European Journal of Combinatorics 30 (2009) 1810–1824

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The 3-connected binary matroids with circumference 6 or 7

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ARTICLE INFO

Article history:
Available online 13 January 2009

ABSTRACT

In this paper, we construct all 3-connected binary matroids with circumference equal to 6 or 7 having large rank.

1. Introduction

In this paper, we assume familiarity with matroid theory. The notation and terminology used in this article follow Oxley [1]. For a matroid M that has a circuit, $\mathrm{circ}(M)$ denotes the *circumference* of M, that is, the maximum cardinality of a circuit of M. In recent years, the circumference of a matroid has appeared in some bounds, for example, in an upper bound for the size of a minimally n-connected matroid and in a lower bound for the size of an n-connected matroid having a circuit whose deletion is also n-connected, for $n \in \{2, 3\}$ (see [2–4]). Using these bounds and results about matroids with small circumference, it is possible to improve some bounds found in the literature. In this paper, we construct all 3-connected binary matroid with circumference 6 or 7 (and large rank). In [5], we use the main results of this paper to improve a lower bound due to Lemos and Oxley [4] for the size of a 3-connected binary matroid having a circuit whose deletion originates also a 3-connected matroid.

The 3-connected matroids having small circumference must have small rank. Lemos and Oxley [4] proved that:

Theorem 1.1. *Suppose that M is a 3-connected matroid. If $r(M) \geq 6$, then* $\mathrm{circ}(M) \geq 6$.

By this result, every 3-connected matroid with circumference at most 5 has rank at most 5. Maia and Lemos [6] proved that a 3-connected matroid having rank at most 5 is Hamiltonian, unless it is

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doi:10.1016/j.ejc.2008.12.001

isomorphic to $U_{1,1}$, F_7^*, AG(3, 2), J_9, or J_{10}, where J_{10} is the matroid whose representation over $GF(2)$ is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and J_9 is the matroid obtained from J_{10} by deleting the last column.

Maia [7] constructs all the matroids with circumference at most five. With the knowledge of all matroids with circumference c, for example, one can calculate all the Ramsey numbers $n(c+1, y)$ for matroids, for every value of y (for a definition of $n(x, y)$ see Reid [8]). These numbers were completely determined by Lemos and Oxley [9] using a sharp bound for the number of elements of a connected matroid as a function of its circumference and cocircumference.

Before the description of all the 3-connected binary matroids with circumference 6 or 7, we need to give some definitions. Let $T_1^*, T_2^*, \ldots, T_m^*$ be pairwise disjoint triads of a 3-connected binary matroid M. There is a unique binary matroid N over $E(M) \cup \{e_1, e_2, \ldots, e_m\}$, where $\{e_1, e_2, \ldots, e_m\}$ is an m-element set disjoint from $E(M)$, such that $N \setminus \{e_1, e_2, \ldots, e_m\} = M$ and, for every $i \in \{1, 2, \ldots, m\}$, $Q_i = T_i^* \cup e_i$ is a circuit of N. Moreover, Q_i is a cocircuit of M. (There is a cocircuit C_i^* of M such that $T_i^* \subseteq C_i^* \subseteq T_i^* \cup \{e_1, e_2, \ldots, e_m\}$. By orthogonality with Q_j, for $j \neq i$, $e_j \notin C_i^*$ and so $C_i^* \in \{T_i^*, Q_i\}$. But $|C_i^* \cap Q_i|$ is even. Thus $C_i^* = Q_i$.) Following Geelen and Whittle [10], we say that a 4-element circuit-cocircuit of a matroid is a *quad*. Therefore $Q_1, Q_2, \ldots, Q_m$ are pairwise disjoint quads of N. We say that *N is obtained from M by completing the triads $T_1^*, T_2^*, \ldots, T_m^*$ to quads*. It is easy to see that N is 3-connected.

Suppose that l, m and n are integers such that $0 \leq l \leq 3 \leq n$ and $0 \leq m \leq n$. Let $\{U, V\}$ be a partition of the vertices of the complete bipartite graph $K_{3,n}$ such that U and V are stable sets, $|U| = 3$ and $|V| = n$, say $V = \{v_1, v_2, \ldots, v_n\}$. Let $K_{3,n}^{(l)}$ be the simple graph obtained from $K_{3,n}$ by adding l edges joining two vertices belonging to U. (These l edges are referred as *special edges* of $K_{3,n}^{(l)}$. When $l = 3$, this set of edges is called the *special triangle* of $K_{3,n}^{(l)}$.) We define $M_{n,m,l}$ to be the binary matroid obtained from $M(K_{3,n}^{(l)})$ by completing the triads $st(v_1), st(v_2), \ldots, st(v_m)$ to quads. We prove that:

Theorem 1.2. *Let M be a 3-connected binary matroid such that $r(M) \geq 8$. Then, $\mathrm{circ}(M) = 6$ if and only if M is isomorphic to $M_{n,m,l}$, for some integers l, m and n such that $0 \leq l \leq 3$, $6 \leq n$ and $0 \leq m \leq n$.*

Theorem 1.3. *For a 3-connected binary matroid M such that $r(M) \geq 9$, the following statements are equivalent:*

(i) $\mathrm{circ}(M) = 7$.
(ii) *There is a 3-connected rank-4 binary matroid N having a Hamiltonian circuit C and a triangle T satisfying $|T \cap C| = 2$ such that $T = E(N) \cap E(K_{3,r(M)-4}^{(3)})$ is the special triangle of $K_{3,r(M)-4}^{(3)}$ and M is obtained from $M' \setminus X$ by completing a set of pairwise disjoint triads of $M(K_{3,r(M)-4}^{(3)})$ to quads, where M' is the generalized parallel connection of $M(K_{3,r(M)-4}^{(3)})$ with N and $X \subseteq T$.*

We think that it is very difficult to construct all 3-connected matroids with circumference 6 or 7 (and large rank). To construct all the 3-connected binary matroids with circumference 8 looks to be hard as well.

2. Contracting a maximum size circuit

Let M be a matroid. For $F \subseteq E(M)$, an *F-arc* (see Section 3 of [11]) is a minimal non-empty subset A of $E(M) - F$ such that there exists a circuit C of M with $C - F = A$ and $C \cap F \neq \emptyset$. Such a circuit C is called an *F-fundamental for A*. Let A be an F-arc and $P \subseteq F$. Then $A \to P$ if there is an F-fundamental for A contained in $A \cup P$. Thus $A \not\to P$ denotes that there is no such Z-fundamental. The next result is a consequence of (3.8) of [11].

Lemma 2.1. *Suppose that M is a connected matroid. Let X and Y be non-empty subsets of $E(M)$ such that $M|X$ and $M|Y$ are both connected. If $M|(X \cup Y) = (M|X) \oplus (M|Y)$, then there is a circuit C of M such that $C \cap X \neq \emptyset$, $C \cap Y \neq \emptyset$ and $C - (X \cup Y)$ is contained in a series class of $M|(X \cup Y \cup C)$.*

The next lemma is likely to be known but we do not have a reference for it.

Lemma 2.2. *Let M be a connected matroid. If $\emptyset \neq F \subseteq E(M)$, $M|F$ is connected and $\mathrm{circ}(M/F) \geq 3$, then there is a circuit C of M/F such that C is an F-arc and $|C| \geq 3$.*

Proof. Assume that this result is not true. Let C be a circuit of M/F such that $|C| = \mathrm{circ}(M/F)$. Hence C is a circuit of M and $M|(C \cup F) = (M|C) \oplus (M|F)$. By Lemma 2.1, there is a circuit D of M such that $D \cap C \neq \emptyset$, $D \cap F \neq \emptyset$ and $D - (C \cup F)$ is contained in a series class of $M|(C \cup D \cup F)$. If $e \in D - (C \cup F)$ and $f \in C - D$, then $(C \cup D) - (\{e, f\} \cup F)$ is independent in M/F. Therefore $D - F$ is a circuit of M/F. Hence $|D - F| = 2$, say $D - F = \{e, g\}$, where $g \in C \cap D$. As $(M/g)|[F \cup (C - g)] = [(M/g)|F] \oplus [(M/g)|(C - g)]$ and F spans e in M/g, it follows that $C - g$ is a series class of $M|(C \cup D \cup F)$. Thus $C' = C \triangle D = (C \cup D) - g$ is a circuit of M. But $C' - F$ is a circuit of M/F such that $C' - F \rightarrow F$. Therefore $2 = |C' - F| = |e \cup (C - g)|$. Hence $|C - g| = 1$ and so $|C| = 2$; a contradiction. $\square$

We say that L is a *Tutte-line* of a matroid M, when L is the union of circuits of M and $r^*(M|L) = 2$. Every Tutte-line has a partition $\{L_1, L_2, \ldots, L_k\}$, which is called *canonical*, such that C is a circuit of M contained in L if and only if $C = L - L_i$, for some $i \in \{1, 2, \ldots, k\}$. We say that a Tutte-line L is *connected* provided $M|L$ is connected. When a Tutte-line L is connected, its canonical partition has at least three sets.

In general, when C is a maximum size circuit of a connected matroid M, the circumference of M/C is at most $|C| - 2$. (This sharp result due to Seymour is a consequence of Lemma 2.1.) We reduce this upper bound substantially in a special case. The next proposition plays a central role in the proofs of the main results of this paper.

Proposition 2.1. *Suppose that M is a 3-connected binary matroid such that $\mathrm{circ}(M) \in \{6, 7\}$ and $r(M) \geq \mathrm{circ}(M) + 2$. If C is a maximum size circuit of M, then the rank of every connected component of M/C is at most one.*

Proof. It is enough to show that $\mathrm{circ}(M/C) \leq 2$ because a connected matroid with circumference 1 or 2 is isomorphic to $U_{0,1}$ or $U_{1,n}$, for some $n \geq 2$, respectively. Assume that $\mathrm{circ}(M/C) \geq 3$. By Lemma 2.2, there is a circuit A of M/C such that $|A| \geq 3$ and A is a C-arc. Hence $L = C \cup A$ is a connected Tutte-line of M. Suppose that the canonical partition of L is equal to $\{X_1, X_2, X_3\}$. So $A = X_i$, for some $i \in \{1, 2, 3\}$, say $A = X_1$. As $C = L - A$ is a circuit of M having maximum size, it follows that $3 \leq |A| \leq |X_i|$, for every $i \in \{1, 2, 3\}$. Thus $|A| = 3$ and $\{|X_2|, |X_3|\} = \{3, |C| - 3\}$ because

$$7 \geq |C| = |L - A| = |X_2| + |X_3| \geq 2|A| \geq 6.$$

Suppose that $|X_2| = 3$.

