2.1, for each $A' \in \mathcal{A}' - \mathcal{A}''$, there is a circuit $C_{A'}$ of M such that $A' = C_{A'} - L$ and $(|C_{A'} \cap X_1|, |C_{A'} \cap X_2|, |C_{A'} \cap X_3|) = \gamma$, where $\gamma \in \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$. Choose $C_{A'}$ so that $|C_{A'} \cap X_1|$ is minimum. Now, we prove that

$$(|C_{A'} \cap X_1|, |C_{A'} \cap X_2|, |C_{A'} \cap X_3|) = (1, 2, 2). \tag{2.10}$$

If (2.10) does not hold, then $|C_{A'} \cap X_j| = 1$, for some $j \in \{2, 3\}$. By(T7), $C_{A'} \cup (X_1 \cup X_j)$ is a connected T-line of M. By (T6), M has a circuit D such that $C_{A'} \triangle (X_1 \cup X_j) \subseteq D$. Hence

$$\begin{aligned} |D| &\geq |C_{A'} - (X_1 \cup X_j)| + |(X_1 \cup X_j) - C_{A'}| \\ &= [|A'| + |C_{A'} \cap X_{5-j}|] + [|X_1 - C_{A'}| + |X_j - C_{A'}|] \\ &= [1 + 2] + [1 + 2] = 6. \end{aligned}$$

We must have equality along the last display. Thus $(|D \cap X_1|, |D \cap X_2|, |D \cap X_3|) = \gamma$, for $\gamma = (1, 2, 2)$. We arrive at a contradiction since $D - L = A'$. Thus (2.10) holds.

For $A_1 \in \mathcal{A}_1$, let D_{A_1} be a circuit of M such that $A_1 = D_{A_1} - L$ and $D_{A_1} \subseteq X_1 \cup A_1$. For each $A' \in \mathcal{A}' - \mathcal{A}''$ and $A_1 \in \mathcal{A}_1$, we establish that:

$$D_{A_1} = A_1 \cup (X_1 - C_{A'}). \tag{2.11}$$

Assume that (2.11) does not hold. Now, we establish that $|D_{A_1} \cap X_1| \geq 2$. If $|D_{A_1} \cap X_1| = 1$, then $D_{A_1} \triangle (X_1 \cup X_2)$ is a circuit of M. Therefore $|D_{A_1} \triangle (X_1 \cup X_2)| \leq 6$ and so $|D_{A_1}| \leq 2$; a contradiction because M is 3-connected and so is simple. Hence $|D_{A_1} \cap X_1| \geq 2$. Therefore D_{A_1} intercepts both sets belonging to $\{X_1 - C_{A'}, X_1 \cap C_{A'}\}$. In particular,

$$1 \leq |(C_{A'} \triangle D_{A_1}) \cap X_1|. \tag{2.12}$$

As $C_{A'} \cap D_{A_1} \neq \emptyset$, it follows that $C_{A'} \cup D_{A_1}$ is a connected T-flat of M. Observe that $C_{A'} \cup D_{A_1}$ is a T-line of M because A_1 and A' are disjoint series classes of $M|(L \cup A_1 \cup A')$ and $(D_{A_1} \cup C_{A'}) - (A_1 \cup A')$ is an independent set of $M|L$. By (T6), there is a circuit D of M such that $D_{A_1} \triangle C_{A'} \subseteq D$. Thus

$$|A_1| + |A'| + |C_{A'} \cap (X_2 \cup X_3)| + |(C_{A'} \triangle D_{A_1}) \cap X_1| \leq |D|.$$

By (2.12), $3 + |C_{A'} \cap (X_2 \cup X_3)| \leq |D|$ and so $7 \leq |D|$; a contradiction. Therefore (2.11) holds.

Let X be a subset of X_1 such that $D_{A_1} = A_1 \cup X$, for some $A_1 \in \mathcal{A}_1$. By (2.11), for every $A' \in \mathcal{A}' - \mathcal{A}''$, $X \cap C_{A'} = \emptyset$. As $\mathcal{A}' - \mathcal{A}'' \neq \emptyset$, by Lemma 2.2, it follows that X is uniquely determined. Hence $D_{A_1} = X \cup A_1$, for every $A_1 \in \mathcal{A}_1$. Note that $\{X, L - X\}$ is a 2-separation of $M|L$ such that

(i) $A_1 \to X$, for every $A_1 \in \mathcal{A}_1$; and
(ii) $A'' \to L - X$, for every $A'' \in \mathcal{A} - \mathcal{A}_1$.

(Note that (ii) occurs when: $A'' \in \mathcal{A}_2 \cup \mathcal{A}_3$ or $A'' \in \mathcal{A}''$ because $X_2 \cup X_3 \subseteq L - X$; $A'' \in \mathcal{A}' - \mathcal{A}''$ because $C_{A''} - A'' \subseteq L - X$.) By (3.8) of [14], there is a 2-separation $\{X', Y'\}$ of M such that $X \subseteq X'$ and $L - X \subseteq Y'$; a contradiction. Therefore Lemma 2.3 follows. □

By Lemma 2.3, $\mathcal{A} = \mathcal{A}'$. By Lemma 2.1, $L = C \cup A$ spans M and so $r(M) = 7$. We arrive at a contradiction. The result follows. □

3. Local structural results

For a circuit C of a matroid M, let A be C-arc. Observe that $C \cup A$ is a connected T-line of M. Hence there is a partition $\Pi_M(A, C) = \{W_1, W_2, \ldots, W_k\}$ of C, for some integer k exceeding 1, such that $(C - W_1) \cup A, (C - W_2) \cup A, \ldots, (C - W_k) \cup A$ are circuits of M. (Equivalently $\{A, W_1, W_2, \ldots, W_k\}$ is the canonical partition of $C \cup A$.) For C-arcs A_1 and A_2, we say that:

(i) A_1 and A_2 are *strongly disjoint* provided $A_1 \cap A_2 = \emptyset$, $\min\{|A_1|, |A_2|\} \geq 2$ and $(M/C)|(A_1 \cup A_2) = [(M/C)|A_1] \oplus [(M/C)|A_2]$; and
(ii) W_1 and W_2 *cross* provided $\emptyset \notin \{W_1 \cap W_2, W_1 - W_2, W_2 - W_1, C - (W_1 \cup W_2)\}$, where $W_i \in \Pi_M(A_i, C)$, for each $i \in \{1, 2\}$.

Lemma 3.1. *Let C be a circuit of a matroid M such that $|C| = \operatorname{circ}(M) = 6$. If A_1 and A_2 are strongly disjoint C-arcs and, for $i \in \{1, 2\}$, $W_i \in \Pi_M(A_i, C)$, then:*

(i) $|W_1 - W_2| = |W_2 - W_1| = 1$, *when* W_1 *and* W_2 *cross.*
(ii) $|W_1| = |W_2| = 2$, *when* W_1 *and* W_2 *cross. Moreover,* $3 = |\Pi_M(A_1, C)| = |\Pi_M(A_2, C)|$.
(iii) *If* $|W_1| = 3$, *then, for* $i \in \{1, 2\}$, *there is* $W_i' \in \Pi_M(A_i, C)$ *such that* $W_1' \subseteq W_2'$. *Moreover,* $2 = |\Pi_M(A_1, C)| = |\Pi_M(A_2, C)|$.

