# Lagrangian and Hamiltonian mechanics on Lie algebroids 

Mechanics and Lie Algebroids

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Lisbon, September 112007

## Abstract

I will review the most relevant ideas and results about mechanical systems defined on Lie algebroids.

## Lie Algebroids

A Lie algebroid structure on the vector bundle $\tau: E \rightarrow M$ is given by
$\square$ a Lie algebra structure $(\operatorname{Sec}(E),[]$,$) on the set of sections of E$, and
$\square$ a morphism of vector bundles $\rho: E \rightarrow T M$ over the identity, such that

$$
\begin{aligned}
& \triangleright \rho([\sigma, \eta])=[\rho(\sigma), \rho(\eta)] \\
& \quad \triangleright[\sigma, f \eta]=f[\sigma, \eta]+(\rho(\sigma) f) \eta, \\
& \text { where } \rho(\sigma)(m)=\rho(\sigma(m)) .
\end{aligned}
$$

The first condition is actually a consequence of the second and the Jacobi identity.

## Examples

- Tangent bundle.
$E=T M$,
$\rho=\mathrm{id}$,
$[]=$, bracket of vector fields.
- Integrable subbundle.
$E \subset T M$, integrable distribution
$\rho=i$, canonical inclusion
$[]=$, restriction of the bracket to vector fields in $E$.
- Lie algebra.
$E=\mathfrak{g} \rightarrow M=\{e\}$, Lie algebra (fiber bundle over a point)
$\rho=0$, trivial map (since $T M=\left\{0_{e}\right\}$ )
$[]=$, the bracket in the Lie algebra.

Atiyah algebroid.
Let $\pi: Q \rightarrow M$ a principal $G$-bundle.
$E=T Q / G \rightarrow M$, (Sections are equivariant vector fields)
$\rho([v])=T \pi(v)$ induced projection map
$[]=$, bracket of equivariant vectorfields (is equivariant).

Transformation Lie algebroid.
Let $\Phi: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on $M$.
$E=M \times \mathfrak{g} \rightarrow M$,
$\rho(m, \xi)=\Phi(\xi)(m)$ value of the fundamental vectorfield
$[]=$, induced by the bracket on $\mathfrak{g}$.

## Mechanics on Lie algebroids

Lie algebroid $E \rightarrow M$.
$L \in C^{\infty}(E)$ or $H \in C^{\infty}\left(E^{*}\right)$
$\square E=T M \rightarrow M$ Standard classical Mechanics
$\square E=\mathcal{D} \subset T M \rightarrow M$ (integrable) System with holonomic constraints
$\square E=T Q / G \rightarrow M=Q / G$ System with symmetry (ej. Classical particle on a Yang-Mils field)
$\square E=\mathfrak{g} \rightarrow\{e\}$ System on a Lie algebra (ej. Rigid body)
$\square E=M \times \mathfrak{g} \rightarrow M$ System on a semidirect product (ej. heavy top)

## Structure functions

A local coordinate system $\left(x^{i}\right)$ in the base manifold $M$ and a local basis of sections $\left(e_{\alpha}\right)$ of $E$, determine a local coordinate system $\left(x^{i}, y^{\alpha}\right)$ on $E$.

The anchor and the bracket are locally determined by the local functions $\rho_{\alpha}^{i}(x)$ and $C_{\beta \gamma}^{\alpha}(x)$ on $M$ given by

$$
\begin{aligned}
\rho\left(e_{\alpha}\right) & =\rho_{\alpha}^{i} \frac{\partial}{\partial x^{i}} \\
{\left[e_{\alpha}, e_{\beta}\right] } & =C_{\alpha \beta}^{\gamma} e_{\gamma} .
\end{aligned}
$$

The function $\rho_{\alpha}^{i}$ and $C_{\beta \gamma}^{\alpha}$ satisfy some relations due to the compatibility condition and the Jacobi identity which are called the structure equations:

$$
\begin{gathered}
\rho_{\alpha}^{j} \frac{\partial \rho_{\beta}^{i}}{\partial x^{j}}-\rho_{\beta}^{j} \frac{\partial \rho_{\alpha}^{i}}{\partial x^{j}}=\rho_{\gamma}^{i} C_{\alpha \beta}^{\gamma} \\
\sum_{\operatorname{cyclic}(\alpha, \beta, \gamma)}\left[\rho_{\alpha}^{i} \frac{\partial C_{\beta \gamma}^{\nu}}{\partial x^{i}}+C_{\beta \gamma}^{\mu} C_{\alpha \mu}^{\nu}\right]=0
\end{gathered}
$$

## Lagrange's equations

Given a function $L \in C^{\infty}(E)$, we define a dynamical system on $E$ by means of a system of differential equations, which in local coordinates reads

$$
\begin{aligned}
& \frac{d}{d t}\left(\frac{\partial L}{\partial y^{\alpha}}\right)+\frac{\partial L}{\partial y^{\gamma}} C_{\alpha \beta}^{\gamma} y^{\beta}=\rho_{\alpha}^{i} \frac{\partial L}{\partial x^{i}} \\
& \dot{x}^{i}=\rho_{\alpha}^{i} y^{\alpha} .
\end{aligned}
$$

