Reduction and Lagrangian mechanics on Lie algebroids

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Mechanics and Lie algebroids, one day workshop

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Outline

1. Reduction of Lie algebroids & Poisson reduction
2. Reduction of Lagrangian mechanics on Lie algebroids
3. Hamel symbols and nonholonomic mechanics on Lie algebroids

Outline

Reduction of Lie algebroids

Notation

- ✓ (A, p, M), $(\widehat{A}, \widehat{p}, \widehat{M})$ vector bundles
- ✓ $(\Pi, \pi) : (A, p, M) \longrightarrow (\widehat{A}, \widehat{p}, \widehat{M})$ epimorphism

Definition (Reduced Lie algebroid)

Let A and $\widehat{A}$ be Lie algebroids with exterior derivatives d_A and $d_{\widehat{A}}$ respectively. The bundle $\widehat{A}$ is a *reduced Lie algebroid* of A if (Π, π) is a Lie algebroid homomorphism[a], i.e. $d_A \circ \Pi^* = \Pi^* \circ d_{\widehat{A}}$.

[a] This definition of Lie algebroid morphism is equivalent to the inicial one given by *Higgins-Mackenzie*, J. Algebra 129 (1990).

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Definition (Section of an epimorphism)

A section of Π is a set of maps between fibers

$$S = \left\{ S_x^{\widehat{x}} : \widehat{A}_{\widehat{x}} \to A_x \mid \pi(x) = \widehat{x}, x \in M \right\}$$

such that $\Pi_x \circ S_x^{\widehat{x}} = \mathrm{id}_{\widehat{A}_{\widehat{x}}}$, for all $x \in M$.

Notes

1. The set S does not have to define a vector bundle morphism, but when it does S is a section in the usual sense.
2. A section S defines a map $\mathbf{S} : \Gamma(\widehat{A}) \to \Gamma(A)$ in the following way

$$\mathbf{S}(\widehat{v})(x) = S_x^{\widehat{x}}(\widehat{v}_{\widehat{x}}),$$

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Reduction theorem

- ✓ $(A, \rho, [\cdot,\cdot]_A)$ Lie algebroid over M
- ✓ $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ surjective submersion

1. $\exists\ (\widetilde{\Pi}, \pi) : (A^*, \tau, M) \to (\widehat{A}^*, \widehat{\tau}, \widehat{M})$ surjective submersion, such that $S = \left\{ S_x^{\widehat{x}} := (\widetilde{\Pi}_x)^* : \widehat{A}_{\widehat{x}} \to A_x \mid \pi(x) = \widehat{x},\ x \in M \right\}$ is a section of Π
2. $\operatorname{Im} \mathbf{S}$ is a subalgebra of $(\Gamma(A), [\cdot,\cdot]_A)$
3. $d_A \circ \Pi^* \circ \widetilde{\Pi} = \Pi^* \circ \widetilde{\Pi} \circ d_A$, where $\widetilde{\Pi} : \Omega^{\widetilde{\Pi}}(A) \to \Omega(\widehat{A})$ is defined by

$$\widetilde{\Pi}(\alpha)(\widehat{v}_1, \ldots, \widehat{v}_k) = \alpha(\mathbf{S}(\widehat{v}_1), \ldots, \mathbf{S}(\widehat{v}_k))$$

for all $\widetilde{\Pi}$-projectable A-k-form α.

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Reduction theorem

Then, $\widehat{A}$ is a reduced Lie algebroid of A with exterior derivative

$$d_{\widehat{A}} := \widetilde{\Pi} \circ d_A \circ \Pi^*,$$

and Lie algebroid structure $(\widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$:

- $\widehat{\rho} = T\pi \circ \rho \circ S$
- $[\cdot,\cdot]_{\widehat{A}} \circ \pi = \Pi \circ [\mathbf{S}(\cdot), \mathbf{S}(\cdot)]_A$

Note

An equivalent version of this theorem was proved by *David Iglesias, Juan Carlos Marrero, David Martín de Diego, Eduardo Martínez and Edith Padrón*, SIGMA 3 (2007).