Let $\mathcal{A}$ be the set of L-arcs. For $k \in \{1, 2, 3\}$, we define $\mathcal{A}_k = \{A' \in \mathcal{A} : A' \rightarrow X_k\}$ and $\mathcal{A}' = \mathcal{A} - (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$. We divide the proof in some steps.

Step 1. *If $A' \in \mathcal{A}'$, then $|A'| = 1$. Moreover, there is a circuit $C_{A'}$ of M such that $A' = C_{A'} - L$ and $(|C_{A'} \cap X_1|, |C_{A'} \cap X_2|, |C_{A'} \cap X_3|) = \gamma$,*

(i) *for some $\gamma \in \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$, when $|C| = 6$; or*
(ii) *for some $\gamma \in \{(1, 2, 3), (2, 1, 3), (2, 2, 2)\}$, when $|C| = 7$.*

We argue by contradiction. Assume that $|A'| \geq 2$ or, when $|A'| = 1$, $C_{A'}$ does not exist. Let D be a circuit of $M|(L \cup A')$ such that $A' = D - L$. Assume that

$$|D \cap X_r| \leq |D \cap X_s| \leq |D \cap X_t|,$$

where $\{r, s, t\} = \{1, 2, 3\}$ (when possible, take s to be equal to 3). As $A' \notin \mathcal{A}_t$, it follows that $|D \cap X_s| \geq 1$. First, we prove that:

$$|D \cap X_r| \leq 1 \quad \text{and} \quad |D \cap X_s| \leq 2. \tag{2.1}$$

If (2.1) does not hold, then

$$7 \geq |D| = |A'| + |D \cap X_1| + |D \cap X_2| + |D \cap X_3| \geq |A'| + 6.$$

Hence $|A'| = 1$ and $|C| = 7$. Moreover, $(|D \cap X_1|, |D \cap X_2|, |D \cap X_3|) = \gamma$, where $\gamma \in \{(2, 2, 2), (0, 3, 3)\}$; a contradiction unless $\gamma = (0, 3, 3)$. As $D \triangle (X_s \cup X_t)$ is a union of pairwise disjoint circuits of M, it follows that M has a circuit with at most two elements; a contradiction. Therefore (2.1) holds.

In this paragraph, we establish that

$$X_t \subseteq D. \tag{2.2}$$

If $|D \cap X_t| < |X_t|$, then, by (2.1), $(X_r \cup X_s) \cup D$ is a connected Tutte-line of M. So $D_1 = (X_r \cup X_s) \triangle D$ is a circuit of M. But

$$7 \geq |D_1| = |A'| + |X_r - D| + |X_s - D| + |D \cap X_t|. \tag{2.3}$$

Observe that

$$|X_s - D| + |D \cap X_t| = |X_s| + (|D \cap X_t| - |D \cap X_s|) \geq |X_s| \geq 3. \tag{2.4}$$

Now, we prove that

$$|D \cap X_r| \neq 0. \tag{2.5}$$

If $|D \cap X_r| = 0$, then, by (2.3), $4 - |A'| \geq |X_s - D| + |D \cap X_t|$. By (2.3) and (2.4), $|D_1| = 7$, $|A'| = 1$, $|X_r| = |X_s| = 3$, $|D \cap X_t| = |D \cap X_s|$ and $|X_s - D| + |D \cap X_t| = 3$. In particular, $t = 3$. We arrive at a contradiction because s can be taken to be equal to 3. Therefore (2.5) follows. By (2.1) and (2.5),

$$|D \cap X_r| = 1. \tag{2.6}$$

Now, we prove that $|A'| = 1$. Suppose that $|A'| \geq 2$. By (2.3) and (2.4), $|D_1| = 7$, $|A'| = 2$, $|X_r| = |X_s| = 3$, $|D \cap X_t| = |D \cap X_s|$ and $|X_s - D| + |D \cap X_t| = 3$. In particular, $t = 3$. Again, we arrive at a contradiction because s can be taken to be equal to 3. Hence $|A'| = 1$. Next, we establish that

$$|D \cap X_s| = 2. \tag{2.7}$$

If (2.7) does not hold, then, by (2.1) and (2.6), $|D \cap X_s| = 1$. By (2.3), $|D \cap X_t| \leq 2$. If $|D \cap X_t| = 2$, then $(|D_1 \cap X_1|, |D_1 \cap X_2|, |D_1 \cap X_3|) = (2, 2, 2)$; a contradiction. If $|D \cap X_t| = 1$, then $s = 3$ and so $(|D_1 \cap X_1|, |D_1 \cap X_2|, |D_1 \cap X_3|) \in \{(1, 2, 2), (2, 1, 2)\}$, when $|C| = 6$, or $(|D_1 \cap X_1|, |D_1 \cap X_2|, |D_1 \cap X_3|) \in \{(1, 2, 3), (2, 1, 3)\}$, when $|C| = 7$; a contradiction. Therefore (2.7) holds. By (2.6) and (2.7) and the choice of A', $|D \cap X_t| = 2$. In particular, $|C| = 7$ and $3 \in \{r, s\}$. We arrive at a contradiction because $(|D_1 \cap X_1|, |D_1 \cap X_2|, |D_1 \cap X_3|) \in \{(2, 2, 2), (1, 2, 3), (2, 1, 3)\}$. Therefore (2.2) follows.

By (2.2), $X_t \subseteq D$. Choose $i \in \{r, s\}$ so that $3 \in \{i, t\}$. Observe that $L' = D \cup (X_i \cup X_t) = D \cup X_i$ is a connected Tutte-line of M. If $X \subseteq X_i \cup X_t$ belongs to the canonical partition of L', then $D_X = L' - X$ is a circuit of $M|(L \cup A')$ such that $D_X - L = A'$. By (2.2) applied to D_X, D_X contains X_j, for some $j \in \{1, 2, 3\}$. Therefore $X_i \subseteq D_X$ or $X_t \subseteq D_X$. In particular, $X \subseteq X_i$ or $X \subseteq X_t$. Assume that $t = 3$. (We need to replace D by D_X, for some $X \subseteq D_t$, when $i = 3$.) Assume also that $D \cap X_i \neq \emptyset$. (We are free to choose i in $\{r, s\}$ because $t = 3$.) As $X \subseteq X_i$ or $X \subseteq X_t$, for each $X \subseteq X_i \cup X_t$ belonging to the canonical partition of L', it follows that X_i and X_t belong to the canonical partition of L'. (Each Tutte-line in a binary matroid has at most three sets in its canonical partition.) We arrive at a contradiction because $X_i - D$ belongs to the canonical partition of L'. Therefore Step 1 follows.

By Step 1, for each $A' \in \mathcal{A}'$, there is a circuit $C_{A'}$ of M such that $A' = C_{A'} - L$ and $(|C_{A'} \cap X_1|, |C_{A'} \cap X_2|, |C_{A'} \cap X_3|) = \gamma$, where

(i) $\gamma \in \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}$, when $|C| = 6$; or
(ii) $\gamma \in \{(1, 2, 3), (2, 1, 3), (2, 2, 2)\}$, when $|C| = 7$.

Choose $C_{A'}$ so that $|C_{A'} \cap X_1|$ is minimum. Now, we prove that

$$\gamma = (1, 2, 2), \quad \text{when } |C| = 6, \quad \text{and} \quad \gamma \in \{(1, 2, 3), (2, 2, 2)\}, \quad \text{when } |C| = 7. \tag{2.8}$$

If (2.8) does not hold, then $|C_{A'} \cap X_j| = 1$, for some $j \in \{2, 3\}$. Observe that $D = C_{A'} \triangle (X_1 \cup X_j)$ is a circuit of M because $C_{A'} \cup (X_1 \cup X_j)$ is a connected Tutte-line of M. Hence $(|D \cap X_1|, |D \cap X_2|, |D \cap X_3|) = \gamma$, for $\gamma = (1, 2, 2)$, when $|C| = 6$, or $\gamma = (1, 2, 3)$, when $|C| = 7$. We arrive at a contradiction since $D - L = A'$. Thus (2.8) holds.

Step 2. $\mathcal{A}' \neq \emptyset$.

Assume that $\mathcal{A}' = \emptyset$. Hence $A' \to X_1$ or $A' \to (X_2 \cup X_3)$, for every L-arc A'. As $\{X_1, X_2 \cup X_3\}$ is a 2-separation of $M|L$, it follows, by (3.8) of [11], that there is a 2-separation $\{X, Y\}$ of M such that $X_1 \subseteq X$ and $X_2 \cup X_3 \subseteq Y$; a contradiction. Therefore Step 2 follows.

Step 3. $\mathcal{A}_i = \emptyset$, *for each* $i \in \{1, 2, 3\}$, *when* $|C| = 6$, *or for each* $i \in \{1, 2\}$, *when* $|C| = 7$.

Suppose that $\mathcal{A}_i \neq \emptyset$, say $i = 1$. For $A_1 \in \mathcal{A}_1$, let D_{A_1} be a circuit of M such that $A_1 = D_{A_1} - L$ and $D_{A_1} \subseteq X_1 \cup A_1$. For each $A' \in \mathcal{A}'$ and $A_1 \in \mathcal{A}_1$, we prove that

(iii) $D_{A_1} = A_1 \cup (X_1 - C_{A'})$, when $|C_{A'} \cap X_1| = 1$; or
(iv) $D_{A_1} = A_1 \cup (X_1 \cap C_{A'})$, when $|C_{A'} \cap X_1| = 2$.

