Orbifolds and Wallpaper Patterns [1]

João Guerreiro [2]

LMAC
Instituto Superior Técnico
2009

[1]This project was developed under the supervision of Professor Rui Loja Fernandes as part of the scholarship *Programa Gulbenkian Novos Talentos em Matemática 2008/2009*

[2]jleitaoguerreiro@gmail.com

Contents

1 Introduction

An orbifold, a generalization of the concept of manifold, is a topological space with some additional structure. Recall that a manifold is a space locally modeled in $\mathbb{R}^n$.

A wallpaper pattern is exactly what you imagine it is, a decoration or a tiling of a wall with a high degree of symmetry. Mathematically speaking it is a tiling of the euclidean plane which is invariant by a vertical and an horizontal translation. For each wallpaper pattern there is a discrete group of isometries associated, the group of symmetries of that pattern.

The goal of these notes is to use our knowledge of orbifolds to classify all the wallpaper patterns. This classification is made by considering the symmetries that a given wallpaper pattern has. We are not interested whether the wallpaper is painted orange and purple or green and white.

In section 2, we define manifolds and orbifolds. In section 3, some algebraic topology concepts such as covering, homotopy and fundamental group are introduced. In section 4, we show a theorem that provides a great deal of examples of orbifolds and in section 5 some results about orbifold coverings are stated. The last two sections are dedicated to wallpaper patters, its classification is made in section 6 and section 7 is a collection of some nice images of wallpaper patterns.

2 Basic definitions

Definition 2.1 (Topological manifold) *A* ***manifold*** *is a Hausdorff topological space X with a collection of open sets $\{U_j\}$ closed under finite intersections such that:*

- $\bigcup_j U_j = X$.
- *For every U_j there is a homeomorphism $\varphi_j : U_j \to \varphi_j(U_j) \subset \mathbb{R}^n$ (coordinate chart).*
- *For every pair φ_{j_1}, φ_{j_2} such that $U_{j_1} \cap U_{j_2} \neq \emptyset$ there is a homeomorphism $\varphi_{j_1}\varphi_{j_2}^{-1} : \varphi_{j_2}(U_{j_1} \cap U_{j_2}) \to \varphi_{j_1}(U_{j_1} \cap U_{j_2})$ (transition maps).*

When the transition maps are diffeomorphisms we get a ***differentiable manifold****.*

The definition of orbifold is similar to the one of manifold. Instead of a covering of sets homeomorphic to open sets in $\mathbb{R}^n$, there is a covering of sets homeomorphic to quotients of open sets in $\mathbb{R}^n$. This means we have to be a bit more careful with the definition of the transition maps.

Before giving the formal defintion of orbifold we'll give some examples of orbifolds.

Examples 2.2

1. *Every topological manifold is an orbifold! (We shall prove it later.)*
2. *The quotient space $\mathbb{R}^3/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflection on the xy-plane.*
3. *The quotient space $S^2/\mathbb{Z}_{73}$ where $\mathbb{Z}_{73}$ acts by rotation on S^2.*
4. *The quotient M/G where M is a topological manifold and G is a compact Lie group with finite isotropy groups.*
5. *The quotient space $(\mathbb{R}^+)^3/\cong$ where $\cong$ is the following equivalence relation: $(x, y, z) \cong (x', y', z')$ if the triangles with length sides $(x+y, x+z, y+z)$ and $(x'+y', x'+z', y'+z')$, respectively, are similar.*

Finally, here's the definition of orbifold.

Definition 2.3 (Orbifold) *An **orbifold** is a Hausdorff topological space X with a collection of open sets $\{U_j\}$ closed under finite intersections such that:*

- $\bigcup_j U_j = X$.
- *For every U_j there is a homeomorphism $\varphi_j : U_j \to \tilde{U}_j/\Gamma_j$ where $\tilde{U}_j$ is a neighbourhood of $\mathbb{R}^n$ and Γ_j is a finite group acting on $\tilde{U}_j$.*
- *For every $U_i \subset U_j$ there is an injective homomorphism $\pi_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\varphi}_{ij} : \tilde{U}_i \to \tilde{U}_j$ such that:*
  - $\tilde{\varphi}_{ij}(\gamma x) = \pi_{ij}(\gamma)\tilde{\varphi}_{ij}(x), \ \gamma \in \Gamma_i$.
  - *the following diagram commutes* [3]*:*

$$
\begin{tikzcd}
\tilde{U}_i \arrow[r, "\tilde{\varphi}_{ij}"] \arrow[d] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r, "\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i"] & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] \arrow[r, phantom, "\subset"] & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}
$$

[3] Diagram taken from [1].

3 Some algebraic topology

We'll do a brief introduction to some topological notions which wil be useful later.

Definition 3.1 (Covering) *Let X be a topological space. A **covering** of X is a topological space Y and a continuous surjective projection $p : Y \to X$ such that for every point $x \in X$ there is an open neighbourhood U of x for which $p^{-1}(U)$ is a disjoint union of "copies" of U. More precisely, $p^{-1}(U)$ is a disjoint union of open sets which are mapped homeomorphically by p to U.*

An open set U in the conditions of the previous definition is called **evenly covered**. If X is evenly covered then the covering is called **trivial**.

Examples:

- The identity map $i : X \to X$ is a covering of X which is trivial.
- The mapping $p : \mathbb{R} \to S^1$ which "wraps" $\mathbb{R}$ around S^1.
- The double covering (because the preimage of each points has 2 points) that consists of identifying antipodal points in S^n, $f : S^n \to \mathbb{R}P^n$.

For a better understanding of a topological space it is often useful to study its paths and how they can be deformed (continuously) into each other, which brings us to the definition of homotopy.

Definition 3.2 (Path) *A **path** in a topological space X is a continuous function $f : [0,1] \to X$.*

A path which starts and ends in the same point ($x = f(0) = f(1)$) is called a **loop** based at x. If for every pair of points in a space X there is a path from one of them to the other then X is called **path-connected**.

Definition 3.3 (Homotopy of paths) *Let f_1 and f_2 be paths in a topological space X. A **homotopy** between f_1 and f_2 is a continuous function $H : [0,1] \times [0,1] \to X$ such that $H(0,t) = f_1(t)$, $H(1,t) = f_2(t)$ and the endpoints are fixed ($H(s,0)$ and $H(s,1)$ are constant).*

Theorem 3.4 (Path lifting property) *Let $p : Y \to X$ be a covering of topological spaces and γ a path in X. Let $y \in Y$ be such that $p(y) = \gamma(0)$. There is a unique path σ in Y such that: $\sigma(0) = y$ and $p \circ \sigma = \gamma$.*

Theorem 3.5 (Homotopy lifting property) *Let $p : Y \to X$ be a covering of topological spaces and H a homotopy in X. Let γ be the path in X such that $\gamma(t) = H(0,t)$ and σ a lifting of γ in Y. There is a unique homotopy H' such that both conditions hold: $\sigma(t) = H'(0,t)$ and $p \circ H' = H$.*

For a proof of these two theorems see [2]. In both cases, when the path or the homotopy lie in an evenly covered open set the proof is quite immediate.