Proof. First, we show (i). By definition, $\emptyset \notin \{W_1 \cap W_2, W_1 - W_2, W_2 - W_1, C - (W_1 \cup W_2)\}$. As $P = C \cup A_1 \cup A_2$ is a T-plane of M, it follows that $[(C - W_1) \cup A_1] \cup [(C - W_2) \cup A_2] = P - (W_1 \cap W_2)$ is a connected T-line of M. Therefore M has a circuit C_1 so that

$$(A_1 \cup A_2) \cup (W_1 - W_2) \cup (W_2 - W_1) = [(C - W_1) \cup A_1] \triangle [(C - W_2) \cup A_2] \subseteq C_1. \tag{3.1}$$

Hence

$$|A_1| + |A_2| + |W_1 - W_2| + |W_2 - W_1| \le |C_1| \le 6. \tag{3.2}$$

As $\emptyset \notin \{W_1 - W_2, W_2 - W_1\}$, it follows, by (3.2), that $|W_1 - W_2| = |W_2 - W_1| = 1$. Thus (i) follows.

Now, we establish the first part of (ii). By (i), $|W_1| = |W_2|$ because, for $i \in \{1, 2\}$,

$$|W_i| = |W_1 \cap W_2| + |W_i - W_{3-i}| = |W_1 \cap W_2| + 1. \tag{3.3}$$

If $|W_1| = |W_2| \ge 3$, then $|\Pi_M(A_1, C)| = |\Pi_M(A_2, C)| = 2$. Therefore $C - W_2 \in \Pi_M(A_2, C)$ and $C - W_2$ crosses W_1. By (i) applied to W_1 and $C - W_2$,

$$|W_1 - (C - W_2)| = |W_1 \cap W_2| = 1. \tag{3.4}$$

We arrive at a contradiction because, by (3.3) and (3.4), $|W_1| = |W_2| = 2$. Thus the first part of (ii) follows. To establish second part of (ii), it is enough to show that

$$|\Pi_M(A_i, C)| \ge 3, \quad \text{for each } i \in \{1, 2\}. \tag{3.5}$$

Suppose that (3.5) does not hold for some i, say $i = 2$. Therefore $\Pi_M(A_2, C) = \{W_2, W_2'\}$, where $W_2' = C - W_2$. As W_1 cross both W_2 and W_2', it follows, by the first part of (ii), that $2 = |W_1| = |W_2| = |W_2'|$; a contradiction since $6 = |C| = |W_2| + |W_2'|$.

Next, we prove (iii). Observe that $\Pi_M(A_1, C) = \{W_1, C - W_1\}$ because $|W_1| = 3$. By (ii), no set belonging to $\Pi_M(A_2, C)$ crosses W_1 or $C - W_1$ and so $|\Pi_M(A_2, C)| = 2$. If $\Pi_M(A_1, C) = \Pi_M(A_2, C)$, then (iii) follows for $W_1' = W_2' = W_1$. Assume that $\Pi_M(A_1, C) \ne \Pi_M(A_2, C)$. There is $W_2' \in \Pi_M(A_2, C)$ such that $|W_2'| = 4$. Note that W_2' contains W_1 or $C - W_1$. Again (iii) holds. □

Lemma 3.2. *Let* C *be a circuit of a matroid* M *such that* $|C| = \mathrm{circ}(M) = 6$. *If* Z *is a* 3*-subset of* $E(M) - C$ *such that* A *is a* C*-arc satisfying* $|\Pi_M(A, C)| = 2$, *for every* 2*-subset* A *of* Z, *then:*

(i) $M|(C \cup Z)$ *is binary; or*
(ii) Z *is a circuit of* M.

Proof. Suppose that $Z = \{a_1, a_2, a_3\}$. For $i \in \{1, 2, 3\}$, $L_i = C \cup (Z - a_i)$ is a T-line of M which contains exactly 3 circuits of M, by hypothesis. If (i) does not hold, then, by Tutte's characterization of binary matroids, $M|(C \cup Z)$ has a T-line L containing at least 4 circuits. Hence $L \ne L_i$, for every $i \in \{1, 2, 3\}$. By (T2), there is a unique circuit C_i of M such that $C_i \subseteq L \cap L_i$. Let D be a circuit of $M|L$ different from C_1, C_2 and C_3. In particular, $Z \subseteq D$. By (T4), $M|(C \cup Z)$ has a T-line L' such that $D \subseteq L'$ and $L' \ne L$. If D_i is the circuit of M contained in $L' \cap L_i$, for $i \in \{1, 2, 3\}$, then $D_i \ne C$, otherwise, by (T5), $C \cup Z \subseteq C \cup D = D_i \cup D = L'$. Moreover, $D_i \ne C_i$, otherwise, by (T5), $L' = D \cup D_i = D \cup C_i = L$. Therefore $\mathcal{C}(M|L_i) = \{C, C_i, D_i\}$.

Now, we show that L_1, L_2, L_3, L and L' are the only T-lines of $M|(C \cup Z)$. Let L'' be a T-line of $M|(C \cup Z)$ such that $L'' \notin \{L_1, L_2, L_3\}$. Hence $C \not\subseteq L''$. In particular, for each $i \in \{1, 2, 3\}$, C_i or D_i is contained in L'' because, by (T2), $L'' \cap L_i$ contains a circuit of M. Therefore L'' contains at least two of C_1, C_2, C_3 (and so $L'' = L$) or L'' contains at least two of D_1, D_2, D_3 (and so $L'' = L'$). That is, L_1, L_2, L_3, L and L' are the only T-lines of $M|(C \cup Z)$. As L_1, L_2 and L_3 are the only T-lines of $M|(C \cup Z)$ avoiding D, it follows, by (T1), that

$$D = [(C \cup Z) - L_1] \cup [(C \cup Z) - L_2] \cup [(C \cup Z) - L_3] = \{a_1, a_2, a_3\} = Z.$$

Thus (ii) follows. □

Lemma 3.3. *Let* C *be a circuit of a matroid* M *such that* $|C| = \mathrm{circ}(M) = 6$. *Suppose that* A' *is a* C*-arc such that* $|W| = 3$, *for each* $W \in \Pi_M(A', C)$. *If* Z *is an independent* 3*-subset of* $E(M)$ *such that* A *is a* C*-arc strongly disjoint from* A', *for each* 2*-subset* A *of* Z, *then the cosimplification* $M|(C \cup Z)$ *is isomorphic to* F_7^*. *Moreover,* $M|(C \cup Z)$ *has just one non-trivial series class* S *and* $S \in \Pi_M(A', C)$.