Assume that both (iii) and (iv) do not hold. Observe that $|D_{A_1} \cap X_1| \geq 2$ because $\mathrm{circ}(M) = |X_1 \cup X_3|$. Therefore D_{A_1} intercepts both sets belonging to $\{X_1 - C_{A'}, X_1 \cap C_{A'}\}$. In particular,

$$|(C_{A'} \triangle D_{A_1}) \cap X_1| \geq 1. \tag{2.9}$$

Moreover, $C_{A'} \triangle D_{A_1}$ is a circuit of M because $D_{A_1} \cup C_{A'}$ is a connected Tutte-line of M. Thus

$$|C| \geq |A_1| + |A'| + |C_{A'} \cap (X_2 \cup X_3)| + |(C_{A'} \triangle D_{A_1}) \cap X_1|.$$

By (2.9), $|C| \geq 3 + |C_{A'} \cap (X_2 \cup X_3)|$ and so $|C| = 7$, $|C_{A'} \cap (X_2 \cup X_3)| = 4$, $|A_1| = 1$, $|C_{A'} \cap X_1| = 2$ and $X_1 \subseteq D_{A_1}$. As $(X_1 \cup X_3) \cup C_{A'}$ is a connected Tutte-line of M, it follows that $D_{A'} = C_{A'} \triangle (X_1 \cup X_3)$ is a 6-element circuit of M. But $D_{A'} \cup D_{A_1}$ is a connected Tutte-line of M. Thus $D_{A'} \triangle D_{A_1}$ is an 8-element circuit of M; a contradiction. Therefore (iii) or (iv) holds.

Let X be a subset of X_1 such that $D_{A_1} = A_1 \cup X$, for some $A_1 \in \mathcal{A}_1$. By (iii) and (iv), for every $A' \in \mathcal{A}'$, $X \cap C_{A'} = \emptyset$, when $|C_{A'} \cap X_1| = 1$, or $X \cap D_{A'} = \emptyset$, when $|C_{A'} \cap X_1| = 2$. As $\mathcal{A}' \neq \emptyset$, it follows that X is uniquely determined. Hence $D_{A_1} = X \cup A_1$, for every $A_1 \in \mathcal{A}_1$. Note that $\{X, L - X\}$ is a 2-separation of $M|L$ such that

(v) $A_1 \to X$, for every $A_1 \in \mathcal{A}_1$; and
(vi) $A'' \to L - X$, for every $A'' \in \mathcal{A} - \mathcal{A}_1$.

(Note that (vi) occurs when: $A'' \in \mathcal{A}_2 \cup \mathcal{A}_3$ because $X_2 \cup X_3 \subseteq L - X$; $A'' \in \mathcal{A}'$ and $|C_{A''} \cap X_1| = 1$ because $C_{A''} - A'' \subseteq L - X$; $A'' \in \mathcal{A}'$ and $|C_{A''} \cap X_1| = 2$ because $D_{A''} - A'' \subseteq L - X$.) By (3.8) of [11], there is a 2-separation $\{X', Y'\}$ of M such that $X \subseteq X'$ and $L - X \subseteq Y'$; a contradiction. Therefore Step 3 follows.

Step 4. $\mathcal{A}_3 \neq \emptyset$. *In particular*, $|C| = 7$.

If $\mathcal{A}_3 = \emptyset$, then, by Step 3, $\mathcal{A} = \mathcal{A}'$. By Step 1, $|A'| = 1$, for every $A' \in \mathcal{A}$. As each element e belonging to $E(M) - L$ is contained in some L-arc, it follows that $\{e\}$ is an L-arc. Therefore L spans M and $r(M) = |L| - 2 = |C| + 1$; a contradiction to hypothesis. Hence $\mathcal{A}_3 \neq \emptyset$. By Step 3, $|C| = 7$.

To finish the proof of this proposition, it suffices to establish the next step:

Step 5. $\mathcal{A}_3 = \emptyset$.

Assume that $\mathcal{A}_3 \neq \emptyset$. For $A_3 \in \mathcal{A}_3$, let D_{A_3} be a circuit of M such that $A_3 = D_{A_3} - L$ and $D_{A_3} \subseteq X_3 \cup A_3$. For each $A' \in \mathcal{A}'$ and $A_3 \in \mathcal{A}_3$, we prove that

(vii) $D_{A_3} \cap X_3 \subseteq C_{A'}$, when $|C_{A'} \cap X_1| = 1$; or
(viii) $D_{A_3} \in \{A_3 \cup (X_3 \cap C_{A'}), A_3 \cup (X_3 - C_{A'})\}$, when $|C_{A'} \cap X_1| = 2$.

If $D_{A_3} \cap C_{A'} = \emptyset$, then (viii) holds because $|D_{A_3} \cap X_3| \geq 2$ and $|C_{A'} \cap X_3| \geq 2$. Assume that $D_{A_3} \cap C_{A'} \neq \emptyset$. If $D_{A_3} \cap X_3 \subseteq C_{A'}$, then (vii) or (viii) follows. We may also assume that $[D_{A_3} - C_{A'}] \cap X_3 \neq \emptyset$. As $C_{A'} \cup D_{A_3}$ is a connected Tutte-line of M, it follows that $D = C_{A'} \triangle D_{A_3}$ is a circuit of M. Hence

$$|A_3| + |[D_{A_3} - C_{A'}] \cap X_3| \leq |D_{A_3} \cap C_{A'}| \tag{2.10}$$

because $|D| \leq |C_{A'}| = |C|$. As $C_{A'} \triangle (X_1 \cup X_3)$ is a circuit of M and $[C_{A'} \triangle (X_1 \cup X_3)] \cup D_{A_3}$ is a connected Tutte-line of M, it follows that $D' = [C_{A'} \triangle (X_1 \cup X_3)] \triangle D_{A_3}$ is a circuit of M. Observe that

$$|D'| = |C_{A'} \triangle (X_1 \cup X_3)| + |A_3| + |D_{A_3} \cap C_{A'}| - |[D_{A_3} - C_{A'}] \cap X_3|.$$

By (2.10), $|D_{A_3} \cap C_{A'}| - |[D_{A_3} - C_{A'}] \cap X_3| \geq |A_3|$ and so

$$|D'| \geq |C_{A'} \triangle (X_1 \cup X_3)| + 2|A_3| = 6 + 2|A_3| \geq 8;$$

a contradiction. Thus (vii) or (viii) follows.

We define $\mathcal{Z} = \{D_{A_3} \cap X_3 : A_3 \in \mathcal{A}_3\}$. First, we show that

$$Z_1 \cap Z_2 \neq \emptyset, \quad \text{when } \{Z_1, Z_2\} \subseteq \mathcal{Z}. \tag{2.11}$$

If (2.11) does not hold, then $|Z_1| = |Z_2| = 2$ and $X_3 = Z_1 \cup Z_2$ has 4 elements. Therefore $|C_{A'} \cap X_3| = 2$, for every $A' \in \mathcal{A}'$, by (vii). By Step 2 and (viii), $\mathcal{Z} = \{Z_1, Z_2\}$. Note that $A' \to L - Z_1$, for every $A' \in \mathcal{A}'$, since $C_{A'} - A' \subseteq L - Z_1$ or $[C_{A'} \triangle (X_1 \cup X_3)] - A' \subseteq L - Z_1$. By Step 3 and (3.8) of [11], there is a 2-separation $\{W_1, W_2\}$ of M such that $Z_1 \subseteq W_1$ and $L - Z_1 \subseteq W_2$; a contradiction and so (2.11) follows. Next, we establish that

$$|\mathcal{Z}| = 1. \tag{2.12}$$

If $\{Z_1, Z_2\}$ is a 2-subset of $\mathcal{Z}$, then, by (2.11), (vii) and (viii), $Z_1 \cup Z_2 = C_{A'} \cap X_3$, for every $A' \in \mathcal{A}'$. By (vii), $A_3 \to Z_1 \cup Z_2$, for every $A_3 \in \mathcal{A}_3$. By (vii), $A' \to L - (Z_1 \cup Z_2)$, for every $A' \in \mathcal{A}'$, because $[C_{A'} \triangle (X_1 \cup X_3)] - A' \subseteq L - (Z_1 \cup Z_2)$. By Step 3 and (3.8) of [11], there is a 2-separation $\{W_1, W_2\}$ of M such that $(Z_1 \cup Z_2) \subseteq W_1$ and $L - (Z_1 \cup Z_2) \subseteq W_2$; a contradiction and so (2.12) follows. By (2.12), $|\mathcal{Z}| = 1$, say $\mathcal{Z} = \{Z\}$. By (vii) and (viii), $A' \to L - Z$, for every $A' \in \mathcal{A}'$, because $C_{A'} - A' \subseteq L - Z$ or $[C_{A'} \triangle (X_1 \cup X_3)] - A' \subseteq L - Z$. By Step 3 and (3.8) of [11], there is a 2-separation $\{W_1, W_2\}$ of M such that $Z \subseteq W_1$ and $L - Z \subseteq W_2$; a contradiction and the proposition follows. □

3. Local structural results

For a circuit C of a binary matroid M, let A be C-arc. Observe that $C \cup A$ is a connected Tutte-line of M. Hence there is a partition $\{C_1, C_2\}$ of C such that $C_1 \cup A$ and $C_2 \cup A$ are circuits of M. For $i \in \{1, 2\}$, we say that C_i is a *projection of* A *over* C. For C-arcs A_1 and A_2, we say that:

(i) A_1 and A_2 are *strongly disjoint* provided $A_1 \cap A_2 = \emptyset$, $\min\{|A_1|, |A_2|\} \geq 2$ and $(M/C)|(A_1 \cup A_2) = [(M/C)|A_1] \oplus [(M/C)|A_2]$; and
(ii) A_1 and A_2 *cross* provided $C_{i1} \cap C_{j2} \neq \emptyset$, for every $\{i, j\} \subseteq \{1, 2\}$, where C_{1k} and C_{2k} are the projections of A_k over C, for $k \in \{1, 2\}$.

Lemma 3.1. *Let C be a circuit of a binary matroid M such that $|C| = \mathrm{circ}(M) \in \{6, 7\}$. If A_1 and A_2 are strongly disjoint C-arcs, then A_1 and A_2 do not cross.*

Proof. Assume that A_1 and A_2 cross. For $k \in \{1, 2\}$, let C_{1k} and C_{2k} be the projections of A_k over C. As A_1 and A_2 cross, it follows that $(A_1 \cup C_{i1}) \cup (A_2 \cup C_{j2})$ is a connected Tutte-line, for every $\{i, j\} \subseteq \{1, 2\}$. Hence $D_{ij} = (A_1 \cup C_{i1}) \triangle (A_2 \cup C_{j2})$ is a circuit of M. But $C \subseteq D_{11} \cup D_{12}$, $A_1 \cup A_2 \subseteq D_{11} \cap D_{12}$ and so

$$2|C| \geq |D_{11}| + |D_{12}| = |D_{11} \cup D_{12}| + |D_{11} \cap D_{12}| \geq |C| + 2(|A_1| + |A_2|);$$

a contradiction since $\min\{|A_1|, |A_2|\} \geq 2$ and $|C| \leq 7$. Thus A_1 and A_2 do not cross. □

Let C be a circuit of a 3-connected binary matroid M such that $|C| = \mathrm{circ}(M) \in \{6, 7\}$. A 3-subset Z of $E(M)$ is said to be a *star with respect to* C provided Z is contained in a connected component of M/C. Let $\pi(C, Z)$ be the series classes of $M|(C \cup Z)$ contained in C. Note that $\pi(C, Z)$ is a partition of C. A star Z' with respect to C is said to be *strongly disjoint* from Z provided $(M/C)|(Z \cup Z') = [(M/C)|Z] \oplus [(M/C)|Z']$.

