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Removing circuits in 3-connected binary matroids

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Abstract

For a k-connected graph or matroid M, where k is a fixed positive integer, we say that a subset X of $E(M)$ is *k-removable* provided $M\backslash X$ is k-connected. In this paper, we obtain a sharp condition on the size of a 3-connected binary matroid to have a 3-removable circuit.

Keywords: Matroid; Binary matroid; 3-connected matroid; Circuit; Removable circuit

1. Introduction

Removable circuits and cocircuits play an important role in studying the structure of graphic matroids (see [11,12,24,25]). There has been much interest in the study of removable circuits and cocircuits in graphs and matroids lately (see [1,4–8,10,12–14,16–18,21,22]).

Hobbs conjectured that every 2-connected graph with minimum degree at least 4 has a 2-removable circuit. Robertson and Jackson independently gave a counter-example to this conjecture (see [7]). Mader [17] proved this conjecture for simple graphs. Goodyn, van der Heuvel and McGuinness established it for graphs without a Peterson Graph as a minor. For more results on graphs that extend this conjecture see [7,14,22]. Inspired by this conjecture, Oxley [20] proposed the following problem: does a simple 2-connected binary matroid with cogirth at least 4 have a 2-removable circuit? Lemos and Oxley [14] constructed a cographic matroid that provides a negative answer to this question.

For a 2-connected graph G with having minimum degree at least four, we have that

$$|E(G)| \geq 2|V(G)|. \tag{1}$$

If M is the graphic matroid associated with G, then this inequality translates as

$$|E(M)| \geq 2r(M) + 2. \tag{2}$$

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For a 2-connected matroid, a condition on the size of the cogirth does not guarantee the existence of a 2-removable circuit, but a condition on its number of elements does.

Theorem 1. *Let M be a 2-connected matroid. If M is non-empty, then M has a 2-removable circuit provided:*
(i) *(Lemos and Oxley [14]) $|E(M)| \geq 3r(M)$; or*
(ii) *(Junior [8]) $r(M) \geq 3$ and $|E(M)| \geq 3r(M) - 1$; or*
(iii) *(Junior [8]) M is simple, $r(M) \geq 7$ and $|E(M)| \geq 3r(M) - 3$.*

Each item of the previous result is sharp. Lemos and Oxley [15] proved that:

Theorem 2. *If M is a 3-connected matroid such that $r(M) \geq 6$ and $|E(M)| \geq 4r(M) - 5$, then M has a 3-removable circuit.*

This result is sharp. Lemos and Oxley [15] construct an infinite family of matroids that attain this bound. But all the matroids in this family are non-binary. For binary matroids, in this paper, we prove the following result (it was conjectured in [9]):

Theorem 3. *If M is a 3-connected binary matroid such that $r(M) \geq 10$ and $|E(M)| \geq 4r(M) - 8$, then M has a 3-removable circuit.*

Theorem 3 is sharp even for graphs as the next example shows. Let $\{U, V\}$ be a partition of the vertices of the complete bipartite graph $K_{4,n}$, for $n \geq 3$, such that U and V are stable sets, $|U| = 4$, and $|V| = n$. Let $K^{(3)}_{4,n}$ be a simple graph obtained from $K_{4,n}$ by adding a set with 3 edges P joining vertices belonging to U so that P is a path. Note that $M(K^{(3)}_{4,n})\backslash C$ is not 3-connected, for every circuit C of $M(K^{(3)}_{4,n})$. Moreover,

$$|E(M(K^{(3)}_{4,n}))| = 4n + 3 = 4r(M(K^{(3)}_{4,n})) - 9. \tag{3}$$

For more detail in removable circuits in graphs and matroids, we recommend Oxley's excellent survey [19]. For notation and terminology in matroid theory, we follow Oxley's book [20].

2. Known theorems

In this section, we state some theorems from other papers that are used in the proof of Theorem 3. Let M be a matroid. We define $\Lambda_1(M)$ to be the set of connected components of M. We set $\lambda_1(M) = |\Lambda_1(M)|$. Now M can be constructed from a collection $\Lambda_2(M)$ of 3-connected matroids by using the operations of 1-sum and 2-sum. It follows from results of Cunningham and Edmonds (see [3]) that $\Lambda_2(M)$ is unique up to isomorphism. We denote by $\lambda_2(M)$ the number of matroids in $\Lambda_2(M)$ that are not isomorphic to $U_{1,3}$. Theorem 1.3 of [15] can be stated as:

Theorem 4. *Let M be a 3-connected matroid other than $U_{1,3}$. If N is a non-empty spanning restriction of M, then M has a 3-connected restriction K such that $E(N) \subseteq E(K)$ and*

$$|E(K)| \leq |E(N)| + \lambda_1(N) + \lambda_2(N) - 2, \tag{4}$$

unless N is a circuit of size at least four, in which case, $|E(K)| \leq 2r(N)$.