An important concept in topology is the notion of **fundamental group**, which consists of all loops based at one point, up to homotopy. Before we give the formal definition of the fundamental group we'll need to define an operation between loops and to show that the homotopy relation is an equivalence relation.

Definition 3.6 (Composition of loops) *The* ***composition*** *of two loops γ and σ, denoted by $\gamma \cdot \sigma$, is given by:*

$$\gamma \cdot \sigma(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \sigma(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Theorem 3.7 *Homotopy is an equivalence relation.*

Proof:

- Reflexivity: If γ is a path then $H(s,t) = \gamma(t)$ is a homotopy between γ and itself.

- Symmetry: If γ is homotopic to σ then there is a homotopy $H(s,t)$ such that $H(0,t) = \gamma(t)$ and $H(1,t) = \sigma(t)$. Clearly, $G(s,t) = H(1-s,t)$ is a homotopy between σ and γ.

- Transitivity: Let γ, σ and δ be paths such that $H_1(s,t)$ is a homotopy between γ and σ and $H_2(s,t)$ a homotopy between σ and δ. Then the following function is a homotopy between γ and δ:

$$H(s,t) = \begin{cases} H_1(2s,t), & 0 \leq s \leq \frac{1}{2} \\ H_2(2s-1,t), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

□

We are now in conditions to give a formal definition of the fundamental group.

Definition 3.8 (Fundamental Group) *Let X be a path-connected topological space and $x \in X$. The* ***fundamental group*** *of X based at x, usually denoted by $\pi_1(X,x)$, is the quotient of the set of loops based at the point x by the homotopy relation.*

To legitimately call this quotient set a group we'll need to justify our claim. Here follows a proof of our definition!

Proof: We'll start by defining an operation between homotopy classes of loops. Denoting the equivalence class of γ by $[\gamma]$, define $[\gamma] \bullet [\sigma] = [\gamma \cdot \sigma]$. To show that this function is well-defined we need to show that $[\gamma \cdot \sigma]$ is independent of the paths γ and σ chosen as long as they belong to the same homotopy class. Let γ' and σ' be paths homotopic to γ and σ by H_1 and H_2, respectively. The following function is a homotopy between $\gamma \cdot \sigma$ and $\gamma' \cdot \sigma'$:

$$H(s,t) = \begin{cases} H_1(s,2t), & 0 \leq t \leq \frac{1}{2} \\ H_2(s,2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

It is now necessary to show that the homotopy classes set with this operation is indeed a group:

- Associativity:
  Let $[\gamma]$, $[\sigma]$, $[\delta] \in \pi_1(X,x)$.
  We'll show that $\gamma \cdot (\sigma \cdot \delta)$ and $(\gamma \cdot \sigma) \cdot \delta$ are homotopic so the classes $[\gamma] \bullet ([\sigma] \bullet [\delta]) = [\gamma \cdot (\sigma \cdot \delta)]$ and $([\gamma] \bullet [\sigma]) \bullet [\delta] = [(\gamma \cdot \sigma) \cdot \delta]$ are the same.
  The following function is a homotopy between those paths:

$$H(s,t) = \begin{cases} \gamma(\frac{4t}{2-s}), & 0 \leq t \leq \frac{2-s}{4} \\ \sigma(4t+s-2), & \frac{2-s}{4} \leq t \leq \frac{3-s}{4} \\ \delta(\frac{4t+s-3}{1+s}), & \frac{3-s}{4} \leq t \leq 1 \end{cases}$$

- Existence of Identity:

  Let ϵ_x be the path defined by $\epsilon_x(t) = x, \ \forall t \in [0,1]$. We'll show that $[\epsilon_x]$ is the identity of $\pi_1(X,x)$.

  Let γ be a path in X. We'll show that γ, $\epsilon_x \cdot \gamma$ and $\gamma \cdot \epsilon_x$ are homotopic which shows that $[\gamma] = [\gamma] \bullet [\epsilon_x] = [\epsilon_x] \bullet [\gamma]$.

  The following functions provide the desired homotopies:

$$H_{left}(s,t) = \begin{cases} x, & 0 \leq t \leq \frac{s}{2} \\ \gamma(\frac{2t-s}{2-s}), & \frac{s}{2} \leq t \leq 1 \end{cases}$$

$$H_{right}(s,t) = \begin{cases} \gamma(\frac{2t}{2-s}), & 0 \leq t \leq \frac{2-s}{2} \\ x, & \frac{2-s}{2} \leq t \leq 1 \end{cases}$$

- Existence of Inverse: Let $[\gamma] \in \pi_1(X,x)$. We'll show that $[\gamma^{-1}] = [\gamma]^{-1}$ where $\gamma^{-1}(t) = \gamma(1-t), \ \forall t \in [0,1]$.

  We'll do it by making homotopies between ϵ_x, $\gamma \cdot \gamma^{-1}$ and $\gamma^{-1} \cdot \gamma$:

$$H_1(s,t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{s}{2} \\ \gamma(s) = \gamma^{-1}(1-s), & \frac{s}{2} \leq t \leq \frac{2-s}{2} \\ \gamma^{-1}(2t-1), & \frac{2-s}{2} \leq t \leq 1 \end{cases}$$

$$H_2(s,t) = \begin{cases} \gamma^{-1}(2t), & 0 \leq t \leq \frac{s}{2} \\ \gamma^{-1}(s) = \gamma(1-s), & \frac{s}{2} \leq t \leq \frac{2-s}{2} \\ \gamma(2t-1), & \frac{2-s}{2} \leq t \leq 1 \end{cases}$$

□

If the topological space is path-connected then the fundamental group is the same for all base points, up to group isomorphism. In this case, we usually simply refer to the fundamental group of the whole space. When the fundamental group is the trivial group (the group with just one element) then we say the topological space is **simply connected**. As we shall see later, these are the largest covering spaces. Some examples of simply connected spaces are S^2 and $\mathbb{R}^n$ ($n \geq 2$). On the other hand, the torus T^2 isn't simply connected as its fundamental group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Theorem 3.9 (Fundamental Group of a Covering) *Let $p : Y \to X$ be a covering of topological spaces with the additional assumption that both spaces are path-connected. Let $y \in Y$ and $x = p(y) \in X$. Then $\pi_1(Y, y)$ is isomorphic to some subgroup of $\pi_1(X, x)$ where the isomorphism is induced by the projection p.*

Proof:

For any continuous map $p : Y \to X$ and any point $y \in Y$, there is an induced group homomorphism $p_* : \pi_1(Y, y) \to \pi_1(X, x)$, where $x = p(y)$:

$$p_*([\gamma]) = [p \circ \gamma].$$

This function is well-defined because a continuous map preserves homotopies, i.e. if H is a homotopy between γ_1 and γ_2 then $p \circ H$ is a homotopy between $p \circ \gamma_1$ and $p \circ \gamma_2$. As $(p \circ \gamma_1) \cdot (p \circ \gamma_2) = p \circ (\gamma_1 \cdot \gamma_2)$ it follows that p_* is a group homomorphism.