Lemma 3.2. *Let C be a circuit of a* 3*-connected binary matroid M such that $|C| = \mathrm{circ}(M) \in \{6, 7\}$. If Z is a star with respect to C, then Z is independent and:*

(i) *The cosimplification of* $M|(C \cup Z)$ *is isomorphic to* $M(K_4)$*. In this case,* $|S| \in \{2, 3\}$*, for every* $S \in \pi(C, Z)$*. Or*
(ii) *The cosimplification of* $M|(C \cup Z)$ *is isomorphic to* F_7^**.*

When (i) happens, we say that Z is a *simple* star with respect to C. When (ii) occurs, we say that Z is *non-simple*.

Proof. Let Z be a star with respect to C. By Proposition 2.1, Z is contained in a connected component of M/C whose rank is equal to one. Therefore each 2-subset of Z is a C-arc of M. In particular, $M|(C \cup Z)$ is connected and each element of Z belongs to a trivial series class of $M|(C \cup Z)$. As Z is a cocircuit of the simple matroid $M|(C \cup Z)$, it follows, by orthogonality, that Z is independent. Observe that $H = [M|(C \cup Z)]^*$ is a plane having Z as a 3-point line. Let $P_1, P_2, \ldots, P_k$ be the parallel classes of H avoiding Z. As H is connected, it follows that $k \geq 2$. Now, we establish that $k \geq 3$. Assume that $k = 2$. Hence $W \cup P_1$ is a cocircuit of H for any 2-subset W of Z. In particular, when W' and W'' are different 2-subsets of Z, $(W' \cup P_1) \triangle (W'' \cup P_1)$ is a cocircuit of H and so a circuit of $M|(C \cup Z)$; a contradiction since $|(W' \cup P_1) \triangle (W'' \cup P_1)| = 2$. Therefore $k \geq 3$. The cosimplification of H is isomorphic to $M(K_4)$ or to F_7 because H is binary. □

Lemma 3.3. *Let C be a circuit of a 3-connected binary matroid M such that* $|C| = \mathrm{circ}(M) \in \{6, 7\}$*. If Z and Z' are strongly disjoint stars with respect to C, then:*

(i) *Z and Z' are both simple and* $\pi(C, Z) = \pi(C, Z')$*; or*
(ii) $|C| = 7$*, exactly one of Z or Z' is simple, say Z, and there is* $S \in \pi(C, Z)$ *and* $S' \in \pi(C, Z')$ *such that* $|S| = 3$, $|S'| = 4$ *and* $C = S \cup S'$*; or*
(iii) *Z and Z' are both non-simple and there is* $S \in \pi(C, Z)$ *and* $S' \in \pi(C, Z')$ *such that* $\{|S|, |S'|\} \subseteq \{3, 4\}$ *and* $C = S \cup S'$*.*

Proof. Assume that this result is not true. By Lemma 3.2, we obtain that:

(a) The cosimplification of $M|(C \cup Z)$ is isomorphic to $M(K_4)$ and $\pi(C, Z) = \{Z_1, Z_2, Z_3\}$ with $|Z_1| = |Z_2| = 2$ and $|Z_3| \in \{2, 3\}$. Moreover, the elements of Z can be labeled as z_1, z_2, z_3 so that, for each $i \in \{1, 2, 3\}$, $(Z - z_i) \cup (C - Z_i)$ is a circuit of $M|(C \cup Z)$. Or
(b) The cosimplification of $M|(C \cup Z)$ is isomorphic to F_7^* and $\pi(C, Z) = \{Z_1, Z_2, Z_3, Z_4\}$. Moreover, there is $r \in \{1, 2, 3, 4\}$ so that $|Z_r| = 1$, say $r = 4$, since $|C| \leq 7$. The elements of Z can be labeled as z_1, z_2, z_3 so that, for each $i \in \{1, 2, 3\}$, $(Z - z_i) \cup [C - (Z_i \cup Z_4)]$ is a circuit of $M|(C \cup Z)$.

By the previous paragraph applied to Z' instead of Z, we conclude that:

(c) The cosimplification of $M|(C \cup Z')$ is isomorphic to $M(K_4)$ and $\pi(C, Z) = \{Z'_1, Z'_2, Z'_3\}$ with $|Z'_1| = |Z'_2| = 2$ and $|Z'_3| \in \{2, 3\}$. Moreover, the elements of Z' can be labeled as z'_1, z'_2, z'_3 so that, for each $i \in \{1, 2, 3\}$, $(Z' - z'_i) \cup (C - Z'_i)$ is a circuit of $M|(C \cup Z')$. Or
(d) The cosimplification of $M|(C \cup Z)$ is isomorphic to F_7^* and $\pi(C, Z) = \{Z'_1, Z'_2, Z'_3, Z'_4\}$. Moreover, we can label these sets so that $Z_4 \subseteq Z'_4$, when (b) occurs, and $|Z'_4| = 1$, when (a) occurs. The elements of Z' can be labeled as z'_1, z'_2, z'_3 so that, for each $i \in \{1, 2, 3\}$, $(Z' - z'_i) \cup [C - (Z'_i \cup Z'_4)]$ is a circuit of $M|(C \cup Z')$.

Now, we divide the proof in three steps.

Step 1. (b) and (d) cannot occur simultaneously.

Suppose that (b) and (d) occur simultaneously. Assume that $Z_4 = \{a\}$. First, we show that

$$Z_4 \in \pi(C, Z'), \quad \text{that is, } Z_4 = Z'_4 = \{a\}. \tag{3.1}$$

If $Z_4 \notin \pi(C, Z')$, then $|Z'_4| \geq 2$. Choose $b \in Z'_4 - Z_4$. We may assume that $b \in Z_1$. Let $\{r, s\}$ and t be respectively a 2-subset of $\{1, 2, 3\}$ and an element of $\{2, 3\}$. By (b) and (d), $C_{1t} = \{z_1, z_t\} \cup Z_1 \cup Z_t$ and $C'_{rs} = \{z'_r, z'_s\} \cup Z'_r \cup Z'_s$ are circuits of M. Observe that

$$a \notin C_{1t} \cup C'_{rs} \quad \text{and} \quad b \in C_{1t} - C'_{rs}. \tag{3.2}$$

As $\{z_1, z_t\}$ and $\{z'_r, z'_s\}$ are strongly disjoint C-arcs, it follows, by Lemma 3.1, that $\{z_1, z_t\}$ and $\{z'_r, z'_s\}$ do not cross. Therefore, by definition and (3.2),

$$C_{1t} \cap C'_{rs} = \emptyset \quad \text{or} \quad [C'_{rs} - C_{1t}] \cap C = \emptyset. \tag{3.3}$$

Thus

$$(Z_1 \cup Z_t) \cap (Z'_r \cup Z'_s) = \emptyset \quad \text{or} \quad (Z_1 \cup Z_t) \supseteq (Z'_r \cup Z'_s). \tag{3.4}$$

As (3.4) holds for every 2-subset $\{r, s\}$ of $\{1, 2, 3\}$, it follows that

$$(Z_1 \cup Z_t) \cap (Z'_1 \cup Z'_2 \cup Z'_3) = \emptyset \quad \text{or} \quad (Z_1 \cup Z_t) \supseteq (Z'_1 \cup Z'_2 \cup Z'_3). \tag{3.5}$$

From (3.5) for $t = 2$ and $t = 3$, it is not difficult to show that there is $k \in \{1, 2, 3, 4\}$ such that

$$(Z'_1 \cup Z'_2 \cup Z'_3) \subseteq Z_k. \tag{3.6}$$

In particular, $|Z_k| \geq 3$ and $C = Z_k \cup Z'_4$. As $Z_i \subseteq Z'_4$, for every $i \in \{1, 2, 3, 4\} - k$, it follows that $|Z'_4| \geq 3$; a contradiction because (iii) happens for $S = Z_k$ and $S' = Z'_4$. Therefore (3.1) holds.

Now, we prove that

$$|Z_1| = |Z_2| = |Z_3| = 2. \tag{3.7}$$

Assume that (3.7) does not hold. As $|C| \leq 7$, it follows that $|Z_i| = 1$, for some $i \in \{1, 2, 3\}$, say $i = 3$ and $Z_3 = \{b\}$. By (3.1), $\{b\} \in \pi(C, Z')$, say $Z'_3 = \{b\}$. By (b) and (d), $C_{12} = \{z_1, z_2\} \cup Z_1 \cup Z_2$ and $C'_{23} = \{z'_2, z'_3\} \cup Z'_2 \cup Z'_3$ are circuits of M. (In this paragraph, we assume also that $|Z'_2| \geq |Z'_1|$.) Note that

$$a \notin C_{12} \cup C'_{23} \quad \text{and} \quad b \in C'_{23} - C_{12}. \tag{3.8}$$

As $\{z_1, z_2\}$ and $\{z'_2, z'_3\}$ are strongly disjoint C-arcs, it follows, by Lemma 3.1, that $\{z_1, z_2\}$ and $\{z'_2, z'_3\}$ do not cross. Therefore, by definition and (3.8),

$$C_{12} \cap C'_{23} = \emptyset \quad \text{or} \quad [C_{12} - C'_{23}] \cap C = \emptyset. \tag{3.9}$$

Observe that $|C_{12} \cap C| = |C| - 2$ and $|C'_{23} \cap C| \geq \frac{|C|}{2}$. (Remember that, in this paragraph, we are assuming that $|Z'_2| \geq |Z'_1|$.) Hence $|C'_{23} \cap C| + |C_{12} \cap C| \geq |C| + 1$. In particular, $C_{12} \cap C'_{23} \cap C \neq \emptyset$. By (3.9), $[C_{12} - C'_{23}] \cap C = \emptyset$. We arrive at a contradiction because $|C_{12}| \geq |C'_{23}|$ and $b \in C'_{23} - C_{12}$. Thus (3.7) follows.