Proof. By Lemma 3.1(iii), $|\Pi_M(A, C)| = 2$, for every 2-subset A of Z. As Z is not a circuit of M, by hypothesis, it follows, by Lemma 3.2, that $M|(C \cup Z)$ is binary. Let N be the cosimplification of $M|(C \cup Z)$. Observe that $C \cap E(N)$ is a circuit-hyperplane of N and so $r(N) = |C \cap E(N)|$. As Z is an independent set of N, it follows that $|E(N) - Z| \ge 3$. Therefore N is isomorphic to $M(K_4)$ or F_7^* since $r^*(N) = 3$. If $N \cong M(K_4)$, then there is a partition $\{W_1, W_2, W_3\}$ of C such that $\Pi_M(Z - a_i, C) = \{W_i, C - W_i\}$, for each $i \in \{1, 2, 3\}$, where $Z = \{a_1, a_2, a_3\}$. In particular, $|W_1| = |W_2| = |W_3| = 2$; a contradiction to Lemma 3.1(ii) since there is $i \in \{1, 2, 3\}$ such that W_i crosses W, for each $W \in \Pi_M(A, C)$. Therefore $N \cong F_7^*$. If S_1, S_2, S_3, S_4 are the series classes of $M|(C \cup Z)$ which are contained in C, say $|S_1| \geq 2$, then, for each $i \in \{2, 3, 4\}$, there is a $j \in \{1, 2, 3\}$ such that $\Pi_M(Z - a_j, C) = \{S_1 \cup S_i, C - (S_1 \cup S_i)\}$. By Lemma 3.1(iii), there is $W_i' \in \Pi_M(A', C)$ such that $C - W_i' \subseteq S_1 \cup S_i$ because $|W_i'| = |C - W_i'| = 3$ and $|S_1 \cup S_i| \geq 3$. Thus, for each $i \in \{2, 3, 4\}$, $C - (S_1 \cup S_i) \subseteq W_i'$. As $|W_i'| = 3$, $2 \leq |C - (S_1 \cup S_i)|$ and $|S_1| = 2$, it follows that $S_1 \cap W_i' = \emptyset$ because, by Lemma 3.1(ii), W_i' and $S_1 \cup S_i$ do not cross. Hence $W_1' = W_2' = W_3'$ and so $S_2 \cup S_3 \cup S_4 = W_1'$. In particular, $S_1 = C - W_1' \in \Pi_M(A', C)$. □

4. An auxiliary lemma

Let C be a Hamiltonian circuit of a matroid M. If $e \in E(M) - C$, then $C \cup e$ is a T-line of M. The element e is said to be *C-large* provided $|\pi(C \cup e)| \geq 4$. That is, $M|(C \cup e)$ contains at least 4 circuits or equivalently $M|(C \cup e)$ is non-binary.

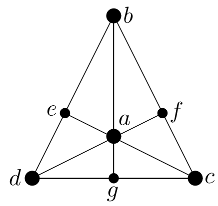
Lemma 4.1. *Let C be a Hamiltonian circuit of a connected matroid M such that*

$$|D \cap C| + 2|D - C| \leq 6, \quad \text{for every circuit } D \text{ of } M. \tag{4.1}$$

If $|E(M) - C| = 3$ and $|C| = 6$, then

(i) *$E(M) - C$ is a dependent set of M; or*
(ii) *there is a graph G such that $M = M(G)$ and $E(M) - C$ is contained in the star of a vertex of G; or*
(iii) *the elements belonging to $E(M) - C$ can be labeled as e, f and g, where e and g are C-large, and the elements belonging to $C - X$ can be labeled as a, b, c and d, where X is a 2-element series class of M, such that the geometric representation of $(M \setminus X)^*$ is given in the next figure.*

Moreover, when (iii) *happens, f is not C-large and M is the parallel connection of $M \setminus X$ and a matroid isomorphic to $U_{2,3}$ having $X \cup f$ as its ground set.*



Proof. Observe that $C \cup e$ is a connected T-line of M, for each $e \in E(M) - C$. Moreover, $\{e\} \in \pi(C \cup e)$ and $\pi(C \cup e) - \{e\}$ is a partition of C. As $(C \cup e) - X$ is a circuit of M, for each $X \in \pi(C \cup e)$, it follows, by (4.1), that $|X| \geq 2$, when $X \neq \{e\}$. In particular, $|\pi(C \cup e)| = 3$ or $|\pi(C \cup e)| = 4$ and, in the last case, $|X| = 2$, for every $X \in \pi(C \cup e)$ such that $X \neq \{e\}$. We need the next two sublemmas.

Sublemma 4.1. *Suppose that $E(M) - C = \{e, f, g\}$ is an independent set of M. If $|\pi(C \cup e)| = |\pi(C \cup f)| = 4$, then:*

(i) *For each $X \in \pi(C \cup e)$ and $Y \in \pi(C \cup f)$ satisfying $|X \cap Y| = 1$, $\{e, f\} \cup (X \triangle Y)$ is a circuit of M. Moreover, $E(M) - [(X \cap Y) \cup g]$ is a T-line of M having $\{(X - Y) \cup e, (Y - X) \cup f, C - (X \cup Y)\}$ as canonical partition.*
(ii) *$\pi(C \cup e) \cap \pi(C \cup f) \neq \emptyset$.*
(iii) *If $X \in \pi(C \cup e) \cap \pi(C \cup f)$, then $|\pi(C \cup g)| = 3$ or $X \in \pi(C \cup g)$. Moreover, when $|\pi(C \cup g)| = 4$, Lemma 4.1*(iii) *follows.*

Proof. (i) Note that $|X| = |Y| = 2$ because, by hypothesis, $X \cap Y \neq \emptyset$ and so $X \neq \{e\}$ and $Y \neq \{f\}$. As $X \in \pi(C \cup e)$ and $Y \in \pi(C \cup f)$, it follows that $(C \cup e) - X = (C - X) \cup e$ and $(C - Y) \cup f$ are different circuits of M. Note that the T-flat $L = [(C - X) \cup e] \cup [(C - Y) \cup f] = \{e, f\} \cup [C - (X \cap Y)]$ is a T-line of M because it is properly contained in the T-plane $\{e, f\} \cup C$ of M. By (T6), there is a circuit D of $M|L$ such that $\{e, f\} \cup (X \triangle Y) = [(C - X) \cup e] \triangle [(C - Y) \cup f] \subseteq D$. By (4.1), $D = \{e, f\} \cup (X \triangle Y)$. Moreover, the sets belonging to $\pi(L)$ are $L - D = C - (X \cup Y)$, $L - [(C - X) \cup e] = f \cup (Y - X)$ and $L - [(C - Y) \cup f] = e \cup (X - Y)$. Thus (i) follows.

(ii) Assume that $\pi(C \cup e) \cap \pi(C \cup f) = \emptyset$. It is possible to label the elements of C by a, x, b, y, c, z so that $\pi(C \cup e) = \{\{a, x\}, \{b, y\}, \{c, z\}\}$ and $\pi(C \cup f) = \{\{x, b\}, \{y, c\}, \{z, a\}\}$. By (i) applied when (X, Y) is equal to $(\{a, x\}, \{x, b\})$ or $(\{b, y\}, \{y, c\})$ or $(\{c, z\}, \{z, a\})$, we conclude respectively that $\{e, f, a, b\}$, $\{e, f, b, c\}$ and $\{e, f, c, a\}$ are circuits of M. As $|\{e, f, a, b\} - \{e, f, b, c\}| = 1$, it follows that $L = \{e, f, a, b\} \cup \{e, f, b, c\} = \{a, b, c, e, f\}$ is a T-line of M. Note that $L - \{e, f, a, b\} = \{c\}$, $L - \{e, f, b, c\} = \{a\}$ and $L - \{e, f, c, a\} = \{b\}$ belongs to $\pi(L)$. The other sets of $\pi(L)$ are contained in $\{e, f\}$. Thus $\{e, f\} \in \pi(L)$ or both $\{e\}$ and $\{f\}$ belong to $\pi(L)$. If $\{e, f\} \in \pi(L)$, then $L - \{e, f\} = \{a, b, c\}$ is a circuit of M properly contained in C; a contradiction. If $\{e\} \in \pi(L)$, then $L - \{e\} = \{a, b, c, f\}$ is a circuit of M. Hence $(C \cup f) - \{a, b, c, f\} = \{x, y, z\}$ belongs to $\pi(C \cup f)$; a contradiction because every set in $\pi(C \cup f)$ contains 1 or 2 elements. Therefore (ii) follows.

