FROM THE TODA LATTICE TO THE VOLTERRA LATTICE AND BACK

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Abstract. We discuss the relationship between the multiple Hamiltonian structures of the generalized Toda lattices and that of the generalized Volterra lattices.

Contents

1. Introduction

The Volterra lattice is the system of o.d.e.'s

$$\dot{a}_i = a_i(a_{i-1} - a_{i+1}) \qquad (i = 1, \dots, n), \tag{1}$$

where $a_{n+1} = a_0 = 0$. This system was first studied by Kac, van-Moerbeke and Moser in two foundational papers ([17, 20]) for the modern theory of integrable systems. They showed, for example, that this system arises as a finite dimensional approximation of the famous KdV equation. In this paper we shall refer to system (1) as the A_n-*Volterra system*.

Another well known discretization of the KdV equation is the Toda lattice [25]. The Toda system can be written in the form:

$$\begin{cases} \dot{a}_i = a_i(b_i - b_{i+1}), \\ \dot{b}_i = a_{i-1} - a_i, \end{cases} \qquad (i = 1, \dots, n) \tag{2}$$

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with $a_0 = b_{n+1} = 0$. We shall refer to this system as the A_{n-1}-*Toda lattice*. Generalizations of both equations to root systems other than A_n were obtained by Bogoyavlenskij in [4, 5].

For the A_n-Toda lattice, quadratic and cubic brackets were constructed, respectively, by Adler in [1] and by Kupershmidt in [19]. Multiple Hamiltonian structures for these systems were introduced by the first author ([8]) in Flaschka coordinates and by the second author ([14]) in natural (q, p) coordinates. The analogous results for B_n-Toda were computed in [10] in Flaschka coordinates, and in [7] in (q, p) coordinates. The C_n case is in [11] in Flaschka coordinates, and in [7] in natural (q, p) coordinates. Finally, the D_n case can be found in [12]. The construction of these multiple Hamiltonian structures for the exceptional root systems is an open problem.

For the Volterra system, multiple Hamiltonian structures were constructed recently by Kouzaris ([18]) for B_n, C_n and D_n systems, which generalize the A_n-case. For the A_n case, it was proved in [15] that the (periodic) Volterra lattice is an algebraic completely integrable (a.c.i.) system. In [16], the hyperelliptic systems were introduced similar to the even and odd Mumford systems (see [21, 22]), and the following link was established: there is a commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi} & \mathcal{M} \\ \uparrow & & \uparrow \\ \mathcal{K} & \xrightarrow[\Phi]{} & \mathcal{P} \end{array}$$

in which $\mathcal{M}$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{K}$ are (in that order) the phase spaces of the (even) Mumford system, the hyperelliptic Prym system (odd or even), the periodic A_n-Toda lattice and the periodic A_n-Volterra system. The vertical arrows are natural inclusion maps exhibiting for both spaces the subspace as fixed points varieties and the horizontal arrows are injective maps that map every fiber of the momentum map on the left injectively into (but not onto) a fiber of the momentum map on the right. In addition, Poisson structures were constructed so that this diagram has a meaning in the Poisson category. These give a precise geometric description, in the A_n case, of the well known connection between the Toda and Volterra systems (see also [23, 24]).

In this paper we initiate the study of the relationship between the Toda and Volterra systems, for root systems other than A_n. Here we shall restrict our attention mainly to the B_n case, for which we give a complete description. The analysis for this case is simplified since we can view this system as a subsystem of the A_{2n}-system. We exhibit the phase space of the B_n-system as a fixed point set of a finite group action by Poisson automorphisms. We prove a general result which allow us to reduce the Poisson structures in such situations, and hence we can relate the multiple Hamiltonian structures of the Volterra and Toda systems.

The plan of this paper is as follows: In Section 2, we discuss several conditions under which the fixed point set of a Poisson action inherits a Poisson bracket, generalizing the Poisson involution theorem (see [16, 27]). In Section 3, we recall the definition of the Toda systems, and give the multiple Hamiltonian structure for the B_n system. These were discovered in [10] (see also [7]) and we use here a new approach based on the Poisson involution theorem to deduce them. In Section 4, we first recall the construction of the Volterra system and its generalizations for any root system associated with a simple Lie algebra. Then, we give the multiple Hamiltonian structure for the B_n (and C_n) cases and, by applying a symmetry approach, we explain the relation between the Poisson structures for Volterra and Toda lattices. In the final section, we describe the connections we have been able to find between the two systems, including a generalization of Moser's recipe (see [20]) to pass from the Volterra to the Toda lattices.