Replacing (Z, Z') by (Z', Z), Eq. (3.7) becomes

$$|Z'_1| = |Z'_2| = |Z'_3| = 2. \tag{3.10}$$

If $\{r, s\}$ is a 2-subset of $\{1, 2, 3\}$, then, by (b) and (d), $C_{rs} = \{z_r, z_s\} \cup Z_r \cup Z_s$ and $C'_{rs} = \{z'_r, z'_s\} \cup Z'_r \cup Z'_s$ are circuits of M. By (3.7) and (3.10), $|C_{rs}| = |C'_{rs}| = 6$. We can label z_1, z_2, z_3 so that $C_{12} \cap C \neq C'_{12} \cap C$. In particular, $[C_{12} - C'_{12}] \cap C \neq \emptyset$ and $[C'_{12} - C_{12}] \cap C \neq \emptyset$. By construction, $a \notin C_{12} \cup C'_{12}$. As $|C_{12} \cap C| + |C'_{12} \cap C| = 8$, it follows that $[C_{12} \cap C'_{12}] \cap C \neq \emptyset$. Therefore $\{z_1, z_2\}$ and $\{z'_1, z'_2\}$ cross; a contradiction to Lemma 3.1 and so Step 1 follows.

By Step 1, (b) and (d) cannot occur simultaneously. Thus (a) or (c) happens, say (a). That is, Z is simple. We arrive at the final contradiction by proving the next two steps.

Step 2. (d) *cannot happen.*

Suppose that (d) happens. By (d), $|Z'_4| = 1$, say $Z'_4 = \{a\}$. By (a), there is $i \in \{1, 2, 3\}$ such that $a \in Z_i$ and there is $b \in Z_i - a$ because $|Z_i| \geq 2$. Observe that $b \in Z'_j$, for some $j \in \{1, 2, 3\}$, say $j = 3$. As $Z - z_i$ and $Z' - z'_k$, for $k \in \{1, 2\}$, are strongly disjoint C-arcs, it follows, by Lemma 3.1, that $Z - z_i$ and $Z' - z'_k$ do not cross. By (a) and (d),

(e) the projections of $Z - z_i$ over C are Z_i and $C - Z_i$; and
(f) the projections of $Z' - z'_k$ over C are $Z'_4 \cup Z'_k$ and $C - (Z'_4 \cup Z'_k)$.

But $a \in Z_i \cap [Z'_4 \cup Z'_k]$, $b \in Z_i \cap [C - (Z'_4 \cup Z'_k)]$ and so, for $k \in \{1, 2\}$,

$$[C - Z_i] \cap [Z'_4 \cup Z'_k] = \emptyset \quad \text{or} \quad [C - Z_i] \cap [C - (Z'_4 \cup Z'_k)] = \emptyset. \tag{3.11}$$

Now, we show that

$$[C - Z_i] \cap [Z'_4 \cup Z'_k] = \emptyset, \quad \text{for some } k \in \{1, 2\}, \text{ say } k = 1. \tag{3.12}$$

If (3.12) does not hold, then, by (3.11), $[C - Z_i] \cap [C - (Z'_4 \cup Z'_k)] = \emptyset$, for each $k \in \{1, 2\}$. Hence $C - (Z'_4 \cup Z'_k) \subseteq Z_i$, for $k \in \{1, 2\}$. Hence Z_i contains Z'_l, for every $l \in \{1, 2, 3\}$; a contradiction because $|Z_i| \leq 3$ and $|Z'_1| + |Z'_2| + |Z'_3| = |C| - |Z'_4| \geq 5$. Therefore (3.12) holds. By (3.12), $Z'_4 \cup Z'_1 \subseteq Z_i$. By (3.11) for $k = 2$, Z'_2 or Z'_3 is a subset of Z_i. As $|Z_i| \leq 3$ and $b \in Z_i \cap Z'_3$, it follows that $|Z_i| = 3$, $Z'_3 \subseteq Z_i$ and $|Z'_1| = |Z'_3| = |Z'_4| = 1$. Therefore (ii) holds; a contradiction and Step 2 follows.

Step 3. (c) *cannot happen.*

Assume that (c) happens. For each $e \in C$, let Z_e and Z'_e be the elements of $\pi(C, Z)$ and $\pi(C, Z')$ respectively so that $e \in Z_e \cap Z'_e$. By (a) and (c), for each $e \in C$, there are circuits C_e and C'_e of M such that $C_e \subseteq C \cup Z$, $C'_e \subseteq C \cup Z'$, $C_e \cap C = C - Z_e$ and $C'_e \cap C = C - Z'_e$. Observe that $e \notin C_e \cup C'_e$ and $C_e \cap C_e \neq \emptyset$ because $|C \cap C_e| \geq 4$ and $|C \cap C'_e| \geq 4$. As $C_e - C$ and $C'_e - C$ are strongly disjoint C-arcs, it follows, by Lemma 3.1, that $C_e - C$ and $C'_e - C$ do not cross and so

$$C_e \cap C \subseteq C'_e \cap C \quad \text{or} \quad C'_e \cap C \subseteq C_e \cap C.$$

Hence

$$Z_e \subseteq Z'_e \quad \text{or} \quad Z'_e \subseteq Z_e, \quad \text{for each } e \in C. \tag{3.13}$$

Now, we prove that

$$Z_e = Z'_e, \quad \text{for each } e \in C. \tag{3.14}$$

By (3.13), we may assume that $Z_e \subseteq Z'_e$. If (3.14) does not hold, then $|Z'_e - Z_e| = 1$, say $Z'_e = Z_e \cup f$. As $Z'_f = Z'_e$ and $\pi(C, Z)$ is a partition of C, it follows that $Z'_f \not\subseteq Z_f$. By (3.13), $Z_f \subseteq Z'_f$ and so $Z_f = \{f\}$; a contradiction and (3.14) follows. By (3.14), $\pi(C, Z) = \pi(C, Z')$ and (i) holds; a contradiction. Therefore both Step 3 and this lemma follow. □

4. Global structural results

In the only result of this section, we describe the structure of the matroid obtained from a 3-connected binary matroid having circumference 6 or 7 after the deletion of all the elements belonging to $\text{cl}(C) - C$, where C is one of its maximum size circuits.

Proposition 4.1. *Suppose that M is a 3-connected binary matroid such that $\text{circ}(M) \in \{6, 7\}$ and $r(M) \geq \text{circ}(M) + 2$. Let C be a maximum size circuit of M. If $K_1, K_2, \ldots, K_n$ are the connected components of M/C having non-zero rank, then $n \geq 3$ and, for every $i \in \{1, 2, \ldots, n\}$, $|E(K_i)| \geq 3$ and $r(K_i) = 1$. Moreover, when Z_i is a 3-subset of $E(K_i)$, for $i \in \{1, 2, \ldots, n\}$, then:*

(i) *There is a partition T_1, T_2, T_3 of C such that $|T_1| = |T_2| = 2$ and T_1, T_2, T_3 are series classes of $M|(C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n)$.*
(ii) *The cosimplification of $M|(C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n)$ is isomorphic to $M(K^{(3)}_{3,n})$ (and $Z_1, Z_2, \ldots, Z_n$ are the stars of the vertices of $K^{(3)}_{3,n}$ having degree 3).*
(iii) *For $i \in \{1, 2, \ldots, n\}$, $E(K_i)$ is a triad or a quad of M.*
(iv) *The cosimplification of $M \setminus [\text{cl}_M(C) - C]$ is isomorphic to $M_{n,l,3}$, where $l = |\{i \in \{1, 2, \ldots, n\} : E(K_i) \text{ is a quad of } M\}|$.*

Proof. By Proposition 2.1, each connected component of M/C has rank equal to 0 or 1. Hence, for every $i \in \{1, 2, \ldots, n\}$, $r(K_i) = 1$ and so

$$n = \sum_{i=1}^{n} r(K_i) = r(M/C) = r(M) - [|C| - 1].$$

By hypothesis, $r(M) \geq |C| + 2$. Consequently,

$$n \geq 3. \tag{4.1}$$

To finish the proof of the first part of this proposition, we need to show that $|E(K_i)| \geq 3$, for every $i \in \{1, 2, \ldots, n\}$. This happens because $E(K_i)$ is a cocircuit of both M/C and M. (Remember that M is 3-connected.) Now, we need to establish (i), (ii), (iii) and (iv). Note that:

$$\text{for } i \in \{1, 2, \ldots, n\}, \text{ any 3-subset of } E(K_i) \text{ is a star with respect to } C. \tag{4.2}$$

By (4.2), for each $i \in \{1, 2, \ldots, n\}$, we can choose stars Z_i and Z_i' with respect to C such that $Z_i \cup Z_i' \subseteq E(K_i)$. We next establish that

$$Z_i \text{ is simple if and only if } Z_i' \text{ is simple.} \tag{4.3}$$

By (4.2), it is enough to prove (4.3) when $|Z_i - Z_i'| = 1$, say $Z_i = \{a, b, c\}$ and $Z_i' = \{b, c, d\}$. Assume that (4.3) does not hold. So exactly one of Z_i or Z_i' is simple, say Z_i. (Consequently, Z_i' is non-simple.) By Lemma 3.3(ii), there are $S \in \pi(C, Z_i)$ and $S' \in \pi(C, Z_i')$ such that $|S| = 3$, $|S'| = 4$, $S \cap S' = \emptyset$ and $S \cup S' = C$. Let D be the circuit of M such that $D - C = \{b, c\}$ and $|D|$ is minimum. Note that $|D| = 4$ because D is a circuit of both $M|(C \cup Z_i)$ and $M|(C \cup Z_i')$. As Z_i is simple, it follows that $D \cap C \in \pi(C, Z_i)$. Hence $(D \cap C) \cap S = \emptyset$ because $|S| = 3$ and $|D \cap C| = 2$. Therefore $D \cap C \subsetneq S'$. We arrive at a contradiction because S' is a series class of $M|(C \cup Z_i')$. Thus (4.3) follows.