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Example: Poisson manifold

- ✓ (M, Λ) Poisson manifold and $(T^*M, \Lambda^\sharp, [\cdot,\cdot]_{T^*M})$ associated Lie algebroid
- ✓ $\pi : M \to \widehat{M}$ surjective submersion and $\sigma : \widehat{M} \to M$ is such that $\pi \circ \sigma = \mathrm{id}_{\widehat{M}}$
- ✓ $\Pi = (T\sigma)^* : T^*M \to T^*\widehat{M}$ and $\widetilde{\Pi} = T\pi : TM \to T\widehat{M}$ are submersions over π
- ✓ $\mathrm{Im}\,\mathbf{S}$ is a Lie subalgebra of $(\Gamma(T^*M), [\cdot,\cdot]_{T^*M})$ and the space of sections of $C_\sigma = \mathrm{Ker}\,(T\sigma)^*$ is an ideal of this algebra.

The reduced structure on $T^*\widehat{M}$:

(i) $\widehat{\rho} = T\pi \circ \Lambda^\sharp \circ S, \quad$ with $S_x^{\pi(x)} = (T_x\pi)^* : T^*_{\pi(x)}\widehat{M} \to T^*_x M;$

(ii) $[\cdot,\cdot]_{\widehat{A}} \circ \pi = \Pi \circ [\mathbf{S}(\cdot), \mathbf{S}(\cdot)]_{T^*M}, \quad$ with $\mathbf{S} = \pi^* : \Gamma(T^*\widehat{M}) \to \Gamma(T^*M).$

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Poisson reduction

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- ✓ $(A, \rho, [\cdot,\cdot]_A)$ Lie algebroid over M
- ✓ $(A^*, \{\cdot,\cdot\}_{A^*})$ associated Lie co-algebroid

Poisson reduction in the sense of Marsden-Ratiu '86

Let C be a subbundle of TA^* such that:

1. it defines a surjective submersion $(\overline{\Pi}, \pi) : (A^*, \tau, M) \longrightarrow (\widehat{A^*}, \widehat{\tau}, \widehat{M})$
2. $\forall F, G \in C^\infty(A^*)$ such that $dF, dG \in C^0$, then $d\{F, G\}_{A^*} \in C^0$

$$\Downarrow$$

$(\widehat{A^*}, \widehat{\tau}, \widehat{M})$ is endowed with a Poisson structure $\{\cdot,\cdot\}_\wedge$ that satisfies:

$$\left\{\widehat{F} \circ \overline{\Pi}, \widehat{G} \circ \overline{\Pi}\right\}_{A^*} = \left\{\widehat{F}, \widehat{G}\right\}_\wedge \circ \overline{\Pi}, \quad \forall \widehat{F}, \widehat{G} \in C^\infty(\widehat{A^*})$$

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Theorem

$$(A, \rho, [\cdot,\cdot]_A) \xrightarrow{LAR} (\widehat{A}, \widehat{\rho}, [\cdot,\cdot]_{\widehat{A}}) \Longrightarrow (A^*, \Lambda_{A^*}) \xrightarrow{PR} (\widehat{A}^*, \Lambda_{\widehat{A}^*})$$

Proposition

If $(A^*, \Lambda_{A^*}) \xrightarrow{PR} (\widehat{A^*}, \Lambda_{\widehat{A^*}})$ then $\widehat{A} := (\widehat{A^*})^*$ is a Lie algebroid:

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Example: Lie group G

Let G be finite dimensional Lie group with Lie algebra $\mathfrak{g}$.

- $(\Pi, \pi) : (TG, p, G) \to (\mathfrak{g}, \widehat{p}, \{\cdot\})$ is the canonical projection defined by the tangent representation $\Phi(g) = TL_g$ of the action of G on itself by left translations, with $TG \equiv G \times \mathfrak{g}$ we have $\Pi(g, X) = X$ for all $X \in \mathfrak{g}$ and $g \in G$.
- With $T^*G \equiv G \times \mathfrak{g}^*$ the projection $\widetilde{\Pi} : T^*G \to \mathfrak{g}^* = T^*G/G$ defined by $\widetilde{\Pi}(g, \alpha) = \alpha$, for all $g \in G$ and $\alpha \in \mathfrak{g}^*$, is a surjective submersion over $\pi : G \to G/G = \{\cdot\}$.

By Marsden–Ratiu $\mathfrak{g}^*$ is endowed with a linear Poisson structure such that $\widetilde{\Pi} : T^*G \to \mathfrak{g}^*$ is a Poisson morphism. Then, we can prove that the conditions of the reduction theorem are satisfied and $\mathfrak{g}$, with its usual structure of Lie algebroid, is a reduced Lie algebroid of TG.