To conclude the proof, we'll show that $\pi_1(Y, y)$ is isomorphic to $p_*(\pi_1(Y, y))$ by showing that the kernel of p_* is trivial.

Suppose $p_*[\gamma] = [p \circ \gamma] = [\epsilon_x]$ with $[\gamma] \in \pi_1(Y, y)$. By Theorem 1.5, we can lift the homotopy between $p \circ \gamma$ and ϵ_x to a homotopy between γ and ϵ_y. Therefore, we conclude that $[\gamma] = [\epsilon_y]$, as desired.

$\square$

Intuitively, the previous theorem says that (under the assumptions of the theorem) the structure of the fundamental group of a covering is "simpler" than the one of the covered space. If we construct a covering of the covering space we'll get an even simpler fundamental group, and so on.

How far can we do this? Till the fundamental group is trivial (with a few extra conditions on the topological space we start with).

Definition 3.10 *Let Y and X be connected topological spaces. A covering $p : Y \to X$ is called a **universal covering** if Y is simply connected.*

A space X is called **semilocally simply connected** if for every point $p \in X$ there is an open neighbourhood U of p such that every loop in U is homotopic to a constant loop.

Theorem 3.11 *Let X be a path-connected and semilocally simply connected space. Then X has a univeral covering, which is unique up to homeomorphism.*

For a proof of this theorem, see [2].

The universal covering has a very important property, the **universal property**: if $p : \tilde{X} \to X$ is the universal covering of X and $q : Y \to X$ is another covering of X (with Y path-connected), then there is a unique covering $r : \tilde{X} \to Y$ such that $p = q \circ r$. A proof of this fact can be found in [4].

We shall present a proof of a weaker result, which will be enough for our study of orbifolds.

Theorem 3.12 *Let X be a path-connected and locally trivial space, i.e., for every point $x \in X$ there is a neighbourhood U such that for every covering p, $p^{-1}(U)$ is a union of disjoint open sets homeomorphic to U. Then there is a covering $p : Y \to X$ with the universal property.*

Proof: Consider all connected coverings of X, $p_\alpha : X_\alpha \to X$. In each space X_α fix a point x_α such that $p_\alpha(x_\alpha) = x$, $x \in X$.

Let $\Pi_X X_\alpha = \{y \in \Pi X_\alpha : p_\alpha(y_\alpha) \text{ is constant for every } \alpha\}$ with the subspace topology induced by the product topology in ΠX_α ($\Pi_X X_\alpha$ is the **fiber product** of the maps p_α). Then we can define $p : \Pi_X X_\alpha \to X$ as $p(y) = p_\alpha(y_\alpha)$ because the point $p_\alpha(y_\alpha)$ is independent of α.

Let Y be the connected component of $\Pi_X X_\alpha$ that contains (x_α). We'll show that $p : Y \to X$ is a covering which satisfies the universal property.

First, we'll show that p is surjective. Let $x' \in X$ and γ a path between x and x'. For each α, there is a lifting $\tilde{\gamma_\alpha}$ of γ starting in x_α and ending in x'_α (note that $p_\alpha(x'_\alpha) = x'$). Thus we can construct a path γ_C by taking the product of all paths $\tilde{\gamma_\alpha}$. This path γ_C connects (x_α) and (x'_α). As $p((x'_\alpha)) = x'$ we conclude that p is surjective.

Now consider a neighbourhood U of $x' \in X$ that is trivially covered. Then $p_\alpha^{-1}(U)$ is a union of disjoint open sets homeomorphic to U. If we take the fiber product of the restrictions of p_α to each of these sets we still have a union of disjoint open sets homeomorphic to U. By considering only the components that belong to Y we get $p^{-1}(U)$, so p is a covering.

It remains to show that the covering p has the universal property. Let $q : Z \to X$ be a connected covering of X. Then $q = p_\alpha$ and $Z = X_\alpha$ for some α.

Consider the canonical projection π_α from ΠX_α to X_α. Let $r : Y \to X_\alpha$ be the restriction of π_α to Y. By definition of p, $p_\alpha \circ r = p$.

We claim that $r : Y \to Z = X_\alpha$ is a covering.

To show that r is surjective consider a point $x'_\alpha \in X_\alpha$ and a path γ_α that connects x_α and x'_α. Let γ be the projection of γ_α to a path in X. Taking the product of all liftings os γ that start in x_α we get a path in Y that connects (x_α) and (x'_α). As $r((x'_\alpha)) = x'_\alpha$ we conclude that r is surjective.

Let $x' = p_\alpha(x'_\alpha$ with $x'_\alpha \in X_\alpha$. Let U be a neighbourhood of x' that is trivially covered. The component of $p_\alpha^{-1}(U)$ that contains x'_α, V, is homeomorphic to U. As $r^{-1}(V)$ consists of the components in $p^{-1}(U)$ that are projected to V we conclude that $r^{-1}(V)$ is a disjoint union of open sets homeomorphic to V, so r is a covering.

$\square$

4 Examples of orbifolds

How can one produce orbifolds? A simple way is by taking certain quotients of smooth actions.

Theorem 4.1 *Let M be a differentiable manifold and Γ a group acting properly discontinuously and smoothly on M, then M/Γ has an orbifold structure. (An action of a group G is* ***properly discontinuous*** *if for every x in$M$ there is a neighbourhood U of x such that $g(U) \cap U \neq \emptyset$ for only a finite number of $g \in G$)*

Proof: Let $x \in M/\Gamma$ and $\tilde{x} \in M$ which projects to x. Let $I_{\tilde{x}}$ be the isotropy group of $\tilde{x}$. We'll start by finding a neighbourhood of $\tilde{x}$ which is disjoint from its translates by elements not in $I_{\tilde{x}}$ and invariant by elements in $I_{\tilde{x}}$.

As the action is properly discontinuous we pick a neighbourhood $\tilde{V}$ of $\tilde{x}$ such that $\gamma(\tilde{V}) \cap \tilde{V} \neq \emptyset$ for only a finite number of elements in Γ. Let $\{\gamma_1, ..., \gamma_n\}$ be those elements which are not in $I_{\tilde{x}}$. Notice that this also implies that $I_{\tilde{x}}$ is finite.

For each γ_j, let V_1 and V_2 be open sets such that $\tilde{x} \in V_1$, $\gamma_j \cdot \tilde{x} \in V_2$ and $V_1 \cap V_2 = \emptyset$ (they exist because M is Hausdorff). Let $W_j = \tilde{V} \cap V_1 \cap \gamma_j^{-1}(V_2)$ be a neighbourhood of $\tilde{x}$. We can easily check that $W_j \cap \gamma_j(W_j) = \emptyset$. Taking $W = \bigcap W_j$ we get a neighbourhood of $\tilde{x}$ such that $W \cap \gamma(W) \neq \emptyset$ if and only if $\gamma \in I_{\tilde{x}}$.