The equation $\dot{x}^{i}=\rho_{\alpha}^{i} y^{\alpha}$ is the local expression of the admissibility condition: A curve $a: \mathbb{R} \rightarrow E$ is said to be admissible if

$$
\rho \circ a=\frac{d}{d t}(\tau \circ a) .
$$

## Exterior differential

## On 0-forms

$$
d f(\sigma)=\rho(\sigma) f
$$

On $p$-forms $(p>0)$

$$
\begin{aligned}
& d \omega\left(\sigma_{1}, \ldots, \sigma_{p+1}\right)= \\
& \quad=\sum_{i=1}^{p+1}(-1)^{i+1} \rho\left(\sigma_{i}\right) \omega\left(\sigma_{1}, \ldots, \widehat{\sigma}_{i}, \ldots, \sigma_{p+1}\right) \\
& \quad-\sum_{i<j}(-1)^{i+j} \omega\left(\left[\sigma_{i}, \sigma_{j}\right], \sigma_{1}, \ldots, \widehat{\sigma}_{i}, \ldots, \widehat{\sigma}_{j}, \ldots, \sigma_{p+1}\right)
\end{aligned}
$$

## Exterior differential-local

Locally determined by

$$
d x^{i}=\rho_{\alpha}^{i} e^{\alpha}
$$

and

$$
d e^{\alpha}=-\frac{1}{2} C_{\beta \gamma}^{\alpha} e^{\beta} \wedge e^{\gamma}
$$

The structure equations are

$$
d^{2} x^{i}=0 \quad \text { and } \quad d^{2} e^{\alpha}=0
$$

## Admissible maps and Morphisms

A bundle map $\Phi$ between $E$ and $E^{\prime}$ is said to be admissible map if

$$
\Phi^{\star} d f=d \Phi^{\star} f
$$

A bundle map $\Phi$ between $E$ and $E^{\prime}$ is said to be a morphism of Lie algebroids if

$$
\Phi^{\star} d \theta=d \Phi^{\star} \theta .
$$

Obviously every morphism is an admissible map.

## Variational description

Solutions of Lagrange equations are the critical points of the action functional

$$
\mathcal{S}(a)=\int_{t_{0}}^{t_{1}} L(a(t)) d t
$$

defined on an adequate infinite dimensional manifold of curves whose tangent vectors (variations of the curve $a(t)$ ) are of the form

$$
\delta x^{i}=\rho_{\alpha}^{i} \sigma^{\alpha} \quad \delta y^{\alpha}=\dot{\sigma}^{\alpha}+C_{\beta \gamma}^{\alpha} a^{\beta} \sigma^{\gamma}
$$

for some curve $\sigma(t)$ such that $\tau(a(t))=\tau(\sigma(t))$.
We use the notation

$$
\Xi_{a}(\sigma)=\rho_{\alpha}^{i} \sigma^{\alpha} \frac{\partial}{\partial x^{i}}+\left[\dot{\sigma}^{\alpha}+C_{\beta \gamma}^{\alpha} a^{\beta} \sigma^{\gamma}\right] \frac{\partial}{\partial y^{\alpha}} .
$$

for the variation vector field.

## E-Homotopy

(Crainic and Fernandes 2003)
Let $I=[0,1]$ and $J=\left[t_{0}, t_{1}\right]$, and $(s, t)$ coordinates in $\mathbb{R}^{2}$.

Definition 1 Two E-paths $a_{0}$ and $a_{1}$ are said to be E-homotopic if there exists a morphism of Lie algebroids $\Phi: T I \times T J \rightarrow E$ such that

$$
\begin{aligned}
\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(0, t)}\right) & =a_{0}(t) & & \Phi\left(\left.\frac{\partial}{\partial s}\right|_{\left(s, t_{0}\right)}\right)=0 \\
\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(1, t)}\right) & =a_{1}(t) & & \Phi\left(\left.\frac{\partial}{\partial s}\right|_{\left(s, t_{1}\right)}\right)=0
\end{aligned}
$$

It follows that the base map is a homotopy (in the usual sense) with fixed endpoints between the base paths.

## Homotopy foliation

The set of $E$-paths

$$
\mathcal{A}(J, E)=\left\{a: J \rightarrow E \left\lvert\, \rho \circ a=\frac{d}{d t}(\tau \circ a)\right.\right\}
$$

is a Banach submanifold of the Banach manifold of $C^{1}$-paths whose base path is $C^{2}$. Every $E$-homotopy class is a smooth Banach manifold and the partition into equivalence classes is a smooth foliation. The distribution tangent to that foliation is given by $a \in \mathcal{A}(J, E) \mapsto F_{a}$ where

$$
F_{a}=\left\{\Xi_{a}(\sigma) \in T_{a} \mathcal{A}(J, E) \mid \sigma\left(t_{0}\right)=0 \quad \text { and } \quad \sigma\left(t_{1}\right)=0\right\}
$$

and the codimension of $F$ is equal to $\operatorname{dim}(E)$. The $E$-homotopy equivalence relation is regular if and only if the Lie algebroid is integrable (i.e. it is the Lie algebroid of a Lie groupoid).

## Variational description

The $E$-path space with the appropriate differential structure is

$$
\mathcal{P}(J, E)=\mathcal{A}(J, E)_{F} \text {. }
$$

Fix $m_{0}, m_{1} \in M$ and consider the set of $E$-paths with such base endpoints

$$
\mathcal{P}(J, E)_{m_{0}}^{m_{1}}=\left\{a \in \mathcal{P}(J, E) \mid \tau\left(a\left(t_{0}\right)\right)=m_{0} \quad \text { and } \quad \tau\left(a\left(t_{1}\right)\right)=m_{1}\right\}
$$

It is a Banach submanifold of $\mathcal{P}(J, E)$.

Theorem 1 Let $L \in C^{\infty}(E)$ be a Lagrangian function on the Lie algebroid $E$ and fix two points $m_{0}, m_{1} \in M$. Consider the action functional $S: \mathcal{P}(J, E) \rightarrow \mathbb{R}$ given by $S(a)=\int_{t_{0}}^{t_{1}} L(a(t)) d t$. The critical points of $S$ on the Banach manifold $\mathcal{P}(J, E)_{m_{0}}^{m_{1}}$ are precisely those elements of that space which satisfy Lagrange's equations.