A circuit C of a matroid M is said to be *Hamiltonian* provided $|C| = r(M) + 1$. If M has at least one circuit, then $\mathrm{circ}(M)$ denotes the *circumference* of M, that is, the maximum cardinality of a circuit of M. The 3-connected matroids having small circumference must have small rank. Lemos and Oxley [16] proved that:

Theorem 5. *Suppose that M is a 3-connected matroid. If $r(M) \geq 6$, then $\mathrm{circ}(M) \geq 6$.*

Cordovil, Junior and Lemos [2] constructed all the 3-connected binary matroids having circumference equal to 6 or 7 with large rank. These matroids are central in the proof of the next result (see [2]):

Theorem 6. *Let M be a 3-connected binary matroid such that $\mathrm{circ}(M) \in \{6, 7\}$ and $r(M) \geq 10$. If $M\backslash C$ is not 3-connected for every circuit C of M, then $|E(M)| < 4r(M) - 8$.*

Using Theorems 5 and 6, we conclude that a counter-example for Theorem 3 must have circumference at least eight. Using the main result of the next section, we conclude that the circumference of this counter-example must be eight.

3. Two auxiliary functions

For a matroid M, we consider the following function

$$\delta(M) = 3r(M) - |E(M)| - \lambda_1(M) - \lambda_2(M). \tag{5}$$

First, we show that δ is both 1-additive and 2-additive. (A function f defined in the class of matroids is called *k-additive* when

$$f(N) = f(N_1) + f(N_2) + \cdots + f(N_n) \tag{6}$$

provided the matroid N is the k-sum of matroids $N_1, N_2 \ldots, N_n$.)

Lemma 1. *If the matroid M is the 1-sum of matroids $M_1, M_2 \ldots, M_n$, then*

$$\delta(M) = \delta(M_1) + \delta(M_2) + \cdots + \delta(M_n). \tag{7}$$

Proof. This result holds because all the functions involved in the definition of δ are 1-additive. □

Following Seymour [23], we consider the 2-sum of matroids M_1 and M_2 having e as a common element only when the connected component of e in M_i has at least three elements, for both $i \in \{1, 2\}$.

Lemma 2. *If the matroid M is the 2-sum of matroids $M_1, M_2 \ldots, M_n$, then*

$$\delta(M) = \delta(M_1) + \delta(M_2) + \cdots + \delta(M_n). \tag{8}$$

Proof. We need to prove this result only when two matroids are involved. When $n = 2$, we have:

$$\begin{aligned}
&r(M) = r(M_1) + r(M_2) - 1\\
&|E(M)| = |E(M_1)| + |E(M_2)| - 2\\
&\lambda_1(M) = \lambda_1(M_1) + \lambda_1(M_2) - 1\\
&\lambda_2(M) = \lambda_2(M_1) + \lambda_2(M_2).
\end{aligned}$$

The result follows easily from these identities. □

Observe that:

Lemma 3. *If M is a coloop, then $\delta(M) = 0$.*

For a matroid M, we define the following function:

$$\Delta(M) = \max\{\delta(N) : N \text{ is a restriction of } M\}. \tag{9}$$

Now, we prove the main result of this section:

Proposition 1. *If M is a 3-connected matroid such that $r(M) \geq 2$, then M has a 3-connected spanning restriction N such that*

$$|E(N)| \leq \begin{cases} 3r(M) - \Delta(M) - 2 & \text{when } M \text{ is not Hamiltonian;}\\ 2r(M) & \text{when } M \text{ is Hamiltonian.}\end{cases} \tag{10}$$

Proof. If M has a Hamiltonian circuit, then the result follows by Theorem 4. Assume that M is not Hamiltonian. Let H be a restriction of M so that $\delta(H) = \Delta(M)$. Choose a basis B of M such that $B \cap E(H)$ spans $E(H)$ in M. Hence $B - E(H)$ is a set of coloops of $K = M|[E(H) \cup B]$. Therefore $K = H \oplus [M|(B - E(H))]$. By Lemmas 1 and 3, $\delta(K) = \delta(H)$. In particular, $\delta(K) = \Delta(M)$. By Theorem 4, M has a 3-connected restriction N such that $N|E(K) = K$ and

$$|E(N)| \leq |E(K)| + \lambda_1(K) + \lambda_2(K) - 2 = 3r(K) - \delta(K) - 2. \tag{11}$$

Thus $|E(N)| \leq 3r(M) - \Delta(M) - 2$ and the result follows. □

The previous result shows the importance of the function Δ. This function is hard to compute, but for our application, we just need an upper bound for it such as $\delta(N)$, when N is a restriction of M with small corank. For example:

Lemma 4. *If M is a circuit with at least 3 elements, then $\delta(M) = |E(M)| - 2$.*

4. Special pairs

We say that (M, C) is a *special pair* provided M is a connected matroid having C as a circuit, $E(M) \neq C$ and, for every $K \in \Lambda_1(M/C)$, $r_{M/C}(E(K)) = 1$ and $E(K)$ is an independent set of M having at least 3 elements. In this section, we establish some properties about special pairs.

In the next section, we prove that every connected component of M/C has rank equal to 0 or 1, when C is a largest circuit of a counter-example M of Theorem 3. Moreover, $(M|[\cup_{H\in\Lambda_1(M/C):r(H)=1}(C \cup B_H)], C)$ is a special pair, where B_H is a basis of $M|E(H)$. Therefore the results obtained in this section will be fundamental to conclude the proof of Theorem 3.

Lemma 5. *Let (M, C) be a special pair. If $\{Z, W\}$ is a 2-separation of M, then:*

(i) *If $K \in \Lambda_1(M/C)$, then $E(K) \subseteq Z$ or $E(K) \subseteq W$.*
(ii) *$C \cap Z \neq \emptyset$ and $C \cap W \neq \emptyset$.*

Moreover, $r(Y) = |C \cap Y| + |\{H \in \Lambda_1(M/C) : E(H) \subseteq Y\}|$, for each $Y \in \{Z, W\}$.

