Lie bialgebroids and reduction

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Mechanics and Lie algebroids
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J. Marsden, A. Weinstein: Reduction of symplectic manifolds with symmetry., Rep. Math. Phys. 5, 121 (1974).

- A symplectic action of Lie group G on a symplectic manifold (M, Ω) with equivariant momentum map $J : M \to \mathfrak{g}^*$
- $\mu \in \mathfrak{g}^*$ a regular value of J

  $$\Downarrow$$

  $J^{-1}\mu$ is a submanifold

  G_μ acts on $\mathrm{J}^{-1}(\mu)$

- The space of orbits $J^{-1}\mu/G_\mu$ is an smooth manifold and the canonical projection $\pi_\mu : J^{-1}\mu \to J^{-1}\mu/G_\mu$ defines a principal G_μ-bundle

Theorem (Symplectic reduction)

There exists a unique symplectic 2-form Ω_μ on $J^{-1}\mu/G_\mu$ such that

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$$

where $\pi_\mu : J^{-1}\mu \to J^{-1}\mu/G_\mu$ is the projection and $i_\mu : J^{-1}\mu \to M$ is the canonical inclusion

1.1 Some tools in Lie algebroid Theory

$\tau_A : A \to M$ a Lie algebroid over M

$$([\![\cdot,\cdot]\!], \rho) \text{ a Lie algebroid structure on } A$$

$$\Gamma(A) \equiv C^\infty(M) - \text{modulo of sections of } A$$

$$(A, [\![\cdot,\cdot]\!], \rho) \text{ Lie algebroid} \Leftrightarrow A^* \text{ is a linear Poisson manifold}$$

The differential of the Lie algebroid A

$$d^A : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*)$$

The Lie derivative with respect to $X \in \Gamma(A)$

$$\mathcal{L}^A_X : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^k A^*)$$

$$\mathcal{L}^A_X = i_X \circ d^A + d^A \circ i_X$$

The Schouten bracket

$$[\![\cdot,\cdot]\!] : \Gamma(\wedge^p A) \times \Gamma(\wedge^q A) \longrightarrow \Gamma(\wedge^{p+q-1} A)$$

- $[\![X, f]\!] = \rho(X)(f)$
- $[\![P, Q \wedge R]\!] = [\![P, Q]\!] \wedge R + (-1)^{q(p+1)} Q \wedge [\![P, R]\!]$
- $[\![P, Q]\!] = (-1)^{pq+p+q} [\![Q, P]\!]$

$f \in C^\infty(M)$, $X \in \Gamma(A)$, $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$, $R \in \Gamma(\wedge^* A)$.

Vertical and complete lifts

The complete and vertical lift to A of $f : M \to \mathbb{R}$

- $f^c : A \to \mathbb{R}, \quad f^c(a) = \rho(a)(f)$
- $f^v : A \to \mathbb{R}, \quad f^v(a) = f(\tau(a)), \quad \forall a \in A.$

The complete and vertical lift to A of $X \in \Gamma(A)$

- $X^c \in \mathfrak{X}(A)$
  - $X^c(f \circ \tau) = \rho(X)(f) \circ \tau, \quad f \in C^\infty(M)$
  - $X^c(\widehat{\alpha}) = \widehat{\mathcal{L}^A_X \alpha}, \quad \alpha \in \Gamma(A^*)$

  $\widehat{\alpha} : A \to \mathbb{R}$ the linear function induced by α
- $X^v \in \mathfrak{X}(A) \qquad X^v(a_x) = (X(x))^v_{a_x}, \quad a_x \in A_x$

  where ${}^v_a : A_{\tau(a)} \to T_a(A_{\tau(a)})$ is the canonical isomorphism of vector spaces.

Lie algebroid morphism

A vector bundle morphism

$$\begin{array}{ccc} A & \xrightarrow{F} & A' \\ {\scriptstyle \tau_A}\downarrow & & \downarrow{\scriptstyle \tau_{A'}} \\ M & \xrightarrow{f} & M' \end{array}$$

$$d^A((F,f)^*\alpha') = (F,f)^*(d^{A'}\alpha'), \text{ for } \alpha' \in \Gamma(\wedge^k(A')^*)$$

$$((F,f)^*\alpha')_x(a_1,\dots,a_k) = \alpha'_{f(x)}(F(a_1),\dots,F(a_k))$$

1.1 Some tools in Lie algebroids Theory

$\widetilde{\pi} : A \longrightarrow A'$ vector bundle epimorphism

$X \in \Gamma(A)$ $\widetilde{\pi}$*-projectable*

$$\begin{tikzcd} A \arrow[r, "\widetilde{\pi}"] & A' \\ M \arrow[u, "X"] \arrow[r, "\pi"] & M' \arrow[u, "X'"'] \end{tikzcd}$$