Now consider the following neighbourhood of $\tilde{x}$: $\tilde{U} = \bigcap_{\gamma \in I_{\tilde{x}}} \gamma(W)$. As $\tilde{U} \subset W$ then $\tilde{U} \cap \gamma(\tilde{U}) = \emptyset$ if $\gamma \notin I_{\tilde{x}}$. On the other hand, if $\sigma \in \Gamma$ then $\sigma(\tilde{U}) = \bigcap_{\gamma \in I_{\tilde{x}}} \sigma(\gamma(W)) = \bigcap_{\gamma \in I_{\tilde{x}}} (\sigma \cdot \gamma)(W) = \bigcap_{\gamma \in I_{\tilde{x}}} \gamma(W)$. We have found a neighbourhood of $\tilde{x}$ with the desired properties. We can suppose that $\tilde{U}$ is contained in some coordinate chart, thus it homeomorphic to some open set in $\mathbb{R}^n$.

Let $Z = \cup_{\gamma \in \Gamma} \gamma(\tilde{U})$ and let $U_x = Z/\Gamma$. By restricting this projection to $\tilde{U}$ we get a homeomorphism between U_x and $\tilde{U}/I_{\tilde{x}}$ where the action of $I_{\tilde{x}}$ is the restriction of the action of Γ on $\tilde{U}$. We'll show that U_x and its finite intersections form a cover of M/Γ.

Let $U_{x_1}, ..., U_{x_k}$ such that $U_{x_1} \cap ... \cap U_{x_k} \neq \emptyset$. This means that there are $\gamma_1, ..., \gamma_k \in \Gamma$ such that $\gamma_1(\tilde{U}_{x_1}) \cap ... \cap \gamma_k(\tilde{U}_{x_k}) \neq \emptyset$.

Consider the following subgroup of Γ, $G = \gamma_1 I_{x_1} \gamma_1^{-1} \cap ... \cap \gamma_k I_{x_k} \gamma_k^{-1}$.

Let $g \in G$, then $g = \gamma_i \sigma_i \gamma_i^{-1}$ where $\sigma_i \in I_{x_i}$. So, $g(\gamma_1(\tilde{U}_{x_1}) \cap ... \cap \gamma_k(\tilde{U}_{x_k})) = (g \cdot \gamma_1)(\tilde{U}_{x_1}) \cap ... \cap (g \cdot \gamma_k)(\tilde{U}_{x_k}) = (\gamma_1 \cdot \sigma_1 \cdot \gamma_1^{-1} \cdot \gamma_1)(\tilde{U}_{x_1}) \cap ... \cap (\gamma_k \cdot \sigma_k \cdot \gamma_k^{-1} \cdot \gamma_1)(\tilde{U}_{x_k}) = \gamma_1(\tilde{U}_{x_1}) \cap ... \cap \gamma_k(\tilde{U}_{x_k})$.

By similar calculations we can show that for $g \notin G$, we have $(\gamma_1(\tilde{U}_{x_1}) \cap ... \cap \gamma_k(\tilde{U}_{x_k})) \cap g(\gamma_1(\tilde{U}_{x_1}) \cap ... \cap \gamma_k(\tilde{U}_{x_k})) = \emptyset$, thus $\gamma_1(\tilde{U}_{x_1}) \cap ... \cap \gamma_k(\tilde{U}_{x_k})/G$ is homeomorphic to $U_{x_1} \cap ... \cap U_{x_k}$ with the homeomorphism given by the projection. $\square$

Examples 4.2

1. *Every differentiable manifold without boundary.*

2. *Every quotient of a differentiable manifold by a smooth action of a finite group.*

This explains why the examples in the second section are orbifolds. We will see later that there are orbifolds which are not quotients of differentiable manifolds by smooth actions. But, locally, every orbifold is of this form, and we introduce:

Definition 4.3 (Singular point) *For every $x \in M$, let Γ_x be the smallest group such that there is neighbourhood of x, $U = \tilde{U}/\Gamma_x$. A point x is called **singular** if Γ_x is not the trivial group. The group Γ_x is called the **isotropy group** of x.*

5 Coverings of orbifolds

As with topological spaces we can define a covering of orbifolds:

Definition 5.1 (Covering of orbifolds) *A **covering orbifold** of an orbifold O is an orbifold $\tilde{O}$ with a projection $p : X_{\tilde{O}} \to X_O$ between the underlying spaces with the following property: for every $x \in X_O$ there is a neighbourhood $U \cong \tilde{U}/\Gamma$ ($\tilde{U}$ is an open set in $\mathbb{R}^n$) such that each connected component of $p^{-1}(U)$ is homeomorphic to $\tilde{U}/\Gamma_i$ for some subgroup Γ_i of Γ. In addition, the homeomorphism has to respect both projection (p and the canonic one between $\tilde{U}/\Gamma_i$ and $\tilde{U}/\Gamma$).*

Definition 5.2 *An orbifold is **good** if it is covered by a manifold. Otherwise, it is a **bad** orbifold.*

Examples 5.3

1. *Every manifold is trivially a good orbifold as it is covered by itself.*

2. *An example of a bad orbifold is the teardrop. The teardrop has S^2 as its underlying topological space and a unique singular point whose neighbourhood is homeomorphic to $U/\mathbb{Z}^n$, where U is a neighbourhood of the origin of $\mathbb{R}^2$ and $\mathbb{Z}^n$ acts by rotations around the origin.*

   To see that it is a bad orbifold, suppose there is a covering $p : M \to O$ where M is a manifold and O is the teardrop. Then there is neighbourhood $U \cong D^2/\mathbb{Z}^n$ of the singular point, thus $p^{-1}(U)$ is a union of disjoint open disks. On the other hand, $X_O - U = S^2 - D^2 \cong \overline{D^2}$. As the closed disk has only trivial coverings (because it is simply connected and locally path-connected, see [2]) then $p^{-1}(X_O - U)$ is a union of disjoint closed disks. Therefore, $M = X_M = p^{-1}(X_O)$ is obtained by "gluing" together pairs of open disks and closed disks. This implies that M is a union of copies of S^2. As M is connected then $M = S^2$ and so there is only one pair open disk/closed disk. Consider an open disk V that overlaps U but doesn't contain p. Then if $x \in U \cap V$ we know that $p^{-1}(x)$ consists of n points because it belongs to U but it has to be just one point as $x \in V$. Contradiction.

Later on we shall introduce a tool that allows us to easily conclude that the teardrop (and some other orbifolds) are bad orbifolds.

In a similar way to the case of topological spaces we can define the universal cover of an orbifold. As we don't have defined here paths in orbifold (altough that is also possible!) we'll to rely on the universal property to define universal covers:

Definition 5.4 (Universal cover of an orbifold) *A connected orbifold $\tilde{O}$ is a universal cover of O if there is a projection $p : X_{\tilde{O}} \to X_O$ with non-singular base points $\tilde{x}$ and $x = p(\tilde{x})$ which has the univesal property (i.e. if $p' : X_{O'} \to X_O$ is a covering of orbifold with $x = p'(x')$ then there is a covering $q : X_{\tilde{O}} \to X_{O'}$ such that $q(\tilde{x}) = x'$ and $q \circ p' = p$).*

Theorem 5.5 (Universal cover) *Every connected orbifold O has a universal cover.*

See [1] for a draft of the proof.