(iii) To establish the first part of (iii), we argue by contradiction. Assume that $|\pi(C \cup g)| = 4$ and $X \notin \pi(C \cup g)$. There is $Y \in \pi(C \cup g)$ such that $Y \cap X \neq \emptyset$, say $X = \{a, b\}$ and $Y = \{b, c\}$. By (i) applied to (e, g, X, Y) or (f, g, X, Y), we conclude respectively that $\{e, g, a, c\}$ and $\{f, g, a, c\}$ are circuits of M. As $|\{e, g, a, c\} - \{f, g, a, c\}| = 1$, it follows that $L = \{e, g, a, c\} \cup \{f, g, a, c\} = \{e, f, g, a, c\}$ is a T-line of M. Observe that $L - \{e, g, a, c\} = \{f\}$ and $L - \{f, g, a, c\} = \{e\}$

belongs to $\pi(L)$. By hypothesis, $\{e,f\}$ is not a circuit of M and so $\{a,c,g\} \notin \pi(L)$. Hence there is $Z \in \pi(L)$ such that $g \notin Z \subseteq \{a,c,g\}$. Thus $L-Z$ is a circuit of M that contains $\{e,f,g\}$. By (4.1), $L-Z = \{e,f,g\}$; a contradiction, since, by hypothesis, $E(M)-C$ is independent. We have established the first part of (iii). Now, we prove the second part of (iii). Suppose that $|\pi(C\cup g)| = 4$ and so $X \in \pi(C\cup g)$. For $h \in \{e,f,g\}$, consider $\Pi_h = \pi(C\cup h) - \{h\}$. If $Y \neq X$ and $Y \in [\Pi_f\cap\Pi_g]\cup[\Pi_e\cap\Pi_g]\cup[\Pi_e\cap\Pi_f]$, then, by the first part of (iii), $Y \in \Pi_e\cap\Pi_f\cap\Pi_g$. Hence $Z = C-(X\cup Y) \in \Pi_e\cap\Pi_f\cap\Pi_g$ because $X, Y \in \Pi_e\cap\Pi_f\cap\Pi_g$. In this case, $\Pi_e = \Pi_f = \Pi_g$. Therefore X, Y, Z are the non-trivial series classes of M. Moreover, $co(M)$ is a line. Thus $E(M)-C$ is a dependent set of $co(M)$ and so of M; a contradiction to hypothesis. Hence Y does not exist. That is, $[\Pi_f\cap\Pi_g]\cup[\Pi_e\cap\Pi_g]\cup[\Pi_e\cap\Pi_f] = \{X\}$. We can label the elements belonging to $C-X$ by a,b,c,d so that $\pi(C\cup e) = \{\{e\}, X, \{a,c\}, \{b,d\}\}$, $\pi(C\cup f) = \{\{f\}, X, \{a,d\}, \{b,c\}\}$ and $\pi(C\cup g) = \{\{g\}, X, \{a,b\}, \{c,d\}\}$. By (i), $\{e,f,a,b\}$, $\{e,f,c,d\}$, $\{e,g,b,c\}$, $\{e,g,a,d\}$, $\{f,g,b,d\}$ and $\{f,g,a,c\}$ are circuits of M. We have the second part of (iii). $\square$

Sublemma 4.2. *Suppose that $E(M)-C = \{e,f,g\}$ is an independent set of M. If $|\pi(C\cup e)| = 4$ and $|\pi(C\cup f)| = 3$, then Lemma 4.1(iii) follows.*

Proof. By the main result of Bixby [1] (see Proposition 11.3.7 of Oxley [10]), $M|(C\cup\{e,f\})$ has a minor H isomorphic to $U_{2,4}$ having f as one of its elements. If $H = M|(C\cup\{e,f\})\setminus X/Y$, where X and Y are disjoint subsets of $C\cup\{e,f\}$ and X is chosen with maximum cardinality, then $L = (C\cup\{e,f\})-X$ is a T-line of $M|(C\cup\{e,f\})$ containing at least 4 circuits and having f as one of its elements. As $L \neq C\cup f$, since $|\pi(C\cup f)| = 3$, it follows that $e \in L$. Thus $\{e,f\} \subseteq L$. Let X_e and X_f be elements of $\pi(L)$ containing respectively e and f. Note that $X_e \neq X_f$, otherwise the circuit $L-X_e$ of M is properly contained in C. As $L-X_f$ is a circuit of $M|(L\cup e)$ that contains e, it follows that $|(L-X_f)\cap C| = 4$, since $|\pi(C\cup e)| = 4$. In particular, $|L| \geq 6$. Assume that $\pi(L) = \{X_e, X_f, X_1, X_2, \ldots, X_k\}$, for disjoint subsets $X_1, X_2, \ldots, X_k$ of C satisfying $|X_1| \leq |X_2| \leq \cdots \leq |X_k|$, where $k \geq 2$. But $L-X_1$ is a circuit of M containing $\{e,f\}$ and so, by (4.1),

$$2+|X_1| \leq |X_e|+|X_f|+|X_1| \leq |X_e|+|X_f|+|X_2|+\cdots+|X_k| = |L-X_1| = |L|-|X_1| \leq 4.$$

Thus $|X_1| = 2$, $|L| = 6$ and we have equality along the previous display. In particular, $X_e = \{e\}$, $X_f = \{f\}$, $k = 2$ and $|X_2| = 2$. Moreover, $\pi(L) = \{\{e\}, \{f\}, X_1, X_2\}$ and $X_1\cup X_2\cup f = L-e$ is a circuit of $M|(C\cup f)$. If $X_3 = C-(X_1\cup X_2)$, then $\pi(C\cup f) = \{\{f\}, X_1\cup X_2, X_3\}$. Hence X_1, X_2, X_3 are series classes of $M|(C\cup\{e,f\})$ and $\pi(C\cup e) = \{\{e\}, X_1, X_2, X_3\}$. If $X_i = \{x_i, y_i\}$, for $i \in \{1,2,3\}$, then f and y_3 are in parallel in $M|(C\cup\{e,f\})/\{x_1,x_2,x_3\}$ and $M|(C\cup\{e,f\})/\{x_1,x_2,x_3\}\backslash f \cong U_{2,4}$.