## Morphisms and reduction

Given a morphism of Lie algebroids $\Phi: E \rightarrow E^{\prime}$ the induced map $\hat{\Phi}: \mathcal{P}(J, E) \rightarrow \mathcal{P}\left(J, E^{\prime}\right)$ given by $\hat{\Phi}(a)=\Phi \circ a$ is smooth and $T \hat{\Phi}\left(\Xi_{a}(\sigma)\right)=$ $\Xi_{\Phi \circ a}(\Phi \circ \sigma)$.
$\square$ If $\Phi$ is fiberwise surjective then $\hat{\Phi}$ is a submersion.
$\square$ If $\Phi$ is fiberwise injective then $\hat{\Phi}$ is a immersion.
Consider two Lagrangians $L \in C^{\infty}(E), L^{\prime} \in C^{\infty}\left(E^{\prime}\right)$ and $\Phi: E \rightarrow E^{\prime}$ a morphism of Lie algebroids such that $L^{\prime} \circ \Phi=L$.

Then, the action functionals $S$ on $\mathcal{P}(J, E)$ and $S^{\prime}$ on $\mathcal{P}\left(J, E^{\prime}\right)$ are related by $\hat{\Phi}$, that is

$$
S^{\prime} \circ \hat{\Phi}=S
$$

Theorem 2 (Reduction) Let $\Phi: E \rightarrow E^{\prime}$ be a fiberwise surjective morphism of Lie algebroids. Consider a Lagrangian L on E and a Lagrangian $L^{\prime}$ on $E^{\prime}$ such that $L=L^{\prime} \circ \Phi$. If $a$ is a solution of Lagrange's equations for $L$ then $a^{\prime}=\Phi \circ a$ is a solution of Lagrange's equations for $L^{\prime}$.

Proof. From $S^{\prime} \circ \hat{\Phi}=S$ we get

$$
\langle d S(a), v\rangle=\left\langle d S^{\prime}(\hat{\Phi}(a)), T_{a} \hat{\Phi}(v)\right\rangle=\left\langle d S^{\prime}\left(a^{\prime}\right), T_{a} \hat{\Phi}(v)\right\rangle
$$

Since $T_{a} \Phi(v)$ surjective, if $d S(a)=0$ then $d S^{\prime}\left(a^{\prime}\right)=0$.

Theorem 3 (Reconstruction) Let $\Phi: E \rightarrow E^{\prime}$ be a morphism of Lie algebroids. Consider a Lagrangian $L$ on $E$ and a Lagrangian $L^{\prime}$ on $E^{\prime}$ such that $L=L^{\prime} \circ \Phi$. If $a$ is an $E$-path and $a^{\prime}=\Phi \circ a$ is a solution of Lagrange's equations for $L^{\prime}$ then a itself is a solution of Lagrange's equations for $L$.

Proof. We have

$$
\langle d S(a), v\rangle=\left\langle d S^{\prime}\left(a^{\prime}\right), T_{a} \hat{\Phi}(v)\right\rangle
$$

If $d S^{\prime}\left(a^{\prime}\right)=0$ then $d S(a)=0$.

Theorem 4 (Reduction by stages) Let $\Phi_{1}: E \rightarrow E^{\prime}$ and $\Phi_{2}: E^{\prime} \rightarrow E^{\prime \prime}$ be fiberwise surjective morphisms of Lie algebroids. Let $L, L^{\prime}$ and $L^{\prime \prime}$ be Lagrangian functions on $E, E^{\prime}$ and $E^{\prime \prime}$, respectively, such that $L^{\prime} \circ \Phi_{1}=L$ and $L^{\prime \prime} \circ \Phi_{2}=L^{\prime}$. Then the result of reducing first by $\Phi_{1}$ and later by $\Phi_{2}$ coincides with the reduction by $\Phi=\Phi_{2} \circ \Phi_{1}$.

## Prolongation

Given a Lie algebroid $\tau: E \rightarrow M$ and a submersion $\mu: P \rightarrow M$ we can construct the $E$-tangent to $P$ (the prolongation of $P$ with respect to $E$ ). It is the vector bundle $\tau_{P}^{E}: \mathcal{T} E P \rightarrow P$ where the fibre over $p \in P$ is

$$
\mathcal{T}_{p}^{E} P=\left\{(b, v) \in E_{m} \times T_{p} P \mid T \mu(v)=\rho(b)\right\}
$$

where $m=\mu(p)$.
Redundant notation: $(p, b, v)$ for the element $(b, v) \in \mathcal{T}_{p}^{E} P$.
The bundle $\mathcal{T}^{E} P$ can be endowed with a structure of Lie algebroid. The anchor $\rho^{1}: \mathcal{T}^{E} P \rightarrow T P$ is just the projection onto the third factor $\rho^{1}(p, b, v)=v$. The bracket is given in terms of projectable sections $(\sigma, X),(\eta, Y)$

$$
[(\sigma, X),(\eta, Y)]=([\sigma, \eta],[X, Y])
$$

Prolongation of maps: If $\Psi: P \rightarrow P^{\prime}$ is a bundle map over $\varphi: M \rightarrow M^{\prime}$ and $\Phi: E \rightarrow E^{\prime}$ is a morphism over the same map $\varphi$ then we can define a morphism $\mathcal{T}^{\Phi} \Psi: \mathcal{T}^{E} P \rightarrow \mathcal{T}^{E^{\prime}} P^{\prime}$ by means of

$$
\mathcal{T}^{\Phi} \Psi(p, b, v)=\left(\Psi(p), \Phi(b), T_{p} \Psi(v)\right) .
$$

