UNIVERSIDAD TÉCNICA DE LISBOA
INSTITUTO SUPERIOR TÉCNICO

Lie Algebroids over Quotient Spaces

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Thesis approved in public session to obtain the PhD
Degree in
Mathematics
Jury final classification: Pass with Merit

Jury
Chairperson: Chairman of the IST scientific board

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Funding Institution:
Fundacão para a Ciência e a Tecnologia

2011

Acknowledgements

I thank my advisor, Professor Rui Loja Fernandes, for providing me with the opportunity to work in mathematics, for sharing his mathematical insight, and for trusting in my capability to learn new things . Also, I would like to thank FCT - Fundaçã o para a Ci e ncia e Tecnologia - for the financial support granted (grant reference SFRH/BD /30574/2006).

Abstract

In this thesis we study Lie algebroids under actions of Lie groups and Lie groupoids, we require that the actions respect the Lie algebroid structure in the appropriate sense. In the case of Lie groups acting on Lie algebroids by Lie algebroid automorphisms, we present some known results on non-singular actions as well as new results on singular actions concerning the behaviour of the Lie algebroid regarding the fixed point set of the action and the orbit type bundles in the quotient space. Similarly, we study actions of Lie groups on Lie bialgebroids by Lie bialgebroid morphisms, and present similar results to those found in the case of actions of Lie groups on Lie algebroids. We apply our results on Lie algebroids and Lie bialgebroids to recover reduction results in Poisson and Dirac geometries, and we study a reduction process on mechanical systems on Lie algebroids. In the case of Lie groupoids acting on Lie algebroids, we first consider the particular case of algebroids over orbifolds, which allows us to generalize the definition of a Lie algebroid to some singular spaces modelled by groupoids with a special type of foliation. In particular, the notion of a Lie algebroid over a singular space modelled by a Lie groupoid should be invariant under Morita equivalence.

Keywords: Lie algebroid, Lie groupoid, Morita equivalence, étale groupoid, orbifold, Lie group action, Lie bialgebroid, Poisson structure, isotropy type manifold, Dirac structure.

Resumo

Nesta tese estudamos algebróides de Lie sob acções de grupos de Lie e de grupóides de Lie que respeitem a estrutura algebróide de Lie no sentido apropriado. No caso de grupos de Lie a atuar em algebróides de Lie por automorfismos de algebróides de Lie, apresentamos alguns resultados conhecidos sobre ações não singulares, bem como novos resultados sobre ações singulares em relação ao comportamento do algebróide de Lie sobre o conjunto dos pontos fixos da acção e dos fibrados de tipo de órbita no espaço quociente. Da mesma forma, estudamos as ações de grupos de Lie em bialgebróides de Lie por morfismos de bialgebróides de Lie, e apresentamos resultados semelhantes aos encontrados no caso de ações de grupos de Lie em algebróides de Lie. Aplicamos também nossos resultados sobre algebróides de Lie e bialgebróides de Lie para recuperar alguns resultados sobre redução de estructuras de Poisson e estructuras de Dirac, assim como a um processo de redução de sistemas mecânicos em algebróides de Lie. No caso de grupóides de Lie a atuar em algebróides de Lie, primeiro consideramos o caso particular de algebróides sobre orbifolds, o que nos permite generalizar a definição de um algebróide de Lie para certos espaços singulares modelados por grupóides, munidos de uma foliação com caracteristicas especiais. Em particular, a noção de um algebróide de Lie sobre um espaço singular modelado por um grupóide de Lie deve ser invariante sob equivalência de Morita.

Palavras-chave: Algebróide de Lie, grupóide de Lie, equivalência de Morita, grupóide étale, orbifold, ação por um grupo de Lie, bialgebróide de Lie, estructura de Poisson, Variedade de tipo de isotropia, estructura de Dirac.

Contents

Introduction

Lie algebroids appear in different contexts in geometry, ranging from equivalence problems to foliation theory. One of the most important and basic applications of Lie algebroids is to classical mechanics and Poisson geometry. In fact, to any Poisson manifold there is associated a Lie algebroid structure on its cotangent bundle. This connection played an important role in recent developments of Lie algebroid theory. Conversely, Lie algebroid (and groupoid) theory has contributed to a deeper understanding of Poisson geometry.

It is well known that symmetries are an important tool to understand classical mechanics. Similarly, symmetries play an important role in Lie algebroid theory. For a proper and free action of a Lie group G on a Lie algebroid A, by Lie algebroid automorphisms, the quotient A/G inherits a Lie algebroid structure, this kind of reduction has been studied by several authors ([16],[3], [19], [15]). In this thesis, we look at proper, but non-free actions. As one could expect, the orbit type stratification plays here a key role.

Eventually our aim is to study Lie algebroids over singular spaces. The simplest situation occurs when the base space is an orbifold. But, more generally, we are interested on orbispaces, i.e., singular spaces which are orbit spaces of proper group actions or even proper Lie groupoids.

This thesis is organized as follows. In Chapter 1 we introduce the basic concepts to be used in this thesis. We begin with Lie groupoids and Lie algebroids and state their intrinsic relation. An introduction to Cartan calculus on Lie algebroids is presented, as well as examples of Lie groupoids and Lie algebroids that will be relevant in the subsequent chapters. Then we introduce background material on Poisson geometry that will be relevant later as it will inspire many of the of the results. To end the chapter we present a classical description of orbifolds that will be the base for understanding a generalization of Lie algebroids over orbifolds and more general singular spaces.

Chapter 2 presents the relevant concepts to understand the category of orbifolds in terms of an appropriate category of Lie groupoids. We first introduce the different notions of equivalence of groupoids, starting with essential equivalence and reaching Morita equivalence, realizing in this way orbifolds as Morita equivalence classes of proper étale lie groupoids. To finalize the chapter we present the different notions of maps between Lie groupoids that enlarge the usual definition of groupoid morphism, these are generalized maps and Hilsum-Skandalis maps. We will see there is an equivalence of categories for groupoids with these classes of maps, we present both definitions as depending on the context is useful to see a map one way or the other.

Chapter 3 is dedicated to Lie algebroids over quotient and singular spaces. We present reduction results for Lie group actions on Lie algebroids and Lie bialgebroids by appropriate morphisms, first addressing the non-singular case and then concentrating on the isotropy submanifolds of the action. Later in the chapter we generalize the notion of Lie algebroid to cover Lie algebroids over orbifolds and then, using the description of orbifolds as groupoids, to more general singular spaces (orbispaces).

The last chapter, Chapter 4 deals with applications of the singular reduction results obtained in Chapter 3. We present known reduction results in Poisson and Dirac geometries and describe their relation to singular Lie algebroid reduction. To conclude, we present an application of singular reduction to mechanical systems on Lie algebroids.

Chapter 1

Basic Concepts

In this first chapter we recall some of the basic facts about Lie groupoids and Lie algebroids, as well as the definition of orbifold, that will be important for us later.

1.1 Groupoids

In this section we introduce the notion of Lie groupoid, we follow the works of Moerdijk [20] and Crainic and Fernandes [7].

Definition 1.1.1. A *groupoid* $\mathcal{G}$ is a small category in which every arrow is invertible.

To clarify concepts and to fix notation, a groupoid $\mathcal{G}$ over M, denoted $\mathcal{G} \rightrightarrows M$, consists of a set of arrows $\mathcal{G}$ and a set of objects M, together with five structure maps

$$\mathcal{G} \times_M \mathcal{G} \xrightarrow{m} \mathcal{G} \underset{\mathbf{t}}{\overset{\mathbf{s}}{\rightrightarrows}} M \xrightarrow{u} \mathcal{G} \xrightarrow{i} \mathcal{G}$$

The maps $\mathbf{s}$ and $\mathbf{t}$ are called *source* and *target*. An element $g \in \mathcal{G}$ with $\mathbf{s}(g) = x$ and $\mathbf{t}(g) = y$ is an arrow from x to y and will be denoted by $g : x \longrightarrow y$ or $x \xrightarrow{g} y$

The set

$$\mathcal{G} \times_M \mathcal{G} = \{(h, g) \in \mathcal{G} \times \mathcal{G} | \mathbf{s}(h) = \mathbf{t}(g)\}$$

consist of composable arrows, and m is called the *composition* or *multiplication* map. For a pair (h, g) of composable arrows, the composition is denoted $m(g, h) = hg$.

The map u is called the unit map and we write $u(x) = 1_x$, and the map i is called the inversion map and we write $i(g) = g^{-1}$

The names of the maps become clear as they must satisfy the following conditions

(i) $\mathbf{s}(hg) = \mathbf{s}(g)$, $\mathbf{t}(hg) = \mathbf{t}(h)$

(ii) $k(hg) = (kh)g$

(iii) $1_{t(g)}g = g = g1_{\mathbf{s}}(g)$

(iv) $\mathbf{s}(g^{-1}) = \mathbf{t}(g)$, $\mathbf{t}(g^{-1}) = \mathbf{s}(g)$, $g^{-1}g = 1_{\mathbf{s}(g)}$, $gg^{-1} = 1_{\mathbf{t}(g)}$

for any k, h, $g \in \mathcal{G}$ with $\mathbf{s}(k) = \mathbf{t}(h)$ and $\mathbf{s}(h) = \mathbf{t}(g)$.

If there is no chance for confusion we will denote the groupoid $\mathcal{G} \rightrightarrows M$ simply by $\mathcal{G}$.

A groupoid $\mathcal{G}$ is called a *Lie (topological) groupoid* if $\mathcal{G}$, M are smooth (topological) manifolds, the maps $\mathbf{s}$, $\mathbf{t}$, m u, i are smooth (continuous) and $\mathbf{s}$, $\mathbf{t}$ are submersions (open maps).

The fact that $\mathbf{s}$, $\mathbf{t}$ are submersions implies that $\mathcal{G} \times_M \mathcal{G}$ is a manifold, and the smoothness of m should be taken with respect to this manifold structure.

We take manifolds to be Hausdorff, second countable spaces, one possible exception to this will be the total space of a groupoid G that can be taken to be non-Hausdorff. Although in the case of interest in this work $\mathcal{G}$ will be a Hausdorff space.

Remark 1.1.2. We will be working almost exclusively in the smooth category, so by a manifold we mean a smooth manifold unless otherwise stated. The same apply to maps between manifolds.

We will be using the following notation for a groupoid $\mathcal{G} \rightrightarrows M$: if $x \in M$, then the sets $\mathbf{s}^{-1}(x)$, $\mathbf{t}^{-1}(x)$ are called the $\mathbf{s}$-*fiber* at x, and the $\mathbf{t}$-*fiber* at x, respectively. The inverse map induces a natural bijection between these two sets:

$$i : s^{-1}(x) \longrightarrow t^{-1}(x).$$

Given $g : x \longrightarrow y$, the right multiplication by g is only defined on the $\mathbf{s}$-fiber at y, and induces a bijection

$$R_g : \mathbf{s}^{-1}(y) \longrightarrow \mathbf{s}^{-1}(x).$$

Similarly, the left multiplication by g induces a map from the $\mathbf{t}$-fiber at x to the $\mathbf{t}$-fiber at y. The intersection of the $\mathbf{s}$ and $\mathbf{t}$-fiber at $x \in M$,

$$\mathcal{G}_x = \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$$

together with the restriction of the groupoid multiplication, is a group called the *isotropy* group at x.

The set $\mathcal{O}_x = \mathbf{t}(\mathbf{s}^{-1}(x)) \subset M$ is called the orbit of x. The orbits of $\mathcal{G}$ define an equivalence relation on M

$$x \sim y \Longleftrightarrow y \in \mathcal{O}_x$$

and the quotient space for this relation is called the orbit space of $\mathcal{G}$ and denoted $M/\mathcal{G}$ or $|\mathcal{G}|$.

From the definition of Lie groupoid we can deduce that the unit map u is an embedding. Also, the fact that the source and target maps are submersions have several implications in the smoothness of the objects defined above, among them, the $\mathbf{s}$, $\mathbf{t}$-fibers and the isotropy groups $\mathcal{G}_x$ are smooth manifolds. However, note that the quotient space $M/\mathcal{G}$ can be singular.

The orbits of a Lie groupoid $\mathcal{G}$ are immersed submanifolds of M, they form a partition of M, as the dimensions of the orbits may vary they do not form a foliation, rather they form a singular foliation of M.

A few basic examples that will be playing a role in the rest of this work,

Example 1.1.3. *Unit groupoid.* Let M be a manifold. Consider the groupoid $\mathcal{G} \rightrightarrows M$ with $\mathcal{G} = M$. This is a Lie groupoid whose arrows are all units, called the unit groupoid. The orbit space $M/\mathcal{G}$ is again the manifold M.

Example 1.1.4. *Pair groupoid.* Let M be a manifold. Consider the groupoid $\mathcal{G} \rightrightarrows M$ with $\mathcal{G} = M \times M$. This is a Lie groupoid with exactly one arrow from any object x to any object y, called the pair groupoid. The orbit space $M/\mathcal{G}$ has only one point.

Example 1.1.5. *Group.* Let G be a Lie group. Let $\bullet$ be the set with one point. Consider the groupoid $\mathcal{G} \rightrightarrows M$ with $M = \bullet$ and $\mathcal{G} = G$. In this way the concept of Lie groupoid extends the concept of Lie group.

Example 1.1.6. *Action groupoid.* Let K be a Lie group acting (on the left) on a manifold M. Consider the groupoid $\mathcal{G} \rightrightarrows M$ with $\mathcal{G} = K \times M$. This is a Lie groupoid with arrows $(k, x) : x \longrightarrow kx$ called the action groupoid and denoted $K \ltimes M$. The orbit space $M/\mathcal{G}$ is the orbit space of the action M/K.

Example 1.1.7. *Groupoid associated to a principal bundle.* Let G be a Lie group and $P \longrightarrow M$ a principal K-bundle. The quotient of the pair groupoid $P \times P$ by the diagonal action of G, denoted $P \otimes_G P$, gives rise to a groupoid $P \otimes_G P \rightrightarrows M$ with the obvious structure maps.

Example 1.1.8. *Fundamental groupoid of a manifold.* Let M be a connected manifold. Consider the Lie groupoid $\Pi_1(M) \rightrightarrows M$. Where

$$\Pi_1(M) = \{[\gamma] : \gamma : [0,1] \to M\}.$$

the set of homotopy classes of paths in M. For an arrow $[\gamma] \in \Pi_1(\mathcal{F})$ its source and target are given by

$$\mathbf{s}([\gamma]) = \gamma(0), \quad \mathbf{t}([\gamma]) = \gamma(1),$$

while the composition of two arrows is just concatenation of paths

$$[\gamma_1][\gamma_2] = [\gamma_1 \cdot \gamma_2].$$

When M is simply-connected, $\Pi_1(M)$ is isomorphic to the pair groupoid $M \times M$.

Related to the last example we have

Example 1.1.9. *Fundamental groupoid of a foliation.*Let $\mathcal{F}$ be a foliation of a connected manifold M. Consider the Lie groupoid $\Pi_1(\mathcal{F}) \rightrightarrows M$. Where

$$\Pi_1(M) = \{[\gamma] : \gamma : [0,1] \to M\} \text{ a path lying in a leaf.}$$

the set of homotopy classes of paths in M lying inside leaves of $\mathcal{F}$. The structure maps of $\Pi_1(\mathcal{F})$ are defined as before. The set of orbits of $\Pi_1(\mathcal{F})$ coincide with the leaves of $\mathcal{F}$.

As the definition of groupoid can be put in categorical terms it is natural to define morphisms between groupoids in the following way

Definition 1.1.10. A *homomorphism* between groupoids $\mathcal{H} \rightrightarrows N$ and $\mathcal{G} \rightrightarrows M$ is a functor $\phi : \mathcal{H} \longrightarrow \mathcal{G}$

The functor ϕ is given by a map $N \longrightarrow M$ on objects and a map $\mathcal{H} \longrightarrow \mathcal{G}$ on arrows, both denoted by ϕ, which together preserve the groupoid structure, i.e. $\phi(\mathbf{s}(h)) = \mathbf{s}(\phi(h))$, $\phi(\mathbf{t}(h)) = \mathbf{t}(\phi(h))$, $\phi(1_x) = 1_{\phi(x)}$ and $\phi(hk) = \phi(h)\phi(k)$ (this implies that also $\phi(h^{-1}) = \phi(h)^{-1}$), for any h, $k \in \mathcal{H}$ with $\mathbf{s}(h) = \mathbf{t}(k)$ and any $x \in N$.

Remark 1.1.11. Groupoids and their morphisms form a category. In the subsequent sections we will define other interesting categories which we will use to define our main object of interest in Section 3.

There are some interesting classes of Lie groupoids which we shall now recall. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, $\mathcal{G}$ is called source simply connected, or $\mathbf{s}$-simply connected, if the $\mathbf{s}$ fibers are simply connected. These groupoids are very important as they play the role of simply connected Lie groups in Lie theory. Another interesting class inspired by Lie group theory, is

Definition 1.1.12. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called *proper* if $\mathcal{G}$ is Hausdorff and the map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \to M \times M$ is a proper map.

Some of the special properties of proper groupoids are parallel to the properties of proper group actions, reflected on the properties of the action groupoid $G \ltimes M$ for a Lie group G acting on a manifold M. The following proposition can be found as an exercise in [7].

Proposition 1.1.13. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then*

(i) All isotropy groups of $\mathcal{G}$ are compact.

(ii) All orbits of $\mathcal{G}$ are closed submanifolds.

(iii) The orbit space $M/\mathcal{G} := |\mathcal{G}|$ is Hausdorff.

Along with proper groupoids another special kind of groupoids that will be used in subsequent sections are

Definition 1.1.14. A Lie groupoid $\mathcal{G}$ is called *étale* if its source map $\mathbf{s}$ is a local diffeomorphism.

The relations between the structure maps of $\mathcal{G}$ imply that if $\mathcal{G}$ is étale then all the structure maps are local diffeomorphisms. In particular we see that all the $\mathbf{s}$-fibers and $\mathbf{t}$-fibers must be discrete. The most basic example of étale groupoid is the action groupoid $G \ltimes M$ for G a discrete Lie group.

1.2 Lie algebroids

Intimately related to Lie groupoids are Lie algebroids, these play a similar role in Lie groupoid theory as Lie algebras play in Lie group theory. We follow the exposition in Crainic and Fernandes [7].

Definition 1.2.1. A *Lie algebroid* over a manifold M consists of a vector bundle A together with a bundle map $\rho : A \longrightarrow TM$ and a Lie bracket $[\ ,\ ]$ on the space of sections $\Gamma(A)$, satisfying the Leibniz identity

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathcal{L}_{\rho(\alpha)}(f)\beta$$

for all α, $\beta \in \Gamma(A)$ and all $f \in C^\infty(M)$.

When there is no chance of confusion we will denote the Lie derivative simply as $\rho(\alpha)f$. The bundle map ρ is called the anchor map of A

Using the Jacobi and Leibniz identities we can see that

$$\rho : \Gamma(A) \longrightarrow \mathfrak{X}(M)$$

is a Lie algebra homomorphism.

For each Lie groupoid $\mathcal{G} \rightrightarrows M$ there is an associated Lie algebroid A over M, that has as fibers $A_x = T_{1_x}\mathbf{s}^{-1}(x)$. The bracket on $\Gamma(A)$ is inherited from the Lie bracket on the right invariant vector fields in $\mathcal{G}$ (which are necessarily tangent to the $\mathbf{s}$-fibers), denoted by $\mathfrak{X}^R(\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$, from the bijective correspondence

$$\begin{aligned} \Gamma(A) &\longrightarrow \mathfrak{X}^R(\mathcal{G}) \\ \alpha &\longmapsto \tilde{\alpha} \end{aligned}$$

where $\tilde{\alpha}_g = \mathrm{d}_{\mathbf{t}(g)} R_g(\alpha_{\mathbf{t}(g)})$. Here $R_g : s^{-1}(y) \longrightarrow s^{-1}(x)$ denotes the right multiplication by $g : x \longrightarrow y$ in $\mathcal{G}$. The anchor map is defined by $\rho = \mathrm{d}\mathbf{t}|_{u(M)}$, the restriction of the differential of the target map $\mathbf{t}$ of $\mathcal{G}$.

Remark 1.2.2. It is important to remark a that Lie's first and second theorem are valid for Lie groupoids and Lie algebroids (replacing the simply connectedness of the group by the the simply connectedness of the $\mathbf{s}$-fibers). Lie's third theorem is not. There are Lie algebroids A for which there is no Lie groupoid $\mathcal{G}$ that has A as its associated Lie algebroid. If such a $\mathcal{G}$ exists then A is called *integrable.*

We seldom use this construction in this work, it is presented for completion and [7] has a more detailed construction of the correspondence as well as a complete characterization of integrable Lie algebroids.

Similarly to Lie groupoids, one has the notion of isotropy and orbits for Lie algebroids, which are defined as follows.

There is a well defined Lie algebra structure on the set $\mathfrak{g}_x(A) = \mathrm{Ker}\,\rho_x$ (or simply $\mathfrak{g}_x$), for $x \in M$. Indeed, if α, $\beta \in \Gamma(A)$ with α_x, $\beta_x \in \mathfrak{g}_x$, the Leibniz identity implies that $[\alpha, f\beta]_x = f(x)[\alpha, \beta]_x$. The value of $[\alpha, \beta]$ depends only of α_x and β_x there is a well defined bracket on $\mathfrak{g}_x$ such that

$$[\alpha, \beta]_x = [\alpha_x, \beta_x].$$

The Lie algebra $\mathfrak{g}_x$ is called the *isotropy Lie algebra* at x. If A is the Lie algebroid associated to a Lie groupoid $\mathcal{G}$, $\mathfrak{g}_x$ is the Lie algebra of the isotropy group $\mathcal{G}_x$ of x in $\mathcal{G}$.

Associated with a Lie algebroid A we have a distribution $\mathrm{Im}\,(\rho)$ on M. The fact that $\rho : \Gamma(A) \longrightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism makes this distribution involutive. The integrability of $\mathrm{Im}\,(\rho)$ is not straight forward because the dimension of $\mathrm{Im}\,(\rho)_x$ may vary and the distribution may be singular, but it is guaranteed by a theorem of Sussmann [31]. The hypothesis of the theorem are met as $\mathrm{Im}\,(\rho)$ is locally finitely generated by the images of a basis of sections of A and ρ as a map of sections is a Lie algebra morphism.

Is important to remark that the dimension of $\mathfrak{g}_x$ may also vary with x, in fact it is related to the dimension of $\mathrm{Im}\,(\rho)_x$ in the obvious way.

The leaves of the foliation $\mathcal{O}$ defined by $\mathrm{Im}\,(\rho)$ are called the *orbits* of the Lie algebroid A. For $x \in \mathcal{O} \subset M$, the orbit through x we have

$$T_x\mathcal{O} = \mathrm{Im}\,(\rho)_x.$$

If A is the Lie algebroid associated to a source connected Lie groupoid $\mathcal{G}$, the orbits in M of both objects coincide.

Now we define morphisms between Lie algebroids

Definition 1.2.3. Let $A \longrightarrow M$ and $B \longrightarrow N$ be Lie algebroids. A **morphism of Lie algebroids** is a vector bundle map

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow[\phi]{} & N \end{array}$$

which is compatible with the anchors and the brackets.

If there is no risk of confusion we will write $\Phi : A \longrightarrow B$, or simply Φ, to denote a Lie algebroid morphism.

By the compatibility with the anchors of Φ we mean the commutativity of the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \rho_B} \\ TM & \xrightarrow[\mathrm{d}\phi]{} & TN \end{array}$$

By compatibility with the bracket we would like to have an identity resembling

$$\Phi[\alpha, \beta] = [\Phi\alpha, \Phi\beta] \tag{1.2.1}$$

where α, $\beta \in \Gamma(A)$. In general this is not possible as sections of A cannot be pushed forward to sections of B. Instead we have to work at the level of the pull-back bundle ϕ^*B.

First note that from sections α of A or β of B, we can produce new sections $\Phi(\alpha)$ and $\phi^*(\beta)$ of ϕ^*B by

$$\Phi(\alpha) = \Phi \circ \alpha, \ \ \phi^*(\beta) = \beta \circ \phi.$$

Now, given any section $\alpha \in \Gamma(A)$, we can express its image under Φ as a (non-unique) finite combination

$$\Phi(\alpha) = \sum_i a_i \phi^*(\alpha_i),$$

where $a_i \in C^\infty(M)$ and $\alpha_i \in \Gamma(B)$. By compatibility with the brackets we mean that, if $\alpha, \beta \in \Gamma(A)$ are sections such that their images are expressed as finite combinations as above, then their bracket is a section whose image can be expressed as:

$$\Phi([\alpha,\beta]_A) = \sum_{i,j} a_i b_j \phi^*[\alpha_i,\beta_j]_B + \\ + \sum_j \mathcal{L}_{\rho(\alpha)}(b_j)\phi^*(\beta_j) - \sum_i \mathcal{L}_{\rho(\beta)}(a_i)\phi^*(\alpha_i). \quad (1.2.2)$$

Notice that, in the case where the sections $\alpha, \beta \in \Gamma(A)$ can be pushed forward to sections $\alpha', \beta' \in \Gamma(B)$, so that $\Phi(\alpha) = \alpha' \circ \phi$ and $\Phi(\beta) = \beta' \circ \phi$, we have

$$\Phi[\alpha,\beta] = [\alpha',\beta'] \circ \phi.$$

Defining the push forward of a section α by

$$\Phi_*\alpha = \Phi(\alpha) \circ \phi^{-1}$$

the compatibility condition takes the form $\Phi_*[\alpha,\beta] = [\Phi_*\alpha, \Phi_*\beta]$ which is close to (1.2.1).

Remark 1.2.4. Lie algebroids and their morphisms form a category.