Proposition

Suppose that $([\![\cdot,\cdot]\!],\rho)$ is a Lie algebroid structure on A. Then, there is a Lie algebroid structure on A' such that $\widetilde{\pi}$ is a Lie algebroid epimorphism if and only if the following conditions hold:

1. $[\![X,Y]\!]$ is a $\widetilde{\pi}$-projectable section of A, for all $X, Y \in \Gamma(A)$ $\widetilde{\pi}$-projectable sections of A.
2. $[\![X,Y]\!] \in \Gamma(\ker \widetilde{\pi})$, for all $X, Y \in \Gamma(A)$ with $X \in \Gamma(A)$ $\widetilde{\pi}$-projectable section of A and $Y \in \Gamma(\ker \widetilde{\pi})$.

D. Iglesias, J.C. Marrero, D. Martín de Diego, E. Martínez, E. Padrón: Reduction of symplectic Lie algebroids by a Lie subalgebroid and a symmetry Lie group, *Symmetry, Integrability and Geometry: Methods and Applications* **3** (2007) 049, 28 pages

Cariñena J.F., Nunes Da Costa J.M., Santos, P., Reduction of Lie algebroid structures *Int. J. Geom. Methods Mod. Phys.* V.2 (2005) (5) 965-991

$$(A, [\![\cdot,\cdot]\!], \rho) \text{ Lie algebroid over } M$$

$$(A^*, [\![\ ,\ ]\!]_*, \rho_*) \text{ Lie algebroid over } M$$

$$\Downarrow$$

(A, A^*) is a *Lie bialgebroid* if

$$d^{A^*}[\![X, Y]\!] = [\![X, d^{A^*}Y]\!] - [\![Y, d^{A^*}X]\!], \ \ \forall X, Y \in \Gamma(A)$$

K. Mackenzie, P. Xu: Lie bialgebroids and Poisson groupoids, *Duke Math. J.*, **73** (1994), 415-452

(A, A^*) a Lie bialgebroid on M

$$\Downarrow$$

$\{\cdot,\cdot\}_M : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a Poisson bracket on M

$$\{f, h\}_M =< d^A f, d^{A^*} h > \quad \forall h \in C^\infty(M)$$

Examples

Lie bialgebras

$\mathfrak{g}$ Lie algebra + $\mathfrak{g}^*$ Lie algebra

$$d_*[\xi_1, \xi_2]_{\mathfrak{g}} = [\xi_1, d_*\xi_2]_{\mathfrak{g}} - [\xi_2, d_*\xi_1]_{\mathfrak{g}}$$

Poisson algebroids

$(A, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid on M, *A is Poisson* if $\Lambda \in \Gamma(\wedge^2 A)$

Properties:

$$[\![\Lambda, \Lambda]\!] = 0.$$

- $\#_\Lambda : A^* \to A, \quad \alpha_x \to (\#_\Lambda)_x(\alpha_x) = (i_\alpha \Lambda)(x)$ homomorphism of vector bundles
- $([\![\cdot,\cdot]\!]_*, \rho_*)$ Lie algebroid structure on A^*

$$[\![\alpha, \beta]\!]_* = \mathcal{L}^A_{\#_\Lambda \alpha}\beta - \mathcal{L}^A_{\#_\Lambda \beta}\alpha - d^A(\Lambda(\alpha,\beta)), \quad \rho_*(\alpha) = \rho(\#_\Lambda \alpha)$$

$$\Downarrow$$

(A, A^*) Lie bialgebroid

- Poisson bracket on M : $\{f, g\} = \Lambda(d^A f, d^A g)$
- The linear Poisson structure on A: Λ^c

SYMPLETIC ALGEBROIDS

$(A, [\![\cdot,\cdot]\!], \rho)$ Lie algebroid on M, *A is symplectic* if closed nondegenerated $\Omega \in \Gamma(\wedge^2 A^*)$

$$\flat_\Omega : \Gamma(A) \longrightarrow \Gamma(A^*)$$

$$\Lambda(\alpha, \beta) = \Omega(\flat_\Omega^{-1}(\alpha), \flat_\Omega^{-1}(\beta))$$