Proof. For $Y \in \{Z, W\}$, $r(C \cap Y) = |C \cap Y| - \delta_Y$, where $\delta_Y = 0$, when $C \not\subseteq Y$, and $\delta_Y = 1$, when $C \subseteq Y$. If X_Y is a subset of $Y - C$ such that, for each $H \in \Lambda_1(M/C)$, $|X_Y \cap E(H)| = 1$, when $Y \cap E(H) \neq \emptyset$, and $|X_Y \cap E(H)| = 0$, when $Y \cap E(H) = \emptyset$, then

$$r(Y) \geq r(Y \cap C) + |X_Y| = |Y \cap C| + |X_Y| - \delta_Y \tag{12}$$

because X_Y is a set of coloops of M/C. There is ϵ_Y such that $\epsilon_Y \geq 0$ and

$$r(Y) = |Y \cap C| + |X_Y| + \epsilon_Y - \delta_Y. \tag{13}$$

Therefore

$$1 + r(M) = r(Z) + r(W) = |C| + |X_Z| + |X_W| + (\epsilon_Z + \epsilon_W) - (\delta_Z + \delta_W). \tag{14}$$

As $|\Lambda_1(M/C)| \leq |X_Z| + |X_W|$, say, for some $\epsilon \geq 0$,

$$|\Lambda_1(M/C)| = |X_Z| + |X_W| - \epsilon, \tag{15}$$

and $r(M) = |C| - 1 + |\Lambda_1(M/C)|$, it follows that

$$0 \leq \epsilon_Z + \epsilon_W + \epsilon = \delta_Z + \delta_W \leq 1. \tag{16}$$

If (i) does not hold, then $E(K) \cap Z \neq \emptyset$ and $E(K) \cap W \neq \emptyset$. In particular, $\epsilon \geq 1$. By (16), $\epsilon = 1$ and $\{\delta_Z, \delta_W\} = \{0, 1\}$, say $\delta_W = 1$. Thus $C \subseteq W$, since $\delta_W = 1$, and $E(H) \subseteq Z$ or $E(H) \subseteq W$, for each $H \in \Lambda_1(M/C) - \{K\}$, since $\epsilon = 1$. (That is, (i) holds for each element of $\Lambda_1(M/C)$ other than K.) Moreover, by (16), $\epsilon_Z = \epsilon_W = 0$. There is $H \in \Lambda_1(M/C)$ such that $|E(H) \cap Z| \geq 2$ because $C \cap Z = \emptyset$ and (i) holds for every $H \in \Lambda_1(M/C) - \{K\}$. We arrive at a contradiction because $[E(H) \cap Z] \cup X_Z$ is an independent set of M and so $\epsilon_Z \geq 1$. Therefore (i) follows. In particular, $\epsilon = 0$.

Suppose that (ii) does not hold. Then $C \cap Z = \emptyset$ or $C \cap W = \emptyset$, say $C \subseteq W$. In particular, $\delta_Z = 0$ and $\delta_W = 1$. By (i), there is $H \in \Lambda_1(M/C)$ such that $E(H) \subseteq Z$. As $E(H) \cup X_Z$ is an independent set of M and $|E(H) \cup X_Z| \geq |X_Z| + 2$, it follows that $\epsilon_Z \geq 2$; a contradiction to (16). Therefore (ii) follows and so $\delta_Z = \delta_W = 0$. By (16), $\epsilon_Z = \epsilon_W = 0$. Thus $r(Y) = |Y \cap C| + |X_Y|$, for each $Y \in \{Z, W\}$. With this, we conclude the proof of this lemma. □

Lemma 6. *Let (M, C) be a special pair. If $\{Z, W\}$ is a 2-separation of M, then there are matroids M_Z and M_W such that:*

(i) *$E(M_Z) = Z \cup e$ and $E(M_W) = W \cup e$, where e is a new element.*
(ii) *$M = M_Z \oplus_2 M_W$.*
(iii) *For $Y \in \{Z, W\}$, M_Y is a circuit or $(M_Y, (C \cap Y) \cup e)$ is a special pair.*
(iv) *$\Lambda_1(M/C) = \Lambda_1(M_Z/C_Z) \cup \Lambda_1(M_W/C_W)$.*

Proof. Observe that (i) and (ii) follow from Section 2 of Seymour [23]. We need to prove only (iii) and (iv). Observe that M_Z and M_W are connected because M is connected. By Lemma 5(ii), $C_Y = (C \cap Y) \cup e$ is a circuit of M_Y, for each $Y \in \{Z, W\}$. By Lemma 5(i), $E(K) \subseteq Z$ or $E(K) \subseteq W$, for each $K \in \Lambda_1(M/C)$, say $E(K) \subseteq Z$. Observe that $E(K)$ is independent in M_Z. We need to show that K is a connected component of M_Z/C_Z. If $f \in C \cap W$, then M_Z is obtained from $M\backslash(W - C)/[(W \cap C) - f]$ by renaming f by e. Thus $M_Z/C_Z = M\backslash(W - C)/C$. As K is a connected component of M/C and $E(K) \subseteq Z$, it follows that K is a connected component of M_Z/C_Z. Hence $\Lambda_1(M/C) \subseteq \Lambda_1(M_Z/C_Z) \cup \Lambda_1(M_W/C_W)$. The equality holds because

$$E(M) - C = [E(M_Z) - C_Z] \cup [E(M_W) - C_W]. \tag{17}$$

With this identity we conclude the proof of this lemma. □

As a consequence of this lemma, we have the following decomposition (use induction):