6 Wallpaper patterns classification

In this section we shall classify the 17 wallpaper patterns by making use of our knowledge about orbifolds. How is that possible? Notice that for each wallpaper pattern we can construct an orbifold, the quotient of $\mathbb{R}^2$ by the discrete group of symmetries of the pattern. As this group contains both a vertical and an horizontal translation the constructed orbifold is compact.

We'll start by noting that the singular points belong to one of three classes.

Theorem 6.1 *Every singular point of a 2-orbifold has its neighbourhood modeled by one of these classes:*

- *Mirror:* $\mathbb{R}^2/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *acts by reflection in the x-axis.*
- *Elliptic point:* $\mathbb{R}^2/\mathbb{Z}_n$ *where* $\mathbb{Z}_n$ *acts by rotations.*
- *Corner reflector:* $\mathbb{R}^2/D_n$ *where* D_n *is the dihedral group of order* $2n$*, which is generated by reflections in the lines which meet the x-axis at angle* $\frac{2k\pi}{n}$.

Proof: Let O be a 2-orbifold and $x \in O$ a singular point. Let U be a neighbourhood of x which is diffeomorphic to $\tilde{U}/\Gamma$, $\tilde{U} \subset \mathbb{R}^2$. We can choose U small enough such that $\tilde{x}$ is the only fixed point of the action of Γ in $\tilde{U}$.

Let g be a Riemannian metric on $\tilde{U}$, it can be the usual euclidean metric. We construct a metric g', invariant by the action of Γ, by averaging it under Γ:

$$g'(v,w) = \sum_{\gamma\in\Gamma} g(D\gamma \cdot v, D\gamma \cdot w).$$

Indeed,

$$\begin{aligned}
\sigma_* g'(v,w) &= \sum_{\gamma\in\Gamma} \sigma_* g(D\gamma \cdot v, D\gamma \cdot w) \\
&= \sum_{\gamma\in\Gamma} g(D(\sigma\cdot\gamma) \cdot v, D(\sigma\cdot\gamma) \cdot w) \\
&= \sum_{\gamma\in\Gamma} g(D\gamma \cdot v, D\gamma \cdot w)
\end{aligned}$$

for some $\sigma \in \Gamma$.

Note that we can define action of Γ in $T_{\tilde{x}}\tilde{U}$ by considering the function $D\sigma : T_{\tilde{x}}\tilde{U} \to T_{\tilde{x}}\tilde{U}$. They form a group of isometries for the metric $g'_{\tilde{x}}$.

Let $\sigma \in \Gamma$, $v \in T_{\tilde{x}}\tilde{U}$ and f a geodesic such that $\dot{f}(0) = v$. As $\sigma : \tilde{U} \to \tilde{U}$ is an isometry then $\sigma \circ f$ is also a geodesic and $(\sigma \dot{\circ} f)(0) = D\sigma \cdot v$. This means that $\sigma(\exp_{\tilde{x}}(v)) = \sigma(f(1)) = (\sigma \circ f)(1) = \exp_{\tilde{x}}(D\sigma \cdot v)$. In conclusion, the actions of Γ commute with the diffeomorphism given by the exponential map. This gives an diffeomorphism between a neighbourhood of x in O and V/Γ where V is neighbourhood of the origin of $\mathbb{R}^2$. As Γ acts by isometries then Γ is a finite subgroup of O_2, the orthogonal group of order 2. Finally, we conclude that the neighbourhood of x is modeled by the three classes defined above.

□

Note that an open neighbourhood in each of these models is homeomorphic to an open neighbourhood of the half-plane. This means that every 2-orbifold has a topological surface with boundary (2-manifold with boundary) as its underlying space.

An important tool for our classification of (some) 2-orbifolds the orbifold Euler number, generalizing the usual Euler number.

Definition 6.2 *Let O be an orbifold and consider a cell-division (triangulation in the 2-dimensional case) of X_O such that the isotropy group of points in the interior of each cell is constant. We define the **Euler number**, $\chi(O)$, by the following formula:*

$$\chi(O) = \sum_{cells,c} \frac{(-1)^{dim(c)}}{|\Gamma(c)|}.$$

Notice that this definition equals the original one for manifolds if O is a manifold.

Theorem 6.3 *We say that $p : \tilde{O} \to O$ is a covering of k sheets if the number of preimages of a non-singular point is k. In that case, we have*

$$\chi(\tilde{O}) = k\chi(O)$$

Proof: Let $x \in O$ and let $U \cong V/\Gamma_x$ be a well-covered neighbourhood of x. Let y be a nonsingular point in U, which corresponds to $|\Gamma_x|$ points in V. Each preimage of y by

p lies in a neighbourhood of a preimage of x of the form $V/\Gamma_{\tilde{x}}$. Thus, in each of those neighbourhoods there are $\frac{|\Gamma_x|}{|\Gamma_{\tilde{x}}|}$ preimages of y. Computing the total number of preimages of y gives:

$$k = \sum_{p^{-1}(x)} \frac{|\Gamma_x|}{|\Gamma_{\tilde{x}}|} \Leftrightarrow \frac{k}{|\Gamma_x|} = \sum_{p^{-1}(x)} \frac{1}{|\Gamma_{\tilde{x}}|}$$

We can construct a cell-division in $\tilde{O}$ by taking the preimage of a cell-division in O. Therefore,

$$k\chi(O) = \sum_{cells,c} \frac{k(-1)^{dim(c)}}{|\Gamma(c)|} = \sum_{\tilde{c}\in p^{-1}(c)} \frac{(-1)^{dim(c)}}{|\Gamma(\tilde{c})|} = \chi(\tilde{O})$$

□

Theorem 6.4 *Let O be a 2-orbifold with underlying space X_O. If O has n elliptic points with orders a_1,...,a_n and m corner reflectors with orders b_1,...,b_m then:*

$$\chi(O) = \chi(X_O) - \sum_{i=1}^{n} \left(1 - \frac{1}{a_i}\right) - \frac{1}{2}\sum_{j=1}^{m} \left(1 - \frac{1}{b_j}\right).$$

Proof: For each elliptic point of order a we add a point (0-dimensional cell) with the group $\mathbb{Z}_a$. This means that addition of such elliptic point corresponds to a decrease of the Euler number in $1 - \frac{1}{a}$.

For each corner reflector of order b we add a point with the group D_b (which has order $2b$) and a 1-dimensional mirror cell with the group $\mathbb{Z}_2$ associated. Thus the addition of such corner reflector correponds to a decrease of the Euler number in $\frac{1}{2} - \frac{1}{2b}$.

Adding several elliptic points and corner reflectors yields the formula above.