Let g be the element in $E(M)-(C\cup\{e,f\})$. If X_1, X_2 and X_3 are series classes of $M|(C\cup\{e,g\})$ or $M|(C\cup\{f,g\})$, then X_1, X_2 and X_3 are also series classes of $M|(C\cup\{e,f,g\})$ because, by the previous paragraph, X_1, X_2 and X_3 are series classes of $M|(C\cup\{e,f\})$. Therefore $\{e,f,g\}$ is a dependent set in the rank-2 matroid $M|(C\cup\{e,f,g\})/\{x_1,x_2,x_3\}$. So $\{e,f,g\}$ is a dependent set in M; a contradiction. We may assume that at least of of the sets X_1, X_2, X_3 is not a series class of $M|(C\cup\{e,g\})$ and of $M|(C\cup\{f,g\})$. First, we establish that $|\pi(C\cup g)| = 4$. Assume that $|\pi(C\cup g)| = 3$. Replacing f by g in the argument of the previous paragraph, we conclude that X_1, X_2 and X_3 are also series classes of $M|(C\cup\{e,g\})$; a contradiction. Thus $|\pi(C\cup g)| = 4$. Replacing e by g in the argument in the previous paragraph, we conclude that $X_3 \in \pi(C\cup g)$, since X_3 is the unique 2-element set belonging to $\pi(C\cup f)$. As X_3 is a series class of M that spans f, it follows that M is the parallel connection of $M_1 = M|(X_3\cup f)$ and $M_2 = M\setminus X_3$. Observe that M_1 is a triangle and that $r(M_2) = 4$. Moreover, $L_e = (C-X_3)\cup\{e,f\}$ and $L_g = (C-X_3)\cup\{f,g\}$ are T-lines of M_2 having respectively canonical partitions $\{\{e\}, \{f\}, \{a,b\}, \{c,d\}\}$ and $\{\{f\}, \{g\}, \{a,c\}, \{b,d\}\}$, where a,b,c,d label the elements of $C-X_3$. Observe that $E(M_2)$ is a T-plane of M_2. For each $z \in \{a,b,c,d\}$, there is $X \in \{\{a,b\}, \{c,d\}\}$ and $Y \in \{\{a,c\}, \{b,d\}\}$ such that $z \notin X\cup Y$. By Lemma 4.1(i), $(L_e-X)\cup(L_f-Y) = E(M_2)-z$ is a T-line of M_2 having canonical partition $\{(X-Y)\cup e, (Y-X)\cup g, \{f,z\}\}$. That is, we conclude that $E(M_2)-a$, $E(M_2)-b$, $E(M_2)-c$, $E(M_2)-d$ are T-lines of M_2 having canonical partitions $\{\{e,c\}, \{g,b\}, \{f,d\}\}$, $\{\{e,d\}, \{g,a\}, \{f,c\}\}$, $\{\{e,a\}, \{g,d\}, \{f,b\}\}$, $\{\{e,b\}, \{g,c\}, \{f,a\}\}$ respectively. There just another T-line of M_2, namely $E(M_2)-f$. Its canonical partition is $\{\{e\}, \{g\}, \{c,b\}, \{a,d\}\}$. We have Lemma 4.1(iii). $\square$

Now, we return to the proof of Lemma 4.1(iii). If $E(M)-C$ is a dependent set of M, then (i) follows. Assume that $E(M)-C$ is an independent set of M. If $|\pi(C\cup e)| = 4$, for every $e \in E(M)-C$, then (iii) follows from Sublemma 4.1(iii). Suppose that $|\pi(C\cup f)| = 3$, for some $f \in E(M)-C$. If $|\pi(C\cup e)| = 4$, for some $e \in E(M)-(C\cup f)$, then (iii) follows from Sublemma 4.2. Thus, we can assume that $|\pi(C\cup e)| = 3$, for every $e \in E(M)-C$.

By the dual of (4.2) of Lemos [5], $M|(C\cup\{e,f\})$ is a binary matroid, for each 2-subset $\{e,f\}$ of $E(M)-C$. First, we show that $M|(C\cup\{e,f\})$ is not a subdivision of $M(K_4)$. If $M|(C\cup\{e,f\})$ is a subdivision of $M(K_4)$, then e and f label edges of a perfect matching of K_4. Therefore there are circuits D_1 and D_2 of $M|(C\cup\{e,f\})$ such that $D_1\cap D_2 = \{e,f\}$ and $C \subseteq D_1\cup D_2$. Hence, by (4.1),

$$12 \geq \sum_{i=1}^{2}[|D_i\cap C|+2|D_i-C|] = \sum_{i=1}^{2}[|D_i\cap C|+2|\{e,f\}|] = 8+\sum_{i=1}^{2}|D_i\cap C| \geq 8+|C|;$$

a contradiction. Thus $M|(C\cup\{e,f\})$ is not a subdivision of $M(K_4)$. In particular, there are $X_e \in \pi(C\cup e)$ and $X_f \in \pi(C\cup f)$ such that $X_e \subseteq X_f$. But this property holds for every 2-subset $\{e,f\}$ of $E(M)-C$. So it is possible to label the elements of $E(M)-C$ by a_1, a_2, a_3 such that, for each $i \in \{1,2,3\}$, there is $X_i \in \pi(C\cup a_i)$ satisfying $X_1 \subseteq X_2 \subseteq X_3$. As both X_i and $C-X_i$

span a_i in M, it follows that $X_1, X_1 \cup a_1, X_2 \cup a_1, X_2 \cup \{a_1, a_2\}, X_3 \cup \{a_1, a_2\}$ and $X_3 \cup \{a_1, a_2, a_3\}$ are all 2-separating sets of M. We have (ii).

To finish the proof of this lemma, when (iii) happens, we need to establish the extra property. We use the labeling fixed in (iii). In M^*, X is a parallel class of cardinality 2. As $r^*(M) = 4$, it follows that each circuit of M is the complement of some plane of M^*. Let Y be a 3-subset of $\{a, b, c, d\}$. By (iii), there is a unique plane P_Y of M^* that contains Y. Moreover, X is not a point of this plane. By (4.1), $P_Y \cap [E(M) - C] \neq \emptyset$. Hence, by (iii), there is a 3-point line L_Y of M^*/X such that $L_Y \subseteq P_Y$. As $L_Y \cup X$ is a plane of M^* and $L_Y \subseteq P_Y \cap (L_Y \cup X)$, it follows that L_Y is contained in a line of M^* and so L_Y is a line of M^*. Now, we prove that there are different 3-subsets Y and Z of $\{a, b, c, d\}$ such that

$$L_Y \neq L_Z \quad \text{and} \quad L_Y \cap L_Z \cap [E(M) - C] \neq \emptyset. \tag{4.2}$$

Assume that (4.2) does not hold. For $Y = \{b, c, d\}$, we may assume that $L_Y = \{c, d, g\}$. As $g \notin L_Z$, when $Z = \{a, b, c\}$, it follows that L_Z is equal to $\{a, c, e\}$ or $\{b, c, f\}$, say $L_Z = \{b, c, f\}$. For $W = \{a, b, d\}$, we conclude that f and g do not belong to L_W and so $e \in L_W$. Therefore $L_W = \{b, d, e\}$ is a line of M^*. That is, $\{c, d, g\}, \{b, c, f\}$ and $\{b, d, e\}$ are lines of M^*. Thus $\{b, c, d, e, f, g\}$ is contained in a plane of M^*; a contradiction because $\{e, f, g\}$ is a plane of M^*. Therefore (4.2) holds. Without loss of generality, we may assume that $g \in L_Y \cap L_Z$, when Y and Z are as described in (4.2). In particular, $\{a, b, g\}$ and $\{c, d, g\}$ are lines of M^*. Hence $\{a, b, c, d, g\}$ is contained in a plane P of M^*. If e or f belongs to P, say e, then M^* is the series connection of M^*/X and N, where N is a matroid isomorphic to $U_{1,3}$ such that $E(N) = X \cup f$. The extra property of M holds, when (iii) happens. We may assume that both e and f does not belong to P. Let P' be the plane of M^* that contains $\{e, f, a\}$. First, we show that $c \notin P'$. Assume that $c \in P'$. As $X \cup \{a, c, e\}$ is a plane of M^*, it follows that $\{a, c, e\} \subseteq P' \cap (X \cup \{a, c, e\})$ is contained in a line of M^* and so $\{a, c, e\}$ is a line of M^*. Hence $e \in P$; a contradiction. Thus $c \notin P'$. Similarly, $d \notin P'$. Note that $b \notin P'$ because $g \notin P'$ — remember that $\{e, f, g\}$ is a plane of M^*. Therefore $P' = \{e, f, a\}$; a contradiction because $E(M) - P'$ is a circuit of M that does not satisfies (4.1). □

Lemma 4.2. *Let $\{Y_1, Y_2, Y_3, C\}$ be a partition of $E(M)$, for a connected matroid M, such that C is a Hamiltonian circuit of M and $|Y_1| = |Y_2| = |Y_3| = 2$. Suppose that*

$$|D \cap C| + 2|D - C| \leq 6, \tag{4.3}$$

for every circuit D of M such that, for each $i \in \{1, 2, 3\}$, $Y_i \not\subseteq D$. If $|C| = 6$ and T is a transversal of (Y_1, Y_2, Y_3), then at least one of the sets $Y_1, Y_2, Y_3, T, T \triangle Y_1, T \triangle Y_2, T \triangle Y_3$ is dependent in M.