Lets look at a few examples of Lie algebroids related to the examples of Lie groupoids in the last section

Example 1.2.5. *Tangent bundles.* Let M be a manifold. Consider the Lie algebroid structure over TM given by the Lie bracket of vector fields and the identity map as an anchor. A Lie groupoid integrating this Lie algebroid is the pair groupoid $M \times M$. The fundamental groupoid $\Pi(M)$ integrates TM since it is locally diffeomorphic to $M \times M$.

Example 1.2.6. *Lie algebras.* Let $\mathfrak{g}$ be a Lie algebra. Considered as a vector bundle over a point it is a Lie algebroid. The simply connected Lie group G integrating $\mathfrak{g}$ can be viewed as a Lie groupoid integrating the Lie algebra.

Example 1.2.7. *Action Lie algebroids.* Let $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ be an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M, i.e., a Lie algebra homomorphism. Consider the Lie algebroid $\mathfrak{g} \ltimes M$, called the action Lie algebroid, as follows. As a vector bundle, it is the trivial vector bundle $M \times \mathfrak{g}$ over M, the anchor is given by the infinitesimal action, while the Lie bracket is uniquely determined by the Leibniz identity and the condition that

$$[c_v, c_w] = c_{[v,w]},$$

for all $v, w \in \mathfrak{g}$, where c_v denotes the constant section of $\mathfrak{g}$. If the infinitesimal action is integrable, i.e., it comes from an action from G, the connected Lie group integrating $\mathfrak{g}$. The action Lie algebroid $\mathfrak{g} \ltimes M$ is the Lie algebroid associated to the action groupoid $G \ltimes M$.

Example 1.2.8. *Atiyah algebroid.* Let G be a Lie group and P a principal G-bundle over M, there is an associated Lie algebroid over M, denoted $A(P)$, and defined as follows. $A(P) := TP/G \longrightarrow M$ as a vector bundle. The anchor is induced by the differential of the projection from P to$M$. Also, since the sections of $A(P)$ correspond to G-invariant vector fields on P, we see that there is a canonical Lie bracket on $\Gamma A(P)$, with these $A(P)$ becomes Lie algebroid and it is the Lie algebroid of $P \otimes_G P$, the groupoid associated with the principal G-bundle P.

Example 1.2.9. *Foliations.* Let $\mathcal{F}$ be a foliation of a connected manifold M given by an involutive distribution A. Then A has the structure of a Lie algebroid with the Lie bracket of vector fields restricted to the sections of A and the inclusion map as anchor. The orbits of A coincide with the leaves of $\mathcal{F}$.

Let us give an example which a priori is not connected to a Lie groupoid, and that gives intuition for some constructions given later.

Example 1.2.10. *Poisson manifolds.* Let (M, π) be a Poisson manifold (see Section 1.3 for definitions). The cotangent bundle T^*M has the structure of a Lie algebroid defined as follows, the anchor map $\pi^\sharp : T^*M \longrightarrow TM$ is defined by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$, for $\alpha, \beta \in \Omega^1(M)$, and the Lie bracket is defined by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - \mathrm{d}(\pi(\alpha, \beta)).$$

This Lie algebroid is called the *cotangent Lie algebroid* of the Poisson manifold (M, π).

Cartan calculus can be generalized to Lie algebroids in the following manner (for details we refer to [2]). For a Lie algebroid $A \longrightarrow M$ with anchor ρ and bracket $[\ ,\ ]$ the exterior differential of A is a map

$$\mathrm{d}_A : \Gamma(\wedge^\bullet A^*) \longrightarrow \Gamma(\wedge^{\bullet+1} A^*)$$

which is defined by the usual Koszul formula

$$\begin{aligned}(\mathrm{d}_A\theta)(v_1, \dots, v_{k+1}) = &\sum_i (-1)^{i+1} \rho(v_i)\theta(v_1, \dots, \hat{v}_i, \dots, v_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \theta([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{k+1}) \end{aligned} \tag{1.2.3}$$

for $v_i \in \Gamma(A)$, $i = 1, \dots, k$.

We will denote d_A simply by d when there is no chance of confusion. The exterior derivative d on A has the following usual properties

- d is $C^\infty(M)$-multilinear,

- $\mathrm{d}^2 = 0$ and
- d is a graded derivation of degree 1, i.e.,

$$\mathrm{d}(\theta \wedge \eta) = \mathrm{d}\theta \wedge \eta + (-1)^{|\theta|}\theta \wedge \mathrm{d}\eta$$

where $|\ |$ denotes the degree of the grading in $\Gamma(\wedge^\bullet A^*)$. $(\Gamma(\wedge^\bullet A^*), \wedge, \mathrm{d})$ forms a differential graded algebra. Conversely,given such a structure, the Lie algebroid structure on A can be recovered by defining the anchor to be

$$\rho(v)f = \mathrm{d}f(v) \text{ for } v \in \Gamma(A) \text{ and } f \in C^\infty(M)$$

and by defining the Lie bracket by setting,

$$\begin{aligned} i_{[\alpha,\beta]}\theta &= \rho(\alpha)\theta(\beta) - \rho(\beta)\theta(\alpha) - \mathrm{d}\theta(\alpha,\beta) \\ &= i_\alpha \mathrm{d}(i_\beta\theta) - i_\beta \mathrm{d}(i_\alpha\theta) - i_{(\alpha\wedge\beta)}\mathrm{d}\theta \end{aligned}$$

for $\alpha, \beta \in \Gamma(A)$ and $\theta \in \Gamma(A^*)$. This establishes a correspondence between Lie algebroid structures and exterior differential operators on A.

The A-Lie derivative of an element of $\wedge^\bullet A^*$ is defined in the usual way by

$$\mathcal{L}_\alpha\theta = \mathrm{d}i_\alpha\theta + i_\alpha \mathrm{d}\theta$$

for $\alpha \in \Gamma(A)$, $\theta \in \Gamma(\wedge^\bullet A^*)$

In analogy to the de Rham cohomology of a manifold, the cohomology groups of the complex $(\Gamma(\wedge^\bullet A^*), \mathrm{d})$ is defined and is called the *Lie algebroid cohomology* of A.

The following proposition, due to Vaintrob [33], characterizes morphisms of Lie algebroids in terms of the exterior differential

Proposition 1.2.11. *Let $\Phi : A \longrightarrow B$ be a bundle map between Lie algebroids A and B. Φ is a Lie algebroid morphism if and only if*

$$\mathrm{d}_A\Phi^*\beta = \Phi^*\mathrm{d}_B\beta \textit{ for } \beta \in \Gamma(B^*)$$

Here the map Φ corresponds to diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow[\phi]{} & N \end{array}$$

and the pull-back Φ^* is defined by

$$(\Phi^*\beta)_x(v) = \beta_{\phi(x)}(\Phi v).$$

The Lie algebroid structure on A gives rise to another construction that will be useful later. Just like the Schouten bracket on multivector fields extends the Lie bracket of vector fields, there is an extension of the bracket from $\Gamma(A)$ to $\Gamma(\wedge^\bullet A)$. This extension is called the *Gerstenhaber bracket* and is the unique $C^\infty(M)$ bilinear operation on $\Gamma(\wedge^\bullet A)$ satisfying the following conditions

- $[\ ,\ ]$ has degree -1,

$$[\ ,\ ] : \Gamma(\wedge^k A) \times \Gamma(\wedge^l A) \longrightarrow \Gamma(\wedge^{k+l-1} A)$$

- $[\ ,\ ]$ is graded commutative,

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]$$

- $[\ ,\ ]$ satisfies the graded Jacobi identity,

$$\begin{aligned}[\alpha,[\beta,\gamma]] + (-1)^{(|\gamma|-1)(|\alpha|+|\beta|)}[\gamma,[\alpha,\beta]] \\ + (-1)^{(|\alpha|-1)(|\beta|+|\gamma|)}[\beta,[\gamma,\alpha]] = 0\end{aligned}$$

- $[\ ,\ ]$ satisfies the graded Leibniz identity,

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(|\alpha|-1)|\beta|}\beta \wedge [\alpha, \gamma]$$

where $|\ |$ denotes the degree of $\Gamma(\wedge^\bullet A)$.

The Gerstenhaber bracket allow us to define the A-Lie derivative of sections of $\wedge^\bullet A$ by the formula

$$\mathcal{L}_\alpha \eta = [\alpha, \gamma]$$

for $\alpha \in \Gamma(A)$, $\gamma \in \Gamma(\wedge^\bullet A)$. For homogeneous elements of $\Gamma(\wedge^\bullet A)$ the bracket can be directly defined by

$$\begin{aligned}i_{[\alpha,\beta]}\theta = (-1)^{(|\alpha|-1)(|\beta|-1)} i_\alpha \mathrm{d}(i_\beta \theta) - i_\beta \mathrm{d}(i_\alpha \theta) \\ - (-1)^{(|\alpha|-1)} i_{(\alpha\wedge\beta)} \mathrm{d}\theta\end{aligned}$$

for $\alpha, \beta \in \Gamma(\wedge^\bullet A)$ and $\theta \in \Gamma(\wedge^{|\alpha|+|\beta|-1} A^*)$.

For a Lie algebroid morphism Φ the Gerstenhaber bracket satisfies a bracket preserving property which is easy to state in the case where Φ is an isomorphism, it is

$$\Phi_*[\alpha, \beta] = [\Phi_*\alpha, \Phi_*\beta]$$

for homogeneous sections $\alpha, \beta \in \Gamma(\wedge^\bullet A)$ where the pull-back is defined in the natural way for elements of $(\wedge^\bullet A)$.

For any Lie algebroid A, the anchor map $\rho : A \longrightarrow TM$ is a Lie algebroid morphism and thus satisfies

$$\mathrm{d}_A \rho^* = \rho^* \mathrm{d},$$

here d denotes the exterior differential on $\Omega^\bullet(M)$.

1.3 Poisson structures

Poisson structures originally appeared in geometry through the theory of mechanical systems but are related in many ways to Lie algebroids. In Example 1.2.10 it was noted that every Poisson manifold (M, π) gives rise to a Lie algebroid structure on the cotangent bundle T^*M. On the other hand, for every Lie algebroid A there is a natural associated Poisson structure on its dual bundle A^* as is explained at the end of the section. We follow the exposition of Cannas and Weinstein in [2].

Definition 1.3.1. Let M be a manifold. M is called a *Poisson manifold* if there exist a Lie bracket on $C^\infty(M)$, denoted $\{\,,\,\}$, satisfying the Leibniz identity

$$\{f, gh\} = g\{f, g\} + h\{f, g\} \text{ for } f, g, h \in C^\infty(M).$$

Equivalently, M is called Poisson if there exists a bivector $\pi \in \Gamma(\wedge^2 TM)$ such that $[\pi, \pi] = 0$ for the Schouten bracket. The relation between the two definitions is given by

$$\{f, g\} = \pi(\mathrm{d}f, \mathrm{d}g).$$

As described in Example 1.2.10, the contraction of π by a form in $\Omega^1(M)$ gives us a map $\pi^\sharp : T^*M \longrightarrow TM$ defined by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$ for $\alpha, \beta \in \Omega^1(M)$. This is the anchor map of a Lie algebroid and as such defines a singular foliation on M. Given L a leaf of this singular foliation $\pi^\sharp$ induces a non-degenerate skew-symmetric bundle map $\pi^\sharp_L : T^*L \longrightarrow TL$ that defines a symplectic structure on the leaf L. This singular foliation is called the *symplectic foliation*.

Example 1.3.2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The dual $\mathfrak{g}^*$ is a Poisson manifold with bracket

$$\{f, g\}(u) = u([\mathrm{d}_u f, \mathrm{d}_u g]_{\mathfrak{g}})$$

where the identification $\mathfrak{g}^{**} \simeq \mathfrak{g}$ is made. The symplectic leaves of $\mathfrak{g}^*$ are the orbits of the coadjoint representation of G in $\mathfrak{g}^*$, where G is the simply connected Lie group integrating $\mathfrak{g}$. This Poisson structure is called the Lie-Poisson structure on $\mathfrak{g}^*$.

The bracket in a Poisson manifold M gives rise to a special kind of vector fields. The Leibniz identity shows that $X_f = \{-, f\}$ is a derivation of $C^\infty(M)$ and thus a vector field called the *Hamiltonian vector field* of f. The Jacobi identity implies that the map

$$\begin{aligned} C^\infty(M) &\longrightarrow \mathfrak{X}(M) \\ f &\longmapsto X_f \end{aligned}$$

is an anti-homomorphism of Lie algebras, this is

$$X_{\{f,g\}} = -[X_f, X_g]$$

Another consequence of the Jacobi identity is that

$$X_h\{f,g\} = \{X_h f, g\} + \{f, X_h g\} \text{ for } f, g, h \in C^\infty(M)$$

this in turn implies that the flow of X_h is a one parameter group of Poisson automorphisms.

Definition 1.3.3. Let (M, π_M), (N, π_N) be Poisson manifolds. A map $\varphi : M \longrightarrow N$ is called a *Poisson map* if

$$\{f, g\}_N \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_M \text{ for } f, g \in C^\infty(N).$$

Equivalently, π_M and π_N are φ-related, $\varphi_* \pi_M = \pi_N$, or yet, for every $x \in M$ the following diagram commutes

$$\begin{array}{ccc} T_x^* M & \xrightarrow{\pi_M^\sharp} & T_x M \\ {\scriptstyle \varphi_x^*} \uparrow & & \downarrow {\scriptstyle \mathrm{d}_x \varphi} \\ T_{\varphi(x)}^* N & \xrightarrow[\pi_N^\sharp]{} & T_{\varphi(x)} N \end{array}$$

A submanifold $i : M \longrightarrow N$ of a Poisson manifold (N, π) is called a *Poisson submanifold* if the inclusion i is a Poisson map.

A Poisson map $\varphi : M \longrightarrow N$ is called *complete* if, for each $h \in C^\infty(N)$, X_h being a complete vector field, implies that $X_{\varphi^* h}$ is also complete.

Definition 1.3.4. Let (M, π) be a Poisson manifold and $\mathfrak{g}^*$ a Lie-Poisson manifold. A complete Poisson map $\mu : M \longrightarrow \mathfrak{g}^*$ is called a *momentum map*.

Let G be the connected Lie group integrating $\mathfrak{g}$. A momentum map μ determines an action of G on M, by integrating the following anti-homomorphism

$$\mathfrak{g} \xrightarrow{h.} C^\infty(\mathfrak{g}^*) \xrightarrow{J^*} C^\infty(M) \xrightarrow{X.} \mathfrak{X}(M)$$

where $h_v(u) = u(v)$ for $u \in \mathfrak{g}^*$ and $X.$ assigns to a function its Hamiltonian vector field. The group G acts on M by Poisson automorphisms and μ is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.

Let $\pi : A \longrightarrow M$ be a Lie algebroid. Let us describe a natural Poisson structure on A^* which extends the construction of the linear Poisson structure on the dual of a Lie algebra. Denote the projection of A^* by p. Consider the following classes functions on A^*

- the *basic functions* $f \circ p$, where $f \in C^\infty(M)$, and
- the *evaluation by a section* $\alpha \in \Gamma(A)$, denoted F_α, and defined by

$$F_\alpha(\xi) := \langle \xi, \alpha \rangle \text{ for } \xi \in A^*.$$

where $\langle\ ,\ \rangle$ denote the natural pairing in $A \oplus A^*$. The functions in $C^\infty(A^*)$ are generated by these classes of functions as can be easily seen in local coordinates. To determine a Poisson structure on A^* is enough to define it for basic functions and evaluation functions. The bracket is defined as follows:

(i) The bracket of basic functions is zero.

(ii) The bracket of two evaluation functions is the evaluation function of their Lie brackets

$$\{F_\alpha, F_\beta\}_{A^*} = F_{[\alpha,\beta]} \text{ for } \alpha, \beta \in \Gamma(A)$$

(iii) The bracket of a basic function and an evaluation function is the basic function given by applying the anchor

$$\{F_\alpha, f \circ p\}_{A^*} = \rho(\alpha)(f) \circ p \text{ for } \alpha \in \Gamma(A),\ f \in C^\infty(M)$$

These definitions are compatible with the Jacobi and Leibniz identities, so they define a Poisson bracket on A^*. This bracket on A^* is fiberwise linear, i.e., the bracket of functions linear on the fibers is a function linear on the fibers.

Conversely, any fiberwise linear Poisson structure on the vector bundle A^* induces a Lie algebroid structure on the dual bundle A, whose associated Poisson bracket is the original one.

Morphisms of Lie algebroids can be characterized by the associated Poisson structures as the next proposition states [17],

Proposition 1.3.5. *Let $\Phi : A \longrightarrow B$ be a bundle map between Lie algebroids A and B, over M, covering a diffeomorphism of M. Φ is a Lie algebroid morphism if and only if $\Phi^* : B^* \longrightarrow A^*$ is a Poisson map.*

The following definition is directly related to the notion of a Poisson map.

Definition 1.3.6. Let (N,π) be a Poisson manifold. A submanifold $i : M \longrightarrow N$ is *coisotropic* if the vanishing ideal $\mathcal{I}_M$ of M given by

$$\mathcal{I}_M := \ker(i^* : C^\infty(N) \longrightarrow C^\infty(M))$$

is a Lie subalgebra of the Poisson algebra $(C^\infty(N), \{\ ,\ \})$ corresponding to the Poisson structure π.

Example 1.3.7. Let (N,π) be a Poisson manifold and M a Poisson submanifold. Then M is a coisotropic.

Example 1.3.8. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ of a (finite dimensional) Lie algebra (over $\mathbb{R}$) is a Lie subalgebra if and only if its annihilator $\mathfrak{h}^\circ$ is a coisotropic submanifold of $\mathfrak{g}^*$ with its linear Poisson structure.

Example 1.3.9. A map $\varphi : (M,\pi_M) \to (N,\pi_N)$ is a Poisson map if and only if its graph is a coisotropic submanifold of the product $(M\times N, \pi_M \oplus (-\pi_N))$

The following proposition was proved by Weinstein in [34]

Proposition 1.3.10. *Let $\varphi : (M,\pi_M) \to (N,\pi_N)$ be a Poisson map. If $C \hookrightarrow (N,\pi_N)$ is a coisotropic submanifold of (N,π_N) which is transversal to φ. Then $\varphi^{-1}(C)$ is a coisotropic submanifold of (M,π_M).*

The Example 1.3.8 is generalized in the next proposition (see [17]):

Proposition 1.3.11. *Let $A \longrightarrow M$ be a Lie algebroid and $B \longrightarrow M$ a subbundle of A. Then B is a Lie subalgebroid of A if and only if B° is a coisotropic subbundle of A^*.*

1.4 Orbifolds

In this section we present the classical description of orbifolds by Satake ([29], [30]),for this we follow the exposition on [20]. We may think of an orbifold as a topological space which is locally homeomorphic to a quotient of an euclidean space by the action of a finite group, in other words, a manifold that has a certain type of singularities.

An action of a finite group G on a manifold M is given by a homomorphism $\Phi : G \longrightarrow \mathrm{Diff}(M)$. For now Φ will be injective and we consider G as subgroup of $\mathrm{Diff}(M)$.

The following facts about finite subgroups of $\mathrm{Diff}(M)$ make the basic properties of orbifolds easy to prove and can be found in [20]. Take a finite subgroup G of $\mathrm{Diff}(M)$, and let $x \in M$. We can choose a G-invariant riemannian metric on M. The exponential map associated to the metric gives us a diffeomorphism from an open ball B centered at 0 in the tangent space $T_x(M)$ to an open neighborhood W of x, $\exp_x : B \longrightarrow W \subset M$. Since the metric is G-invariant, $\mathrm{d}_x g$ is an orthogonal transformation of $T_x(M)$ and $\exp_x \circ \mathrm{d}_x g = g \circ \exp_x$, for any g in the isotropy subgroup G_x. In particular, if $\mathrm{d}_x g = \mathrm{id}$ then $g|_W = \mathrm{id}$.

Lemma 1.4.1. *Let M be a connected manifold and G a finite subgroup of* $\mathrm{Diff}\,(M)$.*Then the set of singular points*

$$\Sigma = \{x \in M | G_x \neq \{e\}\}$$

is closed with empty interior and the differential $d_x : G_x \longrightarrow \mathrm{Aut}(T_x M)$ is injective for each $x \in M$.

Note that the lemma implies that any diffeomorphism of finite order on a connected manifold which fixes an open set is the identity. Denote by $\mathcal{O}_x$ the orbit through x of the G-action.

Lemma 1.4.2. *Let M be a manifold and G a finite subgroup of* $\mathrm{Diff}\,(M)$. *For any smooth map $f : V \longrightarrow M$ defined on a non-empty open connected subset V of M, satisfying $f(x) \in \mathcal{O}_x$, for $x \in V$, there exists a unique $g \in G$ such that $f = g|_V$.*

A connected subset S of M is called G-stable if for every $g \in G$ either $gS = S$ or $gS \cap S = \emptyset$. The *isotropy subgroup* of S is defined by

$$G_S = \{g \in G \mid gS = S\}\,.$$

As G is finite, for any $x \in M$ we can always find an arbitrarily small open G-stable neighborhood S of x such that $G_x = G_S$

Let Q be a topological space. An orbifold chart of dimension $n \geq 0$ on Q is a triple (U, G, ϕ), where U is a connected open subset of $\mathbb{R}^n$, G is a finite subgroup of $\mathrm{Diff}\,(U)$ and $\phi : U \longrightarrow Q$ is an open map which induces a homeomorphism $U/G \longrightarrow \phi(U)$. If (U, G, ϕ) is an orbifold chart on Q and S an open G-stable subset of U, the triple $(U, G_S, \phi|_S)$ is again an orbifold chart called the restriction of (U, G, ϕ) to S. More generally, let (V, H, ψ) be another orbifold chart on Q. An embedding $\lambda : (V, H, \psi) \longrightarrow (U, G, \phi)$ between orbifold charts is an embedding $\lambda : V \longrightarrow U$ such that $\phi \circ \lambda = \psi$. The following proposition states some basic properties of embeddings of orbifold charts, its proof can be found in [20] and uses Lemmas 1.4.1 and 1.4.2.

Proposition 1.4.3. *(i) For any embedding $\lambda : (V, H, \psi) \longrightarrow (U, G, \phi)$ between orbifold charts on Q, the image $\lambda(V)$ is a G-stable open subset of U, and there is a unique isomorphism $\bar{\lambda} : H \longrightarrow G_{\lambda(V)}$ for which $\lambda(hx) = \bar{\lambda}(h)\lambda(x)$.*

(ii) The composition of two embeddings between orbifold charts is an embedding between orbifold charts.

(iii) For any orbifold chart (U, G, ϕ), any diffeomorphism $g \in G$ is an embedding of (U, G, ϕ) into itself, and $\bar{g}(g') = gg'g^{-1}$.

(iv) If $\lambda, \mu : (V, H, \psi) \longrightarrow (U, G, \phi)$ are two embeddings between the same orbifold charts, there exists a unique $g \in G$ with $\lambda = g \circ \mu$.

When there is no chance of confusion we will write λ instead of $\bar{\lambda}$.

We say that two orbifold charts (V, H, ψ) and (U, G, ϕ) of dimension n on Q are compatible if for any $z \in \psi(V) \cap \phi(U)$ there exist an orbifold chart (W, K, θ) on Q with $z \in \theta(W)$ and embeddings between orbifold charts $\lambda : (W, K, \theta) \longrightarrow (V, H, \psi)$ and $\mu : (W, K, \theta) \longrightarrow (U, G, \phi)$.

An *orbifold atlas* of dimension n of a topological space Q is a collection of pairwise compatible orbifold charts $\mathcal{U} = \{(U_i, G_i, \phi_i)\}$ of dimension n on Q such that $\cup_{i \in I} \phi_i(U_i) = Q$. Two orbifold atlases of Q are equivalent if their union is an orbifold atlas.

Definition 1.4.4. An *orbifold* of dimension n is a pair $(Q, \mathcal{U})$, where Q is a second countable Hausdorff topological space and $\mathcal{U}$ is a maximal orbifold atlas of dimension n of Q.

By the remarks at the beginning of this section given an orbifold Q there exists an orbifold atlas $\mathcal{U}$ of Q such that $U = \mathbb{R}^n$ and G is a finite subgroup of $O(n)$, for any orbifold chart $(U, G, \phi) \in \mathcal{U}$.

Definition 1.4.5. Let Q and Q' be two orbifolds. A continuous map $f : Q \longrightarrow Q'$ is an *weak orbifold map* (orbifold maps in Satake [29]) if for any $z \in Q$ there exist orbifold charts (U, G, ϕ) of Q with $z \in \phi(U)$ and (V, H, ψ) of Q' and a smooth map $\tilde{f} : U \to V$ such that $\psi \circ \tilde{f} = \phi \circ f$ ($\tilde{f}$ is called a *local lift* of f), or equivalently the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & V \\ \phi \downarrow & & \downarrow \psi \\ \phi(U) & \xrightarrow[f]{} & \psi(V). \end{array}$$

Further, let $U_1 \xrightarrow{\tilde{f}_1} V_1$ and $U_2 \xrightarrow{\tilde{f}_2} V_2$ two local lifts of f, and let $U_1 \xrightarrow{\lambda} U_2$ an embedding, if there exists an embedding $V_1 \xrightarrow{\mu} V_2$ such that

$$\begin{array}{ccc} U_1 & \xrightarrow{\tilde{f}_1} & V_1 \\ \lambda \downarrow & & \downarrow \mu \\ U_2 & \xrightarrow[\tilde{f}_2]{} & V_2. \end{array}$$

the map f is called a *strong orbifold map.*

In particular, given two embeddings $\lambda : (U', G', \phi') \longrightarrow (U, G, \phi)$ $\mu : (V', H', \psi') \longrightarrow (V, H, \psi)$ such that $\tilde{f}(\lambda(U')) \subset \mu(V')$, the map $\mu^{-1} \circ \tilde{f} \circ \lambda$ is a local lift with respect to (U', G', ϕ') and (V', H', ψ'). The orbifold maps $f : Q \longrightarrow \mathbb{R}$ are called the smooth functions on the orbifold Q, we can see that this is equivalent as f having invariant local lifts.