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Reduction of a Lie algebroid prolongation

✓ $(\Pi, \pi) : (A, p, M) \longrightarrow (\widehat{A}, \widehat{p}, \widehat{M})$ submersion

✓ $\widehat{A}$ reduced Lie algebroid of A

$$\Downarrow$$

$\mathcal{T}\Pi = (\Pi, \Pi, T\Pi) : \mathcal{T}A \to \mathcal{T}\widehat{A}$ is a Lie algebroid surjective homomorphism over $\Pi : A \to \widehat{A}$.

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$\mathcal{T}\widehat{A}$ is a reduced Lie algebroid of $\mathcal{T}A$.

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Prolongation $\longleftrightarrow$ Reduction by symmetry

- ✓ $(A, \rho, [\cdot,\cdot]_A)$ Lie algebroid over M
- ✓ $\Phi : G \to \mathrm{Aut}(A)$ Lie algebroid representation of G on A
- ✓ Φ e Φ^c define proper and free actions of G on A and A^*

$\Downarrow$

A satisfies the reduction theorem conditions, and then A/G is a reduced Lie algebroid of A over M/G.

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$\mathcal{T}(A/G) \simeq (\mathcal{T}A)/G$ is a reduced Lie algebroid of $\mathcal{T}A$.[a]

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Consider a principal fibre bundle $P(M,G)$ and the associated gauge algebroid $(TP/G, p, M)$.

- The canonical projection $\Pi : TP \to TP/G$ is a homomorphism of Lie algebroids over $\pi : P \to M$ that defines the homomorphism of Lie algebroids $\mathcal{T}\Pi : \mathcal{T}(TP) \to \mathcal{T}(TP/G)$ over Π. Note that $\mathcal{T}(TP) \equiv T(TP)$.
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We can prove that $\mathcal{T}\Phi = (\Phi, \Phi, T\Phi)$ is a Lie algebroid representation of G on $\mathcal{T}(TP)$ and then $\mathcal{T}(TP)/G \cong \mathcal{T}(TP/G)$, that is, $T(TP)/G \cong \mathcal{T}(TP/G)$.

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Reduction of Lagrangian mechanics on Lie algebroids

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Theorem

Let $L \in C^{\infty}(A)$ be a Π-invariant Lagrangian of a dynamics Lagrangian system on A, i.e. there exists $l \in C^{\infty}(\widehat{A})$ such that $L = l \circ \Pi$. Then:

- $\widehat{E}_l \circ \Pi = E_L$
- $(\mathcal{T}\Pi)^* \widehat{\theta}_l = \theta_L \Longrightarrow (\mathcal{T}\Pi)^* \widehat{\omega}_l = \omega_L$

When L is regular, then we have:

- l is regular
- $\mathcal{T}\Pi \circ X_L = \widehat{X}_l \circ \Pi$

Therefore, the Lagrangian dynamics on A induced by a regular Π-invariant Lagrangian $L = l \circ \Pi$ reduces to the dynamics on $\widehat{A}$ given by l.

Theorem

Let $L \in C^\infty(A)$ be a Π-invariant Lagrangian of a dynamics Lagrangian system on A, i.e. there exists $l \in C^\infty(\widehat{A})$ such that $L = l \circ \Pi$. Then:

- $\widehat{E}_l \circ \Pi = E_L$
- $(\mathcal{T}\Pi)^* \widehat{\theta}_l = \theta_L \Longrightarrow (\mathcal{T}\Pi)^* \widehat{\omega}_l = \omega_L$

When L is regular, then we have:

- l is regular
- $\mathcal{T}\Pi \circ X_L = \widehat{X}_l \circ \Pi$

Therefore, the Lagrangian dynamics on A induced by a regular Π-invariant Lagrangian $L = l \circ \Pi$ reduces to the dynamics on $\widehat{A}$ given by l.

Outline

Adapted coordinates to nonholonomic restrictions

Notation

- ✓ $(A, \rho, [\cdot, \cdot]_A)$ Lie algebroid over M
- ✓ k nonholonomic linear restritions on A

$$\phi_a(q, \mathbf{v}) = \widehat{\Phi_a}(q, \mathbf{v}) = \phi_{a\beta}(q)\mathbf{v}^\beta,$$

where $\widehat{\Phi_a}$ is a linear function defined by the A-1-form Φ_a

Adapted coordinates

- $\{(q^i, \mathbf{w}^\alpha) \mid i = 1, \dots, n,\ \alpha = 1, \dots, s\}$ local coordinates on A
- $\mathbf{w}^\alpha = \Phi_{\alpha\beta}\, \mathbf{v}^\beta$ the last k coordinates coincide with the restrictions ϕ_a, i.e.