Lemma 7. *Let (M, C) be a special pair. If*

$$\Gamma_1(M) = \{H \in \Lambda_2(M) : E(H) \cap [E(M) - C] \neq \emptyset\} \tag{18}$$

and $\Gamma_2(M) = \Lambda_2(M) - \Gamma_1(M)$, then:

(i) *If $H \in \Gamma_2(M)$, then $H \cong U_{2,3}$ and $E(H) \cap E(M) \subseteq C$.*
(ii) *If $H \in \Gamma_1(M)$ and $C_H = E(H) - [E(M) - C]$, then C_H is a circuit of H and (H, C_H) is a special pair.*
(iii) *$\Lambda_1(M/C) = \cup_{H \in \Gamma_1(M)} \Lambda_1(H/C_H)$.*
(iv) *The matroid obtained by making the 2-sum of the matroids belonging to the family $\{H|C_H : H \in \Gamma_1(M)\} \cup \Gamma_2(M)$ is $M|C$.*

Lemma 8. *If (M, C) is a special pair, then*

$$\delta(M) = \delta(M|C) + \sum_{H \in \Gamma_1(M)} (|C_H| - 3) - \sum_{K \in \Lambda_1(M/C)} (|E(K)| - 3), \tag{19}$$

where $C_H = E(H) - [E(M) - C]$, for $H \in \Gamma_1(M)$.

Proof. By Lemma 7(iv), $M|C$ is the 2-sum of the matroids belonging to $\Gamma_1'(M) \cup \Gamma_2(M)$, where $\Gamma_1'(M) = \{H|C_H : H \in \Gamma_1(M)\}$. By Lemma 2, we have that

$$\delta(M|C) = \sum_{H \in \Gamma_1'(M) \cup \Gamma_2(M)} \delta(H) \quad \text{and} \quad \delta(M) = \sum_{H \in \Lambda_2(M)} \delta(H). \tag{20}$$

By Lemmas 5 and 7(ii), for $H \in \Gamma_1(M)$,

$$\begin{aligned}\delta(H) &= 3\left(r(H|C_H) + \sum_{K \in \Lambda_1(H/C_H)} 1\right) - \left(|C_H| + \sum_{K \in \Lambda_1(H/C_H)} |E(K)|\right) - 1 - 1 \\ &= \delta(H|C_H) - \sum_{K \in \Lambda_1(H/C_H)} (|E(K)| - 3) + \lambda_2(H|C_H) - 1.\end{aligned}$$

Hence

$$\begin{aligned}\delta(M) &= \sum_{H \in \Lambda_2(M)} \delta(H) \\ &= \sum_{H \in \Gamma_1(M)} \delta(H) + \sum_{H \in \Gamma_2(M)} \delta(H)\end{aligned}$$

$$= \sum_{H\in\Gamma_1(M)}\left(\delta(H|C_H) - \sum_{K\in\Lambda_1(H/C_H)}(|E(K)|-3) + \lambda_2(H|C_H) - 1\right) + \sum_{H\in\Gamma_2(M)}\delta(H)$$
$$= \sum_{H\in\Gamma_1'(M)\cup\Gamma_2(M)}\delta(H) + \sum_{H\in\Gamma_1(M)}(\lambda_2(H|C_H)-1) - \sum_{K\in\Lambda_1(M/C)}(|E(K)|-3)$$
$$= \delta(M|C) + \sum_{H\in\Gamma_1(M)}(\lambda_2(H|C_H)-1) - \sum_{K\in\Lambda_1(M/C)}(|E(K)|-3)$$

and the result follows because $\lambda_2(H|C_H) = |C_H| - 2$ (in the passage from the third to the fourth line of this display, we use Lemma 7(iii)). $\square$

A special pair (M, C) is said to be *unitary* provided $|\Lambda_1(M/C)| = 1$. A special pair (M, C) is said to be *strong* provided M is binary and $\Delta(M) = \delta(M|C)$. By Lemma 4, when (M, C) is a strong special pair, C is a largest circuit of M.

Lemma 9. *If (M, C) is a unitary strong special pair, then $|\Gamma_1(M)| = 1$, say $\Gamma_1(M) = \{N\}$, and:*

(i) *If $|E(M) - C| = 3$, then $N \cong M(K_4)$.*
(ii) *If $|E(M) - C| = 4$, then $N \cong M(W_4)$.*
(iii) *If $|E(M) - C| \in \{3, 4\}$, then $E(N) \cap C = \emptyset$.*