As we will work exclusively with strong orbifold maps we will refer to them simply as orbifold maps and no confusion will arise.

Remark 1.4.6. Orbifolds and their morphisms form a category.

For an orbifold chart (U, G, ϕ) the differential of the action at x gives us a representation of G_x in $GL(\mathbb{R}^n)$, denoted $\mathrm{d} : G_x \longrightarrow GL(\mathbb{R}^n)$. Points in the orbit of x give rise to conjugate subgroups of $GL(\mathbb{R}^n)$. If we have an embedding of orbifold chats $\lambda : (V, H, \psi) \longrightarrow (U, G, \phi)$, any point $y \in V$ such that $\lambda(y) = x$ gives rise to a subgroup $\mathrm{d}H_y$ conjugate to $\mathrm{d}G_x$ in $GL(\mathbb{R}^n)$. We now can define

Definition 1.4.7. Let Q be an orbifold. For $z \in Q$ we define the *isotropy group* of z, denoted $\mathrm{Iso_z}$, as the conjugacy class of $\mathrm{d}G_x$ in $GL(\mathbb{R}^n)$, for an orbifold chart (U, G, ϕ) and $x \in U$ such that $\phi(x) = z$

A few examples to conclude the section

Example 1.4.8. Let M be a manifold. M can be seen as an orbifold with trivial isotropy in every point.

Example 1.4.9. Let M be a manifold and G a connected Lie group acting properly on M such that $\dim O_x = \dim G$ for $x \in M$. Then M/G has the structure of an orbifold as can be seen from the slice theorem since all the isotropy groups of the action are finite.

In fact every orbifold can be seen in this way

Proposition 1.4.10. *Any orbifold is isomorphic to the orbifold associated to an action of a compact connected Lie group G with finite isotropy groups.*

Remark 1.4.11. The definition of orbifolds given here actually correspond to *effective* orbifolds, the name reflects the fact that locally we demand the homomorphism $\Phi : G \longrightarrow \mathrm{Diff}\,(M)$ to be injective (the action is effective). Later we will present a characterization of orbifolds as groupoids in which an effective proper étale groupoid is equivalent to an effective orbifold.

Finally, we can define vector bundles over orbifolds, and this will be useful when we consider later Lie algebroids over orbifolds an more general singular spaces.

Definition 1.4.12. Let $(E,\mathcal{U}^*)$ and $(Q,\mathcal{U})$ be orbifolds with a surjective map $E \xrightarrow{\pi} Q$, F a vector space and $\mathbf{G}$ a Lie group acting linearly on F. The set $(E, Q, \pi, F, \mathbf{G})$ is called a *vector bundle* over Q if for every $z \in Q$ there exist orbifold charts (U^*, G^*, ϕ^*) and (U, G, ϕ) of E and Q respectively. with $z \in \phi(Q)$ and $U^* = U \times F$ making the following diagram commute

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi^*} & \phi^*(U^*) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow[\phi]{} & \phi(U) \end{array}$$

where p is the projection on the fist component. Given an embedding of orbifold charts $\lambda : (V, H, \psi) \longrightarrow (U, G, \phi)$ on Q with $\psi(V) \subset \phi(U)$ there exist a corresponding linear embedding $\lambda^* : (V^*, H^*, \psi^*) \longrightarrow (U^*, G^*, \phi^*)$ of charts on E such that $\psi^*(V^*) \subset \phi^*(U^*)$ and for every $(x, v) \in V^* = V \times F$,

$$\lambda^*(x, v) = (\lambda(x), g_\lambda(x)v)$$

with $g_\lambda : V \longrightarrow \mathbf{G}$. The maps g_λ satisfy $g_{\lambda\circ\mu}(x) = g_\lambda(\mu(x))g_\mu(x)$ for composable embeddings μ and λ on $\mathcal{U}$. Furthermore, there is a correspondence between the charts (U^*, G^*, ϕ^*) and (U, G, ϕ) of E and Q as well as a correspondence of embeddings λ and λ^* for any given of charts.

The correspondence of embeddings for charts (U^*, G^*, ϕ^*) and (U, G, ϕ) implies that the groups G and G^* must be isomorphic. Note that $\mathbf{G}$ may act on F in a non-effective way, if this action is effective the equation for g_λ in the definition is satisfied form the definition of orbifold. The space of sections $\Gamma(E)$ of the vector bundle E is defined by the sections of π for which lift locally as sections of the bundle charts.

The definitions of morphisms of vector bundles over orbifolds are straightforward, the requirement being that the local lifts of the morphisms are morphisms of the respective structures.

Chapter 2

Orbifolds as groupoids

This chapter is devoted to present orbifolds using a global approach, i.e. avoiding the use of local charts. For this we will present a broader concept of maps between Lie groupoids, namely generalized maps and Hilsum-Skandalis maps, that will define equivalent bicategories. Orbifolds then will be identified with equivalence classes of proper étale groupoids.

2.1 Equivalences of groupoids

Here we summarize some of the notions of equivalence of groupoids, these notions come directly from category theory, some distinctions have to be made as we work in the smooth case. We follow the exposition on [20] and [21].

First we introduce some notation for groupoids that will be easier to follow as we will be regarding more than two groupoids at the same time. Let $\mathcal{G}$ be a Lie groupoid, the manifolds of objects and arrows of $\mathcal{G}$ will be denoted G_0 and G_1 respectively. The manifold of arrows from x to y in G_0 will be denoted by $G(x, y)$, in particular the isotropy group $\mathcal{G}_x$ can also be denoted by $G(x, x)$

Definition 2.1.1. A *natural transformation* T between two homomorphisms $\phi, \psi : \mathcal{K} \to \mathcal{G}$ is a smooth map $T : K_0 \to G_1$ with $T(x) : \phi(x) \to \psi(x)$ such that for any arrow $h : x \longrightarrow y$ in K_1, the identity $\psi(h)T(x) = T(y)\phi(h)$ holds, i.e. the following square is commutative,

$$\begin{array}{ccc} \phi(x) & \xrightarrow{T(x)} & \psi(x) \\ {\scriptstyle \phi(h)}\downarrow & & \downarrow{\scriptstyle \psi(h)} \\ \phi(y) & \xrightarrow{T(y)} & \psi(y) \end{array}$$

We write $\phi \sim_T \psi$.

For a categories $\mathcal{K}$, $\mathcal{G}$, we can regard the morphisms between them as objects and the natural transformations as morphisms with composition given by $RT(x) = R(x)T(x)$, where $R : \phi \longrightarrow \psi$, $T : \psi \longrightarrow \rho$ for morphisms ϕ, ψ, ρ between $\mathcal{K}$ and $\mathcal{G}$. In fact there is a structure of 2-category that arises in this way with natural transformations as 2-morphisms.

Definition 2.1.2. A morphism $\phi : \mathcal{K} \longrightarrow \mathcal{G}$ of groupoids is called an *equivalence* of groupoids if there exists a morphism $\psi : \mathcal{G} \longrightarrow \mathcal{K}$ of groupoids and natural transformations T and S such that $\psi \circ \phi \sim_T \mathrm{id}_{\mathcal{K}}$ and $\phi \circ \psi \sim_S \mathrm{id}_{\mathcal{G}}$.

In the categorical sense the previous definition of equivalence amounts to the functor ϕ being fully faithful and essentially surjective, in the Lie groupoid case the second description is weaker as we may not be able to choose a smooth inverse for ϕ. This suggest the following notion

Definition 2.1.3. A morphism $\epsilon : \mathcal{K} \to \mathcal{G}$ of groupoids is called an *essential equivalence* of groupoids if

(i) ϵ is essentially surjective in the sense that

$$\mathbf{t} \circ \pi_1 : G_1 \,{}_{\mathbf{s}}\!\times_{\epsilon} K_0 \longrightarrow G_0$$

is a surjective submersion where $G_1 \,{}_{\mathbf{s}}\!\times_{\epsilon} K_0$ is the pull-back along the source $\mathbf{s}$ of $\mathcal{G}$ and $\epsilon : K_0 \longrightarrow G_0$.

(ii) ϵ is fully faithful in the sense that K_1 is the following pull-back of manifolds:

$$\begin{array}{ccc} K_1 & \xrightarrow{\epsilon} & G_1 \\ \downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} & & \downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} \\ K_0 \times K_0 & \xrightarrow{\epsilon\times\epsilon} & G_0 \times G_0 \end{array}$$

Regarding the categorical counterparts of the previous definition, the first condition implies that for any object $y \in G_0$, there exists an object $x \in K_0$ whose image $\epsilon(x)$ can be connected to y by an arrow $g \in G_1$. The second condition implies that for all x, $z \in K_0$, ϵ induces a diffeomorphism $K(x, z) \to G(\epsilon(x), \epsilon(z))$ between the submanifolds of arrows.

As an example on the spirit of the proofs in this section we present,

Proposition 2.1.4. *[20] Every equivalence of Lie groupoids is an essential equivalence.*

Proof. Let $\phi : \mathcal{K} \longrightarrow \mathcal{G}$ be an equivalence of groupoids, with $\psi : \mathcal{G} \longrightarrow \mathcal{K}$ and T and S as in the definition of equivalence. We prove first that the map

$$\mathbf{t} \circ \pi_1 : G_1 \,{}_{\mathbf{s}}\!\times_{\phi} K_0 \longrightarrow G_0$$

of the definition of essential equivalence above is a surjective submersion. Clearly it is surjective because any $y \in G_0$ is the image of $(T(y), \psi(y))$. To see that it is a submersion, we prove that it has a local section through any point $(g_0 : \phi(x_0) \longrightarrow y_0, x_0)$ of $G_1 \,{}_{\mathbf{s}}\!\times_{\phi} K_0$. To this end, consider the arrow

$$T(y_0)^{-1} g_0 : \phi(x_0) \longrightarrow \phi(\psi(y_0))$$

in $\mathcal{G}$. Since ϕ is an equivalence of categories, there is a unique arrow $k_0 : x_0 \longrightarrow \psi(y_0)$ in $\mathcal{K}$ with $\phi(k_0) = T(y_0)^{-1} h_0$. Let $\lambda : U \to K_1$ be a local bisection through k_0 in $\mathcal{K}$, and let $\tilde{\lambda} = \mathbf{t} \circ \lambda : U \longrightarrow K_0$ be the associated diffeomorphism onto an open neighborhood V of $\psi(y_0)$. Let $\kappa : \psi^{-1}(V) \to G_1 \,{}_{\mathbf{s}}\!\times_{\phi} K_0$ be the map

$$\kappa(y) = (T(y)\phi(\lambda(\tilde{\lambda}^{-1}(\psi(y)))), \tilde{\lambda}^{-1}(\psi(y))).$$

Then κ is a section of $\mathbf{t} \circ \pi_1$ through the given point (g_0, x_0). This proves that $\mathbf{t} \circ \pi_1$ is a surjective submersion. In particular, the fibered product $K_0 \,{}_{\phi}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi} K_0$ of $\mathbf{t} \circ \pi_1$ along $\phi : K_0 \longrightarrow G_0$ is a manifold, which fits into a pull-back diagram

$$\begin{array}{ccc} K_0 \,{}_{\phi}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi} K_0 & \xrightarrow{\pi_2} & G_1 \\ \Big\downarrow{\scriptstyle (\pi_1,\pi_3)} & & \Big\downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} \\ K_0 \times K_0 & \xrightarrow{\phi\times\phi} & G_0 \times G_0 \end{array}$$

Since ϕ is an equivalence of categories, the map $\theta : K_1 \longrightarrow K_0 \,{}_{\phi}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi} K_0$, sending k to $(s(k), \phi(k), t(k))$, is a bijection. To prove that this map is a diffeomorphism we notice that the differential of θ is $(\mathrm{d}_k\mathbf{s}, \mathrm{d}_k\phi, \mathrm{d}_k\mathbf{t})$. A vector v in the kernel of $\mathrm{d}\theta$ must satisfy $v \in T_k\mathcal{K}(x, y)$, for $x = \mathbf{s}(k)$ and $y = \mathbf{t}(k)$, but $\mathrm{d}\phi|_{\mathcal{K}(x,y)}$ is a diffeomorphism so v must vanish. This proves that θ is a local diffeomorphism. □

The converse of Proposition 2.1.4 does not hold for Lie groupoids

Example 2.1.5. Let $p : N \longrightarrow M$ be a surjective submersion. Consider the Lie groupoid $N \times_M N \rightrightarrows N$ defined by the pull-back of manifolds

$$\begin{array}{ccc} N \times_M N & \longrightarrow & N \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ N & \xrightarrow{p} & M, \end{array}$$

and the essential equivalence $N \times_M N \longrightarrow M$ induced by p, regarding M as the unit Lie groupoid. Any morphism $M \longrightarrow N \times_M N$ amounts to choose a section of p. If N is a non-trivial principal bundle over M, then such sections do not exist and $N \times_M N \longrightarrow M$ is not an equivalence of Lie groupoids.

We will make extensive use of the following definition

Definition 2.1.6. Let $\psi : \mathcal{K} \to \mathcal{G}$ and $\phi : \mathcal{L} \to \mathcal{G}$ be morphisms of Lie groupoids, the *weak pull-back* $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a groupoid whose space of objects is

$$(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_0 = K_0 \,{}_{\psi}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi} L_0$$

consisting of triples (x, g, y) with $x \in K_0$, $y \in L_0$ and g an arrow in G_1 from $\psi(x)$ to $\phi(y)$. An arrow between (x, g, y) and (x', g', y') is a pair of arrows (k, l) with $k \in K(x, x')$, $l \in L(y, y')$ such that $g'\psi(k) = \phi(l)g$. The space of arrows can be identified with

$$(\mathcal{K}\times_{\mathcal{G}}\mathcal{L})_1 = K_1 \,{}_{\psi\circ\mathbf{s}}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi\circ\mathbf{s}} L_1 = \{(k, g, l)|\psi \circ \mathbf{s}(k) = \mathbf{s}(g), \phi \circ \mathbf{s}(l) = \mathbf{t}(g)\}\,.$$

which can be obtained by two fibered products

$$\begin{array}{ccccc}
K_1 \,{}_{\psi\circ\mathbf{s}}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi\circ\mathbf{s}} L_1 & & \longrightarrow & & L_1 \\
\downarrow & & & & \downarrow{\scriptstyle \mathbf{s}} \\
K_1 \,{}_{\psi\circ\mathbf{s}}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi} L_0 & \longrightarrow & K_0 \,{}_{\psi}\!\times_{\mathbf{s}} G_1 \,{}_{\mathbf{t}}\!\times_{\phi} L_0 & \xrightarrow{\pi_3} & L_0 \\
\downarrow & & \downarrow{\scriptstyle \pi_1} & & \\
K_1 & \xrightarrow{\mathbf{s}} & K_0 & &
\end{array}$$

in which some notation has been omitted.

If at least one of the two morphisms is a submersion on objects or an essential equivalence, then the weak pull-back $\mathcal{K} \times_{\mathcal{G}} \mathcal{L}$ is a Lie groupoid. In this case, the diagram of Lie groupoids

$$\begin{array}{ccc}
\mathcal{K} \times_{\mathcal{G}} \mathcal{L} & \xrightarrow{\pi_3} & \mathcal{L} \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \phi} \\
\mathcal{K} & \xrightarrow{\psi} & \mathcal{G}
\end{array}$$

commutes up to a natural transformation and it is universal with this property [20].

The next proposition describes some properties of essential equivalences and the weak pull-back which will be necessary to define Morita equivalence of groupoids,

Proposition 2.1.7. *[20] Let $\mathcal{G}, \mathcal{K}$ and $\mathcal{L}$ be Lie groupoids.*

(i) For two homomorphisms $\phi, \psi : \mathcal{L} \longrightarrow \mathcal{G}$, if there is a transformation $T : \phi \longrightarrow \psi$ then ϕ is a essential equivalence if and only if ψ is.

(ii) If for a essential equivalence $\phi : \mathcal{L} \longrightarrow \mathcal{G}$ the map $\mathbf{t}\circ\pi_1$ of the essentially surjective condition has a section, then ϕ is an equivalence.

(iii) The composition of two essential equivalences is an essential equivalence.

(iv) For any essential equivalence $\phi : \mathcal{L} \longrightarrow \mathcal{G}$ and any homomorphism $\psi : \mathcal{K} \to \mathcal{G}$ the weak pull-back

$$\begin{array}{ccc} \mathcal{K} \times_{\mathcal{G}} \mathcal{L} & \xrightarrow{\pi_3} & \mathcal{L} \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{K} & \xrightarrow{\psi} & \mathcal{G} \end{array}$$

exists and π_3 is an essential equivalence for which $(\mathcal{K} \times_{\mathcal{G}} \mathcal{L})_0 \longrightarrow L_0$ is a surjective submersion.

2.2 Morita equivalence

In parallel with equivalence of groupoids essential equivalences should give rise to an equivalence relation of Lie groupoids, the first difficulty is that in general essential equivalences cannot be inverted, this motivates the next definition,

Definition 2.2.1. Two Lie groupoids $\mathcal{K}$ and $\mathcal{G}$ are *Morita equivalent* if there exists a Lie groupoid $\mathcal{J}$ and essential equivalences

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\sigma} \mathcal{G}.$$

Using the properties of the weak pull-back we see that Morita equivalence is in fact an equivalence relation. Moreover, we see from the following diagram of weak pull-backs that there exist $\mathcal{J}$ for which ϵ and σ are surjective submersions on objects

$$\begin{array}{ccccc} \mathcal{J} & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G} \\ \big\downarrow & & \big\downarrow & & \big\| \\ \mathcal{L}' & \longrightarrow & \mathcal{J}' & \longrightarrow & \mathcal{G} \\ \big\downarrow & & \big\downarrow & & \\ \mathcal{K} & = & \mathcal{K} & & \end{array}$$

Example 2.2.2. Let M be a connected manifold. Consider $\Pi_1(M)$ the fundamental groupoid of M. There is a Morita equivalence $\Pi_1(M) \sim_M \pi_1(M, x)$ for $x \in M$, where we regard the fundamental group $\pi_1(M, x)$ as a groupoid over the singleton $\{x\}$.

Example 2.2.3. Let G be a Lie group acting freely and properly on a manifold M. Consider the action groupoid $G \ltimes M$ over M. There is a Morita equivalence $G \ltimes M \sim_M M/G$, where we regard M/G as the unit groupoid.

Many properties of Lie groupoids are stable under essential equivalence, and thus under Morita equivalence, we will be interested particularly in two of them. The proof of the proposition can be found in [20].

Proposition 2.2.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be essentially equivalent Lie groupoids. Then*

(i) $\mathcal{G}$ is Hausdorff if and only if $\mathcal{H}$ is.

(ii) $\mathcal{G}$ is proper if and only if $\mathcal{H}$ is.

The proof of the proposition is simple and relies on the fact that for an essential equivalence $\epsilon : \mathcal{K} \longrightarrow \mathcal{G}$ the bottom row of the pull-back of manifolds

$$\begin{array}{ccc} K_1 & \xrightarrow{\epsilon} & G_1 \\ \downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} & & \downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} \\ K_0 \times K_0 & \xrightarrow{\epsilon\times\epsilon} & G_0 \times G_0 \end{array}$$

can be taken as a surjective submersion if it is part of a Morita equivalence. As any essential equivalence ϵ can be taken as a Morita equivalence $\mathcal{K} \xleftarrow{\text{id}} \mathcal{K} \xrightarrow{\sigma} \mathcal{G}$ the proposition follows.

The property of being étale is not invariant under Morita equivalence but directly from the definition of essential equivalence we see that the property of having discrete isotropy groups is. The following result can be found in [20] or [9]

Proposition 2.2.5. *For a Lie groupoid $\mathcal{G}$, the following are equivalent:*

(i) $\mathcal{G}$ is Morita equivalent to a smooth étale groupoid.

(ii) The Lie algebroid $\mathfrak{g}$ of $\mathcal{G}$ has an injective anchor map.

(iii) All isotropy Lie groups of $\mathcal{G}$ are discrete.

Lie groupoids which have discrete isotropy groups are sometimes called *foliation groupoids*. We will use this name with just a brief further comment from its origin. Given a foliation $\mathcal{F}$ of a manifold M, the holonomy groupoid of $\mathcal{F}$ and the groupoid associated to $\mathcal{F}$ in Section 1.1 (called the monodromy groupoid of $\mathcal{F}$) have discrete isotropy groups. The restriction of these groupoids to a complete transversal of $\mathcal{F}$ gives étale groupoids. In fact, the proof of the previous proposition uses the restriction to a complete transversal of the foliation given by the orbits of $\mathcal{G}$ to produce an étale groupoid Morita equivalent to $\mathcal{G}$.

Remark 2.2.6. Let $\mathcal{G}$ and $\mathcal{K}$ be two Morita equivalent étale groupoids, $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\sigma} \mathcal{G}$, as ϵ and σ are essential equivalences we see that $\mathcal{J}$ must be a foliation groupoid, and thus there exist an essential equivalence $\phi : \mathcal{J}' \longrightarrow \mathcal{J}$ with $\mathcal{J}'$ étale such that $\mathcal{K} \xleftarrow{\epsilon\circ\phi} \mathcal{J}' \xrightarrow{\sigma\circ\phi} \mathcal{G}$ represents the Morita equivalence of $\mathcal{G}$ and $\mathcal{K}$.

The last ingredient we need to characterize orbifolds as groupoids is the notion that relates to effective Lie group actions. Let $\mathcal{G}$ be an étale Lie groupoid, there is a well defined morphism of groupoids, called the *effect* of $\mathcal{G}$,

$$\mathrm{Eff} : \mathcal{G} \longrightarrow \Gamma_{\mathrm{G_0}}$$

where Γ_{G_0} is the groupoid of germs of local diffeomorphisms of the manifold G_0. Eff is given by

$$\mathrm{Eff(g)} = \mathrm{germ}_{\mathbf{s}(\mathrm{g})}(\mathbf{t} \circ \mathbf{s}^{-1}|_{\mathrm{U}})$$

where $g \in \mathcal{G}$ and $U \subset G_0$ such that the source map $\mathbf{s}$ restricts to a diffeomorphism. An étale groupoid is called effective if Eff is injective on arrows.

Definition 2.2.7. A foliation groupoid $\mathcal{G}$ is called effective if it is Morita equivalent to an effective étale groupoid.

This definition is consistent since for any two Morita equivalent étale groupoids $\mathcal{H} \sim_M \mathcal{K}$ remark 2.2.6 shows that one is effective if and only if the other one is.

Remark 2.2.8. An essential equivalence of Lie groupoids $\epsilon : \mathcal{K} \to \mathcal{G}$ induces a homeomorphism $|\epsilon| : |\mathcal{K}| \to |\mathcal{G}|$ between orbit spaces.

2.3 Orbifolds as groupoids

The original characterization of orbifolds as Lie groupoids was given by Pronk in [27].

Proposition 2.3.1. *There is a* $1:1$ *correspondence between orbifold structures on a Hausdorff second countable topological space* Q *and Morita equivalence classes of proper effective foliation groupoids* $\mathcal{G}$ *with* $|\mathcal{G}| = Q$*.*

Proof. Let $\mathcal{U}$ be an orbifold atlas for the orbifold structure on Q, the following construction of a proper étale groupoid $\mathcal{G}$ was hinted by Haefliger and can be found in [23]. Let G_0 be the space of pairs (x, U) with $x \in U \in \mathcal{U}$, topologized as the disjoint union of the sets U in $\mathcal{U}$. An arrow $g : (x, U) \longrightarrow (y, V)$ is an equivalence class of triples

$$g = [\lambda, z, \mu] : U \overset{\lambda}{\longleftarrow} W \overset{\mu}{\longrightarrow} V,$$

where $z \in W$ and $\lambda(z) = x,\ \mu(z) = y$. Here W is another chart for Q, and λ, μ are embeddings. The equivalence relation is generated by

$$[\lambda, z, \mu] = [\lambda \circ \nu, z', \mu \circ \nu],$$

for λ, z, μ as above and $\nu : W' \longrightarrow W$ another embedding, with $\nu(z') = z$. The set of arrows G_1 is given a topology identifying a neighborhood of

g with a neighborhood of z in W, this is well defined because, for any other representative of g, ν is an open embedding. it is clear that the source and target maps $\mathbf{s}, \mathbf{t} : G_1 \longrightarrow G_0$ are both étale. The groupoid $\mathcal{G}$ is proper. Given a sequence $\{g_n\}$ in G_1 such that the map $G_1 \longrightarrow G_0 \times G_0$ takes it to a convergent sequence $((x_n, U), (y_n, V)) \longrightarrow ((x, U), (y, V))$, from the definition of the topology in G_1 we note that there exist λ and μ embeddings of a small neighborhood W into U and V respectively such that $g_n = [\lambda, z_n, \mu]$ with $z_n \in W$ and $\lambda(z_n) = x_n,\ \mu(z_n) = y_n$, then the sequence $\{z_n\}$ converges. Note that $\mathcal{G}$ is effective, given $g = [\lambda, z, \mu]$ such that $\lambda \circ \mu^{-1}$ is the identity implies $g = [\mathrm{id}, z, \mathrm{id}]$.