We may reorder the stars $Z_1, Z_2, \ldots, Z_n$ so that $Z_1, Z_2, \ldots, Z_m$ are non-simple and $Z_{m+1}, Z_{m+2}, \ldots, Z_n$ are simple, for some $0 \leq m \leq n$. By definition, when $\{i, j\}$ is a 2-subset of $\{1, 2, \ldots, n\}$, Z_i and Z_j are strongly disjoint stars with respect to C. By Lemma 3.3(ii, iii), there is $S_i \in \pi(C, Z_i)$, for $i \in \{1, 2, \ldots, m\}$, such that $3 \leq |S_i| \leq |C| - 3 \leq 4$. Moreover, by Lemma 3.3(iii), $S_i \cup S_j = C$, when $\{i, j\}$ is a 2-subset of $\{1, 2, \ldots, m\}$. Therefore

$$m \leq 2. \tag{4.4}$$

Now, we show that

$$m \leq 1. \tag{4.5}$$

If (4.5) does not hold, then, by (4.4), $m = 2$. By (4.1), Z_3 exists and so Z_3 is simple. By Lemma 3.3(ii), $|C| = 7$, $S \cup S_1 = S \cup S_2 = C$, where $S \in \pi(C, Z_3)$ and $|S| = 3$. Hence $S_1 = S_2$; a contradiction since $S_1 \cup S_2 = C$. Therefore (4.5) follows.

By Lemma 3.3(i), (4.1) and (4.5),

$$\pi(C, Z_{m+1}) = \pi(C, Z_2) = \pi(C, Z_3) = \cdots = \pi(C, Z_n). \tag{4.6}$$

Now, we establish that:

$$\pi(C, Z_i') = \pi(C, Z_i). \tag{4.7}$$

If Z_i is simple, then replace Z_i by Z_i'. In this case, (4.7) follows from (4.6). If Z_i is non-simple, then, by (4.5), $i = m = 1$. By Lemma 3.3(ii), there is $S \in \pi(C, Z_2)$ such that $|S| = 3$ and $C - S \in \pi(C, Z_i) \cap \pi(C, Z_i')$ (by (4.3), Z_i' is also non-simple). Hence every 1-element subset of S belongs to both $\pi(C, Z_i)$ and $\pi(C, Z_i')$. Thus (4.7) also follows in this case.

To prove this result, we need to show that

$$m = 0. \tag{4.8}$$

If $m > 0$, then, by (4.5), $m = 1$. By Lemma 3.3(ii), $|C| = 7$ and there is $S \in \pi(C, Z_n)$ such that $|S| = 3$. Note that $\{C - S, S\}$ is a 2-separation of $M|C$. By (3.8) of Seymour [11], there is a C-arc Z such that $Z \not\rightarrow S$ and $Z \not\rightarrow C - S$ because M is 3-connected. By (4.7) and (4.6), $Z' \rightarrow C - S$, when Z' is a C-arc such that $Z' \subseteq E(K_i)$, for some $i \in \{2, 3, \ldots, n\}$. Hence $Z \not\subseteq E(K_i)$, for each $i \in \{2, 3, \ldots, n\}$. By (4.7) and Lemma 3.3(ii), $Z'' \rightarrow S$, when Z'' is a C-arc such that $Z'' \subseteq E(K_1)$. Therefore $Z \not\subseteq E(K_i)$, for each $i \in \{1, 2, \ldots, n\}$. In particular, $Z \subseteq \mathrm{cl}_M(C) - C$ and $|Z| = 1$, say $Z = \{e\}$. Let D be a circuit of M such that $e \in D \subseteq C \cup e$ and $|D|$ is minimum. In particular, $|D| \leq 4$. As $Z \not\rightarrow C - S$, it follows that

$D \cap S \neq \emptyset$. Moreover, $|D \cap S| \in \{1, 2\}$ because $D \triangle C$ is also a circuit of M and $Z \not\rightarrow C - S$. Observe that $D - (S \cup e) \neq \emptyset$ since $Z \not\rightarrow S$. Choose 2-subsets X, X' and X'' of Z_1, Z_2 and S respectively such that $D \cap S \subseteq X''$ and both $X \cup X''$ and $X' \cup (C - S)$ are circuits of M. Now, we show that

$$D' = D \triangle (X \cup X'') \triangle (X' \cup (C - S)) \quad \text{is a circuit of } M. \tag{4.9}$$

If C' is a circuit of M such that $C' \subseteq D'$, then

(a) $C' - C \neq \emptyset$ because $C \not\subseteq D'$;
(b) $C' - C \neq X'$ because $S \not\subseteq D'$ and $C - S \not\subseteq D'$;
(c) $C' - C \neq X$ because $X'' \not\subseteq D'$ and $C - X'' \not\subseteq D'$; and
(d) $C' - C \neq \{e\}$ because $D \not\subseteq D'$ and $D \triangle C \not\subseteq D'$.

In particular, $|C' - C| \geq 3$. As $|D' - C| = 5$ and D' is the union of pairwise disjoint circuits of M, it follows that D' is a circuit of M. Therefore (4.9) follows. We arrive at a contradiction because $|D'| \geq 8$. Thus (4.8) holds. In particular, Z_i is simple, for every i.

Now, our goal is to prove that

$$r(E(K_i)) = 3. \tag{4.10}$$

Assume that (4.10) fails for some i. Let B be a maximal independent set of M such that $Z_i \subseteq B \subseteq E(K_i)$. Thus $|B| \geq 4$. Choose a 3-subset Z'_i of B such that $|Z_i \cup Z'_i| = 4$. By (4.3) and (4.8), both Z'_i and Z_i are simple. By (4.7), $\pi(C, Z'_i) = \pi(C, Z_i)$ is the set of series classes of both $M|(C \cup Z_i)$ and $M|(C \cup Z'_i)$ contained in C. Thus $\pi(C, Z'_i) = \pi(C, Z_i)$ is the set of series classes of $M|(C \cup Z_i \cup Z'_i)$ contained in C. If N is the cosimplification of $M|(C \cup Z_i \cup Z'_i)$, then $C \cap E(N)$ is a circuit-hyperplane of N having three elements. So $r(N) = 3$. But each element of $Z_i \cup Z'_i$ is contained in a trivial series class of $M|(C \cup Z_i \cup Z'_i)$. Hence $r_N(Z_i \cup Z'_i) = r(Z_i \cup Z'_i) = |Z_i \cup Z'_i| = 4$; a contradiction. Thus (4.10) follows.

Next, we show (iii), that is,

$$E(K_i) \quad \text{is a triad or a quad of } M. \tag{4.11}$$

If $E(K_i) = Z_i$, then (4.11) follows. Suppose that $E(K_i) \neq Z_i$. By (4.10), for each $e \in E(K_i) - Z_i$, there is a circuit D_e of M so that $e \in D_e \subseteq Z_i \cup e$. As $E(K_i)$ is a cocircuit of M, it follows, by orthogonality, that $|D_e|$ is an even number. Hence $|D_e| = 4$ because M is 3-connected. In particular, $D_e = Z_i \cup e$. As M is simple, it follows that e is unique. Therefore $E(K_i) = Z_i \cup e$ and (4.11) follows.

By (4.6), there is a partition $\{T_1, T_2, T_3\}$ of C such that $|T_1| = |T_2| = 2$ and, for every $i \in \{1, 2, \ldots, n\}, \pi(C, Z_i) = \{T_1, T_2, T_3\}$. We can label the elements of Z_i by a_i, b_i, c_i so that $C_i = \{a_i, b_i\} \cup T_1$ and $D_i = \{a_i, c_i\} \cup T_2$ are circuits of M. Note that $\mathcal{B} = \{C, C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n\}$ spans the cycle space of $M|(C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n)$ because $(C - c) \cup \{a_1, a_2, \ldots, a_n\}$ spans $C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n$, for $c \in C$. In particular, T_1, T_2 and T_3 are series classes of $M|(C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n)$ because every circuit belonging to $\mathcal{B}$ contains T_i or avoids T_i, for every $i \in \{1, 2, 3\}$. Therefore (i) follows.

For $i \in \{1, 2, 3\}$, choose $t_i \in T_i$. By (i), the cosimplification of $M|(C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n)$ is equal to

$$H = [M|(C \cup Z_1 \cup Z_2 \cup \cdots \cup Z_n)]/(C - \{t_1, t_2, t_3\}).$$

Note that $\mathcal{B}' = \{C', C'_1, C'_2, \ldots, C'_n, D'_1, D'_2, \ldots, D'_n\}$ spans the cycle space of H, where $C' = \{t_1, t_2, t_3\}$ and, for $i \in \{1, 2, \ldots, n\}$, $C'_i = \{a_i, b_i, t_1\}$ and $D'_i = \{a_i, c_i, t_2\}$. Hence $H = M(G)$, where G is a simple graph having vertex-set $\{v_1, v_2, \ldots, v_n, w_1, w_2, w_3\}$ whose edges are: t_1 joining w_1 and w_2; t_2 joining w_3 and w_2; t_3 joining w_1 and w_3; and, for every $i \in \{1, 2, \ldots, n\}$, a_i joining v_i and w_2; b_i joining v_i and w_1; and c_i joining v_i and w_3. But $G \cong K^{(3)}_{3,n}$. We have (ii). Note that (iv) is a consequence of (ii) and (iii). $\square$

5. The 3-connected binary matroids with circumference equal to 6

Proof of Theorem 1.2. It is easy to see that $\mathrm{circ}(M_{n,m,l}) = 6$, when $n \geq 3$. Now, assume that M is a 3-connected binary matroid such that $\mathrm{circ}(M) = 6$. Let C be a circuit of M such that $|C| = \mathrm{circ}(M)$.

By Proposition 4.1, $M \setminus [\mathrm{cl}_M(C) - C]$ has three series classes S_1, S_2 and S_3 contained in C. Moreover, $|S_1| = |S_2| = |S_3| = 2$, say $S_1 = \{a, a'\}$, $S_2 = \{b, b'\}$, $S_3 = \{c, c'\}$, and

$$M \setminus [\mathrm{cl}_M(C) - C]/\{a', b', c'\} \cong M_{n',m',3},$$

where $n' = r(M) - 5$. (We also have that $T = \{a, b, c\}$ is the special triangle of $M \setminus [\mathrm{cl}_M(C) - C]/\{a', b', c'\}$.)