Proof. For each $H \in \Gamma_1(M)$, we set $C_H = E(H) - [E(M) - C]$. By Lemma 7(ii), C_H is a circuit of H and (H, C_H) is a special pair. By Lemma 7(iii), $\Lambda_1(M/C) = \cup_{H\in\Gamma_1(M)}\Lambda_1(H/C_H)$ and so

$$1 = |\Lambda_1(M/C)| = \sum_{H\in\Gamma_1(M)}|\Lambda_1(H/C_H)|. \tag{21}$$

As $\Lambda_1(H/C_H) \neq \emptyset$, when $H \in \Gamma_1(M)$, it follows that $|\Gamma_1(M)| = 1$, say $\Gamma_1(M) = \{N\}$. Observe that $C^* = E(N) - C_N$ is a cocircuit of N because N/C_N is a rank-1 connected matroid. Now, we need to prove (i) to (iii). By definition, $\Delta(M) = \delta(M|C)$ and so $0 \geq \delta(M) - \delta(M|C)$. As

$$\sum_{H\in\Gamma_1(M)}(|C_H|-3) - \sum_{K\in\Lambda_1(M/C)}(|E(K)|-3) = (|C_N|-3) - (|C^*|-3), \tag{22}$$

it follows, by Lemma 8, that

$$0 \geq \delta(M) - \delta(M|C) = (|C_N|-3) - (|C^*|-3). \tag{23}$$

Therefore

$$|C_N| \leq |C^*|. \tag{24}$$

As C_N is a circuit-hyperplane of N and C^* is independent in N, it follows that

$$|C^*| \leq r(N) = |C_N|. \tag{25}$$

By (24) and (25),

$$r(N) = |C_N| = |C^*|. \tag{26}$$

Observe that (i) is a consequence of (26) because, up to isomorphism, there is only one 6-element rank-3 3-connected binary matroid, namely $M(K_4)$.

Now, we establish (ii). By (26), N is an 8-element rank-4 3-connected binary matroid. As C^* is an independent cocircuit of N, it follows that N is not isomorphic to $AG(3, 2)$. Therefore N is isomorphic to S_8 or N is regular. If N is regular, then, by (14.2) of [23], N is graphic or cographic and so, by Tutte's characterization of graphic matroids and Kuratowski's Theorem, N is the matroid of a planar graph. Thus N is isomorphic to S_8 or to $M(W_4)$ because W_4 is the unique 3-connected planar graph with 8 edges having a circuit-hyperplane. By Lemma 9(i) applied to $M\backslash a$, for $a \in C^*$, a belongs to a triad T_a^* of N having two elements in common with C_N. Therefore N has at least 4 different triads and so N is isomorphic to $M(W_4)$. Thus (ii) follows.

We argue by contradiction to prove (iii). Assume that $E(N) \cap C \neq \emptyset$, say $c \in E(N) \cap C$. As $c \in C_N$, it follows, by (i) or (ii), that there is a 2-element subset X of C^* such that $c \cup X$ is a triangle of M. Thus $(c \cup X) \triangle C$ is a circuit of M having more elements than C; a contradiction and so (iii) follows. $\square$

Lemma 10. *Let (M, C) be a strong special pair. If $|E(K)| = 3$, for each $K \in \Lambda_1(M/C)$, then $H \cong M(K^{(3)}_{3,|\Lambda_1(H/C_H)|})$ for each $H \in \Gamma_1(M)$.*

Let $\{U, V\}$ be a partition of the vertices of the complete bipartite graph $K_{3,n}$, for $n \geq 1$, such that U and V are stable sets, $|U| = 3$, and $|V| = n$. Let $K^{(3)}_{3,n}$ be a simple graph obtained from $K_{3,n}$ by adding a set with 3 edges joining vertices belonging to U.

Proof. By Lemma 8,

$$\delta(M) = \delta(M|C) + \sum_{H \in \Gamma_1(M)} (|C_H| - 3), \tag{27}$$

where $C_H = E(H) - [E(M) - C]$, for $H \in \Gamma_1(M)$. By hypothesis, $\delta(M) \leq \Delta(M) = \delta(M|C)$ and so

$$0 \geq \sum_{H \in \Gamma_1(M)} (|C_H| - 3). \tag{28}$$

As $|C_H| \geq 3$, for each $H \in \Gamma_1(M)$, it follows that $|C_H| = 3$, for each $H \in \Gamma_1(M)$.

Fix an $H \in \Gamma_1(M)$. Suppose that $\Lambda_1(H/C_H) = \{K_1, K_2, \ldots, K_n\}$. For $i \in \{1, 2, \ldots, n\}$, let $C^*_i = E(K_i)$. If $N = M|(C \cup C^*_1 \cup C^*_2 \cup \cdots \cup C^*_n)$, then (N, C) is a strong special pair such that $\Lambda_1(N/C) = \Lambda_1(H/C_H)$. Moreover,

$$\Gamma_2(N) = \Gamma_2(M) \cup \{L|C_L : L \in \Gamma_1(M) \text{ and } L \neq H\} \quad \text{and} \quad \Gamma_1(N) = \{H\}. \tag{29}$$

For $i \in \{1, 2, \ldots, n\}$, if $N_i = N|C \cup C^*_i = M|C \cup C^*_i$, then (N_i, C) is a unitary strong special pair such that

$$\Gamma_2(N_i) = \Gamma_2(N) \quad \text{and} \quad \Gamma_1(N_i) = \{H|(C_H \cup C^*_i)\} \tag{30}$$

because $|C_H| = 3$. By Lemma 9(i), $H|(C_H \cup C^*_i) \cong M(K_4)$. The elements of C^*_i can be labeled by a_i, b_i, c_i so that $T_i = \{a_i, b_i, x\}$, $S_i = \{a_i, c_i, y\}$, $R_i = \{b_i, c_i, z\}$ are triangles of $H|(C_H \cup C^*_i)$, where $C_H = \{x, y, z\}$. As $\{x, y, a_1, a_2, \ldots, a_n\}$ is a basis of H, it follows that $C_H, T_1, T_2, \ldots, T_n, S_1, S_2, \ldots, S_n$ span the cycle space of H over $GF(2)$. But these sets also span the cycle space of $M(K^{(3)}_{3,n})$ over $GF(2)$, where $K^{(3)}_{3,n}$ has $u, v, w, v_1, v_2, \ldots, v_n$ as vertices and edges: x incident with u and v; y incident with v and w; z incident with u and w; and, for $i \in \{1, 2, \ldots, n\}$, a_i incident with v and v_i; b_i incident with u and v_i; and c_i incident with w and v_i. Therefore $M = M(K^{(3)}_{3,n})$ and the result follows. $\square$