Now let $\mathcal{G}$ be a proper foliation groupoid such that $|\mathcal{G}| = Q$, as every foliation groupoid is Morita equivalent to an étale one, we can assume that $\mathcal{G}$ is étale. Let $x \in G_0$ and U open subset of G_0 containing x invariant under $G = \mathrm{Eff}(\mathcal{G}_{\mathrm{x}})$, which exists because the isotropy group $\mathcal{G}_x$ is compact. G defines a subgroup of $\mathrm{Diff}\,(U)$ which determines every arrow over U by $(\mathrm{Eff(g), x})$ for $x \in U$ and $g \in G$, this produces an isomorphism of groupoids

$$\mathcal{G}|_U \simeq G \ltimes U.$$

which accordingly shows that $U/G \simeq |\mathcal{G}_U|$ is open in Q. Let $V \subset U$ open subset and $y \in V$, set $H = \mathrm{Eff}(\mathcal{G}_{\mathrm{y}})$ and assume that V is H invariant, the inclusion of $\mathcal{G}_V$ in $\mathcal{G}_U$ translates to an equivariant embedding $\lambda : V \longrightarrow U$. Let $\{U_i\}$ be an open cover formed of open sets as before, set $\mathcal{U} = \{(U_i, G_i, \phi_i)\}$ with $\phi_i : U_i \longrightarrow U_i/G_i$. Then $\mathcal{U}$ is an orbifold structure on Q. □

The construction that assigns a foliation groupoid $\mathcal{G}$ to an orbifold $(Q,\mathcal{U})$ is far from being unique. For every effective orbifold there is a foliation groupoid of the form $G \ltimes M$ the action groupoid associated to a certain compact group as is shown in [20].

The category of orbifolds is equivalent to a category of proper foliation groupoids up to Morita equivalence but with a notion of morphism different from the usual notion of groupoid morphism. They are called generalized maps and are closely related to the notion of Morita equivalence itself (see [27]).

A simple example of this construction are

Example 2.3.2. Consider a finite group G acting on a manifold M. The action groupoid $G \ltimes M$ is a proper foliation Lie groupoid that represents the orbifold defined by the quotient M/G. If M is an open set of $\mathbb{R}^n$ then $G \ltimes M$ coincides Lie groupoid defined by the proposition above.

Its worth noting that in order to obtain an étale groupoid representing an orbifold $(Q,\mathcal{U})$ suffices to take a cover of Q by elements of the atlas $\mathcal{U}$. Let $\{U_i\} \subset \mathcal{U}$ be said covering and form H_0, the disjoint union of the elements of $\{U_i\}$. The new manifold of arrows H_1 is obtained by the pull-back of manifolds

$$\begin{array}{ccc} H_1 = (H_0 \times H_0) \times_{G_0 \times G_0} G_1 & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ H_0 \times H_0 & \longrightarrow & G_0 \times G_0 \end{array}$$

We have that $H \longrightarrow G$ is an essential equivalence and H is a groupoid representing the orbifold $(Q,\mathcal{U})$ (see[27]).

Example 2.3.3. The tear-drop orbifold has a natural covering by two orbifold chats, given by two discs $D_0, D_1 \subset \mathbb{R}^2$, and actions of $\{0\}$ and $\mathbb{Z}^2$ by a rotation, respectively. The manifold of arrows consist on three discs D_0, D_1^a, D_1^b and two annuli $A_0 \subset D_0$ and $A_1 \subset D_1$. The source and target maps are given as follows (first column for source and second for target):

$$\begin{aligned} D_0 &\xrightarrow{0} D_0,\ D_0 \xrightarrow{0} D_0 \\ D_1^a &\xrightarrow{0} D_1,\ D_1^a \xrightarrow{0} D_1 \\ D_1^a &\xrightarrow{0} D_1,\ D_1^a \xrightarrow{1} D_1 \\ A_0 &\xrightarrow{0} D_0,\ A_0 \xrightarrow{1} D_1 \\ A_1 &\xrightarrow{1} D_0,\ A_1 \xrightarrow{0} D_1, \end{aligned}$$

where 0 indicates the embedding as a subset, and 1 indicates the action.

2.4 Generalized maps

As an orbifold can be seen as a Morita equivalence class of certain groupoids the notion of maps between orbifolds can also be translated to groupoid language. Since form the point of view of groupoids an orbifold is defined up to an equivalence, the maps between orbifolds must not depend on the representative of the equivalence class. In what follows we will describe a notion of maps for groupoids related to Morita equivalence that restricts to morphisms of orbifolds when working in that category.

Until the end of the chapter we follow closely the exposition in [5].

Definition 2.4.1. A *generalized map* from $\mathcal{K}$ to $\mathcal{G}$ is a pair of morphisms

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

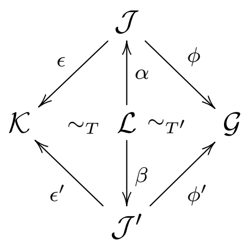
such that ϵ is an essential equivalence. We denote a generalized map by (ϵ, ϕ).

A generalized map from $\mathcal{K}$ to $\mathcal{G}$ morally can be regarded as a groupoid morphism from $\mathcal{K}$ to $\mathcal{G}$ in which the domain can be replaced by an essentially equivalent groupoid $\mathcal{J}$.

Two generalized maps from $\mathcal{K}$ to $\mathcal{G}$, $\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$ and $\mathcal{K} \xleftarrow{\epsilon'} \mathcal{J}' \xrightarrow{\phi'} \mathcal{G}$, are *isomorphic* if there exists a groupoid $\mathcal{L}$ and essential equivalences

$$\mathcal{J} \xleftarrow{\alpha} \mathcal{L} \xrightarrow{\beta} \mathcal{J}'$$

such that the diagram



commutes up to natural transformations. We write $(\epsilon, \phi) \sim (\epsilon', \phi')$.

In other words, there are natural transformations T and T' such that the generalized maps

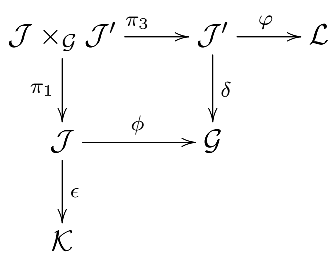
$$\mathcal{K} \xleftarrow{\epsilon\alpha} \mathcal{L} \xrightarrow{\phi\alpha} \mathcal{G}$$
$$\mathcal{K} \xleftarrow{\epsilon'\beta} \mathcal{L} \xrightarrow{\phi'\beta} \mathcal{G}$$

satisfy $\epsilon\alpha \sim_T \epsilon'\beta$ and $\phi\alpha \sim_{T'} \phi'\beta$.

For each groupoid $\mathcal{G}$ a unit arrow $(\mathrm{id}, \mathrm{id})$ is defined as the generalized map $\mathcal{G} \xleftarrow{\mathrm{id}} \mathcal{G} \xrightarrow{\mathrm{id}} \mathcal{G}$. The composition of two arrows $(\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G})$ and $(\mathcal{G} \xleftarrow{\delta} \mathcal{J}' \xrightarrow{\varphi} \mathcal{L})$ is given by the generalized map:

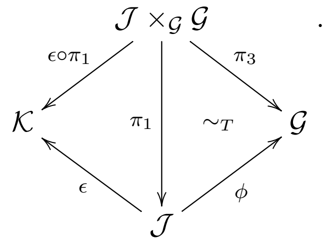
$$\mathcal{K} \xleftarrow{\epsilon\circ\pi_1} \mathcal{J} \times_{\mathcal{G}} \mathcal{J}' \xrightarrow{\varphi\circ\pi_3} \mathcal{L}$$

where π_1 and π_3 are the projections in the following weak pull-back of groupoids:



The morphism π_1 is an essential equivalence since it is the weak pull-back of the essential equivalence δ. Then $\epsilon \circ \pi_1$ is an essential equivalence. This composition is associative up to isomorphism.

The unit arrow is a left and right unit for this multiplication of arrows up to isomorphism. The composition of $(\mathcal{K} \overset{\epsilon}{\longleftarrow} \mathcal{J} \overset{\phi}{\longrightarrow} \mathcal{G})$ with $(\mathcal{G} \overset{\mathrm{id}}{\longleftarrow} \mathcal{G} \overset{\mathrm{id}}{\longrightarrow} \mathcal{G})$ is the generalized map $\mathcal{K} \overset{\epsilon\circ\pi_1}{\longleftarrow} \mathcal{J} \times_{\mathcal{G}} \mathcal{G} \overset{\pi_3}{\longrightarrow} \mathcal{G}$. Then $\pi_3 \sim_T \phi \circ \pi_1$ where T is the natural transformation that makes the weak pull-back square commute and π_1 is an essential equivalence. We have that $(\epsilon \circ \pi_1, \phi \circ \pi_1) \sim (\epsilon, \phi)$ as it is easily seen by the diagram



Remark 2.4.2. Note that the invertible generalized maps are exactly the Morita equivalences.

There is an equivalence relation between the isomorphism diagrams of generalized maps. If we set a *2-isomorphism* as an equivalence class of diagrams, then groupoids, generalized maps and isomorphisms form a bicategory. Vertical and horizontal composition of diagrams are defined in a natural way using weak pull-backs and they satisfy the coherence axioms for a bicategory. The proof of these facts can be found in [25] where one can find also the general construction of a bicategory of fractions. The previous exposition is a particular case of these bicategories.

If we restrict ourselves to the orbifold case it is easy to see from the construction in the last section that an orbifold map $f : Q \longrightarrow Q'$ gives a morphism of proper étale groupoids $F : \mathcal{G} \longrightarrow \mathcal{G}'$. Changing the atlas of the orbifold Q does not change the map f but it will lead to Morita equivalent groupoids representing Q, hence the need for generalized maps.

2.5 Hilsum-Skandalis maps

Another approach to generalize morphisms between Lie groupoids is given by Hilsum-Skandalis maps. Although the relation with 2-morphisms of generalized maps is not immediate, the bicategory arising from this point of view is equivalent. In this setting morphisms are defined by right principal bibundles of Lie groupoids. We follow the exposition in [5].

Definition 2.5.1. Let M be a manifold, $\mathcal{G}$ a groupoid and $\mu : M \longrightarrow G_0$ a smooth map. A *right action* of $\mathcal{G}$ on M is a map

$$\begin{aligned} M \ {}_{\mu}\!\times_{\mathbf{t}} G_1 &\longrightarrow M \\ (x, g) &\longmapsto xg \end{aligned}$$

defined on $M\ {}_\mu\!\times_{\mathbf{t}} G_1$ given by the following pull-back of manifolds along the target map:

$$\begin{array}{ccc} M\ {}_\mu\!\times_{\mathbf{t}} G_1 & \longrightarrow & M \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ G_1 & \xrightarrow{\mathbf{t}} & G_0 \end{array}$$

and satisfying $\mu(xg) = \mathbf{s}(g)$, $x1_x = x$, $(xg)h = x(gh)$ for $(x, g) \in M \times^{\mathbf{t}}_{G_0} G_1$ and $h \in G_1$ composable with g.

Analogously, we have a *left action* by considering the pull-back $G_1\ {}_\mu\!\times_{\mathbf{s}} M$ along the source map and demanding $\mu(gx) = \mathbf{t}(g)$, $1_x x = x$, $h(gx) = (hg)x$ for $(g, x) \in G_1\ {}_\mu\!\times_{\mathbf{s}} M$ and $h \in G_1$ composable with g.

The *action groupoid* $M \rtimes \mathcal{G}$ associated to a right action of $\mathcal{G}$ on M is defined in a similar manner to the case of a group acting on a manifold by $(M \rtimes \mathcal{G})_0 = M$ and $(M \rtimes \mathcal{G})_1 = M\ {}_\mu\!\times_{\mathbf{t}} G_1$ where the source map is given by the action $\mathbf{s}(x, g) = xg$ and the target map is just the projection $\mathbf{t}(x, g) = x$.

A groupoid $\mathcal{G}$ acting on a manifold M may have extra structure

Definition 2.5.2. A *right $\mathcal{G}$-bundle* M over B is a map $\pi : M \to B$ with a right action of $\mathcal{G}$ on M preserving the fibers. A right $\mathcal{G}$-bundle is *principal* if the map

$$\begin{aligned} M\ {}_\mu\!\times_{\mathbf{t}} G_1 &\longrightarrow M \times_B M \\ (x, g) &\longmapsto (xg, x) \end{aligned}$$

is a diffeomorphism and π is a surjective submersion.

Remark 2.5.3. When $\mathcal{G}$ is a group G, this notion corresponds to the usual notion of G-principal bundle.

Definition 2.5.4. A *$\mathcal{KG}$-bibundle* M is a left $\mathcal{K}$-bundle over G_0 as well as a right $\mathcal{G}$-bundle over K_0 for which the actions commute. A $\mathcal{KG}$-bibundle is represented by the following diagram:

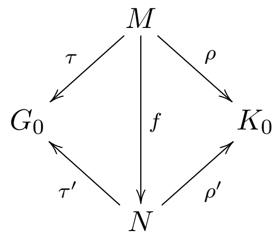
$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle \tau}\swarrow & & \searrow{\scriptstyle \rho} & \\ K_0 & & & & G_0 \end{array}$$

where ρ is a left $\mathcal{K}$-bundle and τ is a right $\mathcal{G}$-bundle. Denote a $\mathcal{KG}$-bibundle M as $(\mathcal{K}, M, \mathcal{G})$ and write $(\mathcal{K}, \rho, M, \mathcal{G}, \tau)$ if the bundle maps need to be specified.

A $\mathcal{KG}$-bibundle M is *right principal* if the right $\mathcal{G}$-bundle $\tau : M \to K_0$ is principal. In this case, $M\ {}_\mu\!\times_{\mathbf{t}} G_1$ is diffeomorphic to $M \times_{K_0} M$ and $M/\mathcal{G}$ is diffeomorphic to K_0. Left principal bundles are defined analogously.

A $\mathcal{KG}$-bibundle M is called *biprincipal* if both left and right bundles ρ and τ are principal.

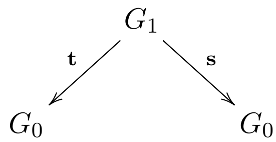
Two $\mathcal{KG}$-bibundles M and N are *isomorphic* if there is a diffeomorphism $f : M \to N$ that intertwines the maps $M \longrightarrow G_0$, $M \longrightarrow K_0$ with the maps $N \longrightarrow G_0$, $N \longrightarrow K_0$ and also intertwines the $\mathcal{K}$ and $\mathcal{G}$-actions. In other words, $f(hxg) = hf(x)g$ and $\tau = \tau' \circ f$, $\rho = \rho' \circ f$ as in the following diagram. We write $(\mathcal{K}, M, \mathcal{G}) \sim (\mathcal{K}, N, \mathcal{G})$.



Definition 2.5.5. A Hilsum-Skandalis map $|(\mathcal{K}, M, \mathcal{G})|$ is an isomorphism class of right principal $\mathcal{KG}$-bibundles.

The collection of groupoids, right principal bibundles and and equivariant diffeomorphisms form a bicategory which can be defined as follows.

For each groupoid $\mathcal{G}$ the unit arrow is defined as the $\mathcal{GG}$-bibundle

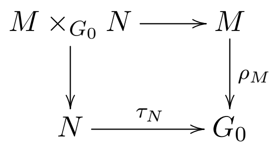


The left and right actions of $\mathcal{G}$ on G_1 are given by the left and right multiplication in the groupoid $\mathcal{G}$.

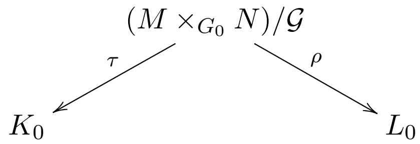
To construct the multiplication of arrows we need the following proposition [20].

Proposition 2.5.6. *Let $\pi : M \longrightarrow B$ be a principal $\mathcal{G}$-bundle, let N be a manifold with a right $\mathcal{G}$-action, and let $f : N \longrightarrow M$ be a submersion preserving the $\mathcal{G}$-action. Then $N/\mathcal{G}$ is a manifold and the quotient map $N \longrightarrow N/\mathcal{G}$ is a principal $\mathcal{G}$-bundle.*

The multiplication of arrows $(\mathcal{K}, M, \mathcal{G})$ and $(\mathcal{G}, N, \mathcal{L})$ is given by the bibundle $(\mathcal{K}, (M \times_{G_0} N)/\mathcal{G}, \mathcal{L})$ where $M \times_{G_0} N$ is the pull-back of manifolds



and in addition, $\mathcal{G}$ acts on the manifold $M \times_{G_0} N$ on the right by $(x, y)g = (xg, g^{-1}y)$. The orbit space is a $\mathcal{KL}$-bibundle

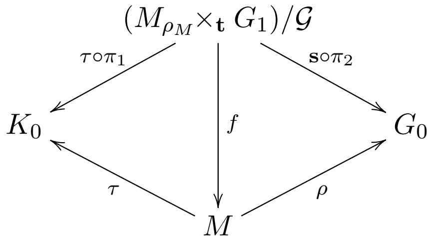


where $\tau([x, y]) = \tau_M(x)$ and $\rho([x, y]) = \rho_N(y)$. The left $\mathcal{K}$-action is given by $k[x, y] = [kx, y]$ and the right $\mathcal{L}$-action by $[x, y]l = [x, yl]$. Note that $(M \times_{G_0} N)/\mathcal{G}$ is a manifold by Proposition 2.5.6. This bibundle is right principal and the multiplication is associative up to isomorphism.

The unit arrow $(\mathcal{G}, G_1, \mathcal{G})$ is a left and right unit for this multiplication of arrows up to isomorphism. We have that the multiplication of $(\mathcal{K}, M, \mathcal{G})$ and $(\mathcal{G}, G_1, \mathcal{G})$ is isomorphic to $(\mathcal{K}, M, \mathcal{G})$ since the map

$$\begin{aligned} (M_{\rho_M}\times_{\mathbf{t}} G_1)/\mathcal{G} &\xrightarrow{f} M \\ [x, g] &\longmapsto xg \end{aligned}$$

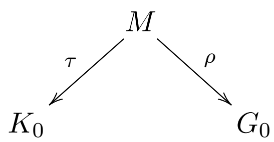
is a diffeomorphism satisfying $f(k[g, x]h) = kf([g, x])h$ and gives a 2-morphism



2.6 Hilsum-Skandalis and generalized maps

We present a process to get from the bicategory of groupoids, generalized maps and isomorphisms to the category of groupoids, right principal bibundles and equivariant diffeomorphisms and vice versa. This process is described in [22] and more details can be found in [5].

Given a right principal $\mathcal{K}\mathcal{G}$-bibundle M:



where ρ is a left $\mathcal{K}$-bundle and τ is a right principal $\mathcal{G}$-bundle, construct a generalized map

$$\mathcal{K} \xleftarrow{\bar{\tau}} \mathcal{K} \ltimes M \rtimes \mathcal{G} \xrightarrow{\bar{\rho}} \mathcal{G}$$

where the *double action groupoid* $\mathcal{K} \ltimes M \rtimes \mathcal{G}$ is defined by $(\mathcal{K} \ltimes M \rtimes \mathcal{G})_0 = M$ and $(\mathcal{K} \ltimes M \rtimes \mathcal{G})_1 = K_1 \,{}_{\mathbf{s}}\times_{\tau} M \,{}_{\rho}\times_{\mathbf{s}} G_1$, this space of arrows is obtained by the following pull-backs of manifolds:

$$
\begin{array}{ccccc}
K_1 \,{}_{\mathbf{s}}\!\times_\tau M \,{}_\rho\!\times_{\mathbf{s}} G_1 & \longrightarrow & & & G_1 \\
\downarrow & & & & \downarrow{\scriptstyle s} \\
K_1 \,{}_{\mathbf{s}}\!\times_\tau M & \xrightarrow{\pi_2} & M & \xrightarrow{\rho} & G_0 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \tau} & & \\
K_1 & \xrightarrow{\mathbf{s}} & K_0 & &
\end{array}
$$

and is has the form $(\mathcal{K} \ltimes M \rtimes \mathcal{G})_1 = \{(k,x,g) \mid s(k) = \tau(x) \text{ and } s(g) = \rho(x)\}$ with $\mathbf{s}(k,x,g) = x$ and $\mathbf{t}(k,x,g) = kxg^{-1}$. The composition of arrows is given by $(k', kxg-1, g')(k,x,g) = (k'k, x, g'g)$ and the groupoid morphisms $\bar{\rho}$, $\bar{\tau}$ are defined on arrows by the projections

$$\bar{\tau}(k,x,g) = k, \qquad \bar{\rho}(k,x,g) = g$$

The notation $\mathcal{K} \ltimes M \rtimes \mathcal{G}$ is used as a mnemonic, a more appropriate notation would be $(\mathcal{K} \times \mathcal{G}) \ltimes M$ as the construction first transforms the right action of $\mathcal{G}$ in M into a left action and then forms the action groupoid for the product groupoid [26]. Since τ is a principal bundle $\bar{\tau}$ becomes an essential equivalence.

If $(\mathcal{K}, M, \mathcal{G}) \sim (\mathcal{K}, N, \mathcal{G})$ then the associated generalized maps (τ, ρ) and (τ', ρ') are isomorphic. Let $f : M \to N$ be the equivariant diffeomorphism that intertwines the bundles. Define

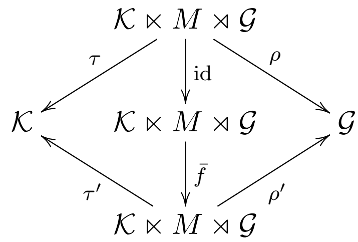
$$\bar{f} : \mathcal{K} \ltimes M \rtimes \mathcal{G} \longrightarrow \mathcal{K} \ltimes N \rtimes \mathcal{G}$$

by $\bar{f} = f$ on objects and $\bar{f}(h,x,g) = (h, f(x), g)$ on arrows. These maps commute with all the structural maps by the equivariance of f. Since $\bar{f}_0$ is a diffeomorphism, it is in particular a surjective submersion and the manifold of arrows $K_1 \,{}_{\mathbf{s}}\!\times_\tau M \,{}_\rho\!\times_{\mathbf{s}} G_1$ is obtained from the following pull-back of manifolds:

$$
\begin{array}{ccc}
K_1 \,{}_{\mathbf{s}}\!\times_\tau M \,{}_\rho\!\times_{\mathbf{s}} G_1 & \xrightarrow{\bar{f}} & K_1 \,{}_{\mathbf{s}}\!\times_{\tau'} N \,{}_{\rho'}\!\times_{\mathbf{s}} G_1 \\
{\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\
M \times M & \xrightarrow{\bar{f} \times \bar{f}} & N \times N
\end{array}
$$

then $\bar{f}$ is an essential equivalence. Also, as f intertwines the bundles, we have that $\rho' = \rho \circ \bar{f}$ and $\tau' = \tau \circ \bar{f}$ and follows that $(\tau, \rho) \sim (\tau', \rho')$.

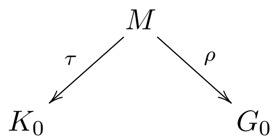
For 2-morphisms $f : M \to N$, consider the following diagram



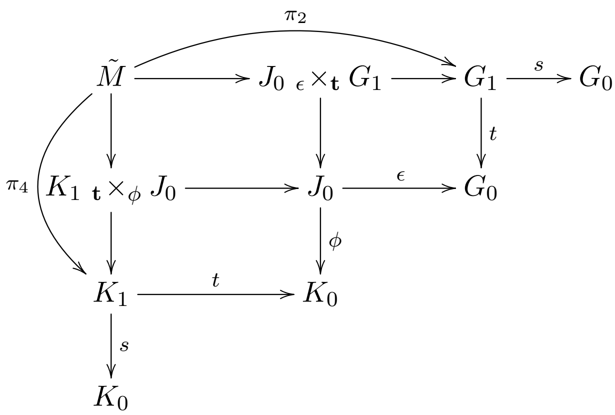
Conversely, given a generalized map from $\mathcal{K}$ to $\mathcal{G}$

$$\mathcal{K} \xleftarrow{\epsilon} \mathcal{J} \xrightarrow{\phi} \mathcal{G}$$

construct an associated right principal $\mathcal{K}\mathcal{G}$-bibundle M



where M is the quotient by the action of $\mathcal{J}$ on $\tilde{M} = K_1 \,{}_{\mathbf{t}}\!\times_{\phi} J_0 \,{}_{\epsilon}\!\times_{\mathbf{t}} G_1$ given by the following pull-backs of manifolds:



The maps τ and ρ are induced in the quotient by $s \circ \pi_4$ and $s \circ \pi_2$. The action of $\mathcal{J}$ on $\tilde{M}$ is given by $((a, b, d), j) \longmapsto (\epsilon(j)a, \mathbf{t}(j), \phi(j)d)$. The left action of $\mathcal{K}$ on $M = \tilde{M}/\mathcal{J}$ is given by

$$K_1 \,{}_{\mathbf{s}}\!\times_{\tau} M \longrightarrow M, \qquad (k, [a, b, d]) \mapsto [ak^{-1}, b, d]$$

and the right action of $\mathcal{G}$ by

$$M \,{}_{\rho}\!\times_{\mathbf{t}} G_1 \to M, \qquad ([a, b, d]), g) \mapsto [a, b, dg].$$

If $(\epsilon, \phi) \sim (\epsilon', \phi')$ then the associated bibundles $(\mathcal{K}, M, \mathcal{G})$ and $(\mathcal{K}, N, \mathcal{G})$ are isomorphic.

It can be proved that the the correspondences give a weak biequivalence of bicategories (composition is only preserved up to 2-morphisms).

Chapter 3

Lie algebroids over quotient spaces

Our main objective is to analyse the behaviour of Lie algebroids over quotient spaces. For this we start presenting reduction results for the non-singular case, then we consider the singular case. For a proper action of a Lie group G on a Lie algebroid A by Lie algebroid automorphisms we give reductions of the Lie algebroid structure to the orbit type manifolds in the quotient M/G. Then we move on to define Lie algebroids over orbifolds in the naive way using charts and transport this definition to the groupoid setting. Finally we define Lie algebroids over singular spaces represented by Lie groupoids which satisfy some specific properties.

3.1 Non-singular quotients of Lie algebroids

A non-singular quotient is defined by a proper and free action of a Lie group G. Here we consider free and proper actions of G on Lie algebroids and Lie bialgebroids.