Proof. We argue by contradiction. Suppose the result does not hold for M. That is, $Y_1, Y_2, Y_3, T, T \triangle Y_1, T \triangle Y_2$ and $T \triangle Y_3$ are independent sets in M. Observe that $M_1 = M|(C \cup T)$ satisfies the hypothesis of Lemma 4.1. As T is independent in M, it follows that Lemma 4.1(ii) or (iii) holds for $M|(C \cup T)$. First, we prove that Lemma 4.1(ii) happens. Assume that Lemma 4.1(iii) occurs. We can label the elements of T by e, f, g so that e and g are C-large. Moreover, $M|(C \cup T)$ has a series class X such that $|X| = 2$ and $X \cup f$ is a triangle of M. If $f \in Y_3$, say $Y_3 = \{f, f'\}$, then $M|[C \cup (T \triangle Y_3)]$ satisfies the hypothesis of Lemma 4.1. As e and g are C-large, it follows that Lemma 4.1(iii) holds for $M_2 = M|[C \cup (T \triangle Y_3)]$. Moreover, X is a series class and $X \cup f'$ is a triangle of M_2. Thus X is a series class of $M|(C \cup T \cup Y_3)$ and f and f' are in parallel in this matroid; a contradiction and Lemma 4.1(iii) cannot happen for $M|(C \cup T)$. Hence Lemma 4.1(ii) occurs for $M|(C \cup T)$.

Assume that $T = \{a_1, a_2, a_3\}$, where $a_i \in Y_i$, for $i \in \{1, 2, 3\}$. By Lemma 4.1(ii) applied to $M|(C \cup T)$, it is possible to relabel the sets Y_i's such that there is $X_i \in \pi(C \cup a_i)$ satisfying

$$X_1 \subsetneq X_2 \subsetneq X_3 \quad \text{and} \quad (|X_1|, |X_2|, |X_3|) = (2, 3, 4). \tag{4.4}$$

Lemma 4.1(ii) also holds for $M|[C \cup (T \triangle Y_i)]$ because a_j is not C-large, for every $j \in \{1, 2, 3\}$. Therefore there is $X_i' \in \pi(C \cup a_i')$, where $Y_i = \{a_i, a_i'\}$, such that X_i can be replaced by X_i' in (4.4). If A is a 2-subset of $Y_1 \cup Y_2 \cup Y_3$, then by the dual of (4.2) of Lemos [5], $M|(C \cup A)$ is binary. If $X_i = X_i'$, then a_i and a_i' are in parallel in M because $M|(C \cup \{a_i, a_i'\})$ is binary; a contradiction. Hence $X_i \neq X_i'$, for every $i \in \{1, 2, 3\}$. As $X_1 \neq X_1'$, $|X_1| = |X_1'| = 2$, $X_1 \cup X_1' \subseteq X_2$ and $|X_2| = 3$, it follows that $X_2 = X_1 \cup X_1'$ and $|X_1 \cap X_1'| = 1$, say $X_1 \cap X_1' = \{e\}$. But $M|(C \cup \{a_1, a_1'\})$ is binary and so $[(C - X_1) \cup a_1] \triangle [(C - X_1') \cup a_1'] = [C - (X_2 - e)] \cup \{a_1, a_1'\}$ is a circuit of M; a contradiction to (4.3). □

Lemma 4.3. *Let C be a circuit of a matroid N such that $|C| = \mathrm{circ}(N) = 6$. Suppose that Z_1, Z_2 and Z_3 are pairwise disjoint 3-subsets of $E(N) - C$ satisfying:*

(i) *Z_i is an independent set of N, for each $i \in \{1, 2, 3\}$; and*
(ii) *$(N/C)|(Z_1 \cup Z_2 \cup Z_3) = [(N/C)|Z_1] \oplus [(N/C)|Z_2] \oplus [(N/C)|Z_3] \cong U_{1,3} \oplus U_{1,3} \oplus U_{1,3}$.*

If A_i is a 2-subset of Z_i, for each $i \in \{1, 2, 3\}$, then $A_1 \cup A_2 \cup A_3$ is a dependent set of N.

Proof. Suppose that $A_1 \cup A_2 \cup A_3$ is an independent set of N. Observe that A_1, A_2 and A_3 are pairwise strongly disjoint C-arcs. First, we establish the next sublemma.

Sublemma 4.3. *For each $i \in \{1, 2, 3\}$, there is $a_i \in A_i$ such that $(A_1 \cup A_2 \cup A_3) \triangle \{a_i, a_i'\}$ is an independent set of N, where $Z_i - A_i = \{a_i'\}$.*

Proof. We argue by contradiction. Assume this result is not true for some i, say $i = 1$. Hence $(Z_1 - a) \cup A_2 \cup A_3$ and $(Z_1-a') \cup A_2 \cup A_3$ are dependent sets of N, where $A_1 = \{a, a'\}$. So there are circuits D and D' of N such that $D \subseteq (Z_1-a) \cup A_2 \cup A_3$ and $D' \subseteq (Z_1 - a') \cup A_2 \cup A_3$. As $A_2 \cup A_3$ is an independent set of N and, for each 2-subset A of Z_1, A is a series class of $N|(C \cup A \cup A_2 \cup A_3)$, it follows that $Z_1 - a \subseteq D$ and $Z_1 - a' \subseteq D'$. Note that $(Z_1 - a) \cap (Z_1 - a') = \{a'_1\}$. Therefore there is a circuit D'' of N such that

$$D'' \subseteq (D \cup D') - a'_1 \subseteq A_1 \cup A_2 \cup A_3;$$

a contradiction because $A_1 \cup A_2 \cup A_3$ is an independent set of N. $\square$

For $i \in \{1, 2, 3\}$, set $Y_i = \{a_i, a'_i\}$ and $Z_i - Y_i = \{z_i\}$, where a_i and a'_i are defined in the statement of Sublemma 4.3. Let $M = N/\{z_1, z_2, z_3\}$. By hypothesis, Y_1, Y_2 and Y_3 are all independent sets of M. Observe that $T = \{a_1, a_2, a_3\}$ is a transversal of (Y_1, Y_2, Y_3) such that T is an independent set of M. By Sublemma 4.3, $T \triangle Y_i$ is an independent set of M, for every $i \in \{1, 2, 3\}$. As the conclusions of Lemma 4.2 do not hold for (M, C, Y_1, Y_2, Y_3), it follows that (4.3) does not hold for (M, C, Y_1, Y_2, Y_3). Let D be a circuit of M such that $Y_i \not\subseteq D$, for every $i \in \{1, 2, 3\}$, and (4.3) does not hold for D. As $D' = D \cup \{z_i : Z_i \cap D \neq \emptyset\}$ is a circuit of N, it follows that

$$6 \geq |D'| = |D \cap C| + |D - C| + |\{z_i : Z_i \cap D \neq \emptyset\}| = |D \cap C| + 2|D - C|;$$

a contradiction since this is the inequality (4.3) for D. Therefore $A_1 \cup A_2 \cup A_3$ is a dependent set of N. The result follows. $\square$