Lemma 11. *Let (M, C) be a strong special pair such that $|C| = 8$. If $|E(K)| = 4$, for some $K \in \Lambda_1(M/C)$, then (M, C) is unitary.*

Proof. Suppose this result is not true. Choose a counter-example (M, C) such that $|E(M)|$ is minimum. Hence $|\Lambda_1(M/C)| \geq 2$. There is $H \in \Lambda_1(M/C)$ such that $H \neq K$. First, we show that $E(M) = C \cup E(H) \cup E(K)$, $|E(H)| = 3$, and $|E(K)| = 4$. If $N = M|(C \cup X \cup Y)$, where X is a 3-subset of $E(H)$ and Y is a 4-subset of $E(K)$, then $\Lambda_1(N/C) = \{H|X, K|Y\}$ and so (N, C) is a strong special pair. By the choice of (M, C), $M = N$. That is, $E(M) = C \cup E(H) \cup E(K)$, $X = E(H)$ and $Y = E(K)$. By Lemma 9(ii) applied to the unitary special pair $(M|(C \cup E(K)), C)$, there is a matroid L such that $L \cong M(W_4)$ and $\Gamma_1(M|(C \cup E(K))) = \{L\}$. By Lemma 9(iii),

$$[E(L) - E(K)] \cap C = \emptyset. \tag{31}$$

Observe that

$$|\Gamma_2(M|(C \cup E(K)))| = 4 \tag{32}$$

because, by Lemma 7(iv), $M|C$ is the 2-sum of the matroids belonging to the family $\Gamma_2(M|(C \cup E(K))) \cup \{L \backslash E(K)\}$. By (31) and (32),

$$|C \cap E(L')| = 2, \quad \text{for every } L' \in \Gamma_2(M|(C \cup E(K))). \tag{33}$$

Now, we show that

$$|\Gamma_1(M)| = 1, \quad \text{say } \Gamma_1(M) = \{L'\}. \tag{34}$$

Assume that (34) is not true. By Lemma 10,

$$2 \le |\Gamma_1(M)| \le |\Lambda_1(M/C)| = |\{H, K\}| = 2, \tag{35}$$

say $\Gamma_1(M) = \{L_H, L_K\}$, where $E(H) \subseteq E(L_H)$ and $E(K) \subseteq E(L_K)$. By Lemma 9(i), $L_H \cong M(K_4)$. Note that

$$\Lambda_2(M|(C \cup E(K))) = [\Lambda_2(M) - \{L_H\}] \cup \{L_H \backslash E(H)\}. \tag{36}$$

Hence $L_H \backslash E(H) \in \Gamma_2(M|(C \cup E(K)))$. By (33), $|C \cap [E(L_H) - E(H)]| = 2$; a contradiction to Lemma 9(iii) applied to the unitary strong special pair $(M|[C \cup E(H)], C)$. Therefore (34) holds.

By (34) and Lemma 8, when $C_{L'} = E(L') - [E(H) \cup E(K)]$,

$$\delta(M) = \delta(M|C) + (|C_{L'}| - 3) - (|E(H)| - 3) - (|E(K)| - 3). \tag{37}$$

Therefore

$$|C_{L'}| = 4 + [\delta(M) - \delta(M|C)] \le 4. \tag{38}$$

But $L \in \Lambda_2(L' \backslash E(H))$ and so $L = L' \backslash E(H)$. By Lemma 9(i) applied to the unitary special pair $(M \backslash E(K), C)$, $L' \backslash E(K) = L_1 \oplus_2 L_2$, where $L_1 \cong U_{2,3}$ and $L_2 \cong M(K_4)$. In particular, $L' \backslash E(K)$ has a unique 2-separation $\{Z, W\}$, say $|Z| = 2$. Choose $e \in E(K)$ such that $r(Z \cup (E(K) - e)) = 4$ and $r((W - E(H)) \cup (E(K) - e)) = 4$. Note that $L' \backslash e$ is 3-connected. Therefore $\{L' \backslash e\} = \Gamma_1(M \backslash e)$; a contradiction to Lemma 10 applied to strong special pair $(M \backslash e, C)$. □

5. There exists no counter-example to Theorem 3

In this section, we prove Theorem 3 by contradiction. Suppose that M is a 3-connected binary matroid such that $r(M) \ge 10$,

$$|E(M)| \ge 4r(M) - 8 \tag{39}$$

and M does not have a circuit C such that $M \backslash C$ is 3-connected.

Lemma 12. $\Delta(M) \le 6$.