$$\begin{aligned} \mathbf{w}^I &= \Phi_{I\beta}\mathbf{v}^\beta, \quad \forall I = 1, ..., (s-k), \\ \mathbf{w}^{s-k+a} &= \phi_a, \quad \forall a = 1, ..., k \end{aligned}$$

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Associated with the new coordinates A, we consider on the prolongation of A the following basis of local sections:

$$\mathcal{X}'_{\alpha}(a) = (a, f_\alpha(p(a)), X_\alpha(a)), \qquad \mathcal{V}'_{\alpha}(a) = \left(a, 0, \left.\frac{\partial}{\partial \mathbf{w}^\alpha}\right|_a\right),$$

where $X_\alpha = \Psi_{\beta\alpha}\,\rho^i{}_\beta\ \left.\partial_{q^i}\right|_{\mathbf{w}}$, for all $\alpha = 1, \ldots, r$, where $\Psi_{\alpha\beta}\Phi_{\beta\gamma} = \delta_{\alpha\gamma}$.

Lie algebroid structure on $\mathcal{T}A$

$$\begin{aligned}
[\mathcal{X}'_\alpha, \mathcal{X}'_\beta]_{\mathcal{T}A} &= \gamma^\epsilon_{\alpha\beta}\mathcal{X}'_\epsilon\,, \qquad [\mathcal{X}'_\alpha, \mathcal{V}'_\beta]_{\mathcal{T}A} = 0\,, \qquad [\mathcal{V}'_\alpha, \mathcal{V}'_\beta]_{\mathcal{T}A} = 0\,,\\
\rho_{\mathcal{T}A}(\mathcal{X}'_\alpha) &= X_\alpha\,, \qquad \rho_{\mathcal{T}A}(\mathcal{V}'_\alpha) = \frac{\partial}{\partial \mathbf{w}^\alpha},
\end{aligned}$$

where $[f_\alpha, f_\beta]_A = \gamma^\epsilon_{\alpha\beta} f_\epsilon$.

Euler-Lagrange equations in adapted coordinates

Let $L \in C^\infty(A)$ be a regular Lagrangian of a dynamical system on the Lie algebroid A with a non-conservative force $\mathcal{Q}$.

Euler-Lagrange generalized equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{w}^\alpha}\right) \quad = \quad \Psi_{\beta\alpha}\,\rho^i{}_\beta \frac{\partial L}{\partial q^i} + \mathbf{w}^\epsilon \gamma^\beta_{\epsilon\alpha} \frac{\partial L}{\partial \mathbf{w}^\beta} + \Upsilon_\alpha,$$

with:

- $\dot{q}^i = \mathbf{w}^\alpha \Psi_{\beta\alpha} \rho^i{}_\beta$, where $\Psi = \Phi^{-1}$
- Υ_α is the α-component of the nonconservative force $\mathcal{Q}$, in the new coordinates
- $\gamma^\beta_{\epsilon\alpha}$ Hamel symbols

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Gauge algebroid TG/G of a Lie group G

Let G be a Lie group and e its neutral element.

- Using the map $TL_{g^{-1}} : TG \rightarrow G \times T_eG$ given by $TL_{g^{-1}}(g, \dot{g}) = (g, \xi)$ we can identify TG with $G \times T_eG$.
- If (ξ^I), for $I = 1, \ldots, \dim G$, is the set of coordinates of $\xi \in T_eG$ with respect to a basis $\{e_I\}$ of T_eG, then we can define a set of quasi-velocities (ξ^I) on TG by $\xi^I e_I = \xi = T_gL_{g^{-1}}(\dot{g})$. If g is a point in G of local coordinates (g^I), then (g^I, ξ^I) defines a set of quasi-coordinates in TG.

A regular G-invariant Lagrangian $\mathcal{L} \in C^\infty(TG)$ in quasi-coordinates is given by $\mathcal{L}(g, \dot{g}) = l(\xi)$, where l is a function on the Lie algebra $\mathfrak{g} = T_eG$. The Euler-Lagrange equations of the gauge algebroid $TG/G \equiv \mathfrak{g}$ are given by:

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \xi^I}\right) = \xi^J c_{JI}{}^K \frac{\partial l}{\partial \xi^K},$$

where $c_{JI}{}^K$ are the structure constants of the Lie algebra $\mathfrak{g}$ of the Lie group G with respect to the basis $\{e_I\}$ of $\mathfrak{g}$.

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