2.1 Actions of a Lie group on a Lie algebroid by complete lifts

- $(A, [\![\cdot,\cdot]\!], \rho)$ a Lie algebroid over M
- G a connected Lie group
- An action of G on A:

$$g \in G \Longrightarrow \begin{array}{ccc} A & \xrightarrow{\Phi_g} & A \\ \tau \downarrow & & \downarrow \tau \\ M & \xrightarrow{\phi_g} & M \end{array}$$ is a commutative diagram

(Φ_g, ϕ_g) is a vector bundle automorphism

- Φ is an action by complete lifts

$$\Updownarrow$$

$\exists \psi : \mathfrak{g} \to \Gamma(A)$ Lie algebra homomorphism

infinitesimal generator of ξ with respect to $\Phi = \xi_A = \psi(\xi)^c, \quad \forall \xi \in \mathfrak{g}$

SOME CONSEQUENCES:

1. infinitesimal generator of ξ with respect to the $\phi = \xi_M = \rho(\psi(\xi))$
2. (Φ_g, ϕ_g) is a Lie algebroid automorphism, $\forall g \in G$

2.1 Actions of a Lie group on a Lie algebroid by complete lifts

G a Lie group, $\quad \cdot : G \times G \longrightarrow G$ the multiplication of G

$$\Downarrow$$

TG is also a Lie group

$T\cdot : TG \times TG \to TG$ is the multiplication of TG

e the identity element of TG

$$\Downarrow$$

$$TG \cong G \times \mathfrak{g}$$

$$TG \to G \times \mathfrak{g}, \quad X_g \in T_g G \to (g, (T_g l_{g^{-1}})(X_g)) \in G \times \mathfrak{g}$$

$$(g, \xi) \cdot (g', \xi') = (gg', \xi' + Ad^G_{(g')^{-1}} \xi)$$

$(e, 0)$ the identity element of $G \times \mathfrak{g}$

The Lie algebra of $TG \cong G \times \mathfrak{g}$

$$T_{(e,0)}(G \times \mathfrak{g}) \cong \mathfrak{g} \times \mathfrak{g}$$

$$[(\xi, \eta), (\xi', \eta')]_{\mathfrak{g}\times\mathfrak{g}} = ([\xi, \xi']_{\mathfrak{g}}, [\xi, \eta']_{\mathfrak{g}} - [\xi', \eta]_{\mathfrak{g}})$$

2.1 Actions of a Lie group on a Lie algebroid by complete lifts

Theorem

Let $((\Phi, \phi), \psi)$ be an action of a connected Lie group G by complete lifts over the Lie algebroid $\tau : A \longrightarrow M$. Then, the map $\Phi^T : (G \times \mathfrak{g}) \times A \longrightarrow A$ given by

$$\Phi^T((g, \xi), a_x) = \Phi_g(a_x) + \Phi_g(\psi(\xi)(x)), \quad \text{for } (g, \eta) \in G \times \mathfrak{g}$$

defines an affine action of $G \times \mathfrak{g} \cong TG$ on A. Moreover, if $(\xi, \eta) \in \mathfrak{g} \times \mathfrak{g}$ then the infinitesimal generator $(\xi, \eta)_A$ of (ξ, η) with respect to the action Φ^T is

$$(\xi, \eta)_A = \psi(\xi)^c + \psi(\eta)^v$$

$$\Phi_g(\psi(Ad_{g^{-1}}\xi)(x)) = \psi(\xi)(\phi_g(x)),$$

for $g \in G, \xi \in \mathfrak{g}$ and $x \in M$

2.2 Momentum maps for Lie bialgebroids

(A, A^*) a Lie bialgebroid over M

$((\Phi, \phi), \psi)$ an action of the Lie group G on A by complete lifts
$J : M \longrightarrow \mathfrak{g}^*$ smooth equivariant map with respect to ϕ

$$Coad_g^G(J(x)) = J(\phi_g(x)), \quad \forall x \in M, \quad \forall g \in G$$

Definition 1

The action Φ is said to be Hamiltonian with momentum map J if

$$\psi(\xi) = d^{A^*}\widehat{J}_\xi, \qquad \text{for } \xi \in \mathfrak{g},$$

$$\widehat{J}_\xi : M \longrightarrow \mathbb{R}, \quad \widehat{J}_\xi(x) = \langle J(x), \xi \rangle, \text{for any } x \in M$$