For $e \in \mathrm{cl}_M(C) - C$, let C_e be a circuit of M such that $e \in C_e \subseteq C \cup e$ and $|C_e|$ is minimum. Hence $|C_e - e| \in \{2, 3\}$. First, we establish that

$$S_i \subseteq C_e, \quad \text{for some } i \in \{1, 2, 3\}. \tag{5.1}$$

If (5.1) is not true, then C_e meets each S_i in 0 or 1 element. In particular, C_e meets at least two S_i's in 1 element, say $C_e \cap S_1 = \{a\}$ and $C_e \cap S_2 = \{b\}$. We have two cases to deal with. If $|C_e| = 3$, then $C_e \cap S_3 = \emptyset$ and $C_e \triangle D$ is a 7-element circuit of M, where D is a circuit of M such that $S_2 \cup S_3 \subseteq D$ and $|D - \mathrm{cl}_M(C)| = 2$; a contradiction. If $|C_e| = 4$, then C_e meets S_3 in 1 element, say $C_e \cap S_3 = \{c\}$. Let D_1 and D_2 be 4-element circuits of M such that $D_i \cap C = S_i$, for $i \in \{1, 2\}$, and $D_1 - C$ and $D_2 - C$ are strongly disjoint C-arcs. We arrive at a contradiction by proving that

$$X = C_e \triangle D_1 \triangle D_2$$

is a circuit of M. (Observe that $|X| = 8$.) If X is not a circuit of M, then $X = C_1 \cup C_2 \cup \cdots \cup C_l$, where $C_1, C_2, \ldots, C_l$ are pairwise disjoint circuits of M, for some $l \geq 2$. Assume that $e \in C_1$. Note that $C_1 - \mathrm{cl}_M(C) \neq \emptyset$, otherwise $C_1 = \{e, a', b', c\}$, by the choice of C_e, and so $C_1 \triangle C_e = \{c, c'\}$. Hence C_1 meets $D_1 - C$ or $D_2 - C$, say $D_1 - C$. But $D_1 - C$ is a series class of $M|(C \cup e \cup D_1 \cup D_2)$. Consequently, $D_1 - C \subseteq C_1$. As C_2 is not a proper subset of C, it follows that $D_2 - C \subseteq C_2$. In particular, $C_2 \cap C$ is a projection of the C-arc $D_2 - C$; a contradiction because $C_2 \cap C$ does not contain any S_i. Therefore (5.1) holds.

By (5.1), for $e \in \mathrm{cl}_M(C) - C$, we can choose C_e so that $|C_e \cap \{a', b', c'\}| = 1$. Therefore the elements belonging to $\mathrm{cl}_M(C) - C$ can be labeled as:

(i) s_i, for $i \in \{1, 2, 3\}$, when $S_i \cup s_i$ is a triangle of M.
(ii) t_{ij}, for a 2-subset $\{i, j\}$ of $\{1, 2, 3\}$, when $S_i \cup \{t, t_{ij}\}$ is a circuit of M, for $t \in S_j \cap \{a, b, c\}$.

In particular, $|\mathrm{cl}_M(C) - C| \leq 9$. Let M' be the binary extension of M obtained by adding all the elements described in (i) or (ii) which do not belong to M (with the dependence described in (i) or (ii)). When $\{1, 2, 3\} = \{i, j, k\}$, $\{t_{ik}, t_{jk}\} \cup S_k$ is a circuit of M'. In particular, $M' \setminus \{t_{12}, t_{23}, t_{31}\} \cong M_{n'+3,m',3}$ and so $M' \cong M_{n'+3,m'+3,3}$. (Observe that $\{s_1, s_2, s_3\}$ is the special triangle of M'.) Hence $M \cong M_{n,m,l}$, where $n = n' + 3$, $m = m' + [|E(M) \cap \{t_{12}, t_{13}, t_{21}, t_{23}, t_{31}, t_{32}\}| - 3]$ and $l = |E(M) \cap \{s_1, s_2, s_3\}|$. (Observe that $|\{t_{ik}, t_{jk}\} \cap E(M)| \geq 1$, when $\{i, j, k\} = \{1, 2, 3\}$, otherwise S_k is a cocircuit of M.) □

6. The 3-connected binary matroids with circumference equal to 7

A quad Q of a matroid M is said to be *special* when $Q \cap C = \emptyset$, for some largest circuit C of M.

Lemma 6.1. *Let M be a 3-connected binary matroid such that* $\mathrm{circ}(M) \in \{6, 7\}$. *If Q is a special quad of M, then there is an element e belonging to Q such that $M \setminus e$ is* 3*-connected.*

Proof. By definition, there is a circuit C of M such that $|C| = \mathrm{circ}(M)$ and $Q \cap C = \emptyset$. As Q is a cocircuit of M/C, it follows, by Proposition 2.1, that $Q \subseteq E(K)$, for a connected component K of M/C such that $r(K) = 1$. Therefore $Q = E(K)$ because $E(K)$ is a cocircuit of M. If $M \setminus e$ is not 3-connected, for every $e \in Q$, then, by Theorem 1 of Lemos [12], Q meets at least two triads of M, say T_1^* and T_2^*. (Remember that Q is also a circuit of M.) As $|T_i^* \cap Q| = 2$, $Q \cap C = \emptyset$ and $|T_i^* \cap C| \neq 1$, it follows that $T_i^* \cap C = \emptyset$. Hence T_1^* and T_2^* are cocircuits of M/C and so T_1^* and T_2^* are also cocircuits of K. We arrive at a contradiction because $T_1^* \subsetneq E(K) = Q$. Thus there is $e \in Q$ such that $M \setminus e$ is 3-connected. □

Lemma 6.2. *Suppose that M is a 3-connected binary matroid such that $\mathrm{circ}(M) \in \{6, 7\}$. Let T^* be a triad of M. If N is an one-element binary extension of M, say $M = N \setminus e$, such that $T^* \cup e$ is a circuit of N, then $T^* \cup e$ is a quad of N and $\mathrm{circ}(N) = \mathrm{circ}(M)$. Moreover, if T'^* is a triad or a quad of M such that $T^* \cap T'^* = \emptyset$, then T'^* is respectively a triad or a quad of N.*

Proof. First, we show that $T^* \cup e$ is a quad of N. There is a cocircuit C^* of N such that $T^* \subseteq C^* \subseteq T^* \cup e$. By orthogonality, the circuit $T^* \cup e$ meets the cocircuit C^* in an even number of elements. Therefore $C^* = T^* \cup e$ and so $T^* \cup e$ is a quad of N.

We argue by contradiction to prove that $\mathrm{circ}(M) = \mathrm{circ}(N)$. If $\mathrm{circ}(M) \neq \mathrm{circ}(N)$, then $\mathrm{circ}(M) < \mathrm{circ}(N)$, since M is a restriction of N. Let C be a maximum size circuit of N. As $\mathrm{circ}(M) < |C|$, it follows that $e \in C$. By orthogonality with the quad $T^* \cup e$, $|C \cap T^*| = 1$ or $T^* \subseteq C$. Observe that $T^* \not\subseteq C$, otherwise $C = T^* \cup e$ and $|C| < \mathrm{circ}(M)$. Hence $|C \cap T^*| = 1$. Let D be a circuit of N such that $D \subseteq C \triangle (T^* \cup e)$. Note that $D \cap (T^* \cup e) \neq \emptyset$ because D is not a proper subset of C. By orthogonality, $|D \cap (T^* \cup e)| \geq 2$ and so $[C \triangle (T^* \cup e)] \cap (T^* \cup e) \subseteq D$. In particular, D is unique. As $C \triangle (T^* \cup e)$ is a union of pairwise disjoint circuits of N, it follows that $C \triangle (T^* \cup e)$ is a circuit of N. But

$$|C| = |C \triangle (T^* \cup e)| > \mathrm{circ}(M);$$

a contradiction because $C \triangle (T^* \cup e)$ is also a circuit of M. Thus $\mathrm{circ}(M) = \mathrm{circ}(N)$.

Now, we show that T'^* is a triad or a quad of N. If T'^* is not respectively a triad or a quad of N, then $T'^* \cup e$ is a cocircuit of N. But the quad $T^* \cup e$ meets the cocircuit $T'^* \cup e$ in just one element, namely e; a contradiction to orthogonality. Consequently, T'^* is respectively a triad or a quad of N. □

Proof of Theorem 1.3. In this paragraph, we show that (ii) implies (i). We construct a sequence of matroids $M_0, M_1, M_2, \ldots, M_m$ such that $M_0 = M' \setminus X$ and, for each $i \in \{1, 2, \ldots, m\}$, M_i is a 1-element binary extension of M_{i-1}, say $M_{i-1} = M_i \setminus e_i$, and $Q_i = T_i^* \cup e_i$ is a circuit of M_i. By induction on i and Lemma 6.2, it is easy to show that:

$$Q_1, \ldots, Q_i \quad \text{are quads of } M_i;\ T_{i+1}^*, \ldots, T_m^* \text{ are triads of } M_i;\ \mathrm{circ}(M_i) = 7. \tag{6.1}$$

Take M to be M_m. The result follows because M_m is 3-connected.

Now, we just need to show that (i) implies (ii). We argue by contradiction. Choose a counter-example M such that $|E(M)|$ is minimum. First, we establish that:

$$M \text{ has no special quad.} \tag{6.2}$$

Suppose that (6.2) does not hold. Let Q be a special quad of M. By definition, there a circuit C of M such that $|C| = \mathrm{circ}(M)$ and $C \cap Q = \emptyset$. By Lemma 6.1, there is $e \in Q$ such that $M \setminus e$ is 3-connected. Observe that $T^* = Q - e$ is a triad of $M \setminus e$ and $|C| = \mathrm{circ}(M \setminus e) \leq \mathrm{circ}(M) = |C|$. Therefore $\mathrm{circ}(M \setminus e) = 7$. By the choice of M, there is a 3-connected rank-4 binary matroid N having a Hamiltonian circuit D and a triangle T satisfying $|T \cap D| = 2$ such that $T = E(N) \cap E(K_{3,r(M)-4}^{(3)})$ is the special triangle of $K_{3,r(M)-4}^{(3)}$ and $M \setminus e$ is obtained from $M' \setminus X$ by completing the set of pairwise disjoint triads $T_1^*, T_2^*, \ldots, T_m^*$ of $M(K_{3,r(M)-4}^{(3)})$ to quads, where M' is the generalized parallel connection of $M(K_{3,r(M)-4}^{(3)})$ with N and $X \subseteq T$. As C is a 7-element circuit of $M \setminus e$, it follows that $[C \cap E(N)] \cup Y$ is a Hamiltonian circuit of N, for some 2-subset Y of T. In particular, T^* is a triad of $M(K_{3,r(M)-4}^{(3)})$. Therefore M is obtained from $M' \setminus X$ by completing the set of pairwise disjoint triads $T_1^*, T_2^*, \ldots, T_m^*, T^*$ of $M(K_{3,r(M)-4}^{(3)})$ to quads; a contradiction and (6.2) follows.