□

Theorem 6.5 (Classification of compact surfaces) *Any connected compact surface M is either homeomorphic to a sphere, a connected sum of tori or a connected sum of projectives planes. Moreover, $\chi(S^2) = 2$, $\chi(\underbrace{T\#...\#T}_{n\ times}) = 2 - 2n$ and $\chi(\underbrace{\mathbb{R}P^2\#...\#\mathbb{R}P^2}_{n\ times}) = 1 - n$.*

Corollary 6.6 *The teardrop is a bad manifold.*

Proof: The teardrop has a single elliptic point (of order $n \geq 2$) so $\chi(teardrop) = \chi(S^2) - (1 - \frac{1}{n}) = \frac{n+1}{n}$. If the teardrop is covered by a manifold M, then $\chi(M) = \frac{k(n+1)}{n}$ and M has to be compact. As the Euler characteristic is an integer for manifolds then n divides k. Thus, $\chi(M) \geq n + 1 \geq 3$ which is impossible, due to the classification theorem.

$\square$

Theorem 6.7 (Classification of compact surfaces with boudary) *Any bordered connected compact surface with boundary N is of the form $M \setminus (D_1 \sqcup ... \sqcup D_k)$ where M is a connected compact surface and D_i are disjoint disks in M. Moreover, $\chi(N) = \chi(M) - k$.*

The proofs of these classification theorems can be found in [3].

With these four theorems we are ready to classify the 17 wallpaper patterns. Each pattern corresponds to a 2-orbifold which is covered by $\mathbb{R}^2$. As the Euler number for orbifold matches the one for manifold when the orbifold is actually a manifold then $\chi(\mathbb{R}^2) = 0$. By theorem 4.3. we conclude that an orbifold O covered by $\mathbb{R}^2$ also has $\chi(O) = 0$. We shall classify those orbifolds.

Theorem 6.8 *The compact 2-orbifolds O with $\chi(O) = 0$ are the following:*

Underlying space (X_O)	*Orders of elliptic points*	*Orders of corner reflectors*
S^2	*2, 3, 6*	-
	2, 4, 4	-
	3, 3, 3	-
	2, 2, 2, 2	-
D^2	-	*2, 3, 6*
	-	*2, 4, 4*
	-	*3, 3, 3*
	-	*2, 2, 2, 2*
	2	*2, 2*
	3	*3*
	4	*2*
	2, 2	-
$\mathbb{R}P^2$	*2, 2*	-
T^2	-	-
Klein bottle	-	-
Annulus	-	-
Möbius band	-	-

Proof: First note that $0 = \chi(O) \leq \chi(X_O)$ where X_O is the base space of O. By the first classification theorem, we know that the only compact surfaces with that property are S^2, T, $\mathbb{R}P^2$ and the Klein bottle (which is $\mathbb{R}P^2 \# \mathbb{R}P^2$). By removing disks from these surfaces we can construct some more appropriate manifolds: D^2 and the annulus arise from removing disks from S^2 and the Möbius band is constructed by taking a disk from $\mathbb{R}P^2$.

We'll now classify the orbifolds by considering the possible base spaces.

<u>$X_O = S^2$:</u>

Notice that compact surfaces cannot have corner reflectors as singular points because they always lie in the border of the orbifold. With this restriction, by theorem 4.4:

$$\chi(O) = \chi(S^2) - \sum_{i=1}^{n} \left(1 - \frac{1}{a_i}\right) \Leftrightarrow \sum_{i=1}^{n} \left(1 - \frac{1}{a_i}\right) = 2.$$

As, $1-\frac{1}{a_i}\in[\frac{1}{2},1[$ then $n=3,4$.

If $n=3$,

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}=1,$$

which has the solutions $(a_1,a_2,a_3)=(2,3,6),(2,4,4),(3,3,3)$.

If $n=4$,

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}=2,$$

which has the solution $(a_1,a_2,a_3,a_4)=(2,2,2,2)$.

$\underline{X_O = D^2}$:

By theorem 4.4:

$$\chi(O)=\chi(D^2)-\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right)-\frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)\Leftrightarrow\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right)+\frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=1.$$

By the argument used before, $n\leq 2$.

If $n=0$,

$$\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=2.$$

We have already found the solutions to this equation: $(b_1,b_2,b_3)=(2,3,6),(2,4,4),(3,3,3)$ and $(b_1,b_2,b_3,b_4)=(2,2,2,2)$.

If $n=1$, then $m\geq 1$ and $\frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)\geq\frac{1}{4}$. This means that $1-\frac{1}{a_1}\leq\frac{3}{4}\Leftrightarrow a_1\leq 4$.

If $a_1=2$, then $\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=1$. As, $1-\frac{1}{a_i}\in[\frac{1}{2},1[$ then $m=2$. Thus we have,

$$\frac{1}{b_1}+\frac{1}{b_2}=1$$

which has the solution $(b_1,b_2)=(2,2)$.

If $a_1=3$, then $\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=\frac{2}{3}$. As, $1-\frac{1}{a_i}\in[\frac{1}{2},1[$ then $m=1$ and the solution is $b_1=3$.

If $a_1=4$, then $\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right)=\frac{1}{2}$. As, $1-\frac{1}{a_i}\in[\frac{1}{2},1[$ then $m=1$ and the solution is $b_1=2$.

If $n = 2$, then $\sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right) \geq 1$ with equality when $a_1 = a_2 = 2$. Thus the only solution for $n = 2$ is with $m = 0$ and $a_1 = a_2 = 2$.

<u>$X_O = \mathbb{R}P^2$:</u>

By theorem 4.4,

$$\chi(O) = \chi(D^2) - \sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right) - \frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right) \Leftrightarrow \sum_{i=1}^{n}\left(1-\frac{1}{a_i}\right) + \frac{1}{2}\sum_{j=1}^{m}\left(1-\frac{1}{b_j}\right) = 1.$$

This is the same equation we had for the disk. However, $\mathbb{R}P^2$ is compact so it cannot have corner reflectors. Therefore, the only solution is $n = 2$, $m = 0$ and $a_1 = a_2 = 2$.