CONSEQUENCE: $\Phi_g : A \longrightarrow A$ and $\Phi^*_{g^{-1}} : A^* \longrightarrow A^*$ Lie algebroid automorphisms over $\phi_g : M \longrightarrow M$

$$\Downarrow$$

Φ_g Lie bialgebroid automorphism

(A, A^*) be a Lie bialgebroid over M

Proposition

Let $\Phi : G \times A \to A$ be a Hamiltonian action of a connected Lie group G over $\phi : G \times M \to M$ with equivariant momentum map $J : M \to \mathfrak{g}^*$. Then, we have that:

(*i*) ϕ is an standard Poisson action of G on the Poisson manifold (M, Π_M) and $J : M \to \mathfrak{g}^*$ is a momentum map for ϕ.

(*ii*) $\Phi^T : TG \times A \to A$ is a Poisson action of the Lie group TG on A and the map $J^T : A \to (\mathfrak{g} \times \mathfrak{g})^* \cong \mathfrak{g}^* \times \mathfrak{g}^*$ given by

$$J^T(a) = ((dJ \circ \rho)(a), J(\tau(a)))$$

is an equivariant momentum map for the action Φ^T. Here, $dJ : TM \to \mathfrak{g}$ denotes the vector bundle morphism defined by $dJ_{|T_xM} = T_xJ$, for all $x \in M$.

3.1 Reduction of Lie bialgebroids by Lie bialgebroid automorphisms

(A, A^*) a Lie bialgebroid, $\Phi : G \times A \longrightarrow A$ be an action by Lie bialgebroid automorphisms of a connected Lie group G
Assume that the action ϕ is free and that M/G is smooth manifold such that the canonical projection is a surjective submersion

$$\Downarrow$$

$$A/G \longrightarrow M/G \text{ is a vector bundle}$$

$$\Gamma(A/G) = \{X \in \Gamma(A)/X \in \ G\text{-invariant}\}$$

$$\pi_A : A \longrightarrow A/G, \quad \pi_{A^*} : A^* \longrightarrow A^*/G \text{ are epimorphisms of vector bundles}$$

$$\{X \in \Gamma(A)/X \text{is } \pi_A\text{-projectable}\} = \{X \in \Gamma(A)/X \text{ is } G\text{-invariant section}\}$$

$$\{\alpha \in \Gamma(A^*)/\alpha \text{is } \pi_{A^*}\text{-projectable}\} = \{\alpha \in \Gamma(A^*)/\alpha \text{ is } G\text{-invariant section}\}$$

$$\ker((\pi_A)_{|A_x}) = \{0\}, \quad \ker((\pi_{A^*})_{|A_x^*}) = \{0\}$$

$$\Downarrow$$

The pair $(A/G, A^*/G)$ is a Lie bialgebroid over M/G

Reduction procedure for Lie bialgebroid analog of the Marsden Weintein reduction for symplectic manifolds

(A, A^*) Lie bialgebroid over $M \Rightarrow (A_\mu, A^*_\mu)$ Lie bialgebroid over $J^{-1}(\mu)/G_\mu$

Proposition

Let (A, A^*) be a Lie bialgebroid over M and $\Phi : G \times A \longrightarrow A$ be a Hamiltonian action of a connected Lie group G on A with momentum map $J : M \longrightarrow \mathfrak{g}^*$. Consider $\mu \in \mathfrak{g}^*$ such that $(0, \mu) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is a regular value of $J^T : A \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$. Then, we have that:

1. $(J^T)^{-1}(0, \mu)$ is a Lie subalgebroid of A over $J^{-1}(\mu)$.
2. The restriction ψ_μ of ψ to the isotropy algebra $\mathfrak{g}_\mu$ of μ with respect to the coadjoint action takes values in $\Gamma((J^T)^{-1}(0, \mu))$.
3. The isotropy Lie group G_μ of μ with respect to the coadjoint action acts on $(J^T)^{-1}(0, \mu)$ by complete lifts with respect to $\psi_\mu : \mathfrak{g}_\mu \longrightarrow \Gamma((J^T)^{-1}(0, \mu))$.
4. The action of G_μ on the Lie subalgebroid $(J^T)^{-1}(0, \mu)$ induces an affine action Φ^T_μ of TG_μ on this subalgebroid.