3.1.1 Group actions on Lie algebroids

We describe some generalities concerning group actions on Lie algebroids by Lie algebroid automorphisms

Let G be a Lie group acting on a Lie algebroid A by algebroid automorphisms, we will denote the action by $\Phi : G \times A \longrightarrow A$ and its restriction to M by ϕ. There is a canonical G-action on A^* associated to Φ given by $\Phi^*_{g^{-1}}$, this is

$$(\Phi^*_{g^{-1}}\xi)_x(v) = \xi_{g^{-1}x}(\Phi_{g^{-1}}v) \text{ for } \xi \in (\Gamma(A^*)).$$

The action of G on A^* covers the same map as the action of G on A as is noted in the following diagram

$$
\begin{array}{ccc}
A^* & \xrightarrow{\Phi^*_{g^{-1}}} & A^* \\
\downarrow & & \downarrow \\
M & \xrightarrow[\phi_g]{} & M.
\end{array}
$$

Moreover, the bundle morphisms $\Phi^*_{g^{-1}}$ are Poisson maps for the natural Poisson structure in A^*.

At the infinitesimal level a G-action on an Lie algebroid A gives rise to a fiberwise linear vector field on A. There is a correspondence between fiberwise linear vector fields on a vector bundle E and $\mathrm{Der}(E)$ the set of derivations of E. Using this correspondence, the infinitesimal counterpart of the action of G on A is given by a Lie algebra morphism

$$\phi_* : \mathfrak{g} \longrightarrow \mathrm{Der}(A),$$

where $\mathfrak{g}$ is the Lie algebra associated to G. The fact that G acts by Lie algebroid morphisms translates to $\phi_*(\xi)$, for $\xi \in \mathfrak{g}$, being not only a derivation of the vector bundle A but also a derivation of the bracket, i.e.

$$\phi_*(\xi)[X, Y] = [\phi_*(\xi)X, Y] + [X, \phi_*(\xi)Y]$$

for $X, Y \in \Gamma(A)$. These are called derivations of the Lie algebroid A and are denoted by $\mathrm{Der}(A)$. The commutator of derivations makes $\mathrm{Der}(A)$ into a Lie algebra.

Definition 3.1.1. Let A be a Lie algebroid and $\mathfrak{g}$ a Lie algebra. An *infinitesimal action* of a Lie algebra $\mathfrak{g}$ on a Lie algebroid A is a Lie algebra homomorphism $\zeta : \mathfrak{g} \longrightarrow \mathrm{Der}(A)$.

If every derivation $\zeta(\xi)$ is complete (i.e., can be integrated to a one parameter group of automorphisms of A), the infinitesimal action ζ can be integrated to an action of G on A by Lie algebroid automorphisms, where G is the simply connected Lie group that integrates $\mathfrak{g}$.

An exposition on actions of Lie groups and Lie algebras on Lie algebroids can be found in [12].

Let $A \longrightarrow M$ be a Lie algebroid and G a connected Lie group, a free and proper action of G on A endows the orbit space of the action with a Lie algebroid structure:

Proposition 3.1.2. *Let $\Phi : G \times A \longrightarrow A$ be a free and proper action of G on A by Lie algebroid morphisms. Then $A/G \longrightarrow M/G$ has a Lie algebroid structure.*

This easy proposition leads to consider the same type of quotients for more complicated structures, like Lie bialgebroids, as well as actions with more structure, as inner actions of Lie groups on Lie algebroids.

3.1.2 Inner actions

In this section, we extend the concept of momentum map from Poisson geometry to the dual of a Lie algebroid, considering an interesting class of infinitesimal actions of a Lie algebra $\mathfrak{g}$ on a Le algebroid A given by:

Definition 3.1.3. An infinitesimal action $\zeta : \mathfrak{g} \longrightarrow \mathrm{Der}(A)$ of $\mathfrak{g}$ on a Lie algebroid A is called an *inner action* if

$$\zeta(\xi) = [\nu(\xi), -]$$

for any $\xi \in \mathfrak{g}$ and a Lie algebra morphism $\nu : \mathfrak{g} \longrightarrow \Gamma(A)$, i.e. ζ and ν commute in the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ \nu\ } & \Gamma(A) \\ & \underset{\zeta}{\searrow} & \downarrow{\scriptstyle ad} \\ & & \mathrm{Der}(A). \end{array}$$

This motivates the following definition:

Definition 3.1.4. Let G be a Lie group acting on an Lie algebroid A by Lie algebroid morphisms. The action is called *inner* if the infinitesimal action is inner with respect to a Lie algebra morphism $\nu : \mathfrak{g} \longrightarrow \Gamma(A)$.

In the presence of an inner action ν of a Lie algebra $\mathfrak{g}$ on a Lie algebroid A a certain subbundle of A inherits a Lie algebroid structure.

Proposition 3.1.5. *Let $A \xrightarrow{\pi} M$ be a Lie algebroid and $\nu : \mathfrak{g} \longrightarrow \Gamma(A)$ a map defining a complete inner action of a Lie algebra $\mathfrak{g}$ on A. If the bundle map ν^* has constant rank and the origin is a regular value, then $(\nu^*)^{-1}(0)^\circ$ is a Lie subalgebroid of A and inherits a G-action where G is the simply connected Lie group integrating $\mathfrak{g}$.*

For the proof we will restate this proposition in the language of Poisson geometry.

The dual map of ν is a vector bundle morphism $\nu^* : A^* \longrightarrow \mathfrak{g}^*$, defined by

$$\langle \nu^*(\alpha), \xi \rangle = \langle \alpha, \nu(\xi)|_{p(\alpha)} \rangle,$$

for $\alpha \in A^*$ and $\xi \in \mathfrak{g}$, where p is the projection of A^* to M. The fact that ν is a Lie algebra morphism implies that ν^* is a Poisson map with respect to the natural Poisson structure on A^* and the Lie-Poisson structure on $\mathfrak{g}^*$. ν^* is a complete Poisson map, i.e., a momentum map, because the inner action of $\mathfrak{g}$ is complete. On the other hand, given a bundle map $\nu^* : A^* \longrightarrow \mathfrak{g}^*$ which is also a complete Poisson map, with respect to the appropriate structures, the dual map on sections, ν, will be a complete inner action of $\mathfrak{g}$ on A.

We can now restate the last proposition in terms of Hamiltonian actions with respect to the natural Poisson structure in A^* and apply proposition 1.3.11 to the kernel of a linear momentum map.

Proposition 3.1.6. *Let A be a Lie algebroid and $\mu : A^* \longrightarrow \mathfrak{g}^*$ a momentum map linear on fibers and of constant rank for which the origin is a regular value. Then $\mu^{-1}(0)^\circ$ is a Lie subalgebroid of A and inherits a G-action by restriction of the G-action on A.*

Proof. As μ is fiberwise linear of constant rank and the origin is a regular value $\mu^{-1}(0)$ is a subbundle of A^*. For the coadjoint action of G on $\mathfrak{g}^*$ the origin is a fixed point, therefore a coadjoint orbit and a symplectic leaf for the Lie-Poisson structure, and thus a coisotropic submanifold. The bundle $\mu^{-1}(0)$ is coisotropic in A^* as an inverse image of a coisotropic submanifold of $\mathfrak{g}^*$. Now we can apply Proposition 1.3.11 and conclude that $\mu^{-1}(0)^\circ$ is a Lie subalgebroid of A. It rests to prove that the G-action lifted to A restrict to $\mu^{-1}(0)^\circ$. Note that the G-action restricts to $\mu^{-1}(0)$ by the equivariance of μ. Given $v \in \mu^{-1}(0)^\circ$

$$\langle \xi, gv \rangle = \langle g^{-1}\xi, v \rangle = 0 \text{ for } \xi \in \mu^{-1}(0)$$

as $g^{-1}\xi \in \mu^{-1}(0)$, this shows that $gv \in \mu^{-1}(0)^\circ$. □

Example 3.1.7. Let G be a Lie group acting freely and properly on a manifold M. Consider the lifted tangent action of G on TM. With respect to the natural Lie algebroid structure on TM, G acts on TM by Lie algebroid automorphisms. For the lifted cotangent G-action on T^*M there is a canonical momentum map with respect to the natural Poisson (symplectic) structure in T^*M,

$$\begin{aligned} j : T^*M &\longrightarrow \mathfrak{g}^* \\ \alpha &\longmapsto (\xi \mapsto \langle \alpha, X_\xi \rangle) \end{aligned}$$

where X_ξ is the infinitesimal generator of the G-action on M. Denote by $\mathcal{O}$ the orbit foliation of the G-action. The subbundle $j^{-1}(0)^\circ \subset TM$ is the tangent distribution to the orbits of the G-action, $T\mathcal{O}$, so its fibers have the form

$$j^{-1}(0)^\circ_x = \{X_\xi|_x \text{ for } \xi \in \mathfrak{g}\}$$

for $x \in M$. We can write the following exact sequence of bundles over M,

$$0 \longrightarrow j^{-1}(0)^\circ \longrightarrow TM \longrightarrow N\mathcal{O} \longrightarrow 0,$$

where $N\mathcal{O}$ denotes the normal bundle to the foliation $\mathcal{O}$. The lifted tangent G-action on TM is free and proper, and, passing to the quotient, we can write the following exact sequence of bundles over M/G,

$$0 \longrightarrow j^{-1}(0)^\circ/G \longrightarrow TM/G \longrightarrow T(M/G) \longrightarrow 0.$$

Note that TM/G is the Atiyah algebroid of the G-principal bundle $M \longrightarrow M/G$.

Let $\mathfrak{g}$ be the Lie algebra of G and let $M \times \mathfrak{g}$ be the associated action algebroid. The map

$$\begin{aligned} \Psi : M \times \mathfrak{g} &\longrightarrow j^{-1}(0)^{\circ} \\ (x, \xi) &\longmapsto X_{\xi}|_x \end{aligned}$$

is an isomorphism of Lie algebroids. Ψ is equivariant with respect to the G-action on $j^{-1}(0)^{\circ}$ (the restriction of the lifted action of G on TM) and the diagonal action of G on $M \times \mathfrak{g}$, where the action on the second factor is the adjoint action of G on $\mathfrak{g}$. Indeed, the identity

$$\mathrm{d}\Phi_g X_{\xi}|_x = X_{\mathrm{Ad}_g \xi}|_{\Phi_g(x)}$$

readily implies the equivariance of Ψ. Passing to the quotient we have isomorphic Lie algebroids

$$j^{-1}(0)^{\circ}/G \simeq \mathrm{Ad}_G \mathfrak{g} := M \times_G \mathfrak{g},$$

where $M \times_G \mathfrak{g}$ denotes the associated bundle, over M/G, with fiber $\mathfrak{g}$ to the principal G-action on M defined by $M \times \mathfrak{g}/G$.

3.1.3 Lie bialgebroids

This section presents results on free and proper Hamiltonian actions on Lie bialgebroids, some of which can be found in [12].

A brief exposition on Lie bialgebroids is necessary for this, and the subsequent, sections.

Definition 3.1.8. Let A be a Lie algebroid. A is called a *Lie bialgebroid* if there exist a degree 1 derivation d_* of the graded algebra $(\Gamma(\wedge^{\bullet} A), \wedge)$ satisfying the following conditions

(i) $\mathrm{d}_*^2 = 0$

(ii) $\mathrm{d}_*[X, Y] = [\mathrm{d}_* X, Y] + [X, \mathrm{d}_* Y]$ for $X, Y \in (\Gamma(A)$

The derivation d_* can be regarded as the exterior differential on the bundle A^* thus it defines a Lie algebroid structure in A^*. The anchor and the bracket of A^* will be denoted ρ_* and $[\ ,\ ]_*$ respectively. A Lie bialgebroid can be regarded as a pair of Lie algebroids (A, A^*) for which the exterior differential of A^* is a derivation of the bracket of A.

The definition of Lie bialgebroid is symmetric, the proof of the next proposition can be found in [17]

Proposition 3.1.9. *Let (A, A^*) be a Lie bialgebroid. Then* *(i)* d_* *is a derivation of the Gerstenhaber bracket of* A*, i.e.,*

$$\mathrm{d}_*[\xi,\eta] = [\mathrm{d}_*\xi,\eta] + [\xi,\mathrm{d}_*\eta] \text{ for } \xi,\eta \in (\Gamma(\wedge^\bullet A)),$$

(ii) (A^*, A) *is a Lie bialgebroid.*

Definition 3.1.10. Let (A, A^*), (B, B^*) be Lie bialgebroids. A map $\Phi : A \longrightarrow B$ is a *Lie bialgebroid morphism* if it is a Lie algebroid morphism and a Poisson map.

Examples of Lie bialgebroids are

Example 3.1.11. A Poisson manifold (M, π) gives rise to a Lie bialgebroid (TM, T^*M) where TM is has the standard Lie algebroid structure and T^*M is the cotangent Lie algebroid of M. The differential of T^*M is given by $\mathrm{d}_\pi = [\pi, -]$.

Example 3.1.12. Let A be a Lie algebroid. Consider the zero Lie algebroid structure on A^*, then (A, A^*) defines a Lie bialgebroid.

A feature of Lie bialgebroids is that they endow the base manifold M with a Poisson structure given by

$$\{f, g\} = \langle \mathrm{d}f, \mathrm{d}_*g \rangle,$$

this is the *Poisson structure induced by* (A, A^*) on M, if we denote its bivector by π_M then we have

$$\pi_M^\sharp = \rho_* \circ \rho^* = -\rho \circ \rho_*^*.$$

As a useful class of Lie bialgebroids come from generalizing a Poisson bivector to the context of Lie algebroids and can be found in [17]

Definition 3.1.13. Let A be a Lie algebroid and $\Lambda \in (\Gamma(\wedge^2 A))$. Λ is called an A-Poisson structure on the Lie algebroid A if $[\Lambda, \Lambda] = 0$ for the Gerstenhaber bracket.

Imitating the constructions for the cotangent algebroid in Example 1.2.10 of a Poisson manifold leads to

Proposition 3.1.14. *Let* Λ *be an* A*-Poisson structure on a Lie algebroid* A*, then* (A, A^*) *is a Lie bialgebroid.*

Proof. The Lie algebroid structure in A is the given one and in A^* we have anchor $\rho_* : A^* \longrightarrow TM$ given by $\rho_* = \rho \circ \Lambda^\sharp$ and bracket

$$[\xi, \eta]_* = \mathcal{L}_{\Lambda^\sharp(\xi)}(\eta) - \mathcal{L}_{\Lambda^\sharp(\eta)}(\xi) - \mathrm{d}(\Lambda(\xi, \eta))$$

for $\xi, \eta \in \Gamma(A^*)$. □

Lie bialgebroids arising this way are called *exact* Lie bialgebroids. The differential of A^* is given by $\mathrm{d}_* = [\Lambda, -]$ and $\Lambda^\sharp$ is a morphism of Lie algebroids.

Remark 3.1.15. The identity $\pi_M^\sharp = \rho_* \circ \rho^*$ implies that the Poisson structure on M defined by (A, A^*) is given in terms of the bisection Λ as

$$\{f, g\} = \Lambda(\mathrm{d}_A f, \mathrm{d}_A g). \tag{3.1.1}$$

Let $F : A \longrightarrow A$ be a Lie algebroid isomorphism over a map φ which is also an A-Poisson morphism, i.e., $F_*\Lambda = \Lambda$. As it would be expected F is a Lie bialgebroid morphism. The fact that F is A-Poisson renders the commutative diagram

$$\begin{array}{ccc} A^* & \xrightarrow{\Lambda^\sharp} & A \\ {\scriptstyle F^*}\uparrow & & \downarrow{\scriptstyle F} \\ A^* & \xrightarrow[\Lambda^\sharp]{} & A. \end{array} \tag{3.1.2}$$

which gives F^* the compatibility with the anchor

$$\begin{array}{ccc} A^* & \xrightarrow{\rho\circ\Lambda^\sharp} & TM \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle \mathrm{d}\varphi^{-1}} \\ A^* & \xrightarrow[\rho\circ\Lambda^\sharp]{} & TM. \end{array}$$

A computation using the definition of $[\ ,\ ]_*$ via d_* proves compatibility with the bracket. This shows that for an exact Lie bialgebroid the notion of automorphism in the categories of Lie bialgebroids and of Lie algebroids endowed with A-Poisson structures coincide.

The following easy proposition gives a Lie bialgebroid structure to a non-singular quotient

Proposition 3.1.16. *Let (A, A^*) be a Lie bialgebroid and $\Phi : G \times A \longrightarrow A$ a free and proper action of G on A by Lie bialgebroid morphisms. Then $(A/G, A^*/G)$ over M/G has a Lie bialgebroid structure and the induced Poisson structure on M/G induced by $(A/G, A^*/G)$ coincides with the one induced by π_M.*

Next we consider the case of Hamiltonian actions on Lie bialgebroids. We follow the exposition in [12] where proofs of many of the following facts can be found.

For the case of a Lie bialgebroid (A, A^*), a group G acting on A defines an action by Lie bialgebroid morphisms if and only if Φ_g and Φ_g^* are Lie algebroid morphisms for A and A^* respectively. In this case we have that the infinitesimal $\mathfrak{g}$-action on A and A^*, denoted by $\phi_* : \mathfrak{g} \longrightarrow \mathrm{Der}(A)$ and $\phi^* : \mathfrak{g} \longrightarrow \mathrm{Der}(A^*)$ respectively, satisfy the duality condition

$$\bar{\phi}(\xi)\langle X, \eta\rangle = \langle \phi_*(\xi)X, \eta\rangle + \langle X, \phi^*(\xi)\eta\rangle$$

for $\xi \in \mathfrak{g}$, $X \in \Gamma(A)$, $\eta \in \Gamma(A^*)$. Where $\bar{\phi}$ denotes the infinitesimal action of $\mathfrak{g}$ on M common to both actions of G on A and A^*.

Definition 3.1.17. Let $\Phi : G \times A \longrightarrow A$ be an action of a Lie group G on a a Lie bialgebroid (A, A^*) by Lie bialgebroid morphisms with infinitesimal action $\phi_* : \mathfrak{g} \longrightarrow \mathrm{Der}(A)$. It is called a *Hamiltonian* action if there exists an equivariant Lie algebroid morphism $j : A \longrightarrow \mathfrak{g}^*$ satisfying the *momentum map* condition:

$$\phi^*(\xi) = -[j^*(\xi), -]_* \text{ for } \xi \in \mathfrak{g}$$

When $\mathfrak{g}^*$ is considered as an abelian Lie algebra, $(\mathfrak{g}^*, \mathfrak{g}^{opp})$ has the structure of a Lie bialgebra (i.e., a Lie bialgebroid over a point), where $\mathfrak{g}^{opp}$ denotes the Lie algebra structure on on the vector space $\mathfrak{g}$ with the opposite sign. Differentiating the equivariance condition

$$j(ga) = \mathrm{Ad}_g^*(a)$$

with respect to the group variable yields

$$\phi^*(\xi) \circ j^* = -j^* \circ \mathrm{ad}_\xi$$

which can be restated as

$$-j^*[\xi, \eta] = [j^*\xi, j^*\eta]_* \text{ for } \xi, \eta \in \mathfrak{g}$$

and implies that $-j$ is a Poisson map with respect to the Poisson structure on A induced by A^*. This proves the following proposition

Proposition 3.1.18. *If $G \times A \longrightarrow A$ is a Hamiltonian G-action on a Lie bialgebroid, the momentum map $j : (A, A^*) \longrightarrow (\mathfrak{g}^*, \mathfrak{g}^{opp})$ is a Lie bialgebroid morphism.*

The momentum map condition for j in Definition 3.1.17 relates to the moment map condition for $-j$ as a Poisson map by noticing that the fiber-wise linear vector field on A defining the derivation $\phi_*(\xi) \in \mathrm{Der}(A)$ is Hamiltonian with Hamiltonian function $a \mapsto \langle j(a), \xi\rangle$.

By definition of Hamiltonian action of G on a Lie bialgebroid (A, A^*) the infinitesimal action on A^* is inner. For exact Lie bialgebroids we have that ϕ_* is also inner

Proposition 3.1.19. *Let Φ be a Hamiltonian action of a Lie group G on a Lie bialgebroid (A, A^*) with momentum map $j : A \longrightarrow \mathfrak{g}^*$. If (A, A^*) is* *an exact Lie bialgebroid then the infinitesimal action* $\phi_* : \mathfrak{g} \longrightarrow \mathrm{Der}(A)$ *is inner with associated map*

$$\begin{aligned} \nu : \mathfrak{g} &\longrightarrow \Gamma(A) \\ \xi &\longmapsto -\pi^\sharp \circ j^*(\xi) \end{aligned}$$

For a Hamiltonian action on Lie bialgebroids the presence of a momentum map leads to the following reduction result.

Proposition 3.1.20. *Let* (A, A^*) *be a Lie bialgebroid over* M *and let* $G \times A \to A$ *be a free Hamiltonian action with equivariant momentum map* $j : A \longrightarrow g^*$*. Then,* $A_{red} := j^{-1}(0)/G \longrightarrow M/G$ *is endowed with a Lie algebroid structure. The dual bundle* A^*_{red} *is also equipped with a Lie algebroid structure and* (A_{red}, A^*_{red}) *is a Lie bialgebroid over* M/G*. Moreover, the Poisson structure on* M/G *induced by the Lie bialgebroid coincides with the reduction of* π_M*.*

For a section X of A_{red} the differential d_{*red} is defined by the projection of the differential d_* for a G-invariant section $\tilde{X}$ of A associated to X, this is

$$\mathrm{d}_{*red}X = \mathrm{proj}_{A_{red}}(\mathrm{d}_*\tilde{X}).$$

Example 3.1.21. Let G be a Lie group acting on a Poisson manifold M by Poisson diffeomorphisms. The lifted action is a Hamiltonian action on the Lie bialgebroid (T^*M, TM) with momentum map $j : T^*M \longrightarrow \mathfrak{g}^*$ given by:

$$\langle j(\alpha), \xi \rangle = \langle \alpha, X_\xi \rangle \text{ for } \alpha \in \Omega^1(M), \xi \in \mathfrak{g}$$

where X_ξ represents the infinitesimal generator of the G-action on M. If the original action is proper and free, so is the lifted action. The reduced Lie bialgebroid (A_{red}, A^*_{red}) given by the last proposition is canonically isomorphic to $(T^*(M/G), T(M/G))$, the Lie bialgebroid determined by the quotient Poisson structure on M/G.

Hamiltonian G-actions on exact Lie bialgebroids have inner infinitesimal action on A, the following definition is inspired on Poisson groupoids as the objects that integrate Lie bialgebroids and will imply that the infinitesimal actions are inner as in the exact Lie bialgebroid case.

Definition 3.1.22. A Hamiltonian G-action on a Lie bialgebroid (A, A^*) with momentum map $j : A \longrightarrow \mathfrak{g}^*$ is called an exact Hamiltonian action if there exists a smooth map $\mu : M \longrightarrow \mathfrak{g}^*$ such that:

$$j = -\mathrm{d}\mu \circ \rho.$$

By abuse notation we also call $\mu : M \longrightarrow \mathfrak{g}^*$ the momentum map. Clearly, if $\mu : M \longrightarrow \mathfrak{g}^*$ is equivariant so is $j : A \longrightarrow \mathfrak{g}^*$. For an exact Hamiltonian action, the infinitesimal actions are both inner:

Proposition 3.1.23. *Let $G \times A \longrightarrow A$ be an exact Hamiltonian action with momentum map $\mu : M \longrightarrow \mathfrak{g}^*$. The infinitesimal actions are given by:*

$$\phi_*(\xi) = -[\mathrm{d}_* \hat{\mu}_\xi, -], \ \phi^*(\xi) = -[\mathrm{d}\hat{\mu}_\xi, -]_* \text{ for } \xi \in \mathfrak{g}$$

where $\hat{\mu}_\xi : M \longrightarrow \mathbb{R}$ is the function $\hat{\mu}_\xi(x) = \langle \mu(x), \xi \rangle$.

An inner action Φ of a Lie group G on a Lie algebroid A induces an action Φ^{TG} of the tangent group $TG \simeq G \times \mathfrak{g}$ on A by

$$\Phi^{TG}((g, \xi), a_x) = \Phi_g(a_x + \nu(\xi)(x))$$

where $\nu : \mathfrak{g} \longrightarrow \Gamma(A)$ defines the inner action. The maps Φ_g^{TG} are affine maps for each $g \in G$ and, as a consequence, Φ^{TG} is not by Lie algebroid automorphisms. However, the map Φ^{TG} is a Lie algebroid morphism from the direct product Lie algebroid $TG \times A$ to A. A reduction for exact Hamiltonian actions is stated in the following proposition:

Theorem 3.1.24. *Let (A, A^*) be a Lie bialgebroid over M and Φ be an exact Hamiltonian action of a Lie group G on A, with equivariant momentum map $\mu : M \longrightarrow \mathfrak{g}^*$. Let μ^T be the map*

$$\begin{aligned} \mu^T : A &\longrightarrow T\mathfrak{g}^* \simeq \mathfrak{g}^* \oplus \mathfrak{g}^* \\ a_x &\longmapsto \mu^T(a_x) = (\mathrm{d}\mu \circ \rho(a_x), \mu(x)) \end{aligned}$$

Then,

(i) $\tilde{A}_c = (\mu^T)^{-1}(0, c)/TG_c \longrightarrow \mu^{-1}(c)/G_c$ is a Lie algebroid.

(ii) The dual bundle $((\mu^T)^{-1}(0, c)/TG_c)^* \longrightarrow \mu^{-1}(c)/G_c$ has a Lie algebroid structure.

(iii) $(\tilde{A}_c; \tilde{A}_c^*)$ is a Lie bialgebroid over $\mu^{-1}(c)/G_c$.

For a proof, we refer to [18].

3.2 Singular quotients of Lie algebroids

This section considers how proper actions of a Lie group G on Lie algebroids and Lie bialgebroids, which are not necessarily free, endow the orbit type manifolds in the quotient with similar structures. First we consider the Lie algebroid case and then we generalize the result to Lie bialgebroids.

3.2.1 Proper group actions

Before proceeding to G-actions on Lie algebroids let us recall the basic results about quotient spaces of group actions on manifolds, for the proofs see [10].

Consider a left smooth proper action of a Lie group G on a manifold M

$$\Phi : G \times M \longrightarrow M,$$

let $\pi : M \longrightarrow M/G$ be the projection to the orbit space.