Let C be a circuit of M such that $|C| = \mathrm{circ}(M)$. By Proposition 4.1, $M \setminus [\mathrm{cl}_M(C) - C]$ has three series classes S_1, S_2 and S_3 contained in C. Moreover, $|S_1| = |S_2| = 2$, say $S_1 = \{a, a'\}$, $S_2 = \{b, b'\}$, $S_3 = \{c, c', c''\}$, and

$$M \setminus [\mathrm{cl}_M(C) - C]/\{a', b', c', c''\} \cong M_{n',m',3},$$

where $n' = r(M) - 6$. (We also have that $T = \{a, b, c\}$ is the special triangle of $M \setminus [\mathrm{cl}_M(C) - C]/\{a', b', c', c''\}$.) Let $K_1, K_2, \ldots, K_{n'}$ be the rank-1 connected components of M/C. By (6.2) and Proposition 4.1(iii), $E(K_1), E(K_2), \ldots, E(K_{n'})$ are triads of M and so $m' = 0$. Choose C-arcs Z_1, Z_2, Z_3 such that $Z_i \cup S_i$ is a circuit of M and $Z_i \subseteq E(K_i)$, for each $i \in \{1, 2, 3\}$.

For $e \in \mathrm{cl}_M(C) - C$, let C_e be a circuit of M such that $e \in C_e \subseteq C \cup e$ and $|C_e \cap S_3|$ is maximum. Hence $2 \le |C_e \cap S_3|$ and $|C_e| \le 6$ because $C_e \triangle C$ is also a circuit of M. First, we establish that

$$C_e = (S_3 \cap C_e) \cup X \cup e \quad \text{where } X \text{ is a subset of } S_i, \text{ for some } i \in \{1, 2\}. \tag{6.3}$$

Assume that (6.3) does not hold. We have two cases to deal with $S_3 \subseteq C_e$ or $S_3 \not\subseteq C_e$. If $S_3 \subseteq C_e$, then $|C_e \cap S_1| = |C_3 \cap S_2| = 1$ because $|C_e| \le 6$. Note that $C_e \triangle (S_2 \cup Z_2)$ is a circuit of M having 8 elements; a contradiction. Thus $S_3 \not\subseteq C_e$ and so $|S_3 \cap C_e| = 2$. Now, we prove that

$$|C_e \cap S_1| = |C_e \cap S_2| = 1. \tag{6.4}$$

If (6.4) does not hold, then $|C_e \cap (S_1 \cup S_2)| = 3$, say $S_1 \subseteq C_e$. Again $C_e \triangle (S_2 \cup Z_2)$ is a circuit of M having 8 elements; a contradiction. Hence (6.4) holds. Observe that $C_e \triangle (S_1 \cup Z_1) \triangle (S_2 \cup Z_2)$ is a circuit of M having 9 elements; a contradiction. Therefore (6.3) happens.

For $i \in \{1, 2\}$, we establish that:

$$|\{g \in \mathrm{cl}_M(C) - C : |C_g \cap S_i| = 1\}| = 1. \tag{6.5}$$

Assume that $i = 1$. Observe that $C' = (C - S_1) \cup Z_1$ is a maximum size circuit of M. The rank-1 connected components of M/C' are $K_1', K_2, \dots, K_{n'}$. Moreover, by (6.3),

$$E(K_1') = S_1 \cup \{g \in \mathrm{cl}_M(C) - C : |C_g \cap S_i| = 1\}.$$

By Proposition 4.1(iii) and (6.2), $E(K_1')$ is a triad of M. So (6.5) follows. By (6.3) and (6.5), for $i \in \{1, 2\}$, there is $e_i \in \mathrm{cl}_M(C) - C$, $s_i \in S_i$ and $X_i \subseteq S_3$ such that $|X_i| \in \{2, 3\}$ and $C_{e_i} = X_i \cup \{e_i, s_i\}$. Moreover, e_i is unique. In this paragraph, we have proved more:

$$S_i \cup e_i \quad \text{is a triad of } M. \tag{6.6}$$

Now, we show that, for $i \in \{1, 2\}$,

$$\text{when } |X_i| = 2, \quad X_i \cup \{g \in \mathrm{cl}_M(C) - C : X_i \not\subseteq C_g\} \text{ is a triad of } M. \tag{6.7}$$

Assume that $i = 1$. Observe that $C'' = (C_{e_1} \triangle C) \triangle (S_1 \cup Z_1)$ is a maximum size circuit of M. The rank-1 connected components of M/C'' are $K_1'', K_2, \dots, K_{n'}$ and $E(K_1'') = X_1 \cup \{g \in \mathrm{cl}_M(C) - C : X_i \not\subseteq C_g\}$. So (6.7) follows from (6.2) and Proposition 4.1(iii).

Let I be the subset of $\{1, 2, 3\}$ so that $i \in I$ if and only if there is $f_i \in E(M)$ such that $f_i \cup S_i$ is a circuit of M. Choose a $(3 - |I|)$-set disjoint of $E(M)$, say $\{f_j : j \in \{1, 2, 3\} - I\}$. Let M' be a 3-connected binary extension of M such that $E(M') = E(M) \cup \{f_j : j \in \{1, 2, 3\} - I\}$ and $f_i \cup S_i$ is a circuit of M', for every $i \in \{1, 2, 3\}$. Now, we divide the proof in three cases.

Case 1. $|X_1| = |X_2| = 2$.

First, assume that $X_1 \ne X_2$. Note that $D = C_{e_1} \triangle C_{e_2} \triangle (f_3 \cup S_3) \triangle \{f_1, f_2, f_3\}$ is a 7-element circuit of M'. Therefore $D \triangle (f_1 \cup Z_1) \triangle (f_2 \cup Z_2)$ is a 9-element circuit of M; a contradiction. So $X_1 = X_2$. Observe that $C_{e_1} \triangle C_{e_2} \triangle (S_1 \cup Z_1) \triangle (S_2 \cup Z_2)$ is an 8-element circuit of M; a contradiction.

Case 2. $|X_1| = 2$ *and* $|X_2| = 3$.

So $C_{e_2} = S_3 \cup \{e_2, s_2\}$. Therefore $C_{e_2} \triangle C = S_1 \cup \{e_2, s_2'\}$ is a circuit of M, where $S_2 = \{s_2, s_2'\}$. Hence $D = C_{e_1} \triangle (S_1 \cup \{e_2, s_2'\}) \triangle (S_2 \cup f_2)$ is a 7-element circuit of M'; a contradiction because $D \triangle (f_2 \cup Z_2)$ is an 8-element circuit of M.

Case 3. $|X_1| = |X_2| = 3$.

For $i \in \{1, 2\}$, $C_{e_i} = S_3 \cup \{e_i, s_i\}$. Therefore $C_{e_i} \triangle C = S_{3-i} \cup \{e_i, s_i'\}$ is a circuit of M, where $S_i = \{s_i, s_i'\}$, and so $(S_{3-i} \cup \{e_i, s_i'\}) \triangle (S_{3-i} \cup f_{3-i}) = \{e_i, s_i', f_{3-i}\}$ is a circuit of M'. If $Y = (S_1 \cup e_1) \cup (S_2 \cup e_2) \cup E(K_1) \cup E(K_2) \cup \dots \cup E(K_{n'})$, then Y is the union of pairwise disjoint triads of M' (use (6.6)). As $M'|[Y \cup \{f_1, f_2, f_3\}] \cong K^{(3)}_{3,n'+2}$, it follows that $\{Y, E(M') - Y\}$ is an exact 3-separation of M'. So M' is the generalized parallel connection of $M'|[Y \cup \{f_1, f_2, f_3\}]$ and $M' \setminus Y$. By (6.3) and (6.5), $M' \setminus Y$ is a rank-4 3-connected binary matroid having $S_3 \cup \{f_1, f_2\}$ as a Hamiltonian circuit and $\{f_1, f_2, f_3\}$ as a triangle. But $M = M' \setminus X$, where $X = \{f_i : i \in I\}$; a contradiction because the result holds for M. $\square$

Now, we prove a result that will be used in [5]:

Corollary 6.1. *Let M be a* 3*-connected binary matroid such that* $\mathrm{circ}(M) \in \{6, 7\}$ *and* $r(M) \ge 10$*. If $M \setminus C$ is not* 3*-connected, for every circuit C of M, then* $|E(M)| < 4r(M) - 8$.

Proof. Suppose that $|E(M)| \geq r(M) - 8$. If $\mathrm{circ}(M) = 6$, then, by Theorem 1.2, $M \cong M_{n,0,l}$. Note that

$$|E(M)| = 3n + l = 3r(M) - 6 + l \geq 4r(M) - 8.$$

Therefore $5 \geq l + 2 \geq r(M)$; a contradiction. Hence $\mathrm{circ}(M) = 7$. By Theorem 1.3, there is a 3-connected rank-4 binary matroid N having a Hamiltonian circuit C and a triangle T satisfying $|T \cap C| = 2$ such that $T = E(N) \cap E(K^{(3)}_{3,r(M)-4})$ is the special triangle of $K^{(3)}_{3,r(M)-4}$ and $M = M' \setminus X$, where M' is the generalized parallel connection of $M(K^{(3)}_{3,r(M)-4})$ with N and $X \subseteq T$. Observe that

$$|E(M)| = 3r(M) - 12 + |E(N)| - |X| \geq 4r(M) - 8.$$

Thus

$$|E(N)| - |X| \geq r(M) + 4 \geq 14. \tag{6.8}$$

As $r(N) = 4$, it follows that $|E(N)| \leq 15$. Moreover, $N \setminus X \cong PG(3,2) \setminus Y$, where $|Y| \leq 1$. Let Z be a 7-element subset of $E(PG(3,2))$ such that $Y \subseteq Z$ and $PG(3,2) \setminus Z \cong AG(3,2)$. If T' is a triangle of $PG(3,2)$ avoiding Y and contained in Z, then $PG(3,2) \setminus (T' \cup Y)$ is 3-connected. So N has a triangle T'' such that $N \setminus (T'' \cup X)$ is 3-connected; a contradiction because $M \setminus T''$ is 3-connected. □

Acknowledgements

The first author's research was supported by FCT (Portugal) through program POCTI. The third author is partially supported by CNPq (Grants no. 476224/04-7 and 301178/05-4) and FAPESP/CNPq (Grant no. 2003/09925-5).

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