5. Proof of the main result

Choose a circuit C of M such that $6 = |C| = \text{circ}(M)$. Let $M_1, \ldots, M_n$ be the connected components of M/C having a rank equal to 1. By Proposition 2.1,

$$n = \sum_{i=1}^{n} r(M_i) = r(M/C) = r(M) - (|C| - 1) \geq 3. \tag{5.1}$$

Now, we establish that, for every $i \in \{1, \ldots, n\}$,

$$r(E(M_i)) \geq 3. \tag{5.2}$$

First, observe that $E(M_i)$ is a cocircuit of M because $M_i = M/[E(M) - E(M_i)]$ and M_i is a connected matroid having a rank equal to 1. As M is 3-connected, it follows that

$$\begin{aligned} 2 &\leq r(E(M_i)) + r^*(E(M_i)) - |E(M_i)| \\ &\leq r(E(M_i)) + [|E(M_i)| - 1] - |E(M_i)| \\ &\leq r(E(M_i)) - 1. \end{aligned}$$

Therefore (5.2) follows. By (5.2), for each $i \in \{1, \ldots, n\}$, there is a 3-subset Z_i of $E(M_i)$ which is independent in M, say $Z_i = \{a_i, b_i, c_i\}$. Note that every 2-subset of Z_i is a C-arc. Moreover, 2-subsets of different Z_i's are strongly disjoint C-arcs.

Lemma 5.1. *If A is a 2-subset of Z_i, for some $i \in \{1, \ldots, n\}$, then $|W| \neq 3$, for every $W \in \Pi_M(A, C)$.*

Proof. Suppose that $|W| = 3$, for some $W \in \Pi_M(A, C)$, say $i = 1$. Therefore $|\Pi_M(A, C)| = 2$ and $|W| = 3$, for every $W \in \Pi_M(A, C)$. For each $j \in \{2, 3\}$, by Lemma 3.3, $M|(C \cup Z_j)$ has just one non-trivial series class, say S_j, and its cosimplification is isomorphic to F_7^*. Moreover, by the last sentence of Lemma 3.3, $S_j \in \Pi_M(A, C)$. First, we establish that

$$S_2 \neq S_3. \tag{5.3}$$

Assume that $S_2 = S_3$. Let W_2 and W_3 be different 2-subsets of $C - S_2$. For $j \in \{2, 3\}$, there is a 2-subset A_i of Z_i such that $W_i \in \Pi_M(A_i, C)$; a contradiction to Lemma 3.1(ii) because W_2 and W_3 cross and $|\Pi_M(A_2, C)| = |\Pi_M(A_3, C)| = 2$. Thus (5.3) follows.

Let A' be a 2-subset of Z_1 such that $A' \neq A$. If $W \in \Pi_M(A', C)$, then, by Lemma 3.1(ii), W does not cross W', for every

$$W' \in \bigcup_{i \in \{2,3\}} \bigcup_{A_i \in \mathcal{P}_2(Z_i)} \Pi_M(A_i, C) = \mathcal{P}_2(Z_2) \cup \mathcal{P}_2(Z_3),$$

where $\mathcal{P}_2(Z)$ denotes the set of all 2-subsets of a set Z. Hence $W = S_2$ or $W = S_3$. That is, $A \cup S_i$ and $A' \cup S_i$ are circuits of M, for both $i \in \{2, 3\}$. In particular, S_2 and S_3 are series classes of $M|(C \cup Z_1)$ and so Z_1 is dependent in M; a contradiction. $\square$

Lemma 5.2. *For a 2-subset $\{i, j\}$ of $\{1, \ldots, n\}$, it is possible to rename the elements of $Z_i - a_i$ and $Z_j - a_j$ so that $\{a_i, b_i, a_j, b_j\}$ is an independent set of M.*

Proof. If this labeling cannot be done, then $\{a_i, b_i, a_j\}$ and $\{a_i, a_j, b_j\}$ span respectively Z_j and Z_i. Hence $\{a_i, b_i, a_j\}$ spans $Z_i \cup Z_j$. Therefore Z_i spans Z_j because

$$3 \leq r(Z_i) \leq r(Z_i \cup Z_j) = r(\{a_i, b_i, a_j\}) \leq 3.$$

We arrive at a contradiction to orthogonality because $Z_j \subseteq E(M_j)$ and $E(M_j)$ is a cocircuit of M such that $E(M_j) \cap Z_i = \emptyset$. $\square$

The next lemma is the core of our proof.

Lemma 5.3. *For $i \in \{1, \dots, n\}$, $r(E(M_i)) = 3$. In particular, $E(M_i)$ is a 3-separating set of M.*

Proof. To establish that $E(M_i)$ is a 3-separating set of M, it is enough to prove that the equality holds in (5.2). If Z_i spans $E(M_i)$, then the equality in (5.2) follows. Assume that Z_i does not span $d_i \in E(M_i) - Z_i$, say $i = 1$. By Lemma 5.2, the elements of $Z_2 - a_2$ and $Z_3 - a_3$ can be labeled so that

$$\{a_2, b_2, a_3, b_3\} \text{ is an independent set of } M. \tag{5.4}$$

By Lemma 4.3, $A \cup \{a_2, b_2, a_3, b_3\}$ is a dependent set of M, for every 2-subset A of $Z_1 \cup d_1$. By (5.4),

$$\{a_1, a_2, b_2, a_3, b_3\} \text{ spans } Z_1 \cup d_1 \text{ in } M. \tag{5.5}$$

Therefore

$$r(\{a_1, b_1, c_1, d_1, a_2, b_2, a_3, b_3\}) = 5. \tag{5.6}$$

As a_2 and a_3 are coloops of $M/[C \cup E(M_1)]|\{a_2, a_3\}$, it follows that $Z_1 \cup \{d_1, a_2, a_3\}$ is an independent set of M; a contradiction to (5.6). Therefore d_1 does not exist and we have equality in (5.2). □

Lemma 5.4. *If $a_i \in E(M_i)$, for $i \in \{1, \dots, n\}$, then $[E(M_1) - a_1] \cup \cdots \cup [E(M_n) - a_n]$ is a line of $M/\{a_1, \dots, a_n\}$. In particular,*

$$r(E(M_1) \cup \cdots \cup E(M_k)) = 2 + k.$$

That is, $E(M_1), \dots, E(M_n)$ is a book of M.