In particular, for $P=E$ we have the $E$-tangent to $E$

$$
\mathcal{T}_{a}^{E} E=\left\{(b, v) \in E_{m} \times T_{a} E \mid T \tau(v)=\rho(b)\right\}
$$

The structure of Lie algebroid in $\mathcal{T}^{E} E$ can be defined in terms of the brackets of vertical and complete lifts

$$
\left[\eta^{\mathrm{c}}, \sigma^{\mathrm{c}}\right]=[\sigma, \eta]^{\mathrm{c}}, \quad\left[\eta^{\mathrm{c}}, \sigma^{\mathrm{v}}\right]=[\sigma, \eta]^{\mathrm{v}} \quad \text { and } \quad\left[\eta^{\mathrm{v}}, \sigma^{\mathrm{v}}\right]=0
$$

## Geometric Lagrangian Mechanics

Associated to $L$ there is a section $\theta_{L}$ of $\left(\mathcal{T}^{E} E\right)^{*}$,

$$
\left\langle\theta_{L}, \eta^{c}\right\rangle=d_{\eta^{v}} L \quad \text { and } \quad\left\langle\theta_{L}, \eta^{v}\right\rangle=0 .
$$

Equivalent conditions:

$$
i_{\Gamma} \omega_{L}=d E_{L}
$$

with $\omega_{L}=-d \theta_{L}$ and $E_{L}=d_{\Delta} L-L$ the energy, or

$$
d_{\Gamma} \theta_{L}=d L
$$

with $\Gamma$ a SODE-section. (Martínez 2001)

## Poisson bracket

The dual $E^{*}$ of a Lie algebroid carries a canonical Poisson structure. In terms of linear and basic functions, the Poisson bracket is defined by

$$
\begin{aligned}
& \{\hat{\sigma}, \hat{\eta}\}=\widehat{[\sigma, \eta]} \\
& \{\hat{\sigma}, \tilde{g}\}=\rho(\sigma) g \\
& \{\tilde{f}, \tilde{g}\}=0
\end{aligned}
$$

for $f, g$ functions on $M$ and $\sigma, \eta$ sections of $E$.
Basic and linear functions are defined by

$$
\begin{gathered}
\tilde{f}(\mu)=f(m) \\
\hat{\sigma}(\mu)=\langle\mu, \sigma(m)\rangle \quad \text { for } \mu \in E_{m}^{*}
\end{gathered}
$$

In coordinates

$$
\left\{x^{i}, x^{j}\right\}=0 \quad\left\{\mu_{\alpha}, x^{j}\right\}=\rho_{\alpha}^{i} \quad\left\{\mu_{\alpha}, \mu_{\beta}\right\}=C_{\alpha \beta}^{\gamma} \mu_{\gamma} .
$$

## Hamiltonian formalism

Consider the prolongation $\mathcal{T}^{E} E^{*}$ of the dual bundle $\pi: E^{*} \rightarrow M$ :

$$
\mathcal{T}^{E} E^{*}=\left\{(\mu, a, W) \in E^{*} \times E \times T E^{*} \mid \mu=\tau_{E^{*}}(W) \quad \rho(a)=T \pi(W)\right\}
$$

There is a canonical symplectic structure $\Omega=-d \Theta$, where the 1 -form $\Theta$ is defined by

$$
\left\langle\Theta_{\mu},(\mu, a, W)\right\rangle=\langle\mu, a\rangle .
$$

In coordinates

$$
\Theta=\mu_{\alpha} \mathcal{X}^{\alpha}
$$

and

$$
\Omega=\mathcal{X}^{\alpha} \wedge \mathcal{P}_{\alpha}+\frac{1}{2} \mu_{\gamma} C_{\alpha \beta}^{\gamma} \mathcal{X}^{\alpha} \wedge \mathcal{X}^{\beta}
$$

The Hamiltonian dynamics is given by the vector field $\rho\left(\Gamma_{H}\right)$ associated to the section $\Gamma_{H}$ solution of the symplectic equation

$$
i_{\Gamma_{H}} \Omega=d H
$$

In coordinates, Hamilton equations are

$$
\frac{d x^{i}}{d t}=\rho_{\alpha}^{i} \frac{\partial H}{\partial \mu_{\alpha}} \quad \frac{d \mu_{\alpha}}{d t}=-\left(\mu_{\gamma} C_{\alpha \beta}^{\gamma} \frac{\partial H}{\partial \mu_{\beta}}+\rho_{\alpha}^{i} \frac{\partial H}{\partial x^{i}}\right) .
$$

The canonical Poisson bracket on $E^{*}$ can be re-obtained by means of

$$
\Omega(d F, d G)=\{F, G\}
$$

for $F, G \in C^{\infty}\left(E^{*}\right)$.
The equations of motion are Poisson

$$
\dot{F}=\{F, H\} .
$$

## Hamilton-Jacobi theory

Let $H \in C^{\infty}\left(E^{*}\right)$ a Hamiltonian function and $\Gamma_{H}$ the Hamiltonian section.
Theorem: Let $\alpha$ be a closed section of $E^{*}$ and let $\sigma=\mathcal{F}_{H} \circ \alpha$. The following conditions are equivalent
$\square$ If $m(t)$ is an integral curve of $\rho(\sigma)$ then $\mu(t)=\alpha(m(t))$ is a solution of the Hamilton equations.
$\square \alpha$ satisfies the equation $d(H \circ \alpha)=0$.