2. Fixed point sets of Poisson actions

We shall see that the relation between the Toda and Volterra systems relies on special symmetries of the phase spaces. In this section, we study conditions under which the fixed point set of a Poisson action inherits a Poisson bracket.

Although we will be interested mainly in finite symmetries, we give the following general result:

Theorem 2.1. *Suppose that $(M, \{\cdot,\cdot\})$ is a Poisson manifold, and G a compact group acting on M by Poisson automorphisms. Let $N = M^G$ be the submanifold of M consisting of the fixed points of the action and let $\iota : N \hookrightarrow M$ be the inclusion. Then N carries a (unique) Poisson structure $\{\cdot,\cdot\}_N$ such that*

$$\imath^* \{F_1, F_2\} = \{\imath^* F_1, \imath^* F_2\}_N \tag{3}$$

for all G-invariant functions $F_1, F_2 \in C^\infty(M)$.

Proof. For $f_1, f_2 \in C^\infty(N)$ we choose $F_1, F_2 \in C^\infty(M)$ such that $f_i = \imath^* F_i$. We may assume that F_1 and F_2 are G-invariant by replacing F_i by

$$\tilde{F}_i = \int_G F_i d\mu,$$

where μ is the Haar measure on G, so that $\int_G 1 d\mu = 1$. We set

$$\{f_1, f_2\}_N \equiv \imath^* \{F_1, F_2\}$$

and show that this definition is independent of the choice of F_i. For this we observe that, since the action of G is Poisson, the Hamiltonian vector field X_F associated with a G-invariant function $F : M \to \mathbb{R}$ is tangent to N. It follows that the Poisson bracket $\{F_1, F_2\}|_N$, where F_1 and F_2 are G-invariant functions, depends only on the restrictions $F_i|_N$.

It is obvious from its definition that $\{\cdot,\cdot\}_N$ is bilinear, and satisfies the Jacobi and Leibniz identities, for these identities hold for $\{\cdot,\cdot\}$. Hence, we obtain a Poisson bracket on N, and it is the only Poisson bracket satisfying (3). □

Remark 2.2. The previous result can be seen as a particular case of Dirac reduction (for the general theorem on Dirac reduction, see Weinstein ([26], Prop. 1.4) and Courant ([6], Thm. 3.2.1). The case where G is a reductive algebraic group is discussed in [22] using a different method.

Since this result applies in particular when G is a finite group, we have:

Corollary 2.3. *Suppose that $(M, \{\cdot,\cdot\})$ is a Poisson manifold, and G is a finite group acting on M by Poisson automorphisms. Then the fixed point set $N = M^G$ carries a (unique) Poisson structure $\{\cdot,\cdot\}_N$ satisfying equation (3).*

Let us consider the special case $G = \mathbb{Z}_2$. Then $G = \{I, \phi\}$, where $\phi : M \to M$ is a Poisson involution. We conclude that $N = M^G = \{x : \phi(x) = x\}$ has a unique Poisson bracket satisfying equation (3). So we see that Theorem 2.1 contains as a special case the following result, which is known as the *Poisson involution theorem* (see [16, 27]).

Corollary 2.4. *Suppose that* $(M, \{\cdot,\cdot\})$ *is a Poisson manifold, and* $\phi : M \to M$ *is a Poisson involution. Then the fixed point set* $N = \{x \in M : \phi(x) = x\}$ *carries a (unique) Poisson structure* $\{\cdot,\cdot\}_N$ *such that*

$$\imath^* \{F_1, F_2\} = \{\imath^* F_1, \imath^* F_2\}_N$$

for all functions $F_1, F_2 \in C^\infty(M)$ *invariant under* ϕ*.*

One should note that, in general, the fixed point set M^G is not a Poisson submanifold of M, since it is not a union of symplectic leaves of M. In other words, underlying Theorem 2.1 there is a true Dirac type reduction, rather than a restriction or Poisson reduction.

3. The Toda lattices

3.1. **The A_n-Toda system.** The phase space $\mathcal{T}_n$ of the (non-periodic) A_{n-1}-Toda lattice is the affine variety of all Lax operators in $\mathfrak{sl}_n$ of the form

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 & 0 \\ 1 & b_2 & a_2 & & & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & & & & b_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & \cdots & 1 & b_n \end{pmatrix}. \tag{4}$$

The reader will notice that we adopt a set of variables (due to Kostant) that differ slightly from the usual Flaschka variables (see [3], Chp. V). This Lax pair has an obvious Lie algebraic interpretation which leads to the generalized Toda Lattices (see next paragraph).