For each closed Lie subgroup H of G define the *isotropy type* set

$$M_H = \{x \in M | G_x = H\}$$

where $G_x = \{g \in G | gx = x\}$ is the isotropy group of $x \in M$. Note that the action being proper implies that the isotropy groups are compact. The sets M_H, where H ranges over the closed Lie subgroups of G, form a partition of M and therefore they define an equivalence relation in M. The normalizer of H in G is the set $N(H) = \left\{g \in G | gHg^{-1} = H\right\}$ which is a closed Lie subgroup of G. H is a normal subgroup of $N(H)$ and therefore the quotient

$$L(H) = N(H)/H$$

is a Lie group.

If $x \in M_H$, we have $G_x = H$ and $G_g x = gHg^{-1}$, for all $g \in G$. Therefore, $gx \in M_H$ if and only if $g \in N(H)$. The action of G on M restricts to an action of $N(H)$ on M_H which induces a free and proper action of $L(H)$ on M_H. Define the *orbit type* set

$$M_{(H)} = \{x \in M | G_x \in (H)\}$$

where (H) is the conjugacy class of H, and

$$M^H = \{x \in M | hx = x, h \in H\}$$

the set of fixed points of H. Then

$$M_H = M_{(H)} \cap M^H$$

and M_H is an open subset of M^H.

The connected components of M^H, M_H and $M_{(H)}$ are embedded submanifolds of M and therefore M_H and $M_{(H)}$ are called isotropy type and orbit type manifolds. Moreover,

$$M_{(H)}/G = \left\{gx | x \in M_{(H)}\right\}/G \simeq M_H/N(H) \simeq M_H/L(H) \qquad (3.2.1)$$

The action of $L(H)$on M_H is free and proper, therefore $M_H/L(H)$ is a manifold. By (3.2.1) the set $M_{(H)}/G$ is a manifold contained in the orbit space M/G and it is called the *orbit type manifold in the quotient* M/G. There is a natural explicit isomorphism between $M_H/L(H)$ and $M_{(H)}/G$ given by

$$\begin{aligned} F_H : M_H/L(H) &\longrightarrow M_{(H)}/G \\ L(H)x &\longmapsto Gx. \end{aligned}$$

The manifolds M_H, $M_{(H)}$ form stratifications of the manifold M and of the space M/G respectively, they are called the isotropy type and orbit type stratifications. Proves of these facts can be found in [24] and [10].

The dimension of the connected components of the manifolds M^H, M_H and $M_{(H)}$ may vary, by an abuse of notation every time we write M^H, M_H and $M_{(H)}$ we are fixing a connected component of the respective manifold.

3.2.2 Lie algebroids

In this section we consider non-free actions of Lie groups on Lie algebroids by Lie algebroid automorphisms and endow the orbit type manifolds of the actions with reduced Lie algebroid structures.

If a Lie group G acts on a Lie algebroid A by Lie algebroid automorphisms, abstracting from the process described in [11] we can endow the orbit type manifolds in the quotient $A_{(H)}/G$ with Lie algebroid structures over $M_{(H)}/G$.

Theorem 3.2.1. *Let $A \longrightarrow M$ be a Lie algebroid and G a Lie group acting properly on A by Lie algebroid morphisms, and let H be an isotropy subgroup of G. Then $A_{(H)}/G \longrightarrow M_{(H)}/G$ has a Lie algebroid structure.*

Note that as the action of G on A is in particular by vector bundle morphisms, this implies that there is a canonical G-action on the base M regarding it as the zero section of A. We denote the action of G on M by Φ as it is showed in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi_g} & A \\ \downarrow & & \downarrow \\ M & \xrightarrow[\phi_g]{} & M. \end{array}$$

The Lie algebroid structures on $A_{(H)}$ fit into a stratified space as they form a stratification of A/G (see [24]). The proof will use the isomorphism $M_{(H)}/G \simeq M_H/L(H)$. The following lemmas will allow us to endow A^H with a Lie algebroid structure.

Lemma 3.2.2. *Let $A \longrightarrow M$ be a Lie algebroid, L a submanifold of M such that the restriction of the anchor to L satisfies $\rho(A|_L) \subset TL$. Then*

$A|_L \longrightarrow L$ is a Lie subalgebroid of A with anchor map ρ and the restriction of the bracket of A.

Proof. Let $\{l_i\}$ be a local basis of sections of $\Gamma(A)$. If $f \in C^\infty(M)$ such that $f|_L = 0$, then the Leibniz identity

$$[l_i, fl_j]_x = f(x)[l_i, l_j]_x + \rho(l_i|_x)(f)l_j|_x$$

for $x \in L$ implies $[l_i, fl_j]|_L = 0$. Here $\rho(l_i|_x)(f)$ represents the Lie derivative of f with respect to the vector $\rho(l_i|_x)$. Let $\alpha = \sum_i a_i l_i$ such that $a^i|_L = 0$, $\beta = \sum_j b_j l_j$ for $\alpha_i,\ \beta_j \in C^\infty(M)$, and let $x \in M$ then $[\alpha, \beta] = 0$, indeed by Leibniz and the argument above we see that

$$\begin{aligned}
[\alpha, \beta]_x &= \sum_{i,j} [a_i l_i, b_j l_j]_x \\
&= \sum_{i,j} b_j(x)[a_i l_i, l_j]_x + \sum_{i,j} a_i(x)\rho(l_i|_x)(b_j)l_j \\
&= 0
\end{aligned}$$

This implies that the value at $x \in L$ of the bracket of $\alpha,\ \beta \in \Gamma(A)$ only depends on the restrictions $\alpha|_L$ and $\beta|_L$. Moreover, given $\tilde{a}_i,\ \tilde{b}_j \in C^\infty(M)$ such that $\tilde{a}_i|_L = a_i$ and $\tilde{b}_j|_L = b_j$ we have

$$\begin{aligned}
[\alpha, \beta] &= [\tilde{\alpha}, \tilde{\beta}]|_L \\
&= \sum_{i,j} [\tilde{a}_i l_i, \tilde{b}_j l_j]|_L \\
&= \sum_{i,j} (\tilde{a}_i \tilde{b}_j [l_i, l_j] + \rho(\tilde{\alpha})(\tilde{b}_j)l_j - \rho(\tilde{\beta})(\tilde{a}_i)l_i)|_L \\
&= \sum_{i,j} a_i b_j [l_i, l_j]|_L + \sum_j \rho(\alpha)(b_j)l_j|_L - \sum_i \rho(\beta)(a_i)l_i|_L
\end{aligned}$$

which proves that $A|_L$ is a Lie subalgebroid of A. ☐

Lemma 3.2.3. *Let $A \longrightarrow M$ be a Lie algebroid, $B \longrightarrow L$ a subbundle of A such that the restriction of the anchor to L satisfies $\rho(B) \subset TL$. If in addition*

$$[\Gamma(B), \Gamma(B)] \subset \Gamma(B)$$

for the bracket in A then $B \longrightarrow L$ is a Lie subalgebroid with anchor map ρ and the restriction of the bracket of A.

Proof. Let $\{l_i\}$ be a local basis of sections of $\Gamma(B)$. By local triviality of A and the fact that L is an immersed submanifold we can extend $\{l_i\}$ to local sections of A. Completing $\{l_i\}$ to a basis of sections of $\Gamma(A)$ and applying the previous lemma show the result holds. $\square$

In order to endow the fixed point set of a G-action on a Lie algebroid A we first show that it has a vector bundle structure

Lemma 3.2.4. *Let H be a compact Lie group acting on a vector bundle $E \xrightarrow{\pi} M$ by vector bundle morphisms. Then $E^H \longrightarrow M^H$ is a vector bundle.*

Proof. Let μ be a bi-invariant measure in G (i.e., a measure which is invariant by left and right translations on the group). Let $U \subset M$ be a trivializing G invariant open set, $x_0 \in U$ a fixed point for the action and φ a diffeomorphism such that

$$\begin{array}{ccc} E|_U & \xrightarrow{\varphi} & U \times E_{x_0} \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \pi_1} & \\ U & & \end{array}$$

Define a new diffeomorphism $\hat{\varphi} : E|_U \longrightarrow U \times E_{x_0}$ by

$$\hat{\varphi}(e) = \int_G g^{-1}\varphi(ge)\mathrm{d}\mu,$$

it is easy to see that $\hat{\varphi}$ is G equivariant, with respect to the diagonal action in the codomain. As G acts on E by vector bundle morphisms, note that $\pi_1 \circ \varphi(ge) = \pi(ge) = g\pi(e)$ hence $\pi_1 \circ \hat{\varphi} = \pi_1 \circ \varphi$ and $\hat{\varphi}$ is a new trivialization. From this we see that the fixed points A^G form a vector bundle over M^G with fiber the vector subspace $E_{x_0}^G \subset E_{x_0}$. $\square$

The next proposition shows that the fixed points of the action A^H have a Lie algebroid structure

Proposition 3.2.5. *Let $A \longrightarrow M$ be a Lie algebroid and H a compact Lie group acting on A by Lie algebroid morphisms. Then $A^H \longrightarrow M^H$ has a Lie algebroid structure.*

Proof. Appealing to Lemma 3.2.3 all we need to prove is that $\rho(A^H) \subset TM^H$ and

$$[\Gamma(A^H), \Gamma(A^H)] \subset \Gamma(A^H).$$

By the invariance of ρ we see that $\rho(A^H) \subset (TM)^H$, where $(TM)^H$ is the fixed point set for the lifted action of H on TM. it is easy to see that $TM^H \subset (TM)^H$. As H is compact the action linearises at points in M^H, by counting dimensions we conclude that $TM^H = (TM)^H$ and the first condition of Lemma 3.2.3 is satisfied.

Denote the action of H on A by $\Phi : H \times A \longrightarrow A$. By hypothesis the map Φ_h is a Lie algebroid automorphism, hence we can write

$$\Phi_h[\alpha, \beta] = [\Phi_h^* \alpha, \Phi_h^* \beta] \circ \phi_h \tag{3.2.2}$$

for $\alpha, \beta \in \Gamma(A)$. Here $\Phi_h^* \alpha$ represents the push-forward of sections defined by

$$(\Phi_h^* \alpha)(x) = \Phi_h \alpha \circ \phi_{h^{-1}}(x).$$

For $\alpha, \beta \in \Gamma(A)$ such that $\alpha|_{M^H}, \beta|_{M^H} \in \Gamma(A^H)$ and $x \in M^H$ we have $(\Phi_h^* \alpha)_x = \alpha_x$ and $(\Phi_h^* \beta)_x = \beta_x$. From (3.2.2) we conclude that $\Phi_h[\alpha, \beta]_x = [\alpha, \beta]_x$ and the proposition follows. □

Now we show that in the setting of Theorem 3.2.1 the bundle $A_{(H)}/G$ over $M_{(H)}/G$ has indeed a Lie algebroid structure.

Proof of Theorem 3.2.1. As H is an isotropy group it must be compact, by Proposition 3.2.5 the bundle $A^H \longrightarrow M^H$ is a Lie algebroid. The action of $L(H)$ is free and proper on M_H, which is an open subset of M^H, therefore the action $L(H)$ on A_H is also free and proper, and by Lie algebroid automorphisms. We conclude that the quotient $A_H/L(H)$ inherits a Lie algebroid structure. We denote by $\Gamma(A^H)^{L(H)}$ the set of $L(H)$ invariant sections of A^H.

Using the diffeomorphism $F_H : A_H/L(H) \longrightarrow A_{(H)}/G$ the Lie algebroid structure of $A_H/L(H)$ is transported to $A_{(H)}/G$. Given $H_1, H_2 \in (H)$ the map $F_{H_2}^{-1} \circ F_{H_1}$ is an isomorphism of Lie algebroids and thus the Lie algebroid structure on $A_{(H)}/G$ is well defined. □

Remark 3.2.6. Note that the bracket on A^H although defined using extensions it is independent from the extensions chosen. Also these need not be invariant extensions in general.

3.2.3 Lie bialgebroids

Similarly to the case of Lie groups acting on Lie algebroids we have the following proposition for the fixed point set of a Lie bialgebroid

Proposition 3.2.7. *Let (A, A^*) be a Lie bialgebroid and G a compact Lie group acting on (A, A^*) by Lie bialgebroid morphisms. Then, the set of fixed points of the G-action, $(A^G, (A^G)^*)$ becomes a Lie bialgebroid, where the Lie bracket is defined by*

$$[X|_{M^G}, Y|_{M^G}]_{A^G} = [X, Y]|_{M^G}$$

and differential operator on $(A^G)^*$ *by*

$$\mathrm{d}_{(A^G)^*} X|_{M^G} = (\mathrm{d}_* X)|_{M^G}$$

for $X, Y \in \Gamma(A)$ *such that* $X|_{M^G}, Y|_{M^G} \in \Gamma(A^G)$.

The last proposition readily implies that there is a Lie bialgebroid structure in $(A_{(H)}/G, (A^*)_{(H)}/G)$, for an isotropy group H. Recall that there is a free and proper action of $L(H) = N(H)/H$ on A_H such that $A_H/L(H) \simeq A_{(H)}/G$. As $N(H)$ acts on A^H by Lie bialgebroid morphisms the next proposition follows:

Proposition 3.2.8. *Let* (A, A^*) *be a Lie bialgebroid and* G *a Lie group acting properly on* (A, A^*) *by Lie bialgebroid morphisms. Then, given an isotropy group* H*, the orbit type bundle* $(A_{(H)}/G, (A^*)_{(H)}/G)$ *has a Lie bialgebroid structure that depends only on the conjugacy class of* H*.*

Before giving the proof of Proposition 3.2.7 we shall prove the following lemma, which extends the case of the tangent bundle considered in [11] and its proof applies virtually unchanged.

Let A and G be as in the hypothesis of Proposition 3.2.7. Fix a G-invariant metric $(\ ,\ )$ on A and let

$$E = \left\{ v \in A_{M^G} | (v, w) = 0, \forall w \in A^G \right\} \subset A_{M^G}$$

be the orthogonal subbundle to A^G.

Lemma 3.2.9. $A_{M^G} = A^G \oplus E$ *and* $E = \left[(A^*_{M^G})^G\right]^\circ$.

Proof. Since $E = (A^G)^\perp$ it is clear that $A_{M^G} = A^G \oplus E$. We should prove

$$E^\circ = (A^*_{M^G})^G.$$

If $v \in A_{M^G}$ we can decompose it as $v = v_G + v_E$, where $v_G \in (A_{M^G})^G$ and $v_E \in E$. Hence, for $\xi \in E^0$ we find

$$\begin{aligned} \Phi^*_{g^{-1}} \xi(v_G + v_E) &= \xi(\Phi_{g^{-1}} v_G + \Phi_{g^{-1}} v_E) = \xi(v_G) + \xi(\Phi_{g^{-1}} v_E) = \xi(v_G) \\ &= \xi(v_G) + \xi(v_E) = \xi(v_G + v_E). \end{aligned}$$

here we used the fact that the bundle E is closed under the action of G as a consequence of the direct sum $A_{M^G} = A^G \oplus E$.

We conclude that $\Phi^*_{g^{-1}} \xi = \xi$ for all $\xi \in E^0$ and hence $E^0 \subset (A^*_{M^G})^G$. The equality follows noticing that the bundles have the same dimension. $\square$

This proves that even though E is defined with a metric it is actually independent from it.

From the exact sequence

$$0 \longrightarrow E \longrightarrow A_{M^G} \xrightarrow{\sigma} A^G \longrightarrow 0$$

is easy to see that $A_{M^G}/E \simeq A^G$ therefore $(A^G)^* \overset{\varphi}{\simeq} E^\circ = (A^*)^G$, where the isomorphism φ is given by

$$\begin{aligned} \varphi : E^\circ &\longrightarrow (A^G)^* \\ \alpha &\longmapsto (v \mapsto \langle \alpha, v \rangle). \end{aligned}$$

We will use this isomorphism to induce a Lie algebroid structure in $(A^G)^*$.

Let $\lambda : E^\circ \longrightarrow A^*$ be the inclusion, then

Lemma 3.2.10. $(\lambda \circ \varphi^{-1})^* X = X|_{M^G}$ *for* $X \in \Gamma(A)$ *such that* $X|_{M^G} \in \Gamma(A^G)$.

Proof. Consider the sequence of maps

$$(A^G)^* \xrightarrow{\varphi^{-1}} E^\circ \xrightarrow{\lambda} A^*,$$

it is not difficult to see that, as a map on sections, $\lambda \circ \varphi^{-1}$ is equal to σ^*, where σ is the projection form $A|_{M^G}$ to A^G. Let $X \in \Gamma(A)$ and $\xi \in \Gamma(A^G)^*$, we see that

$$\begin{aligned} (\lambda \circ \varphi^{-1})^* X(\xi) &= \langle X|_{M^G}, \lambda \circ \varphi^{-1}(\xi) \rangle \\ &= \langle X|_{M^G}, \sigma^* \xi \rangle \\ &= \langle \sigma X|_{M^G}, \xi \rangle. \end{aligned}$$

If $X|_{M^G} \in \Gamma(A^G)$ then $\sigma X|_{M^G} = X|_{M^G}$ which proves the lemma. □

Our proof of Proposition 3.2.7 uses an analogue of a Lie-Dirac submanifold of a Poisson manifold in the context of Lie bialgebroids.

Proof of Proposition 3.2.7. Applying Proposition 3.2.5 independently to A and A^* we find Lie subalgebroid structures on A^G and $(A^*)^G$. We transport the Lie algebroid structure on $(A^*)^G$ to $(A^G)^*$ via the isomorphism φ^{-1}.

Define the differential on $(A^G)^*$ by the formula

$$\mathrm{d}_{(A^G)^*} X|_{M^G} = (\mathrm{d}_* X)|_{M^G},$$

where $X \in \Gamma(A)$ such that $X|_{M^G} \in \Gamma(A^G)$. Lemma 3.2.10 along with the facts that the Lie bracket on A^G is defined by restriction and the maps λ and φ^{-1} are Lie algebroid morphisms imply

$$\begin{aligned} \mathrm{d}_{(A^G)^*}[X|_{M^G}, Y|_{M^G}] &= \mathrm{d}_{(A^G)^*}[X,Y]|_{M^G} = (\lambda\circ\varphi^{-1})^*\mathrm{d}_*[X,Y] \\ &= (\lambda\circ\varphi^{-1})^*([\mathrm{d}_*X,Y]+[X,\mathrm{d}_*Y]) \\ &= [\mathrm{d}_*X,Y]|_{M^G}+[X,\mathrm{d}_*Y]|_{M^G} \\ &= [\mathrm{d}_{(A^G)^*}X|_{M^G}, Y|_{M^G}]+[X|_{M^G}, \mathrm{d}_{(A^G)^*}Y|_{M^G}] \end{aligned}$$

which shows that $(A^G, (A^G)^*)$ is a Lie bialgebroid. □

The Poisson structure on M^G induced by the Lie bialgebroid $(A^G, (A^G)^*)$ has the expression

$$\begin{aligned} \{f|_{M^G}, g|_{M^G}\}_G &= \langle \mathrm{d}_{A^G} f|_{M^G}, \mathrm{d}_{(A^G)^*} g|_{M^G}\rangle \\ &= \langle (\mathrm{d}f)|_{M^G}, (\mathrm{d}_\circ g)|_{M^G}\rangle \end{aligned}$$

for $f, g \in C^\infty(M)^G$, where $\mathrm{d}_\circ$ is the differential operator of the Lie algebroid E°. The requirement of f and g being G-invariant is needed to ensure that $(\mathrm{d}f)|_{M^G}$ and $(\mathrm{d}_\circ g)|_{M^G}$ are sections of A^G and E° respectively.

Example 3.2.11. Let us consider the case of an exact Lie bialgebroid. This situation arises in applications to mechanical systems on Lie algebroids. Let (A, Λ) be a Lie algebroid endowed with an A-Poisson structure. If a compact Lie group G acts on (A, Λ) by A-Poisson automorphisms (i.e., we have $(\Phi_g)_*\Lambda = \Lambda$ for any $g \in G$). Applying Proposition 3.2.7 we obtain a Lie bialgebroid structure on $(A^G, (A^G)^*)$ and the differential on $(A^G)^*$ is given by

$$\mathrm{d}_{(A^G)^*} X|_{M^G} = [\Lambda, X]|_{M^G}$$

for $X \in \Gamma(A)$ such that $X|_{M^G} \in \Gamma(A^G)$.

It can be proved that $(A^G, (A^G)^*)$ is an exact Lie bialgebroid using techniques from Dirac geometry applied to the case of Courant algebroids (see [8] for the Dirac case).

3.3 Lie algebroids over orbifolds

In this section we consider Lie algebroids over orbifolds. These structures may arise from non-free quotients of Lie algebroids by finite groups, or actions with finite isotropies. More generally, we can construct them in a local manner using atlases of orbifold structures.

The most simple case of an orbifold is the quotient of a manifold M by the action of a finite group H, regarding the action as a model for the quotient space M/H one can define the notion of a Lie algebroid structure over M/H as a Lie algebroid $A \longrightarrow M$ provided with a H-action by Lie algebroid morphisms, for instance

Example 3.3.1. Let G be a finite Lie group acting non-freely on (M,π) a Poisson manifold by Poisson automorphisms. Consider the the cotangent Lie algebroid T^*M with the lifted G-action. Then T^*M/G is a Lie algebroid over M/G.

Let us define a Lie algebroid over an orbifold by considering an orbifold atlas $(Q,\mathcal{U})$ (see Definition 1.4.4 for notations):

Definition 3.3.2. A Lie algebroid $(\bar{A},\mathcal{U}^*)$ over an orbifold $(Q,\mathcal{U})$ is a vector bundle $\bar{A} \stackrel{\pi}{\longrightarrow} Q$ such that for every pair of corresponding charts (U^*, G^*, ϕ^*) and (U, G, ϕ) there is a Lie algebroid structure on $U^* \longrightarrow U$ with ρ_{U^*} an equivariant map for the actions of G^* and G and a bracket that is closed for invariant sections, i.e.,

$$[\Gamma(U^*)^G, \Gamma(U^*)^G] \subset \Gamma(U^*)^G.$$

Further, any embedding $\lambda^* : V^* \longrightarrow U^*$ is a morphism of Lie algebroids.

Example 3.3.3. Let $(Q,\mathcal{U})$ be an orbifold. Consider the the tangent bundle TQ as the orbifold over Q with the atlas defined by $(TU, G, \mathrm{d}\phi)$ for every chart $(U, G, \phi) \in \mathcal{U}$ and the lifted action of G on TU. Then TQ has the structure of a Lie algebroid over Q

To look for a coordinate free definition for Lie algebroids we must regard the orbifold $(Q,\mathcal{U})$ as a groupoid. Recall that there is an equivalence from the category of orbifolds to the category of proper foliation groupoids.

In order to proceed we need the following definition

Definition 3.3.4. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A vector bundle $E \stackrel{\pi}{\longrightarrow} M$ is called a *representation* if there is an (left) action of $\mathcal{G}$ on E with momentum map π for which the action restricted to the fibers of π is linear.

Representations will play the role of vector bundles as the canvas in which a Lie algebroid structure is defined. We use right or left representations depending on the situation, when is unambiguous from the context we will not state explicitly the type of representation used.

Given a Lie algebroid $(\bar{A},\mathcal{U}^*)$ over an orbifold $(Q,\mathcal{U})$. In order to patch the local Lie algebroid structures given in terms of the atlas $\mathcal{U}$, take M as the disjoint union of the cover $\mathcal{U}$ and the étale groupoid $\mathcal{G}$ over M constructed in Proposition 2.3.1. The disjoint union of $\mathcal{U}^*$ defines a vector bundle over M denoted by A which is a representation of $\mathcal{G}$. Given $g \in \mathcal{G}$, $g = (\lambda, z, \mu)$, the action on A is given by $ge = \mu^*(\lambda^*)^{-1}e$, this is well defined as there is a correspondence of embeddings form the atlases $\mathcal{U}$ and $\mathcal{U}^*$.

The collection of maps ρ_{U^*} define a morphism of vector bundles $\rho : A \longrightarrow TM$. Define a bracket on $\Gamma(A)$ locally by restriction to the brackets in $\mathcal{U}^*$. For this bracket we have

$$[\Gamma(A)^{\mathcal{G}}, \Gamma(A)^{\mathcal{G}}] \subset \Gamma(A)^{\mathcal{G}}$$

where $\Gamma(A)^{\mathcal{G}}$ are the invariant sections of A defined by

$$\Gamma(A)^{\mathcal{G}} = \left\{\alpha \in \Gamma(A) | \alpha_{\mathbf{t}(g)} = g\alpha_{\mathbf{s}(g)},\ g \in \mathcal{G}.\right\}$$

This follows form the fact that any embedding $\lambda^* : V^* \longrightarrow U^*$ is a Lie algebroid morphism, noticing that for any $\alpha \in \Gamma(A)^{\mathcal{G}}$ we have $\alpha = \lambda^*\alpha \circ \lambda^{-1}$. Compatibility with the bracket in the case of embeddings reduces to

$$\lambda^*[\alpha, \beta] = [\lambda^*\alpha \circ \lambda^{-1}, \lambda^*\beta \circ \lambda^{-1}] \circ \lambda$$

which implies that $\lambda^*[\alpha, \beta] \in \Gamma(A)^{\mathcal{G}}$ if $\alpha,\ \beta \in \Gamma(A)^{\mathcal{G}}$. The Leibniz identity for sections of A is local and as such is satisfied as it comes from the Lie algebroid structures in U^*.