We can try $\alpha=d S$, for $S \in C^{\infty}(M)$ (but notice that closed $\neq$ exact, even locally). In such case if $H \circ d S=0$ then $\frac{d}{d t}(S \circ m)=L \circ \sigma \circ m$, or in other words

$$
S\left(m\left(t_{1}\right)\right)-S\left(m\left(t_{0}\right)\right)=\int_{t_{0}}^{t_{1}} L(\sigma(m(t))) d t
$$

## Lie groupoids

A groupoid over a set $M$ is a set $\boldsymbol{G}$ together with the following structural maps:
$\square$ A pair of maps (source) $s: G \rightarrow M$ and (target) $\boldsymbol{t}: \boldsymbol{G} \rightarrow M$.
$\square$ A partial multiplication $\boldsymbol{m}$, defined on the set of composable pairs

$$
\begin{aligned}
\boldsymbol{G}_{2} & =\{(g, h) \in G \times G \mid \boldsymbol{t}(g)=\boldsymbol{s}(h)\} . \\
& \triangleright \boldsymbol{s}(g h)=\boldsymbol{s}(g) \text { and } \boldsymbol{t}(g h)=\boldsymbol{t}(h) . \\
& \triangleright g(h k)=(g h) k .
\end{aligned}
$$

$\square$ An identity section $\boldsymbol{\epsilon}: M \rightarrow \boldsymbol{G}$ such that
$\triangleright \boldsymbol{\epsilon}(\boldsymbol{s}(g)) g=g$ and $g \boldsymbol{\epsilon}(\boldsymbol{t}(g))=g$.
$\square$ An inversion map $\boldsymbol{i}: G \rightarrow G$, to be denoted simply by $\boldsymbol{i}(g)=g^{-1}$, such that

$$
\triangleright g^{-1} g=\epsilon(\boldsymbol{t}(g)) \text { and } g g^{-1}=\boldsymbol{\epsilon}(\boldsymbol{s}(g)) .
$$



A groupoid is a Lie groupoid if $\boldsymbol{G}$ and $M$ are manifolds, all maps (source, target, inversion, multiplication, identity) are smooth, $s$ and $t$ are submersions (then $\boldsymbol{m}$ is a submersion, $\boldsymbol{\epsilon}$ is an embedding and $\boldsymbol{i}$ is a diffeomorphism).

## The Lie algebroid of a Lie groupoid

The Lie algebroid of a Lie groupoid $\boldsymbol{G}$ is the vector bundle $\tau: E \rightarrow M$ where $E_{m}=\operatorname{Ker}\left(T_{\boldsymbol{\epsilon}(m)} \boldsymbol{s}\right)$ with $\rho_{m}=T_{\boldsymbol{\epsilon}(m)} \boldsymbol{t}$.

The bracket is defined in terms of left-invariant vector fields.
Left and right translation:
$g \in \boldsymbol{G}$ with $\boldsymbol{s}(g)=m$ and $\boldsymbol{t}(g)=n$

$$
\begin{array}{ll}
l_{g}: s^{-1}(n) \rightarrow s^{-1}(m), & l_{g}(h)=g h \\
r_{g}: \boldsymbol{t}^{-1}(m) \rightarrow \boldsymbol{t}^{-1}(n), & r_{g}(h)=h g
\end{array}
$$

Every section $\sigma$ of $E$ can be extended to a left invariant vectorfield $\overleftarrow{\sigma} \in$ $\mathfrak{X}(\boldsymbol{G})$. The bracket of two sections of $E$ is defined by $[\sigma, \eta]=[\overleftarrow{\sigma}, \overleftarrow{\eta}]$.

## Examples

Pair groupoid.
$G=M \times M$ with $\boldsymbol{s}\left(m_{1}, m_{2}\right)=m_{1}$ and $\boldsymbol{t}\left(m_{1}, m_{2}\right)=m_{2}$.
Multiplication is $\left(m_{1}, m_{2}\right)\left(m_{2}, m_{3}\right)=\left(m_{1}, m_{3}\right)$
Identities $\boldsymbol{\epsilon}(m)=(m, m)$
Inversion $\boldsymbol{i}\left(m_{1}, m_{2}\right)=\left(m_{2}, m_{1}\right)$.
The Lie algebroid is $T M \rightarrow M$.

- Lie group.

A Lie group is a Lie groupoid over one point $M=\{e\}$. Every pair of elements is composable.
The Lie algebroid is just the Lie algebra.

Transformation groupoid.
Consider a Lie group $H$ acting on a manifold $M$ on the right. The set $\boldsymbol{G}=M \times H$ is a groupoid over $M$ with $\boldsymbol{s}(m, g)=m$ and $\boldsymbol{t}(m, g)=m g$. Multiplication is $\left(m, h_{1}\right)\left(m h_{1}, h_{2}\right)=\left(m, h_{1} h_{2}\right)$.
Identity $\boldsymbol{\epsilon}(m)=(m, e)$
Inversion $\boldsymbol{i}(m, h)=\left(m h, h^{-1}\right)$
The Lie algebroid is the transformation Lie algebroid $M \times \mathfrak{h} \rightarrow M$.

- Atiyah or gauge groupoid.

If $\pi: Q \rightarrow M$ is a principal $H$-bundle, then $(Q \times Q) / H$ is a groupoid over $M$, with source $\boldsymbol{s}\left(\left[q_{1}, q_{2}\right]\right)=\pi\left(q_{1}\right)$ and target $\boldsymbol{t}\left(\left[q_{1}, q_{2}\right]\right)=\pi\left(q_{2}\right)$.
Multiplication is $\left[q_{1}, q_{2}\right]\left[h q_{2}, q_{3}\right]=\left[h q_{1}, q_{3}\right]$.
Identity $\boldsymbol{\epsilon}(m)=[q, q]$
Inversion $\boldsymbol{i}\left(\left[q_{1}, q_{2}\right]\right)=\left[q_{2}, q_{1}\right]$
(An element of $(Q \times Q) / G$ can be identified with an equivariant map between fibers)