3.2 Reduction of Lie bialgebroids with momentum map

suppose

- The action of G_μ on $J^{-1}(\mu)$ is free
- The space of orbits $J^{-1}(\mu)/G_\mu$ is a smooth manifold such that the projection $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is a surjective submersion

$$\Downarrow$$

$J^{-1}(\mu)/G_\mu$ is a reduced Poisson manifold

Theorem

Let (A, A^*) be a Lie bialgebroid over M and $\Phi : G \times A \to A$ be a Hamiltonian action of a Lie group G on (A, A^*) with momentum map $J : M \to \mathfrak{g}^*$. Assume that μ is an element of $\mathfrak{g}^*$ such that $(0, \mu)$ is a regular value for $J^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*$ and that the space of orbits $J^{-1}(\mu)/G_\mu$ is a quotient manifold. Then,

(i) The space of orbits A_μ of the action of TG_μ on $(J^T)^{-1}(0, \mu)$ is a Lie algebroid over $J^{-1}(\mu)/G_\mu$.

(ii) The dual bundle $A^*_\mu \to J^{-1}(\mu)/G_\mu$ is endowed with a Lie algebroid structure.

(iii) The pair (A_μ, A^*_μ) is a Lie bialgebroid over $J^{-1}(\mu)/G_\mu$.

Sketch of the proof

(*i*) The Lie algebroid structure over $A_\mu = (J^T)^{-1}(0,\mu)/TG_\mu$

$$\psi_\mu(\mathfrak{g}_\mu) \text{ Lie subalgebroid of } (J^T)^{-1}(0,\mu) \text{ over } J^{-1}(\mu)$$

$$+$$

$$G_\mu \text{ acts by Lie algebroid automorphisms on } (J^T)^{-1}(0,\mu) \text{ and } \psi_\mu(\mathfrak{g}_\mu)$$

$$\Downarrow$$

induce Lie algebroid structures on $(J^T)^{-1}(0,\mu)/G_\mu$ and $\psi_\mu(\mathfrak{g}_\mu)/G_\mu$ such that $\psi_\mu(\mathfrak{g}_\mu)/G_\mu$ is a Lie subalgebroid of $(J^T)^{-1}(0,\mu)/G_\mu$

$$\Downarrow$$

$$\psi_\mu(\mathfrak{g}_\mu)/G_\mu \text{ is an ideal of } (J^T)^{-1}(0,\mu)/G_\mu$$

$$\Downarrow$$

$$((J^T)^{-1}(0,\mu)/G_\mu)/(\psi_\mu(\mathfrak{g}_\mu)/G_\mu) \text{ Lie algebroid over } J^{-1}(\mu)/G_\mu$$

$$\Downarrow$$

$$A_\mu := (J^T)^{-1}(0,\mu)/TG_\mu \cong ((J^T)^{-1}(0,\mu)/G_\mu)/(\psi_\mu(\mathfrak{g}_\mu)/G_\mu)$$

Sketch of the proof

$$A_\mu := (J^T)^{-1}(0,\mu)/TG_\mu \cong ((J^T)^{-1}(0,\mu)/G_\mu)/(\psi_\mu(\mathfrak{g}_\mu)/G_\mu)$$

$$\Gamma(A_\mu) \cong \frac{\Gamma((J^T)^{-1}(0,\mu))^{G_\mu}}{\Gamma(\psi_\mu(\mathfrak{g}_\mu))^{G_\mu}}$$

$\Gamma((J^T)^{-1}(0,\mu))^{G_\mu}$ the space of G_μ-invariant section on $(J^T)^{-1}(0,\mu)$

$\Gamma(\psi_\mu(\mathfrak{g}_\mu))^{G_\mu}$ the space of G_μ-invariant section on $\psi_\mu(\mathfrak{g}_\mu)$

Lie algebroid structure $([\![\cdot,\cdot]\!]_{A_\mu}, \rho_{A_\mu})$ is characterized by

$$[\![[X],[Y]]\!]_{A_\mu} = [[\![X,Y]\!]],$$

$$\rho_{A_\mu}([X]) = T\pi_\mu \circ \rho(X), \quad \text{for} X, Y \in \Gamma((J^T)^{-1}(0,\mu))^{G_\mu}.$$

Sketch of the proof

(ii) The Lie algebroid structure over A_μ^*

To prove that there exists a Lie algebroid structure on A_μ^*, we will show that there exists a linear Poisson structure on A_μ