Proof. By Proposition 1, M has a 3-connected spanning minor N such that

$$|E(N)| \le \begin{cases} 3r(M) - \Delta(M) - 2 & \text{when } M \text{ is not Hamiltonian} \\ 2r(M) & \text{when } M \text{ is Hamiltonian.} \end{cases} \tag{40}$$

Observe that

$$|E(M) - E(N)| \ge \begin{cases} r(M) + \Delta(M) - 6 & \text{when } M \text{ is not Hamiltonian} \\ 2r(M) - 8 & \text{when } M \text{ is Hamiltonian.} \end{cases} \tag{41}$$

If $E(M) - E(N)$ contains a circuit C of M, then $M \backslash C$ is 3-connected because N is 3-connected and spanning. Hence $E(M) - E(N)$ is independent and so

$$r(M) \ge |E(M) - E(N)| \ge \begin{cases} r(M) + \Delta(M) - 6 & \text{when } M \text{ is not Hamiltonian} \\ 2r(M) - 8 & \text{when } M \text{ is Hamiltonian.} \end{cases} \tag{42}$$

Thus M is not Hamiltonian and $\Delta(M) \le 6$. □

Lemma 13. *If C is a circuit of M, then $|C| \le 8$. Moreover, $\Delta(M) = \delta(M|C)$, when $|C| = 8$.*

Proof. By Lemma 4, $|C| - 2 = \delta(M|C) \le \Delta(M)$. By Lemma 12, $\Delta(M) \le 6$ and so $|C| \le 8$. Observe that we have equality in all inequations when $|C| = 8$. □

Lemma 14. $\operatorname{circ}(M) = 8$.

Proof. Suppose that $\operatorname{circ}(M) \neq 8$. By Lemma 13, $\operatorname{circ}(M) \leq 7$. By Theorem 5, $\operatorname{circ}(M) \in \{6, 7\}$; a contradiction to Theorem 6. $\square$

We say that L is a *Tutte-line* of a matroid H, when $H|L$ does not have coloops and $r((H|L)^*) = 2$. Observe that every Tutte-line of M is a subdivision of $U_{0,2}$ or $U_{1,3}$ since M is binary. We prove that:

Lemma 15. *If L is a Tutte-line of M, then $\delta(M|L) = |L| - 4$. Moreover, $|L| \leq 10$.*

Proof. We have two cases to consider. If $M|L$ is a subdivision of $U_{0,2}$, then $M|L$ is the 1-sum of matroids $M|L_1$ and $M|L_2$, where L_1 and L_2 are the circuits of $M|L$. Hence, by Lemmas 1 and 4,

$$\delta(M|L) = \delta(M|L_1) + \delta(M|L_2) = (|L_1| - 2) + (|L_2| - 2) = |L| - 4. \tag{43}$$

If $M|L$ is a subdivision of $U_{1,3}$, then

$$\lambda_2(M|L) = |L| - 3. \tag{44}$$

(Remember that, by definition, a matroid belonging to $\Lambda_2(M|L)$ which is isomorphic to $U_{1,3}$ does not contribute to $\lambda_2(M|L)$.) Thus,

$$\delta(M|L) = 3(|L| - 2) - |L| - 1 - (|L| - 3) = |L| - 4. \tag{45}$$

The first part of the result follows. By Lemma 12, we have that

$$|L| - 4 = \delta(M|L) \leq 6 \tag{46}$$

and so $|L| \leq 10$. $\square$

Lemma 16. *If C is a circuit of M such that $|C| = 8$, then every connected component of M/C has rank equal to* 0 *or* 1.

Proof. Let A be a circuit of M/C. Observe that $L = C \cup A$ is a Tutte-line of M. By Lemma 15, $|C \cup A| \leq 10$ and so $|A| \leq 2$. Therefore $\operatorname{circ}(M/C) \leq 2$. The result follows because every connected component of a matroid with circumference at most 2 has rank equal to 0 or 1. $\square$

By Lemma 14, M has a circuit C such that $|C| = 8$. By Lemma 16, each connected component of M/C has rank equal to 0 or 1. Let $M_1, M_2, \ldots, M_n$ be the connected components of M/C having rank equal to 1.

Lemma 17. $r(E(M_i)) \geq 3$*, for every* $i \in \{1, 2, \ldots, n\}$.

Proof. If $r(E(M_i)) \leq 2$, then $\{E(M) - E(M_i), E(M_i)\}$ is a 1- or 2-separation of M because $r(E(M) - E(M_i)) = r(M) - 1$; a contradiction. $\square$

For $i \in \{1, 2, \ldots, n\}$, let B_i be a basis of M_i. If $N = M|(C \cup B_1 \cup B_2 \cup \cdots \cup B_n)$, then (N, C) is a strong special pair because, by Lemma 13, $\delta(M|C) = \Delta(M)$ and, by Lemma 17, $|B_i| \geq 3$, for every $i \in \{1, 2, \ldots, n\}$. By Lemma 11, $|B_1| = |B_2| = \cdots = |B_n| = 3$, since $n = r(M) - 7 \geq 3$.