## Discrete Lagrangian Mechanics

A discrete Lagrangian on a Lie groupoid $\boldsymbol{G}$ is just a function $\boldsymbol{L}$ on $\boldsymbol{G}$. It defines a discrete dynamical system by mean of discrete Hamilton principle.
$\square$ Action sum: defined on composable sequences $\left(g_{1}, g_{2}, \cdots, g_{n}\right) \in \boldsymbol{G}_{n}$

$$
S\left(g_{1}, g_{2}, \ldots, g_{n}\right)=\boldsymbol{L}\left(g_{1}\right)+\boldsymbol{L}\left(g_{2}\right)+\cdots+\boldsymbol{L}\left(g_{n}\right)
$$

$\square$ Discrete Hamilton principle: Given $p \in \boldsymbol{G}$, a solution of a Lagrangian system is a critical point of the action sum on the set of composable sequences with product $p$, i.e. sequences $\left(g_{1}, g_{2} \cdots, g_{n}\right) \in \boldsymbol{G}_{n}$ such that $g_{1} g_{2} \cdots g_{n}=p$

## Discrete Euler-Lagrange equations

We can restrict to sequences of two elements $(g, h)$. Since $g h=p$ is fixed, variations are of the form $g \mapsto g \eta(t)$ and $h \mapsto \eta(t)^{-1} h$, with $\eta(t)$ a curve thought the identity at $m=\boldsymbol{t}(g)=\boldsymbol{s}(h)$ with $\dot{\eta}(0)=a \in E_{m}$. Then the discrete Euler-Lagrange equations are:

$$
\begin{aligned}
\langle\operatorname{DEL}(g, h), a\rangle & =\left.\frac{d}{d t}\left[\boldsymbol{L}(g \eta(t))+\boldsymbol{L}\left(\eta(t)^{-1} h\right)\right]\right|_{t=0} \\
& =\left\langle d^{0}\left(\boldsymbol{L} \circ l_{g}+\boldsymbol{L} \circ r_{h} \circ \boldsymbol{i}\right), a\right\rangle .
\end{aligned}
$$

## Simplecticity

In the case of the pair groupoid, it is well known that the algorithm defined by the discrete Euler-Lagrange equations is symplectic.

In the general case of a Lagrangian system on a Lie groupoid one can also define a symplectic section on an appropriate Lie algebroid which is conserved by the discrete flow. From this it follows that the algorithm is Poisson (In the standard sense).

Such appropriate Lie algebroid is called the prolongation of the Lie groupoid $\mathcal{P} G \rightarrow \boldsymbol{G}$, where

$$
\mathcal{P}_{g} \boldsymbol{G}=\operatorname{Ker}\left(T_{g} \boldsymbol{s}\right) \oplus \operatorname{Ker}\left(T_{g} \boldsymbol{t}\right)
$$

It can be seen isomorphic to

$$
\mathcal{P} \boldsymbol{G}=\{(a, g, b) \in E \times \boldsymbol{G} \times E \mid \tau(a)=\boldsymbol{s}(g) \quad \text { and } \quad \tau(b)=\boldsymbol{t}(g)\}
$$

where $\tau: E \rightarrow M$ is the Lie algebroid of $\boldsymbol{G}$.

## Cartan forms

Given a discrete Lagrangian $\boldsymbol{L} \in C^{\infty}(\boldsymbol{G})$ we define the Cartan 1-sections $\Theta_{L}^{-}$and $\Theta_{L}^{+}$of $\mathcal{P} G^{*}$ by

$$
\Theta_{L}^{-}(g)\left(X_{g}, Y_{g}\right)=-X_{g}(L), \quad \text { and } \quad \Theta_{L}^{+}(g)\left(X_{g}, Y_{g}\right)=Y_{g}(L),
$$

for each $g \in G$ and $\left(X_{g}, Y_{g}\right) \in V_{g} \beta \oplus V_{g} \alpha$.
The difference between them is

$$
d \boldsymbol{L}=\Theta_{\boldsymbol{L}}^{+}-\Theta_{\boldsymbol{L}}^{-}
$$

The Cartan 2-section is

$$
\Omega_{L}=-d \Theta_{L}^{+}=-d \Theta_{L}^{-}
$$

A Lagrangian is said to be regular if $\Omega_{L}$ is a symplectic section.

## Discrete evolution operator

For a regular Lagrangian there exists a locally unique map $\xi: \boldsymbol{G} \rightarrow \boldsymbol{G}$ such that it solves the discrete Euler-Lagrange equations

$$
\operatorname{DEL}(g, \xi(g))=0 \quad \text { for all } g \text { in an open } \mathcal{U} \subset \boldsymbol{G}
$$

One of such maps is said to be a discrete Lagrangian evolution operator.
Given a map $\xi: \boldsymbol{G} \rightarrow \boldsymbol{G}$ such that $s \circ \xi=\boldsymbol{t}$, there exists a unique vector bundle map $\mathcal{P} \xi: \mathcal{P} G \rightarrow \mathcal{P} G$, such that $\Phi=(\mathcal{P} \xi, \xi)$ is a morphism of Lie algebroids.

A map $\xi$ is a discrete Lagrangian evolution operator if and only if

$$
\Phi^{*} \Theta_{\boldsymbol{L}}^{-}-\Theta_{\boldsymbol{L}}^{-}=d \boldsymbol{L}
$$

If $\xi$ is a discrete Lagrangian evolution operator then it is symplectic, that is, $\Phi^{*} \Omega_{L}=\Omega_{L}$.