We have seen that

Proposition 3.3.5. *For any Lie algebroid $(\bar{A}, \mathcal{U}^*)$ over an orbifold $(Q, \mathcal{U})$, there is a representation A of the proper effective étale groupoid $\mathcal{G} \rightrightarrows M$ defined in Proposition 2.3.1 such that A is a Lie algebroid over M with equivariant anchor map ρ and the bracket is closed under invariant sections $\Gamma(A)^{\mathcal{G}}$.*

The next definition is a immediate consequence of the previous discussion

Definition 3.3.6. Let $\mathcal{G} \rightrightarrows M$ be a proper effective étale groupoid. A Lie algebroid over the étale groupoid $\mathcal{G}$ is a representation $A \longrightarrow M$ together with a Lie algebroid structure such that the anchor map ρ is $\mathcal{G}$-equivariant and $\Gamma(A)^{\mathcal{G}}$ is closed under the Lie bracket.

The compatibility relation underlying the definition of Lie algebroid $(A, \mathcal{U}^*)$ over an orbifold $(Q, \mathcal{U})$ states that given another Lie algebroid structure with the same quotient space $(A, \mathcal{V}^*)$ over $(Q, \mathcal{V})$ they are regarded as the same if and only if $(A, \mathcal{U}^* \cup \mathcal{U}^*)$ over $(Q, \mathcal{U} \cup \mathcal{U})$ is again a Lie algebroid. The construction of the étale groupoid $\mathcal{G}$ and the Lie algebroid over it for each atlas fit into the next commutative diagram

$$\begin{array}{ccccc}
A_{\mathcal{U}} & \longleftarrow & A_{\mathcal{U}\cup\mathcal{V}} & \longrightarrow & A_{\mathcal{V}} \\
\downarrow & & \downarrow & & \downarrow \\
M_{\mathcal{U}} & \longleftarrow & M_{\mathcal{U}\cup\mathcal{V}} & \longrightarrow & M_{\mathcal{V}} \\
\upuparrows & & \upuparrows & & \upuparrows \\
\mathcal{G}_{\mathcal{U}} & \underset{\sim}{\longleftarrow} & \mathcal{G}_{\mathcal{U}\cup\mathcal{V}} & \underset{\sim}{\longrightarrow} & \mathcal{G}_{\mathcal{U}}
\end{array}$$

where the top row consists of Lie algebroid morphisms, which are local diffeomorphisms, and the bottom row of essential equivalences.

Given a Morita equivalence $\mathcal{G} \overset{\epsilon}{\longleftarrow} \mathcal{K} \overset{\sigma}{\longrightarrow} \mathcal{H}$ of proper étale Lie groupoids $\mathcal{G}$ and $\mathcal{H}$, we can suppose that ϵ and σ are surjective submersions on objects.

Any essential equivalence between étale groupoids is an étale map (local diffeomorphism) on objects and arrows, so we can further suppose that $\mathcal{K}$ is étale. By a similar argument we can suppose that $\mathcal{K}$ is also proper. For Lie algebroids A and B over $\mathcal{G}$ and $\mathcal{H}$ respectively, using local trivializations, we can define Lie algebroid structures on the pull-back bundles ϵ^*A and σ^*B over K_0, the space of objects of $\mathcal{K}$, pulling back the brackets and anchors locally by ϵ and σ (using the fact that they are local diffeomorphisms). Furthermore, as ϵ and σ are essential equivalences we have isomorphisms

$$\mathcal{G}(\epsilon(x), \epsilon(y)) \simeq \mathcal{K}(x, y) \simeq \mathcal{H}(\sigma(x), \sigma(y))$$

for $x, y \in K_0$, and we define $\mathcal{K}$-actions on ϵ^*A and σ^*B by

$$k(x, a) = (\mathbf{t}(k), \epsilon(k)a) \;\; k(x, b) = (\mathbf{t}(k), \sigma(k)b)$$

for $k \in \mathcal{K}$, $(x, a) \in \epsilon^*A$ and $(x, b) \in \sigma^*B$ such that $\mathbf{s}(k) = x$.

This suggests the following definition

Definition 3.3.7. Let $\mathcal{G} \stackrel{\epsilon}{\longleftarrow} \mathcal{K} \stackrel{\sigma}{\longrightarrow} \mathcal{H}$ be a Morita equivalence of proper étale Lie groupoids. Lie algebroids A and B over $\mathcal{G}$ and $\mathcal{H}$ are *equivalent* if the Lie algebroid structures on the pull-backs ϵ^*A and σ^*B are isomorphic as Lie algebroids over K.

This is an equivalence relation as it boils down to the equivalence of atlases of Lie algebroids over the orbifold structures associated to different coverings of the orbit space of either $\mathcal{G}$ or $\mathcal{H}$.

Remark 3.3.8. Locally any orbifold $(Q, \mathcal{U})$ can be described as a quotient of an open set $U \subset \mathbb{R}^n$ by the action of a finite group, and as such the orbifold $(Q, \mathcal{U})$ has well defined orbit type manifolds in Q. The very definition of an orbifold guarantees the compatibility of this local orbit type manifolds, as for any pair of intersecting charts (U, G, ϕ) and (V, H, ψ) there exists a chart (W, K, θ) and equivariant embeddings $\lambda : W \longrightarrow U$, $\mu : W \longrightarrow V$. Clearly, $\lambda(W^K) = U^{\lambda^* K}|_{\lambda(W)}$, $\mu(W^K) = V^{\mu^* K}|_{\mu(W)}$ and similar relations are satisfied for W_K and $W_{(K)}$.

Using the last remark, one can show that the reduction results of the previous section can be easily applied in the case of a Lie algebroid A over an orbifold $(Q, \mathcal{U})$, obtaining well defined Lie algebroid structures over the orbit type manifolds in Q.

3.4 Lie algebroids over singular spaces

We consider singular spaces defined as orbit spaces of proper groupoids which are not necessarily étale. The definition should take into account an equivalence class of structures over a Morita equivalence class of groupoids as Morita equivalent groupoids give the same quotient space. This takes the form of an equivalence relation of Lie algebroids defined over groupoids.

3.4.1 Definition of a Lie algebroid over a groupoid

The definition of a Lie algebroid A over a groupoid $\mathcal{G} \rightrightarrows M$ is similar to the one for étale groupoids with the main difference that we take into account the fact that for the anchor ρ to be equivariant there is the need of a $\mathcal{G}$-action on TM. We supply the action by means of demanding the existence of a foliation satisfying certain properties.

Let us remark the two cases that canonically give an action of $\mathcal{G}$ in TM. First, for an étale groupoid the effect morphism Eff gives us an action on TM, and second, for an action of a Lie group G on a manifold M, the action groupoid $G \ltimes M$ comes with a natural foliation by global bisections that induces the lifted action of G on TM. These foliations are compatible with the multiplication on $\mathcal{G}$. We will use the following definition from [1]

Definition 3.4.1. A *pseudoaction* or *étalification*([32]) of a groupoid $\mathcal{G} \rightrightarrows M$ is a multiplicatively closed foliation $\mathcal{F}$ of the manifold of arrows such that locally every leaf L of $\mathcal{F}$ is a local bisection of $\mathcal{G}$ and the space of units $\mu(M)$ is a leaf of $\mathcal{F}$.

A foliation $\mathcal{F}$ of $\mathcal{G}$ is multiplicatively closed if for $g = hk \in \mathcal{G}$ such that the leaves L_g, L_h and L_k, through g, h and k respectively, are locally bisections of $\mathcal{G}$ then $L_g \subset L_h L_k$ for appropriate open sets on the leaves.

Remark 3.4.2. Not every Lie groupoid admits an étalification. In [32] a counter example is given by the pair groupoid $S^2 \times S^2 \rightrightarrows S^2$. In fact, groupoids admitting an étalification are not very far from being action groupoids.

Let $\mathcal{F}$ be an étalification of a groupoid $\mathcal{G} \rightrightarrows M$. We can define the effect morphism $\mathrm{Eff} : \mathcal{G} \longrightarrow \Gamma_M$ in a similar manner as for étale groupoids, where Γ_M denotes the groupoid of germs of local diffeomorphisms of M. Let $g \in \mathcal{G}$, define

$$\mathrm{Eff}(g) = \mathrm{germ}_{\mathbf{s}(g)}(\mathbf{t} \circ (\mathbf{s}|_{\mathcal{F}})^{-1}|_U)$$

where U is an open subset of M containing $\mathbf{s}(g)$. The morphism Eff is well defined as $\mathcal{F}$ defines a bisection for U small enough.

An étalification $\mathcal{F}$ of a groupoid $\mathcal{G}$ induces a representation of $\mathcal{G}$ on TM through the differential of the effect, $\mathrm{d}_{\mathbf{s}(g)}\mathrm{Eff}(g) : T_{\mathbf{s}(g)}M \longrightarrow T_{\mathbf{t}(g)}M$.

Definition 3.4.3. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and A a representation of $\mathcal{G}$. Let $\mathcal{F}$ be an étalification of $\mathcal{G}$. A Lie algebroid A over $\mathcal{G}$ consist of a Lie algebroid structure on the bundle A such that

$$[\Gamma(A)^{\mathcal{G}}, \Gamma(A)^{\mathcal{G}}] \subset \Gamma(A)^{\mathcal{G}}$$

and the anchor map $\rho : A \longrightarrow TM$ is equivariant with respect to the actions of $\mathcal{G}$ on A and TM, where the representation of $\mathcal{G}$ on TM is given by Eff the effect morphism.

As no confusion will arise we will denote the morphism Eff simply by juxtaposition gv for $g \in \mathcal{G}$ and $v \in TM$.

The first simple consequence of this definition is given by the equivariance of ρ

Proposition 3.4.4. *The image of the anchor map at a point $x \in M$ is an invariant subspace* $\operatorname{Im} \rho$ *of T_xM by the action of the isotropy group $\mathcal{G}_x$*

This somehow limits the behaviour of the singular foliation defined by A as is illustrated by these simple examples

Example 3.4.5. Consider the group $\mathbb{Z}/2$ acting on $\mathbb{R}^2$ by a reflection r on the subspace $\{y = 0\}$. Let $\mathcal{F}$ be the foliation of $\mathbb{R}^2$ defined by the family of parabolas $y = ax^2 + a$ for $a \in \mathbb{R}$. Consider the Lie algebroid of the foliation $\mathcal{F}$ and the action groupoid $\mathbb{Z}/2 \ltimes \mathbb{R}^2$ acting on it. The action sends orbits to orbits by $r(\{y = ax^2 + a\}) = \{y = -ax^2 - a\}$. For a fixed point $(b, 0)$ its orbit is the line $\{y = 0\}$ which is an invariant subspace for the $\mathbb{Z}/2$-action. On the other hand, consider the foliation $\mathcal{F}$ defined by the family of parabolas $x = ay^2 + a$, $a \in \mathbb{R}$. In this case the orbits of the Lie algebroid are invariant under the action and for points $(x, 0)$ the image of the anchor map coincide with the subspaces $\{x = b\}$ which are invariant under the action of $\mathbb{Z}/2$.

As we can see from the example, the behaviour of the foliation $\operatorname{Im} \rho$ with respect to the action of the groupoid $\mathcal{G}$ is controlled by the isotropies but is not dominated by it as there are many invariant subspaces of T_xM even in the simpler cases. There are examples where the isotropy group determines completely the orbits of the Lie algebroid A at certain points

Example 3.4.6. Consider the action of the group $SO(2)$ on $\mathbb{R}^2$ by rotations, and as in the previous example, let $\mathcal{F}$ be a foliation invariant under the action of $SO(2)$. The origin is a fixed point for the action and the only invariant subspaces for the action are the the origin and the whole of $\mathbb{R}^2$. In the first case the orbit of the Lie algebroid containing the origin has to be the origin itself. In the second case, the rank of the foliation for a Lie algebroid is upper semi continuous, the rank can only increase in a neighborhood of the origin and the orbit containing it must be an open subset of $\mathbb{R}^2$ or $\mathbb{R}^2$ itself.

These examples can be easily generalized to higher dimensions to illustrate that the singular foliation defined by A may be contained in the orbits of the groupoid $\mathcal{G}$ and differ in dimension.

Example 3.4.7. Consider the action of the group $SO(2) \times \mathbb{R}$ on $\mathbb{C} \times \mathbb{R}$ by $(\theta, a)(z, x) = (e^{i\theta} z, x + a)$. Consider also the Lie algebroid defined by the foliation $\mathcal{F}$ of $\mathbb{C} \times \mathbb{R}$ by lines $\{z_0\} \times \mathbb{R}$. The orbits of the groupoid are cylinders with axis $\{0\} \times \mathbb{R}$ and the axis itself. Clearly, the leaves of $\mathcal{F}$ are contained in the orbits of $\mathcal{G}$ and have lower dimension.

3.4.2 Equivalence of Lie algebroids over groupoids

As noticed before, there should be an equivalence relation on Lie algebroids over groupoids that allows us to define algebroids over the space of orbits. In order to define the equivalence relation we recall the notion of pull-back over a map for groupoids and Lie algebroids

Definition 3.4.8. Let $\mathcal{G}$ be a Lie groupoid and $\phi : N \longrightarrow G_0$ a smooth map. The *pull-back groupoid* $\phi^!\mathcal{G}$ over N is formed regarding arrows in $\mathcal{G}$ from $\phi(x)$ to $\phi(y)$, for $x, y \in N$, as arrows form x to y , i.e.

$$(\phi^!\mathcal{G})_1 = N \times_{G_0} G_1 \times_{G_0} N.$$

The multiplication is given by the multiplication in $\mathcal{G}$. The space $(\phi^!\mathcal{G})_1$ can be constructed using two pull-backs as in the diagram

$$\begin{array}{ccccc} (\phi^!\mathcal{G})_1 & & \longrightarrow & & N \\ \downarrow & & & & \downarrow{\scriptstyle \phi} \\ G_1 \times_{G_0} N & \xrightarrow{\pi_1} & G_1 & \xrightarrow{\mathbf{t}} & G_0 \\ \downarrow & & \downarrow{\scriptstyle \mathbf{s}} & & \\ N & \xrightarrow{\phi} & G_0 & & \end{array}$$

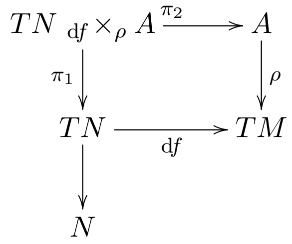
The lower pull-back is smooth because $\mathbf{s}$ is a submersion. If the composition $\mathbf{t} \circ \pi_1$ is also a submersion (note the resemblance with the weak pull-back), the upper pull-back has a smooth structure as well. The diagram

$$\begin{array}{ccc} (\phi^!\mathcal{G})_1 & \longrightarrow & G_1 \\ {\scriptstyle (\mathbf{s},\mathbf{t})}\downarrow & & \downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} \\ N \times N & \xrightarrow{\phi\times\phi} & G_0 \times G_0 \end{array}$$

is a pull-back of manifolds. The induced map $\phi : \phi^!\mathcal{G} \longrightarrow \mathcal{G}$ is a morphism of groupoids.

If $\epsilon : \mathcal{G} \longrightarrow \mathcal{H}$ is an essential equivalence we can easily see that $\mathcal{G} \simeq \epsilon^! H$, this will be relevant to define the equivalence relation among Lie algebroids over groupoids.

In the Lie algebroid setting, let $A \longrightarrow M$ be a Lie algebroid and a smooth map $f : N \longrightarrow M$, if the pull-back



is a manifold, it defines a vector bundle $TN \ {}_{\mathrm{d}f}\!\times_\rho A \overset{\pi}{\longrightarrow} N$, denoted $f^!A$, where π is the composition of π_1 with the projection on the vector bundle TN. A section of π is called projectable if it is given by $(X, \sigma) \in \mathfrak{X}(N) \times \Gamma(A)$ such that $\mathrm{d}f(X) = \rho(\sigma)$. Any section V of the bundle $\operatorname{Ker} \mathrm{d}f$ defines a projectable section $(V, 0)$ of $f^!A$.

Note that if f is a submersion then $f^!A$ is always a manifold. Locally, we can always choose a basis of projectable sections for $f^!A$ by choosing local bases $\{\alpha\}$ for $\Gamma(A)$ and $\{V\}$ for $\operatorname{Ker} \mathrm{d}f$ and form the basis $\left\{(\widetilde{\rho(\alpha)}, \alpha), (V, 0)\right\}$, where $\widetilde{\rho(\alpha)}$ is any lift of $\rho(\alpha) \in \mathfrak{X}(M)$ with respect to $\mathrm{d}f$.

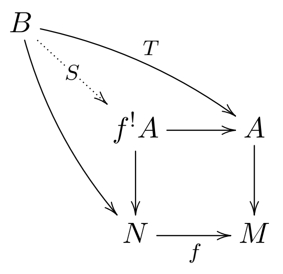
Definition 3.4.9. Let A be a Lie algebroid over the manifold M and $f : N \longrightarrow M$ a submersion. The *pull-back Lie algebroid* over N consists of the vector bundle $f^!A$ together with the Lie algebroid structure given by π_1 as anchor map and for projectable sections $\{(X, \alpha)\}$ and $\{(Y, \beta)\}$ the bracket is

$$[(X, \alpha), (Y, \beta)] = ([X, Y], [\alpha, \beta]).$$

The bracket is extended to arbitrary sections by the Leibniz identity.

It can be proved that the map $\pi_1 : f^!A \longrightarrow A$ is a Lie algebroid morphism. The anchor map of $f^!A$ will be denoted ρ as usual instead of π_1 and no confusion will arise form this fact.

The pull-back Lie algebroid is characterized by the universal property represented in the diagram



which states that every Lie algebroid morphism $T : B \longrightarrow A$ factors through $f^!A$ by a unique Lie algebroid morphism S.

Some properties of the pull-back Lie algebroid are resumed in the next proposition (see [17])

Proposition 3.4.10. *Let A and B be Lie algebroids over manifolds M and N.*

(i) For submersions $\phi : P \longrightarrow M$, $\eta : Q \longrightarrow P$, the pull-back Lie algebroid $(\phi \circ \eta)^! A = \eta^!(\phi^! A)$.

(ii) For $F : A \longrightarrow B$ a Lie algebroid morphism over a map f, submersions ϕ, ψ and a map $\tilde{f}$ making the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & Q \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ M & \xrightarrow[f]{} & N. \end{array}$$

there is a Lie algebroid morphism $F^! : \phi^! A \longrightarrow \psi^! B$.

(iii) If $\mathcal{G}$ is a Lie groupoid with Lie algebroid A, and $\phi : P \longrightarrow M$ is a submersion, then $\phi^! \mathcal{G}$ is a Lie groupoid with Lie algebroid $\phi^! A$.

In particular if A and B are isomorphic Lie algebroids over a manifold M then $\phi^! A \simeq \phi^! B$ for any $\phi : N \longrightarrow M$.

A equivalence relation on Lie algebroids over groupoids is given in the following manner

Definition 3.4.11. Let $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\sigma} \mathcal{H}$ be a Morita equivalence of proper Lie groupoids. Lie algebroids A and B, over $\mathcal{G}$ and $\mathcal{H}$ respectively, are called *Morita equivalent* if there exists a Lie algebroid C over $\mathcal{K}$ such that

(i) $\epsilon^! A \simeq C \simeq \sigma^! B$ as Lie algebroids,

(ii) The maps $\epsilon^! : C \longrightarrow A$ and $\sigma^! : C \longrightarrow B$ are equivariant with respect to ϵ and σ.

We note the equivalence of A and B as $A \simeq B$. The following diagram illustrates the equivalence relation just defined

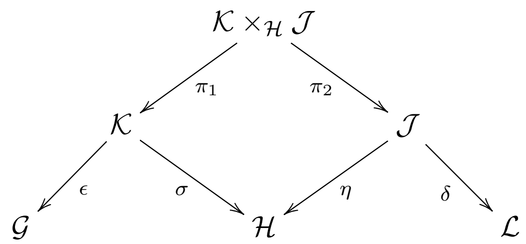
$$\begin{array}{ccccc} A & \xleftarrow{\epsilon^!} & C & \xrightarrow{\sigma^!} & B \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G_0 & \longleftarrow & K_0 & \longrightarrow & H_0 \\ \upuparrows & & \upuparrows & & \upuparrows \\ \mathcal{G} & \xleftarrow[\sim]{\epsilon} & \mathcal{K} & \xrightarrow[\sim]{\sigma} & \mathcal{H} \end{array}$$

where the top row consists on Lie algebroid morphisms and the bottom row is formed by essential equivalences.

Proposition 3.4.12. *Equivalence of Lie algebroids over groupoids is a well defined equivalence relation.*

Proof. It is clear that the relation is reflexive and symmetric. Let $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\sigma} \mathcal{H}$ and $\mathcal{H} \xleftarrow{\eta} \mathcal{J} \xrightarrow{\delta} \mathcal{L}$ be Morita equivalences of groupoids. Let A, B, C, D and E be Lie algebroids over $\mathcal{G}, \mathcal{K}, \mathcal{H}, \mathcal{J}$ and $\mathcal{L}$ respectively such that $A \simeq C$ through B and $C \simeq E$ through D.

From the properties of the pull-back of Lie algebroids and the following diagram



we see that $(\epsilon \circ \pi_1)^! \simeq (\sigma \circ \pi_1)^! = (\eta \circ \pi_2)^! \simeq (\delta \circ \pi_2)^!$.

Endow $\mathcal{K} \times_{\mathcal{H}} \mathcal{J}$ with an étalification whose leaves are given by manifolds

$$L_{(k,h,j)} = \left\{(k', h', j') \in \mathcal{K} \times_{\mathcal{H}} \mathcal{J} | g' \in L_g, h' \in L_h, j' \in L_j\right\}$$

for L_g, L_h and L_j leaves of the étalifications of $\mathcal{G}, \mathcal{H}$ and $\mathcal{J}$ respectively. We notice that π_1 and π_2 send leaves to leaves and thus they are equivariant with respect to the action induced by the manifolds $L_{(k,h,j)}$.

Define $\tau = \sigma \circ \pi_1 = \eta \circ \pi_2$, the pull-back $\tau^! C$ fits in the commutative diagram

$$\begin{array}{ccc} \tau^! C & \longrightarrow & C \\ \downarrow & & \downarrow \\ (\mathcal{K} \times_{\mathcal{H}} \mathcal{J})_0 & \xrightarrow[\tau]{} & H_0 \end{array}$$

Using the expression for the pull-back $\tau^! C = T(\mathcal{K} \times_{\mathcal{H}} \mathcal{J})_0 \,{}_{\mathrm{d}\tau}\!\times_{\rho} C$ and the $\mathcal{K} \times_{\mathcal{H}} \mathcal{J}$-action on $T(\mathcal{K} \times_{\mathcal{H}} \mathcal{J})_0$ we define a $\mathcal{K} \times_{\mathcal{H}} \mathcal{J}$-action on $\tau^! C$ by

$$l(v, c) = (lv, \tau(l)c)$$

for $(v, c) \in \tau^! C$ and $l \in \mathcal{K} \times_{\mathcal{H}} \mathcal{J}$. The projection $\pi_1^! : \tau^! C \longrightarrow B$ sending (v, c) to $(\mathrm{d}\pi_1 v, c)$ becomes an equivariant map, which implies that $(\epsilon \circ \pi_1)^! : \tau^! C \longrightarrow A$ is equivariant. Analogously, $(\delta \circ \pi_2)^! : \tau^! C \longrightarrow E$ is equivariant. This proves that the relation is transitive. ◻

Remark 3.4.13. Let G be a Lie group acting on a Lie algebroid $A \xrightarrow{\pi} M$ by Lie algebroid automorphisms. The G-action on A defines a Lie algebroid structure A over the action groupoid $G \ltimes M$ endowed with the natural étalification. Indeed, the $G \ltimes M$-action on A is defined by $(g, x)a = ga$ for $a \in A$ such that $\pi(a) = x$. The fact that the G-action on A is by Lie algebroid automorphisms implies that the anchor map ρ is equivariant with respect to the actions of $G \ltimes M$ on A and TM, where the action on TM comes from the effect of the étalification of $G \ltimes M$. Also, $(G \ltimes M)$-invariant sections of A are closed under the Lie bracket.

On the other hand, let A be a Lie algebroid over the action groupoid $G \ltimes M$, we can define an action $\Phi : G \times A \longrightarrow A$ by setting $ga = (g, \pi(a))a$ for $g \in G$, $a \in A$. In this way, G acts on A by vector bundle morphisms and the action is compatible with the anchor by the equivariance of ρ with respect to the the actions of $G \ltimes M$ on A and TM. The fact that $\Gamma(A)^{G \ltimes M} = \Gamma(A)^G$ together with the fact that the Lie bracket is closed under invariant sections implies that

$$\Phi_*[\alpha, \beta] = [\Phi_*\alpha, \Phi_*\beta]$$

for any $\alpha, \beta \in \Gamma(A)^G$. But note that this last equation does not necessarily hold for **all** $\alpha, \beta \in \Gamma(A)$. This shows that the notion of an algebroid A over an action groupoid $G \ltimes M$ is weaker than the notion of a G-action on A by Lie algebroid morphisms.

Chapter 4

Applications

In Section 3.2 was presented a reduction process for general Lie algebroids that was inspired in the Poisson singular reduction in [11] by Fernandes, Ortega and Ratiu. In [28], Ratiu and Jotz prove reduction results for Dirac manifolds analogous to the results on [11]. Our results for Lie algebroids can be applied to Poisson and Dirac reduction to recover some of the results in those works. We can also apply our results on exact Lie bialgebroids to study symmetries of Hamiltonian dynamical systems on Lie algebroids.

4.1 Applications to Poisson geometry

In Section 3.1.2, we applied results from Poisson geometry to a Lie algebroid A ,with an inner action of a Lie algebra $\mathfrak{g}$, to obtain a Lie algebroid structure in a subbundle of A. Here we will apply our reduction results on singular actions of Lie groups by Lie algebroid (and Lie bialgebroids) automorphisms to the cotangent Lie algebroid T^*M (and the Lie bialgebroid (TM, T^*M)) of a Poisson manifold (M, Π).

4.1.1 Poisson reduction

We show how to recover some results in [11] concerning singular reduction of Poisson manifolds under the action of symmetry groups.