Proof. By Lemma 5.2, we can label the elements of $Z_1 - a_1$ and $Z_2 - a_2$ so that

$$\{a_1, b_1, a_2, b_2\} \text{ is an independent set of } M. \tag{5.7}$$

By (5.7), $\{b_1, b_2\}$ is an independent set of $M/\{a_1, \dots, a_n\}$. Consider $L = \mathrm{cl}_{M/\{a_1,\dots,a_n\}}(\{b_1, b_2\})$. The first part of this lemma follows provided we establish that

$$[E(M_1) - a_1] \cup \cdots \cup [E(M_n) - a_n] \subseteq L. \tag{5.8}$$

Choose $a \in [E(M_1) - a_1] \cup \cdots \cup [E(M_n) - a_n]$, say $a \in E(M_i) - a_i$. If $i \geq 3$, then, by Lemma 4.3, $\{a_1, b_1, a_2, b_2, a_i, a\}$ is dependent in M and so $\{b_1, b_2, a\}$ is dependent in $M/\{a_1, \dots, a_n\}$. That is, $a \in L$. If $i \leq 2$, say $i = 1$, then, similar to the first paragraph, there is $b_3 \in E(M_3) - a_3$ such that

$$\{a_2, b_2, a_3, b_3\} \text{ is an independent set of } M. \tag{5.9}$$

Hence $L = \mathrm{cl}_{M/\{a_1,\dots,a_n\}}(\{b_2, b_3\})$. The result follows because $\{a_1, a, a_2, b_2, a_3, b_3\}$ is dependent in M. The second part of this lemma follows easily from the first. □

Suppose that Z is a 2-subset of Z_i, for some $i \in \{1, \dots, n\}$, say $i = 1$. If $A \in \Pi_M(Z, C)$ and $|A| = 2$ (it exists by Lemma 5.1), then $D := (C - A) \cup Z$ is a maximum size circuit of M. Such a circuit D is said to be a *cousin* of C. Observe that $M_2, \dots, M_n$ are the rank-1 connected components of $M/(C \cup D)$. By Proposition 2.1, there is a rank-1 matroid M_D such that $M_D, M_2, \dots, M_n$ are the connected components of M/D. Note that $A \subseteq E(M_D)$ because A is a series class of $M|(C \cup D)$. By the previous lemma applied to D, $E(M_D), E(M_2), \dots, E(M_n)$ is a book of M. In particular, $A \cap [\mathrm{cl}_M(E(M_2)) \cap \mathrm{cl}_M(E(M_3))] = \emptyset$.

Lemma 5.5. *If D is a cousin of C, then $E(M_D) \cap E(M_1) = \emptyset$.*

Proof. Assume that $E(M_D) \cap E(M_1) \neq \emptyset$, say $a \in E(M_D) \cap E(M_1)$. By Lemma 5.4,

$$r(E(M_1) \cup E(M_2)) = r(E(M_1) \cup E(M_3)) = r(E(M_1) \cup E(M_2) \cup E(M_3)) - 1 = 4.$$

Therefore

$$r_{M/a}([E(M_1) \cup E(M_2)] - a) = r_{M/a}([E(M_1) \cup E(M_3)] - a) = r_{M/a}([E(M_1) \cup E(M_2) \cup E(M_3)] - a) - 1 = 3.$$

In M/a, $[E(M_1) \cup E(M_2)] - a$ and $[E(M_1) \cup E(M_2)] - a$ are planes. As $r_{M/a}([E(M_1) \cup E(M_2) \cup E(M_3)] - a) = 4$, it follows, by submodularity, that $\{[E(M_1) \cup E(M_2)] - a\} \cap \{[E(M_1) \cup E(M_2)] - a\}$ spans a line L of M/a. In particular, $E(M_1) - a \subseteq L$. Note that $L = \mathrm{cl}_{M/a}(E(M_2)) \cap \mathrm{cl}_{M/a}(E(M_3))$. Replacing M_1 by M_D in the previous argument, we conclude that $E(M_D) - a \subseteq L$. Therefore $r_{M/a}([E(M_D) \cup E(M_1)] - a) = 2$ and so $r(E(M_D) \cup E(M_1)) = 3$. Hence $E(M_1)$ spans $E(M_D)$ in M. In particular, $A = C - D \subseteq E(M_D) - E(M_1)$ is contained in the line $\mathrm{cl}_M(E(M_2)) \cap \mathrm{cl}_M(E(M_3))$; a contradiction. □

By Lemmas 5.4 and 5.5 and Lemma 2.1 of [11], when D is a cousin of C, $E(M_D), E(M_1), \dots, E(M_n)$ is a book of M.

Lemma 5.6. *If D and D' are cousins of C, then $E(M_D) = E(M_{D'})$ or $E(M_D) \cap E(M_{D'}) = \emptyset$.*

Proof. Suppose that $E(M_D) \cap E(M_{D'}) \neq \emptyset$ otherwise the result follows. Similar to the proof of the previous lemma, $E(M_D)$ spans $E(M_{D'})$ in M. As $E(M_{D'}) \cap [\mathrm{cl}_M(E(M_2)) \cap \mathrm{cl}_M(E(M_3))] = \emptyset$ and $\mathrm{cl}_M(E(M_D)) - E(M_D) \subseteq \mathrm{cl}_M(E(M_2)) \cap \mathrm{cl}_M(E(M_3))$, it follows that $E(M_{D'}) \subseteq E(M_D)$. Interchanging the role of D and D' in the previous argument, we obtain $E(M_D) \subseteq E(M_{D'})$. Thus $E(M_D) = E(M_{D'})$. □

Lemma 5.7. *If D and D' are cousins of C such that $E(M_D) \cap E(M_{D'}) = \emptyset$, then the result follows.*

Proof. Applying the comment made before the statement of Lemma 5.6 for both D and D', we get that $E(M_{D'}), E(M_D), E(M_1), \dots, E(M_n)$ is a book of M. If $W = E(M_{D'}) \cup E(M_D) \cup E(M_1) \cup \cdots \cup E(M_n)$, then W is a 3-separating set of M and so

$$r(E(M) - W) \leq r(M) + 2 - r(W) = r(M) + 2 - [2 + (2 + n)] = r(M) - n - 2. \tag{5.10}$$

By (5.1) and (5.10), $r(E(M) - W) = 3$. If $W' = (E(M) - W) - [\mathrm{cl}_M(E(M_2)) \cap \mathrm{cl}_M(E(M_3))]$, then $W', E(M_{D'}), E(M_D), E(M_1), \dots, E(M_n)$ is a book of M having $E(M) - [W' \cup E(M_{D'}) \cup E(M_D) \cup E(M_1) \cup \cdots \cup E(M_n)]$ as its back. The result follows. $\square$

We may assume that $E(M_D) \cap E(M_{D'}) \neq \emptyset$, when D and D' are cousins of C. By Lemma 5.6, $E(M_D) = E(M_{D'})$. As $r(E(M_D)) = 3$ and $(C - D) \cup (C - D') \subseteq E(M_D)$, it follows that $(C - D) \cap (C - D') \neq \emptyset$. That is, if $A_i \in \Pi_M(W_i, C)$ and $A_j \in \Pi_M(W_j, C)$ satisfies $|A_i| = |A_j| = 2$, where W_i and W_j are respectively a 2-subset of Z_i and Z_j, then $A_i \cap A_j \neq \emptyset$. In particular, $|\Pi_M(W, C)| = 2$, for every 2-subset W of some Z_k. By Lemma 3.2, $M|(C \cup Z_k)$ is a binary matroid, for every $k \in \{1, \dots, n\}$. Observe that $M|(C \cup Z_k)$ is a subdivion of $M(K_4)$ or F_7^*. It cannot be a subdivion of $M(K_4)$ otherwise C is partitioned in 3 2-subsets each one belonging to $\Pi_M(W_k, C)$, for some 2-subset W_k of Z_k (these 2-subsets are the non-trivial series classes of $M|(C \cup Z_k)$). Therefore $M|(C \cup Z_k)$ is a subdivion of F_7^* having just one non-trivial series class S (with 3 elements). Moreover, S does not depend on k. We have a contradiction by Lemma 3.1(ii).

Acknowledgements

The first author's research was supported by FCT (Portugal) through program POCTI. The second author is partially supported by CNPq (Grants No. 476224/04-7, 301178/05-4 and 502048/07-7) and FAPESP/CNPq (Grant No. 2003/09925-5).

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