## Hamiltonian formalism

Define the discrete Legendre transformations $\mathcal{F}^{-} \boldsymbol{L}: \boldsymbol{G} \rightarrow E^{*}$ and $\mathcal{F}^{+} \boldsymbol{L}$ : $G \rightarrow E^{*}$ by

$$
\begin{aligned}
\left(\mathcal{F}^{-} L\right)(h)(a) & =-a\left(L \circ r_{h} \circ i\right), \quad \text { for } a \in E_{\boldsymbol{s}(h)} \\
\left(\mathcal{F}^{+} L\right)(g)(b) & =b\left(L \circ l_{g}\right), \quad \text { for } b \in E_{\boldsymbol{t}(g)}
\end{aligned}
$$

The Lagrangian is regular if and only if $\mathcal{F}^{ \pm} \boldsymbol{L}$ is a local diffeomorphism.
If $\Theta$ is the canonical 1-section on the prolongation of $E^{*}$ then

$$
\left(\mathcal{P} \mathcal{F}^{ \pm} \boldsymbol{L}\right)^{*} \Theta=\Theta_{L}^{ \pm},
$$

and

$$
\left(\mathcal{P} \mathcal{F}^{ \pm} \boldsymbol{L}\right)^{*} \Omega=\Omega_{L}
$$

We also have that

$$
\operatorname{DEL}(g, h)=\mathcal{F}^{+} L(g)-\mathcal{F}^{-} L(h)
$$

so that the Hamiltonian evolution operator $\xi_{L}$ is

$$
\xi_{L}=\left(\mathcal{F}^{+} L\right) \circ\left(\mathcal{F}^{-} L\right)^{-1}
$$

which is therefore symplectic

$$
\left(\mathcal{P} \xi_{L}\right)^{*} \Omega=\Omega
$$

## Morphisms and reduction

A morphism of Lie groupoids is a bundle map $(\phi, \varphi)$ between groupoids $G$ over $M$ and $G^{\prime}$ over $M^{\prime}$ such that $\Phi(g h)=\Phi(g) \Phi(h)$.

The prolongation $\mathcal{P} \phi$ of $\phi$ is the map $\mathcal{P} \phi(X, Y)=(T \phi(X), T \Phi(Y))$ from $\mathcal{P} G$ to $\mathcal{P} \boldsymbol{G}^{\prime}$.

Assume that we have a Lagrangian $\boldsymbol{L}$ on $\boldsymbol{G}$ and a Lagrangian $\boldsymbol{L}^{\prime}$ on $\boldsymbol{G}^{\prime}$ related by a morphism of Lie groupoids $\phi$, that is $\boldsymbol{L}^{\prime} \circ \phi=\boldsymbol{L}$. Then
$\square\langle\operatorname{DEL}(g, h), a\rangle=\left\langle D_{D E L} \boldsymbol{L}^{\prime}(\phi(g), \phi(h)), \phi_{*}(a)\right\rangle$
$\square \mathcal{P} \phi^{*} \Theta_{L^{\prime}}^{ \pm}=\Theta_{L}^{ \pm}$
$\square \mathcal{P} \phi^{*} \Omega_{L^{\prime}}=\Omega_{L}$

As a consequence:
Let $(\phi, \varphi)$ be a morphism of Lie groupoids from $G \rightrightarrows M$ to $G^{\prime} \rightrightarrows M^{\prime}$ and suppose that $(g, h) \in G_{2}$.

1. If $(\phi(g), \phi(h))$ is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}^{\prime}=\boldsymbol{L} \circ \Phi$, then $(g, h)$ is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}$.
2. If $\phi$ is a submersion then $(g, h)$ is a solution of the discrete EulerLagrange equations for $\boldsymbol{L}$ if and only if $(\phi(g), \phi(h))$ is a solution of the discrete Euler-Lagrange equations for $\boldsymbol{L}^{\prime}$.

## Thank you!

## Example: Heavy top

Consider the transformation Lie algebroid $\tau: S^{2} \times \mathfrak{s o}(3) \rightarrow S^{2}$ and Lagrangian

$$
L_{c}(\Gamma, \Omega)=\frac{1}{2} \Omega \cdot I \Omega-m g l \Gamma \cdot \mathrm{e}=\frac{1}{2} \operatorname{Tr}\left(\hat{\Omega} \mathbb{\mathbb { }} \hat{\Omega}^{T}\right)-m g l \Gamma \cdot \mathrm{e} .
$$

where $\Omega \in \mathbb{R}^{3} \simeq \mathfrak{s o}(3)$ and $\mathbb{I}=\frac{1}{2} \operatorname{Tr}(I) I_{3}-I$.
Discretize the action by the rule

$$
\hat{\Omega}=R^{T} \dot{R} \approx \frac{1}{h} R_{k}^{T}\left(R_{k+1}-R_{k}\right)=\frac{1}{h}\left(W_{k}-I_{3}\right),
$$

where $W_{k}=R_{k}^{T} R_{k+1}$ to obtain a discrete Lagrangian (an approximation of the continuous action) on the transformation Lie groupoid $L: S^{2} \times$ $S O(3) \rightarrow \mathbb{R}$

$$
L\left(\Gamma_{k}, W_{k}\right)=-\frac{1}{h} \operatorname{Tr}\left(\mathbb{I} W_{k}\right)-h m g l \Gamma_{k} \cdot \mathrm{e} .
$$

The value of the action on a variated sequence is

$$
\begin{aligned}
\lambda(t) & =L\left(\Gamma_{k}, W_{k} e^{t K}\right)+L\left(e^{-t K} \Gamma_{k+1}, e^{-t K} W_{k+1}\right) \\
& =-\frac{1}{h}\left[\operatorname{Tr}\left(\mathbb{I} W_{k} e^{t K}\right)+m g l h^{2} \Gamma_{k} \cdot \mathrm{e}+\operatorname{Tr}\left(\mathbb{I} e^{-t K} W_{k+1}\right)+m g l h^{2}\left(e^{-t K} \Gamma_{k+1}\right)\right.
\end{aligned}
$$

where $\Gamma_{k+1}=W_{k}^{T} \Gamma_{k}$ (since the above pairs must be composable) and $K \in \mathfrak{s o}(3)$ is arbitrary.