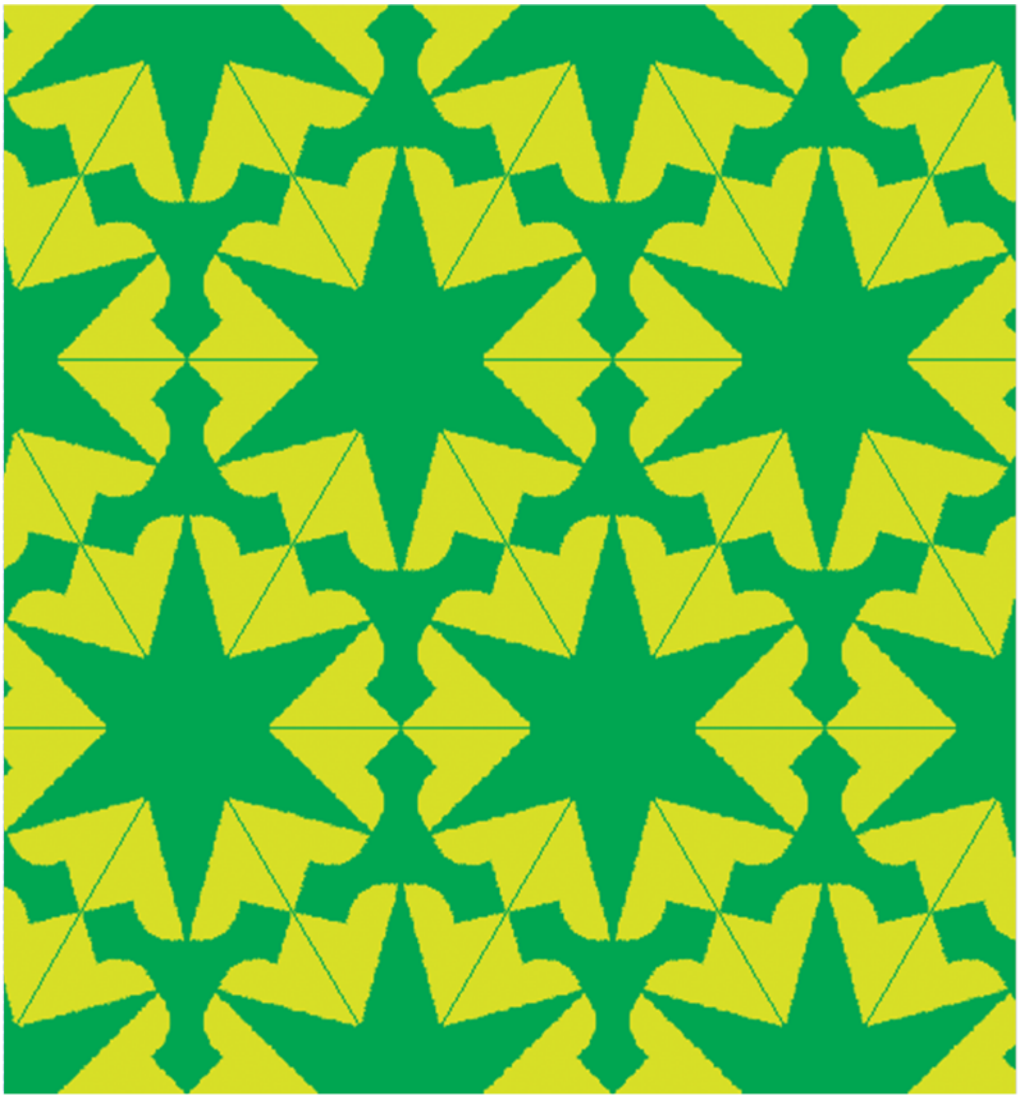
<u>$X_O = T$, Klein bottle, annulus, Möbius band:</u>

As $X_O = 0$, the orbifold O cannot have singular points.