Lemma 18. *For $i \in \{1, 2, \ldots, n\}$, $|E(M_i)| \in \{3, 4\}$. Moreover, when $|E(M_i)| = 4$, $E(M_i)$ is a circuit of M.*

Proof. If $E(M_i) = B_i$, then the result follows. Suppose that $e \in E(M_i) - B_i$. There is a circuit C of M such that $e \in C \subseteq B_i \cup e$. As $E(M_i)$ is a cocircuit of M, it follows, by orthogonality, that $|C \cap E(M_i)|$ is even. But $C \cap E(M_i) = C$ and so $|C| = 4$ because M is 3-connected. In particular, $C = B_i \cup e$. If $e \neq e'$ and $e' \in E(M_i) - B_i$, then $B_i \cup e'$ is a circuit of M; a contradiction because $(B_i \cup e) \triangle (B_i \cup e') = \{e, e'\}$ is a circuit of M. Therefore e' does not exist and the result follows. $\square$

By Lemma 10, for each $H \in \Gamma_1(N)$, $H \cong M(K^{(3)}_{3,|\Lambda_1(H/C_H)|})$, where $C_H = E(H) - (B_1 \cup B_2 \cup \cdots \cup B_n)$. In particular, $|C_H| = 3$ and $C_H \cap C = \emptyset$. By Lemma 7(iv), $M|C = N|C$ is obtained by making the 2-sum of the matroids belonging to the family $\{H|C_H : H \in \Gamma_1(N)\} \cup \Gamma_2(N)$. As every matroid belonging to this family is isomorphic to $U_{2,3}$ and $C_H \cap C = \emptyset$, for every $H \in \Gamma_1(N)$, it follows that $|\Gamma_2(N)| \geq 4$ and so

$$|\Gamma_1(N)| \leq 2. \tag{47}$$

Lemma 19. *If $M\backslash E(M_i)$ is not 3-connected, for some $i \in \{1, 2 \dots, n\}$, then $M\backslash E(M_j)$ is 3-connected, for every $j \in \{1, 2, \dots, n\} - \{i\}$.*

Proof. By (47), the result follows provided we establish that:

$$\text{There is } H \in \Gamma_1(N) \text{ such that } E(H) = B_i \cup C_H. \tag{48}$$

(Remember that $n = r(M) - 7 \geq 3$.)

If (48) is not true, then there is $H \in \Gamma_1(N)$ such that $B_i \subseteq E(H)$ and $E(H) \neq B_i \cup C_H$, say $B_j \subseteq E(H)$, for $j \in \{1, 2, \dots, n\} - \{i\}$. Observe that $\Gamma_2(N) = \Gamma_2(N\backslash B_i)$ and $\Gamma_1(N\backslash B_i) = [\Gamma_1(N) - \{H\}] \cup \{H\backslash B_i\}$. For a matroid L belonging to $\Gamma_1(N) \cup \{H\backslash B_i\}$, let $\hat{L}$ be the unique binary extension of L such that $E(\hat{L}) = E(L) \cup [\cup_{B_i \subseteq E(L)} E(M_i)]$ and, for each $i \in \{1, 2, \dots, n\}$ satisfying $B_i \subseteq E(L)$ and $|E(M_i)| = 4$, $E(M_i)$ is a circuit of $\hat{L}$. We have that:

$$\Lambda_2(M\backslash(\mathrm{cl}_M(C) - C)) = \{\hat{L} : L \in \Gamma_1(N)\} \cup \Gamma_2(N)$$
$$\Lambda_2(M\backslash[(\mathrm{cl}_M(C) - C) \cup E(M_i)]) = \{\hat{L} : L \in [\Gamma_1(N) - \{H\}] \cup \{H\backslash B_i\}\} \cup \Gamma_2(N).$$

In particular, each 2-separation of $M\backslash[(\mathrm{cl}_M(C) - C) \cup E(M_i)]$ is induced by a 2-separation of $M\backslash(\mathrm{cl}_M(C) - C)$. As the elements belonging to $\mathrm{cl}_M(C) - C$ destroy every 2-separation of $M\backslash(\mathrm{cl}_M(C) - C)$, it follows that these elements destroy every 2-separation of $M\backslash[(\mathrm{cl}_M(C) - C) \cup E(M_i)]$. Thus $M\backslash E(M_i)$ is 3-connected; a contradiction. Therefore (48) holds and so the result follows. □

If $|E(M_i)| = 4$, for some $i \in \{1, 2, \dots, n\}$, then $E(M_i)$ is a circuit of M. By hypothesis, $M\backslash E(M_i)$ is not 3-connected. By Lemma 19, there is at most one $i \in \{1, 2, \dots, n\}$ such that $|E(M_i)| = 4$. By Theorem 4, there is a 3-connected spanning restriction M' of M such that $E(M) - [\mathrm{cl}_M(C) - C] \subseteq E(M')$ and

$$|E(M')| - |E(M\backslash[\mathrm{cl}_M(C) - C])| \leq \lambda_1(E(M\backslash[\mathrm{cl}_M(C) - C])) + \lambda_2(E(M\backslash[\mathrm{cl}_M(C) - C])) - 2 = 5. \tag{49}$$

As

$$|E(M\backslash[\mathrm{cl}_M(C) - C])| = |C| + \sum_{i=1}^{n} |E(M_i)| \leq 9 + 3n, \tag{50}$$

it follows that $|E(M')| \leq 3n + 14 = 3r(M) - 7$. Consequently

$$|E(M) - E(M')| \geq [4r(M) - 8] - [3r(M) - 7] = r(M) + 1; \tag{51}$$

a contradiction because $E(M) - E(M')$ contains a circuit C of M and so $M\backslash C$ is 3-connected. With this contradiction, we finish the proof of Theorem 3.

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