Taking the derivative at $t=0$ and after some straightforward manipulations we get the DEL equations

$$
M_{k+1}-W_{k}^{T} M_{k} W_{k}-m g l^{2}(\widehat{\Gamma} \widehat{k+1 \times} \mathrm{e})=0
$$

where $M=W \mathbb{I}-\mathbb{I} W^{T}$.
In terms of the axial vector $\Pi$ in $\mathbb{R}^{3}$ defined by $\hat{\Pi}=M$, we can write the equations in the form

$$
\Pi_{k+1}=W_{k}^{T} \Pi_{k}+m g l h^{2} \Gamma_{k+1} \times \mathrm{e} .
$$

## Examples

Pair groupoid.
Lagrangian: L: $M \times M \rightarrow \mathbb{R}$ Discrete Euler-Lagrange equations:

$$
D_{2} \boldsymbol{L}(x, y)+D_{1} \boldsymbol{L}(y, z)=0
$$

- Lie group.

Lagrangian: L: $G \rightarrow \mathbb{R}$ Discrete Euler-Lagrange equations:

$$
\mu_{k+1}=A d_{g_{k}}^{*} \mu_{k}, \quad \text { discrete Lie-Poisson equations }
$$

where $\mu_{k}=r_{g_{k}}^{*} d \boldsymbol{L}(e)$.

## Action Lie groupoid.

Lagrangian: L: $M \times H \rightarrow \mathbb{R}$ Discrete Euler-Lagrange equations: Defining $\mu_{k}\left(x, h_{k}\right)=d\left(\boldsymbol{L}_{x} \circ r_{h_{k}}\right)(e)$, we have

$$
\mu_{k+1}\left(x h_{k}, h_{k+1}\right)=A d_{h_{k}}^{*} \mu_{k}\left(x, h_{k}\right)+d\left(L_{h_{k+1}} \circ\left(\left(x h_{k}\right) \cdot\right)\right)(e),
$$

where $\left(x h_{k}\right) \cdot: H \rightarrow M$ is the map defined by

$$
\left(x h_{k}\right) \cdot(h)=x\left(h_{k} h\right) .
$$

These are the discrete Euler-Poincaré equations.

## Atiyah groupoid.

Lagrangian: $\boldsymbol{L}:(Q \times Q) / H \rightarrow \mathbb{R}$. Discrete Euler-Lagrange equations: Locally $Q=M \times H$

$$
\begin{align*}
& D_{2} L\left((x, y), h_{k}\right)+D_{1} L\left((y, z), h_{k+1}\right)=0,  \tag{1}\\
& \mu_{k+1}(y, z)=A d_{h_{k}}^{*} \mu_{k}(x, y),
\end{align*}
$$

where

$$
\mu_{k}(\bar{x}, \bar{y})=d\left(r_{h_{k}}^{*} L_{(\bar{x}, \bar{y},)}\right)(e)
$$

for $(\bar{x}, \bar{y}) \in M \times M$.
One can find a global expression in terms of a discrete connection.

## Examples

$\square$ Let $G$ be a Lie group and consider the pair groupoid $G \times G$ over $G$. Consider also $G$ as a groupoid over one point. Then we have that the map

$$
\begin{array}{cccc}
\Phi_{l}: & G \times G & \longrightarrow & G \\
& (g, h) & \mapsto & g^{-1} h
\end{array}
$$

is a Lie groupoid morphism, and a submersion. The discrete Euler-Lagrange equations for a left invariant discrete Lagrangian on $G \times G$ reduce to the discrete Lie-Poisson equations on $G$ for the reduced Lagrangian.
$\square$ Let $G$ be a Lie group acting on a manifold $M$ by the left. We consider a discrete Lagrangian on $G \times G$ which depends on the variables of $M$ as parameters $\boldsymbol{L}_{m}(g, h)$. The Lagrangian is invariant in the sense $\boldsymbol{L}_{m}(r g, r h)=\boldsymbol{L}_{r^{-1} m}(g, h)$.

We consider the Lie groupoid $G \times G \times M$ over $G \times M$ where the elements in $M$ as parameters, and thus $\boldsymbol{L} \in C^{\infty}(G \times G \times M)$ and then $\boldsymbol{L}(r g, r h, r m)=$ $\boldsymbol{L}(g, h, m)$. Thus we define the reduction map (submersion)

$$
\begin{array}{cccc}
\Phi: & G \times G \times M & \longrightarrow & G \times M \\
& (g, h, m) & \mapsto & \left(g^{-1} h, g^{-1} m\right)
\end{array}
$$

where on $G \times M$ we consider the transformation Lie groupoid defined by the right action $m \cdot g=g^{-1} m$.
The Euler-Lagrange equations on $G \times G \times M$ reduces to the Euler-Lagrange equations on $G \times M$.

- A $G$-invariant Lagrangian $L$ defined on the pair groupoid $L: Q \times Q \rightarrow \mathbb{R}$, where $p: Q \rightarrow M$ is a $G$-principal bundle. In this case we can reduce to the Atiyah gauge groupoid by means of the map

$$
\begin{array}{cccc}
\Phi: & Q \times Q & \longrightarrow & (Q \times Q) / G \\
& \left(q, q^{\prime}\right) & \mapsto & {\left[\left(q, q^{\prime}\right)\right]}
\end{array}
$$

Thus the discrete Euler-Lagrange equations reduce to the discrete Lagrange-Poincaré equations.