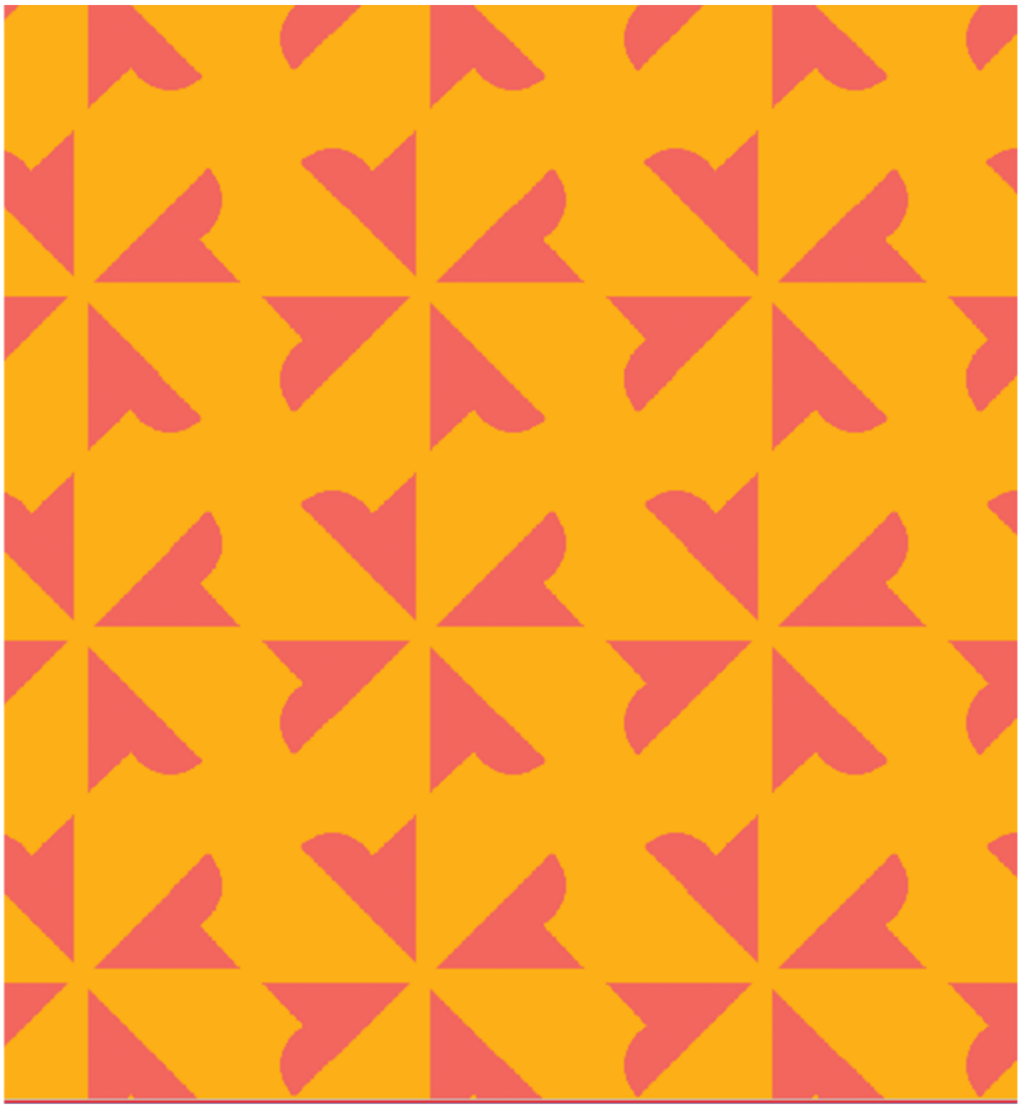
$\square$

7 Images of the wallpaper patterns

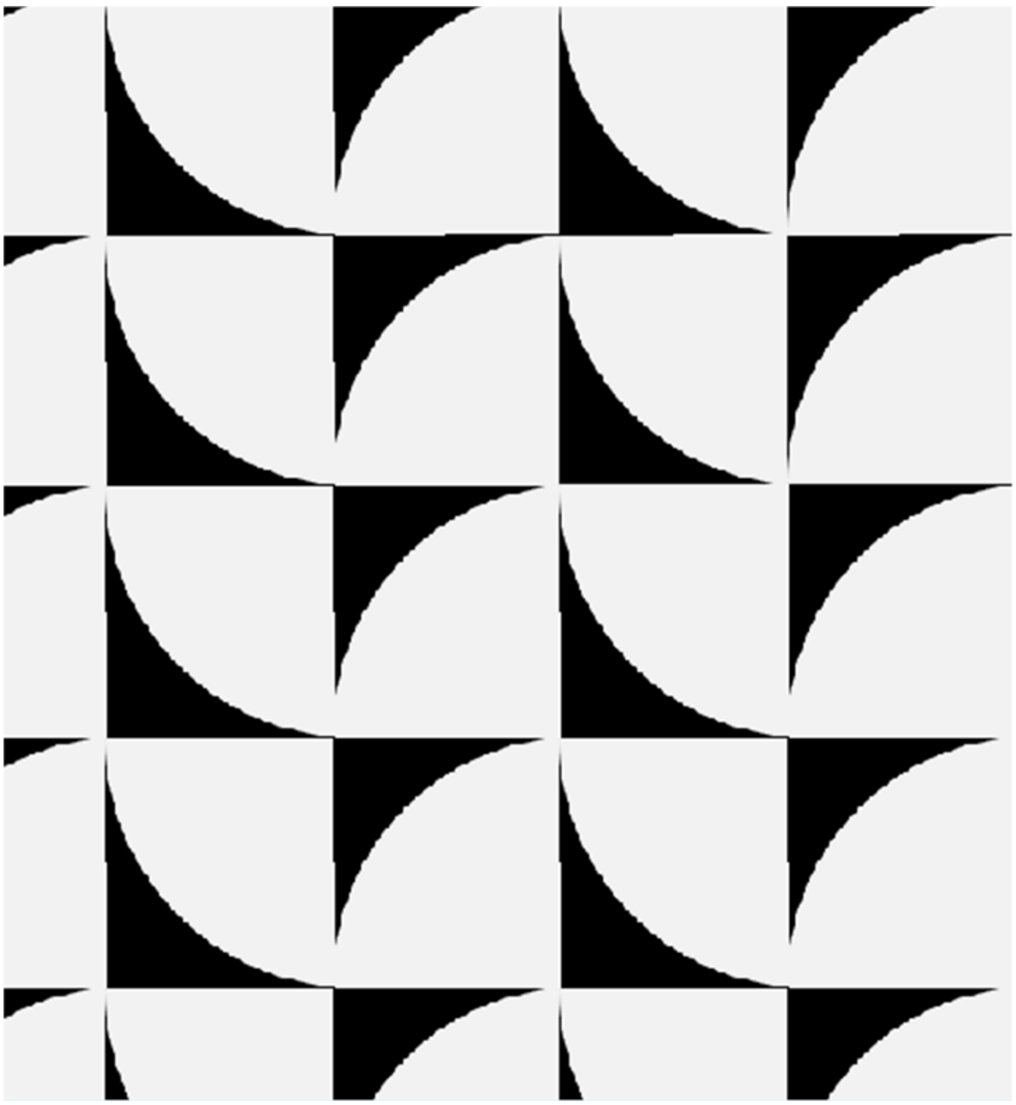
The following images were taken from http://www.atractor.pt/mat/orbifolds/index.html.



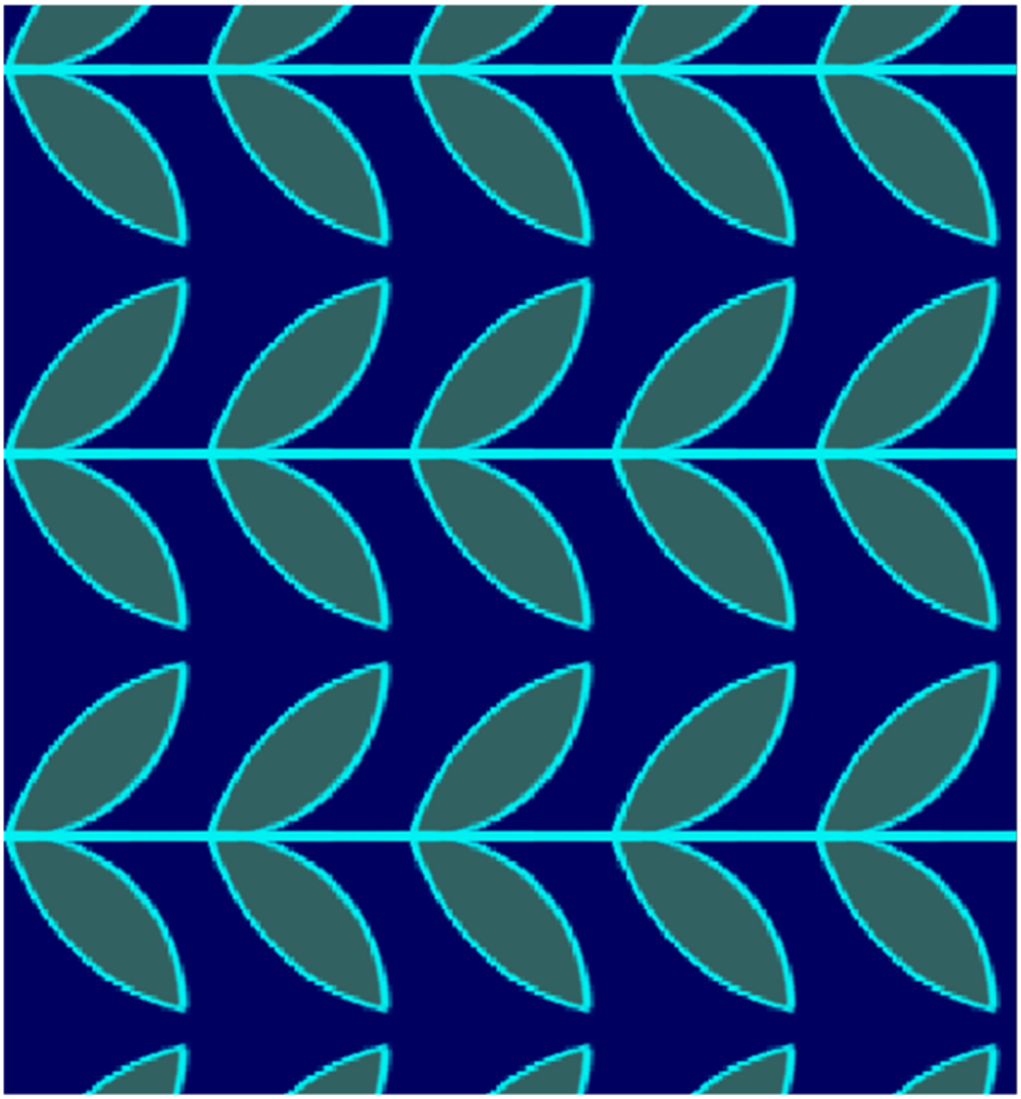
D^2 with corner reflectors of orders 2, 3 and 6.



S^2 with elliptic points of orders 2, 4 and 4.



Klein bottle.



Annulus.

References

[1] Thurston, William P. (1980) *Geometry and Topology of Three-Manifolds, Lecture Notes.*

[2] Fulton, W. (1997) *Algebraic Topology: A First Course*. Springer.

[3] Massey, William S. (1987) *A Basic Course in Algebraic Topology*

[4] Munkres, James R. (2000) *Topology (Second Edition)*