Φ^T is a Hamilton action of TG on (A, Π_A) with momentum map $J^T : A \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*$

$$\Downarrow$$

$$\exists \text{ reduced Poisson structure } \Pi_{A_\mu} \text{ on } (J^T)^{-1}(0,\mu)/(TG)_{(0,\mu)}$$
$$(TG)_{(0,\mu)} = TG_\mu$$

$$\text{the Poisson bracket } \{\cdot,\cdot\}_{\Pi_{A_\mu}} \text{ is linear}$$

$$\Downarrow$$

$$\text{Lie algebroid structure } ([\![\cdot,\cdot]\!]_{A_\mu^*}, \rho_{A_\mu^*})$$

Sketch of the proof

Lie algebroid structure $(\llbracket\cdot,\cdot\rrbracket_{A_\mu^*}, \rho_{A_\mu^*})$

$\alpha_\mu \in \Gamma(A_\mu^*) \Rightarrow \exists\ \alpha \in \Gamma(A^*)$ such that

$$\alpha(\psi(\xi)) = 0 \quad \mathcal{L}^A_{\psi(\xi)}\alpha = 0, \quad \widetilde{\iota}_\mu^*(\alpha) = \widetilde{\pi}_\mu^*(\alpha_\mu) \text{ for all } \xi \in \mathfrak{g}$$

$\widetilde{\iota}_\mu : (J^T)^{-1}(0,\mu) \longrightarrow A$ is the inclusion,

$\widetilde{\pi}_\mu : (J^T)^{-1}(0,\mu) \longrightarrow A_\mu$ is the canonical projection

$$\widetilde{\pi}_\mu^*(\llbracket\alpha_\mu, \beta_\mu\rrbracket_{A_\mu^*}) = \widetilde{\iota}_\mu^*(\llbracket\alpha, \beta\rrbracket_*),$$

for $\alpha_\mu, \beta_\mu \in \Gamma(A_\mu^*)$

$$\rho_{A_\mu^*}(\alpha_\mu)(f_M) \circ \pi_\mu = \rho_*(\alpha)(f) \circ \iota_\mu,$$

for $f_\mu \in C^\infty(J^{-1}(\mu)/G_\mu)$,

$f : M \longrightarrow$ is a real function on M such that $f_\mu \circ \pi_\mu = f \circ \iota_\mu$, with $\iota_\mu : J^{-1}(\mu) \longrightarrow J^{-1}(\mu)/G_\mu$ the canonical inclusion

3.3 Examples: Poisson Lie algebroid

$(A, [\![\cdot,\cdot]\!], \rho, \Lambda)$ Poisson Lie algebroid over M

$\Downarrow$

(A, A^*) is a Lie bialgebroid

$+$

$\Phi : G \times A \to A$ Hamiltonian action of a connected Lie group G over A with momentum map $J : M \to \mathfrak{g}^*$ and associated Lie algebra morphism $\psi : \mathfrak{g} \to \Gamma(A)$

$\Downarrow$

$$\psi(\xi) = \mathcal{H}^{\Lambda}_{\widehat{J(\xi)}}, \quad \text{for all } \xi \in \mathfrak{g}$$

$$(\Phi_g)_*(\Lambda) = \Lambda, \quad \text{for all } g \in G$$

$(0,\mu)$ a regular value of $J^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*$

$(A_\mu, A_\mu^*) = ((J^T)^{-1}(0,\mu)/TG_\mu, ((J^T)^{-1}(0,\mu)/TG_\mu)^*)$ is a Lie bialgebroid over $J^{-1}(\mu)/G_\mu$

$$\Downarrow$$

A_μ admits a linear Poisson structure (the reduced Poisson structure on Λ^c with respect to the action Φ^T and momentum map J^T)

$$\Lambda_\mu \in \Gamma(\wedge^2 A_\mu)$$

$$\Lambda_\mu(\alpha_\mu, \beta_\mu) \circ \pi_\mu = \Lambda(\alpha, \beta) \circ i_\mu \text{ for all } \alpha_\mu, \beta_\mu \in \Gamma(A_\mu)$$

where α, β are TG-invariant sections of A^* satisfying

$$\begin{array}{lll} \widetilde{\pi}_\mu(\alpha_\mu) = \widetilde{i}_\mu^* \alpha, & \mathcal{L}^A_{\psi(\xi)} \alpha = 0, & \alpha(\psi(\xi)) = 0, \\ \widetilde{\pi}_\mu(\beta_\mu) = \widetilde{i}_\mu^* \beta, & \mathcal{L}^A_{\psi(\xi)} \beta = 0, & \beta(\psi(\xi)) = 0 \text{ for all } \xi \in \mathfrak{g} \end{array}$$