Let $\Phi : H \times M \to M$ be an action of a compact Lie group H on the Poisson manifold (M, Π) by Poisson maps. The group H acts on the cotangent algebroid T^*M by Lie algebroid automorphisms through the cotangent lift of the H-action on M

A Lie-Dirac submanifold N of M is defined as follows

Definition 4.1.1. Let M be a Poisson manifold. A submanifold $N \subset M$ is called a *Lie-Dirac submanifold* if there exists a subbundle $E \subset T_N M$ such that

$$T_N M = TN \oplus E$$

and E^0 is a Lie subalgebroid of the cotangent Lie algebroid T^*M.

It can be proved that a Lie-Dirac submanifold N of M is a Poisson-Dirac submanifold, i.e. is itself a Poisson manifold such that its symplectic foliation is given by the intersections of N with the symplectic leaves of M (see [8]).

Let M^H be the fixed point manifold of the action.

Proposition 4.1.2. *Let H be a compact Lie group acting on a Poisson manifold M by Poisson maps. Then $(TM^H, (T^*M^H))$ is a Lie bialgebroid. $T_{M^H}M$ decomposes as*

$$T_{M^H}M = TM^H \oplus E,$$

where $E^\circ = (T^*M)^H$ is a Lie subalgebroid of T^*M, making M^H a Lie-Dirac submanifold of M with Poisson bracket $\{-,-\}_{M^H}$ given by

$$\{f,h\}_{M^H} := \{\widetilde{f},\widetilde{h}\}\Big|_{M^H}, \quad f,h \in C^\infty(M^H), \tag{4.1.1}$$

where $\widetilde{f},\widetilde{h} \in C^\infty(M)^H$ denote arbitrary H-invariant extensions of $f,h \in C^\infty(M^H)$.

Proof. Applying Proposition 3.2.7 we obtain the Lie bialgebroid structure on $(TM^H, (T^*M^H))$ and the Lie algebroid structure on E°, which makes M^H a Lie-Poisson submanifold of M as immediate consequence of the definition. For the expression of the Poisson bracket in M^H, we note that for any H-invariant $f \in C^\infty(M)$ the differential $\mathrm{d}f$ belongs to E°, and the Poisson bracket

$$\{f,g\} = \Pi(\mathrm{d}f, \mathrm{d}g)$$

on M induced by the exact Lie bialgebroid (TM, T^*M) render the expression for the Poisson bracket on M^H. $\square$

Now, let $\Phi : G \times M \to M$ be a proper action by Poisson maps. Denote by $H = G_m$ the *isotropy group* of a point $m \in M$ and let M_H, M^H and $M_{(H)}$ be the orbit type, fixed points and isotropy type sets. The properness of the action guarantees that each H is a compact Lie group and that the connected components of M_H, M^H, and $M_{(H)}$ are embedded submanifolds of M. Recall that M_H is an open subset of M^H and that

$$M_H = M_{(H)} \cap M^H.$$

We can now apply Proposition 3.2.1 to the cotangent Lie algebroid T^*M to associate Lie algebroid structures to the isotropy type bundle and the orbit type bundle in the quotient:

Proposition 4.1.3. *Let $\Phi : G \times M \to M$ be a proper action by Poisson diffeomorphisms, let $H \subset G$ be an isotropy group, and denote by $N(H)$ the normalizer of H in G. The bundles $T^*M_H \longrightarrow M_H$ and $T^*M_{(H)}/G \longrightarrow M_{(H)}/G$ have Lie algebroid structures associated to the cotangent Lie algebroid structure on T^*M.*

The isotropy type manifolds M_H inherit by restriction from M^H the structure of a Lie-Poisson submanifold of M. It can be proved that for a Poisson-Dirac submanifold N of a Poisson manifold M, and a Poisson diffeomorphism ϕ of M, if $\phi(N) = N$ then $\phi|_N$ is a Poisson diffeomorphism of N (see [11]). Recall that the action of $N(H)/H$ on M_H is free and proper. By the last remark the $N(H)/H$-action is also by Poisson diffeomorphisms and we have the following result

Proposition 4.1.4. *Let $\Phi : G \times M \to M$ be a proper action by Poisson difeomorphisms, let $H \subset G$ be an isotropy group. Then the orbit type manifolds in the quotient of the the G-action $M_{(H)}/G$ inherit a Poisson structure from M.*

Clearly, the bundles $T^*M_{(H)}/G$ and $T^*(M_{(H)}/G)$ are different. Although we have reduction of the Poisson structure on M in the presence of a G-action by Poisson diffeomorphisms, the reduction of the cotangent Lie algebroid T^*M can still be interesting as it can retain some properties of the original Poisson structure that are lost in the quotient Poisson structures.

4.1.2 The Toda lattice

The mechanical system named the Toda lattice illustrates how the reduced cotangent Lie algebroid T^*M/G of a Poisson manifold M may inherit interesting characteristics from the original cotangent Lie algebroid that are lost by means of the reduction of the Poisson structure to M/G. We follow [4] for this example.

The Toda lattice is a mechanical system given by the symplectic manifold $(\mathbb{R}^{2n}, \pi_0)$, a Poisson structure π_1 on $\mathbb{R}^{2n}$, compatible with π_0, i.e. $[\pi_1, \pi_0] = 0$, and $h_0, h_1 \in C^\infty((\mathbb{R}^{2n})$ such that

$$\pi_0^\sharp \mathrm{d}h_1 = \pi_1^\sharp \mathrm{d}h_0.$$

There is a recursion operator for the system $N : T\mathbb{R}^{2n} \longrightarrow T\mathbb{R}^{2n}$ defined by

$$N := \pi_1^\sharp \circ (\pi_0^\sharp)^{-1},$$

that satisfies the relation

$$\mathrm{d}h_1 = N^* \mathrm{d}h_0.$$

These structures satisfy certain compatibility conditions which make them into a so-called bi-Hamiltonian system. It follows that, there is a hierarchy of Poisson tensors $\pi_i^\sharp = N^i \pi_0^\sharp$ satisfying the relations $\mathrm{d}h_i = (N^*)^i \mathrm{d}h_0$ and $\pi_i^\sharp \mathrm{d}h_j = \pi_j^\sharp \mathrm{d}h_i$ for Poisson commuting functions $h_i \in C^\infty((\mathbb{R}^{2n})$ defined up to a constant factor (see [13]).

Choosing coordinates in $\mathbb{R}^{2n}$, $(a_1, \dots, a_n, b_1, \dots, b_n)$, the structures of the Toda lattice are described as follows. The first Poisson tensor π_0 is determined by:

$$\begin{aligned} \{a_i, b_i\}_0 &= a_i, && (i = 1, \dots, n-1) \\ \{a_i, b_{i+1}\}_0 &= -a_i, && (i = 1, \dots, n-1) \\ \{a_n, b_n\}_0 &= 1. \end{aligned}$$

while the second Poisson structure π_1 is given by:

$$\begin{aligned} \{a_i, a_{i+1}\}_1 &= -a_i a_{i+1}, && (i = 1, \dots, n-1) \\ \{a_i, b_i\}_1 &= a_i b_i, && (i = 1, \dots, n-1) \\ \{a_n, b_n\}_1 &= b_n \\ \{a_i, b_{i+1}\}_1 &= -a_i b_{i+1}, && (i = 1, \dots, n-1) \\ \{b_i, b_{i+1}\}_1 &= -a_i, && (i = 1, \dots, n-1). \end{aligned}$$

The functions $h_0, h_1 \in C^\infty((\mathbb{R}^{2n})$ are given by

$$\begin{aligned} h_o &= \sum_{i=1}^{n} b_i \\ h_1 &= \sum_{i=1}^{n} \frac{1}{2} b_i^2 + \sum_{i=1}^{n-1} a_i. \end{aligned}$$

The choice of coordinates for the description of the system is called *extended Flaschka coordinates* for the Toda lattice.

Define an proper and free action $\phi : \mathbb{R} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ by

$$\lambda(a_1, \dots, a_n, b_1, \dots, b_n) = (a_1, \dots, a_n + \lambda, b_1, \dots, b_n).$$

We have $\mathrm{d}\phi_\lambda = \mathrm{id}_{Rr^{2n}}$ for any $\lambda \in \mathbb{R}$, and ϕ_λ becomes a Poisson map for any Poisson structure in $\mathbb{R}^{2n}$.

Consider the cotangent Lie algebroid $T^*\mathbb{R}^{2n}$ for the Poisson structures π_i. Recall that the bundle maps $\pi_i^\sharp : T^*\mathbb{R}^{2n} \longrightarrow T\mathbb{R}^{2n}$ are Lie algebroid morphisms, they are equivariant with respect to the $\mathbb{R}$ -action, this makes the recursion operator N equivariant as well. The bundle maps $\pi_i^\sharp$, as well as the recursion operator N, descend to the non singular quotient Lie algebroid $T^*\mathbb{R}^{2n}/\mathbb{R} \longrightarrow \mathbb{R}^{2n}/\mathbb{R}$. Denoting the bundle maps in the quotient by $\bar{\pi}_i^\sharp$ and $\bar{N}$, they satisfy

$$\bar{\pi}_i^\sharp = \bar{N}^i \bar{\pi}_0^\sharp.$$

On the other hand, the Poisson reduction $\mathbb{R}^{2n}/\mathbb{R}$ can be identified with the submanifold $\mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$ defined by $a_n = 0$, it is a Poisson submanifold for π_0 and π_1. The identity $\pi_0^\sharp \mathrm{d}h_1 = \pi_1^\sharp \mathrm{d}h_0$ still holds in the Poisson submanifold $a_n = 0$. However, the tangent space to this submanifold is not left invariant by the tensor N, and one can show that on $\mathbb{R}^{2n-1}$ there is no recursion operator relating the Poisson tensors.

4.2 Dirac structures

Dirac manifolds generalize the concepts of symplectic and Poisson manifolds and are defined by certain type of Lie algebroids. We apply Proposition 3.2.5 and proposition 3.2.1 to obtain the singular reduction of Dirac structures in the presence of a proper Lie group action by appropriate morphisms.

The *generalized tangent bundle* $TM \oplus T^*M$ of a manifold M is endowed with a non-degenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$, called the *pairing*, given by

$$\langle (u,\alpha),(v,\beta)\rangle = \beta(u) + \alpha(v)$$

for all $u, v \in T_mM$ and $\alpha, \beta \in T_m^*M$.

Definition 4.2.1. An *almost Dirac structure* on M is a maximal isotropic subbundle $D \subset TM \oplus T^*M$. That is, D coincides with its orthogonal relative to the pairing (hence its fibers are necessarily $\dim M$-dimensional).

The space $\Gamma(TM \oplus T^*M)$ of local sections of the generalized tangent bundle is endowed with a skew-symmetric bracket given by

$$\begin{aligned}[(X,\alpha),(Y,\beta)] &= ([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}\mathrm{d}\,(\alpha(Y) - \beta(X))) \\ &= ([X,Y], \mathcal{L}_X\beta - i_Y\mathrm{d}\alpha - \frac{1}{2}\mathrm{d}\,\langle (X,\alpha),(Y,\beta)\rangle).\end{aligned}$$

This bracket is $\mathbb{R}$-bilinear and skew symmetric but does not satisfy the Jacobi identity.

If we restrict this bracket to sections of an almost Dirac structure D it reduces to the expression

$$[(X,\alpha),(Y,\beta)] := ([X,Y], \mathcal{L}_X\beta - i_Y\mathrm{d}\alpha)\,,$$

for $(X,\alpha),(Y,\beta) \in \Gamma(D)$. This expression also defines another bracket on $\Gamma(TM \oplus T^*M)$, called the *Courant* bracket, which is not skew-symmetric but satisfies the Jacobi identity.

Definition 4.2.2. An almost Dirac structure is called a *Dirac structure* if it is closed under the Courant bracket, i.e.,

$$[\Gamma(D), \Gamma(D)] \subset \Gamma(D).$$

A Dirac structure is furnished with a Lie algebroid structure, given by the Courant bracket and the projection to TM as anchor.

Example 4.2.3. Let $\Omega \in \Omega^2(M)$ (resp. $\pi \in \chi^2(M)$). Let $L = \mathrm{Graph}(\Omega) \subset TM \oplus T^*M$ (resp. $L = \mathrm{Graph}(\pi)$) be the graph of Ω (resp. of π), it defines a (possibly non-integrable) Dirac structure by

$$L_p = \left\{(v, \alpha) \in T_pM \oplus T_p^*M \mid i_v\Omega = v\right\}$$

for $p \in M$ (similarly for π). In this case, the integrability condition for the Courant bracket in the definition of a Dirac manifold is equivalent to the integrability condition $\mathrm{d}\Omega = 0$ (resp. $[\pi, \pi] = 0$). Hence pre-symplectic and Poisson structures on M are particular cases of Dirac structures.

Let us turn to morphisms of Dirac manifolds. Let (M, D) and (N, E) be smooth Dirac manifolds. A smooth map $\phi : M \longrightarrow N$ is called

- *Forward Dirac map* if

$$E = \phi_* D := \{(\mathrm{d}\phi X, \beta) \in TN \oplus T^*N \mid (X, \phi^*\beta) \in D\}$$

- *Backward Dirac map* if

$$D = \phi^* E := \{(X, \phi^*\beta) \in TM \oplus T^*M \mid (\mathrm{d}\phi X, \beta) \in E\}$$

If ϕ is a diffeomorphism, the notion of forward and backward Dirac maps coincide, i.e., $E = \phi_* D$ if and only if $D = \phi^* E$.

Let G be a Lie group and $\Phi : G \times M \to M$ a smooth G-action. Then G is called a *symmetry Lie group of* D if for every $g \in G$ the condition $(X, \alpha) \in \Gamma(D)$ implies that $\left(\Phi_g^* X, \Phi_g^* \alpha\right) \in \Gamma(\mathsf{D})$. Then, $\Phi_g : (M, D) \to (M, D)$ is a forward and backward Dirac map for all $g \in G$. We say then that the Lie group G acts by Dirac morphisms or that the action of G on M is *Dirac*.

Let G be a Lie group and $\Phi : G \times M \to M$ a proper G-action by Dirac morphisms on (M, D). We can apply Proposition 3.2.5 and 3.2.1 to the Lie algebroid structures associated to Dirac structure D, and find reduced Lie algebroids on M^H, M_H and $M_{(H)}/G$, for an isotropy group H. It turns that this reduced Lie algebroids are again Dirac structures on their respective manifolds. We state the following results similar to the results in [28].

Proposition 4.2.4. *Let (M, D) be a Dirac manifold and G a Lie group acting properly on M by Dirac morphisms, let H be an isotropy subgroup of G. Then the bundles D^H, D_H and $D_H/L(H)$ have reduced Lie algebroid structures associated to reduced Dirac structures on M^H, M_H and $M_{(H)}/G$ respectively.*

Proof. As we saw in Section 3.2, the key step is to prove that D^H is a Dirac structure, the other structures are obtained by restriction to an open set and taking a quotient by a free and proper action. The Dirac structure D is a Lie algebroid over M, applying Proposition 3.2.5 and 3.2.1 we obtain a Lie algebroid structure D^H such that

$$D^H = D \cap (TM^H \oplus E^\circ)$$

where $E = \left[(T^*_{M^H}M)^H\right]^\circ$ and it fits into the exact sequence

$$0 \longrightarrow TM^H \longrightarrow T_{M^H}M \longrightarrow E \longrightarrow 0.$$

The isomorphism

$$\begin{aligned} \varphi : E^\circ &\longrightarrow (TM^H)^* \\ \alpha &\longmapsto (v \mapsto \alpha(v)) \end{aligned}$$

for $v \in TM^H$, allows us to define the map

$$\begin{aligned} \Psi : TM^H \oplus E^\circ &\longrightarrow TM^H \oplus (TM^H)^* \\ (v, \alpha) &\longmapsto (v, \varphi(\alpha)). \end{aligned}$$

It should be clear that Ψ is an isomorphism and preserves the pairing, thus $\Psi(D^H)$ is a maximal isotropic subbundle of $TM^H \oplus (TM^H)^*$. Let $\lambda : M^H \longrightarrow M$ be the inclusion, it is easy to see that $\lambda^*|_{E^\circ} = \varphi$ as a map on sections. The map Ψ preserves the Courant bracket. Indeed,

$$\begin{aligned} \Psi[(X,\alpha),(Y,\beta)] &= ([X,Y], \lambda^*\mathcal{L}_{\tilde{X}}\beta - \lambda^* i_{\tilde{Y}}\mathrm{d}\alpha) \\ &= ([X,Y], \mathcal{L}_X\lambda^*\beta - i_Y\mathrm{d}\lambda^*\alpha) \\ &= [\Psi(X,\alpha), \Psi(Y,\beta)] \end{aligned}$$

for $(X,\alpha),(Y,\beta) \in \Gamma(TM^H \oplus E^\circ)$ and $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)$ such that $\mathrm{d}\lambda X = \tilde{X}|_{M^H}$, $\mathrm{d}\lambda Y = \tilde{Y}|_{M^H}$. This proves that $\Psi(D^H)$ is a Dirac structure on M^H □

4.3 Mechanical systems

The results from Section 3.2 can be directly applied to the formulation of Hamiltonian mechanics on Lie algebroids to obtain a reduction of the mechanical system. Mechanical systems on Lie algebroids are described in [6] as follows.

Throughout this section, let $\pi : A \to M$ be a Lie algebroid and $p : A^* \longrightarrow M$ its dual. Consider the pull-back Lie algebroid

$$\begin{array}{ccc} p^! A & \longrightarrow & A \\ \downarrow & & \downarrow \pi \\ A^* & \xrightarrow[p]{} & M. \end{array}$$

Similarly to the case of Hamiltonian mechanics in T^*M, define the Liouville section $\Theta_A \in \Gamma(p^! A)^*$ by

$$\Theta_A(a^*)(b, v) = a^*(v) \text{ for } a^* \in A^*, \ (b, v) \in p^! A,$$

its differential gives a symplectic section

$$\Omega_A = -\mathrm{d}\Theta_A \in \Gamma(\wedge^2 (p^! A)^*)$$

where d denotes the differential of the Lie algebroid $p^! A$.

Given a function $H \in C^\infty(A^*)$ there is a unique section $\sigma_H \in \Gamma(p^! A)$ such that

$$i_{\sigma_H} \Omega_A = dH.$$

The dynamics of the vector field $\rho(\sigma_H)$ on A^* are called the *Hamiltonian dynamics* of the function H, where ρ denotes the anchor map on $p^! A$, i.e., the second projection.

Example 4.3.1. The tangent bundle. Consider the tangent Lie algebroid TM and $H : T^*M \to \mathbb{R}$ is a Hamiltonian function then the resultant equations are the *classical Hamilton equations* for H.

Example 4.3.2. Real Lie algebras of finite dimension. Consider a real Lie algebra of finite dimension as a Lie algebroid. Then the Hamilton's equations are the *Lie-Poisson equations.*

Let $\Omega_A^\flat : p^! A \longrightarrow (p^! A)^*$ be the contraction by sections of $p^! A$. Define $\Lambda_A^\sharp = (\Omega_A^\flat)^{-1}$ and let Λ_A be the associated section of $\wedge^2(p^! A)$. $\mathrm{d}\Omega_A = 0$ implies $[\Lambda_A, \Lambda_A] = 0$, for the Schouten bracket on $p^! A$, and Λ_A is a Poisson section of $\wedge^2(p^! A)$, in this way the Lie algebroid $p^! A^*$ has a natural structure of an exact Lie bialgebroid.

Let $\{\ ,\ \}_{A^*}$ be the Poisson bracket on A^* associated to the exact Lie bialgebroid $(p^! A, (p^! A)^*)$, it is known that it coincides with the Poisson bracket canonically defined on A^* by the Lie algebroid structure of A (see [16]).The Hamiltonian vector field $\rho(\sigma_H)$ is expressed as $X_H = \{-, H\}_{A^*}$.

Let $\Phi : G \times A \longrightarrow A$ be an action by Lie algebroid morphisms of a Lie group G on the Lei algebroid A. Using Proposition 3.4.10 we pull-back the morphisms Φ_g to morphisms $\Phi_g^! : p^! A \longrightarrow p^! A$ over the maps $\Phi_{g^{-1}}^*$. A calculation proves that

$$(\Phi_g^!)^* \Theta_A = \Theta_A$$

which makes $\Phi_g^!$ morphisms of the exact Lie bialgebroid $(p^!A, (p^!A)^*)$.

Let us consider the case of a free and proper G-action on A by Lie algebroid automorphisms, the exact Lie bialgebroid structure on $(p^!A, (p^!A)^*)$ descends to the quotient by the G-action(see [15]):

Proposition 4.3.3. *Let $\Phi : G \times A \longrightarrow A$ be an free and proper G-action on the Lie algebroid A, over M, by Lie algebroid automorphisms. Then $((p^!A)/G, (p^!A)^*/G)$ has an exact Lie bialgebroid structure, over A^*/G, with a symplectic structure $\Omega_{(p^!A)/G}$ such that*

$$\pi^*_{p^!A}\Omega_{(p^!A)/G} = \Omega_{p^!A},$$

where $\pi^*_{p^!A}$ is the projection form $p^!A$ to $(p^!A)/G$.

Moreover,$((p^!A)/G, (p^!A)^*/G)$ is isomorphic to $(p^!_G(A/G), (p^!_G(A/G)^*)$ with the canonical symplectic structure $\Omega_{p^!_G(A/G)}$, where $p_G : (A/G)^* \longrightarrow M/G$.

Now we consider a proper but not necessarily free G-action on the Lie algebroid A. Applying Proposition 3.2.8 and 3.2.7 to the Lie bialgebroid $(p^!A, (p^!A)^*)$ we get reduced Lie bialgebroids $((p^!A)^K, ((p^!A)^K)^*)$ for any isotropy group K of the G-action. We use this fact in the following proposition:

Proposition 4.3.4. *Let $\Phi : G \times A \longrightarrow A$ be an proper action by Lie algebroid morphisms of a Lie group G on A, let K be an isotropy group of the G-action. Then, the bundles $p^!A^K \longrightarrow (A^K)^*$, $p^!A_K \longrightarrow (A_K)^*$, $p^!A_{(K)}/G \longrightarrow (A_{(K)}/G)^*$ have Lie bialgebroid structures. Moreover, the Lie bialgebroids $((p^!A_{(K)})/G, (p^!A_{(K)})^*/G)$ and $(p^!_G(A_{(K)}/G), (p^!_G(A_{(K)}/G)^*)$ are isomorphic, where $p_G : (A_{(K)}/G)^* \longrightarrow M_{(K)}/G$, which makes $((p^!A_{(K)})/G, (p^!A_{(K)})^*/G)$ an exact Lie bialgebroid.*

Proof. The Lie bialgebroid structures on $(p^!A)^K$, $(p^!A)_K$, $(p^!A)_{(K)}/G$ are obtained by applying Propositions 3.2.7 and 3.2.8 to the Lie bialgebroid $p^!A$ with the appropriate G-action. Recall that the pull-back Lie algebroid $p^!A$ is defined by

$$p^!A = T(A^*) \ {}_{\mathrm{d}p}\times_\rho A$$

and the G-action is defined by acting on each coordinate, therefore

$$\begin{aligned}(p^!A)^K &= (T(A^*))^K \ {}_{\mathrm{d}p}\times_\rho A^K \\ &= T((A^*)^K) \ {}_{\mathrm{d}\bar{p}}\times_\rho A^K \\ &= \bar{p}^! A^K\end{aligned}$$

where $\bar{p} = p|_{(A^*)^K}$ is the restriction of p to $(A^*)^K$, and the bundle $(p^!A)^K$ is isomorphic to the algebroid pull-back of A^K over $\bar{p} : (A^*)^K \longrightarrow M^K$. Let

$\varphi : (A^*)^K \longrightarrow (A^K)^*$ denote the isomorphism described in Section 3.2.3. We now use Proposition 3.4.10 to pull-back the identity morphism of A^K over the map φ, we denote the resulting isomorphism by $\tilde{\varphi}$. By abuse of notation we denote by p the projection of the bundle $(A^K)^* \longrightarrow M^K$ and use $\tilde{\varphi}$ to induce a Lie bialgebroid structure on the bundle $(p^! A)^K \longrightarrow (A^K)^*$.

To endow the bundle $p^! A_K$ with a Lie bialgebroid structure, we note first that for any vector bundle $D \longrightarrow M$ with a G-action by vector bundle morphisms there is an isomorphism between $(D^*)_K$ and $(D_K)^*$. Indeed, we have the following equivalences

$$(D^*)_K = (D^*)^K|_{M_K} \simeq (D^K)^*|_{M_K} \simeq (D^K|_{M_K})^* = (D_K)^*,$$

We now can write

$$\begin{aligned}(p^! A)_K &= (p^! A)^K|_{(A^*)_K} \\ &= (p^! A^K)|_{(A_K)^*} \\ &= p^! A_K\end{aligned}$$

where, similarly to the case of $p^! A^K$, we have transported the Lie bialgebroid structure from the bundle $p^! A^K$ with base $(A^*)_K$ to the bundle with base $(A_K)^*$.

To prove the last assertion of the proposition we can apply Proposition 4.3.3 to obtain the following series of equivalences

$$\begin{aligned}(p^! A)_{(K)}/G \simeq (p^! A)_K/L &\simeq (p^! A_K)/L \\ &\simeq p_L^!(A_K/L) \simeq p_G^!(A_{(K)}/G),\end{aligned}$$

where p_L and p_G are the projections into the quotient spaces by the L and G-actions, and $L = N(K)/K$ acts properly and freely on A_K. $\square$

Finally, note that from Proposition 3.2.8 we see that the expression for the reduced Poisson bracket to $(A^*)^K$ restricts from the Poisson bracket in A^* associated to $(p^! A, (p^! A)^*)$ for invariant extensions, this is, given $f, g \in C^\infty(A^*)^G$

$$\{f|_{(A^*)^K}, g|_{(A^*)^K}\}_{(A^*)^K} = \{f, g\}_{A^*}.$$

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