$\widetilde{\pi}_\mu : (J^T)^{-1}(0,\mu) \to A_\mu \quad \pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$

$\widetilde{i}_\mu : (J^T)^{-1}(0,\mu) \to A \quad i_\mu : J^{-1}(\mu) \to M$

Poisson Lie algebroids

Let $(A, [\![\cdot,\cdot]\!], \rho, \Lambda)$ be a Poisson Lie algebroid over M, $\Phi : G \times A \to A$ be a Hamiltonian action of a connected Lie group G over A with momentum map $J : M \to \mathfrak{g}^*$ and associated Lie algebra morphism $\psi : \mathfrak{g} \to \Gamma(A)$. If $(0, \mu)$ is a regular value of $J^T : A \to (\mathfrak{g} \times \mathfrak{g})^*$, then $A_\mu = (J^T)^{-1}(0, \mu)/TG_\mu$ admits a Poisson Lie algebroid structure Λ_μ over $J^{-1}(\mu)/G_\mu$.

$(A, [\![\cdot,\cdot]\!], \rho, \Omega)$ a symplectic Lie algebroid over M

Λ the corresponding Poisson 2-section

$\Phi : G \times A \to A$ a Hamiltonian action of a connected Lie group G over the induced Lie bialgebroid (A, A^*) with momentum map $J : M \to \mathfrak{g}^*$ and associated Lie algebra morphism $\psi : \mathfrak{g} \to \Gamma(A)$

$$\Downarrow$$

$$i_{\mathcal{H}^{\Omega}_{\widehat{J}_\xi}} \Omega = d^A \widehat{J}_\xi .$$

$$\Downarrow$$

$$\Phi_g^*(\Omega) = \Omega, \quad \forall g \in G,$$

3.3 Examples: Symplectic Lie algebroids

$$\mu \in \mathfrak{g}^* \text{ such that } (0,\mu) \text{ is a regular value of } J^T : M \to \mathfrak{g}^* \times \mathfrak{g}^*$$
$$\Downarrow$$
$$A_\mu \text{ is a Poisson Lie algebroid}$$

$$i_\mu : (J^T)^{-1}(0,\mu) \to A$$

2-section $\widetilde{\Omega}_\mu = i_\mu^*(\Omega)$ on the Lie subalgebroid $(J^T)^{-1}(0,\mu) \to J^{-1}(\mu)$

For all $\widetilde{X_\mu}, \widetilde{Y_\mu} \in \Gamma((J^T)^{-1}(0,\mu))$,

$\widetilde{\Omega}_\mu(\widetilde{X_\mu}, \widetilde{Y_\mu})$ is a π_μ-basic function

$$\Downarrow$$

For all $X_\mu, Y_\mu \in \Gamma(A_\mu) \Rightarrow \exists$ a function $\Omega_\mu(X_\mu, Y_\mu)$ on $J^{-1}(\mu)/G_\mu$ such that

$$\Omega_\mu(X_\mu, Y_\mu) \circ \pi_\mu = \widetilde{\Omega}_\mu(X_\mu, Y_\mu)$$

3.3 Examples: Symplectic Lie algebroids

Symplectic Lie algebroids

Let $(A, [\![\cdot,\cdot]\!], \rho, \Omega)$ be a symplectic Lie algebroid and $\Phi : G \times A \to A$ be a Hamilton action with momentum map $J : M \to \mathfrak{g}^*$ and associated Lie homomorphism $\psi : \mathfrak{g} \to \Gamma(A)$. If μ is an element of $\mathfrak{g}^*$ such that $(0, \mu)$ is a regular of J^T, then $A_\mu = (J^T)^{-1}(0, \mu)/TG_\mu$ is a symplectic Lie algebroid over $J^{-1}(\mu)/G_\mu$

(M, Λ) Poisson manifold $\quad + \quad (\mathfrak{g}, \mathfrak{g}^*)$ Lie bialgebra

$\psi : \mathfrak{g}^* \to \mathfrak{X}(M)$ representation

The Lie bialgebroid $TM \oplus_M M \times \mathfrak{g} \to M$ **over** M

$$(TM, [\cdot,\cdot], 1_{TM}) \quad + \quad (M \times \mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$$

$$\mathfrak{X}(M) \times C^\infty(M, \mathfrak{g}) \to C^\infty(M, \mathfrak{g}) \qquad (X, \xi) \mapsto X(\xi)$$

$$C^\infty(M, \mathfrak{g}) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad (\xi, X) \mapsto 0$$

$$\Downarrow$$

$(TM \oplus_M M \times \mathfrak{g}, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid

$\rho = pr_1 : TM \oplus_M M \times \mathfrak{g} \to TM$

$[\![(X_1, \xi_1), (X_2, \xi_2)]\!] = ([X_1, X_2], [\xi_1, \xi_2]_{\mathfrak{g}} + X_1(\xi_2) - X_2(\xi_1))$

3.3 Examples: Another Example

(M,Λ) Poisson manifold $\quad + \quad (\mathfrak{g},\mathfrak{g}^*)$ Lie bialgebra

$\psi : \mathfrak{g}^* \to \mathfrak{X}(M)$ representation

$$(T^*M, [\cdot,\cdot]_\Lambda, \#_\Lambda) \quad + \quad (M\times\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*}, \psi)$$

$\Omega^1(M)\times C^\infty(M,\mathfrak{g}^*) \to C^\infty(M,\mathfrak{g}^*) \qquad (\alpha,\eta)\mapsto -\#_\Lambda(\alpha)(\eta) + coad_{\psi^*(\alpha)}\eta$

$D : C^\infty(M,\mathfrak{g}^*)\times\Omega^1(M) \to C^\infty(M,\mathfrak{g}) \quad D(\eta,\alpha)(X) = -(\mathcal{L}_{\psi(\eta)}\alpha)(X) + \psi(X(\eta))$

$$\Downarrow$$

$(T^*M\oplus_M M\times\mathfrak{g}^*, [\![\cdot,\cdot]\!]_*, \rho_*)$ is a Lie algebroid

$\rho_* = T^*M\oplus_M M\times\mathfrak{g}^* \to TM \qquad \rho_*(\alpha,\eta) = \#_\Lambda\alpha + \psi(\eta)$

$[(\alpha_1,\eta_1),(\alpha_2,\eta_2)] =$
$([\alpha_1,\alpha_2]_\Lambda + \mathcal{L}_{\psi(\eta_1)}\alpha_2 - \mathcal{L}_{\psi(\eta_2)}\alpha_1, [\eta_1,\eta_2]_{\mathfrak{g}^*} + coad_{\psi^*(\alpha_1)}\eta_2 - coad_{\psi^*(\alpha_2)}(\eta_1))$

(M,Λ) Poisson manifold $\quad + \quad$ $(\mathfrak{g},\mathfrak{g}^*)$ Lie bialgebra

$\psi : \mathfrak{g}^* \to \mathfrak{X}(M)$ representation

The Lie bialgebroid $TM \oplus_M M \times \mathfrak{g} \to M$ **over** M

$\Phi : G \times M \to M$ Poisson action with momentum map $J : M \to \mathfrak{g}^*$

$$\widetilde{\Phi} : G \times TM \oplus_M M \times \mathfrak{g} \to TM \oplus_M M \times \mathfrak{g}, \quad \widetilde{\Phi}(h, X, \xi) = (T\Phi_h(X), -Ad_h \psi^*(d\widehat{J}_\xi))$$

$$\widetilde{\Psi} : \mathfrak{g} \to \mathfrak{X}(M) \times C^\infty(M,\mathfrak{g}) \quad d_*\widehat{J}_\xi = (\#_\Lambda d\widehat{J}_\xi, -\psi^*(d\widehat{J}_\xi))$$

$$\Downarrow$$

$(\widetilde{\Psi}(\xi))^c \equiv$ Infinitesimal generator of ξ with respect $\widetilde{\Phi}$

$(0,0)$ is a regular value for J^T

$$\Downarrow$$

$$(J^T)^{-1}(0,0) = T(J^{-1}(0)) \oplus_{J^{-1}(0)} J^{-1}(0) \times \mathfrak{g}$$

$(TJ^{-1}(0) \oplus J^{-1}(0) \times \mathfrak{g})/TG \to J^{-1}(0)/G$ is a Lie bialgebroid

Thanks !!!!