The Hamiltonians H_k, $(k = 1, \ldots, n)$, defined by

$$H_k = \frac{1}{k} \operatorname{tr}(L^k).$$

are in involution with respect to the linear Poisson bracket $\{\cdot,\cdot\}^1_{\mathcal{T}}$, defined by

$$\begin{aligned} \{a_i, a_j\}^1_{\mathcal{T}} &= \{b_i, b_j\}^1_{\mathcal{T}} = 0 \\ \{a_i, b_j\}^1_{\mathcal{T}} &= a_i(\delta_{ij} - \delta_{i+1,j}). \end{aligned} \tag{5}$$

The commuting vector fields $X_k = \{\cdot\,, H_k\}^1_{\mathcal{T}}$ admit the Lax representation

$$X_k(L) = [L, (L^{k+1})_+], \tag{6}$$

where the subscript $+$ denotes projection into the Lie subalgebra of $\mathfrak{sl}_n$ generated by the positive roots. In [20], it was proved that the non-periodic Toda lattice is a completely integrable system, by applying a finite dimensional analogue of the inverse scattering method. For the periodic case, the flows are linear on the generic fibers of the momentum map $K : \mathcal{T}_n \to \mathbb{C}[x]$, and since these fibers are affine parts of hyperelliptic Jacobians, the periodic Toda lattice is an a.c.i. system (see [2] for details).

The Toda lattice is also Hamiltonian with respect to a quadratic Poisson bracket $\{\cdot,\cdot\}^2_{\mathcal{T}}$, defined by

$$\begin{aligned}(7)\qquad &\{a_i,a_j\}^2_{\mathcal{T}} = a_ia_j(\delta_{i,j+1}-\delta_{i+1,j}), \qquad \{a_i,b_j\}^2_{\mathcal{T}} = a_ib_j(\delta_{i,j}-\delta_{i+1,j}),\\ &\{b_i,b_j\}^2_{\mathcal{T}} = a_i(\delta_{i,j+1}-\delta_{i+1,j}).\end{aligned}$$

The quadratic Toda bracket appeared in a paper of Adler [1] in 1979. The brackets $\{\cdot,\cdot\}^1_{\mathcal{T}}$ and $\{\cdot,\cdot\}^2_{\mathcal{T}}$ are compatible, and are in fact part of a full hierarchy of higher order Poisson brackets [8, 14]. Denote by π_1 and π_2 the Poisson tensors associated with the linear and quadratic brackets. Also, let Z_0 be the Euler vector field

$$Z_0 = \sum_{i=1}^n a_i\frac{\partial}{\partial a_i} + \sum_{i=1}^n b_i\frac{\partial}{\partial b_i}. \tag{8}$$

The following result is proved in [14]:

Proposition 3.1. *There exists a sequence of Poisson tensors π_k and master symmetries Z_k, $k=0,1,2,\dots$ such that*

(i) *The π_k are compatible Poisson tensors and the functions H_k are in involution with respect to all of the π_k;*
(ii) *The vector fields X_k admit the multiple Hamiltonian formulation*

$$X_{k+l-1} = \pi_k dH_l = \pi_{k-1}dH_{l+1};$$

(iii) *The Poisson tensors, the integrals and the Hamiltonian vector fields, satisfy the deformation relations:*

$$\begin{aligned}(9)\qquad &\mathcal{L}_{Z_k}\pi_l = (l-k-2)\pi_{k+l}, \qquad && Z_k(H_l) = (k+l)H_{k+l},\\ &\mathcal{L}_{Z_k}X_l = (l-1)X_{k+l}, && [Z_k,Z_l] = (k-l)Z_{k+l}.\end{aligned}$$

These relations will be important later to deduce some properties of the Volterra lattices. Using these relations we can find explicit formulas for the higher order Poisson brackets. For example, we have that

$$\begin{aligned}(10)\qquad Z_1 = &\sum_{i=1}^{n-1} a_i\left(b_i(1-2i)+b_{i+1}(1+2i)\right)\frac{\partial}{\partial a_i}\\ &+\sum_{i=1}^{n}\left(a_{i-1}(2-2i)+a_i(2+2i)+b_i^2\right)\frac{\partial}{\partial b_i}\end{aligned}$$

so we find that the cubic bracket $\pi_3 = -\mathcal{L}_{Z_1}\pi_2$ is given by (see [10]):

$$\begin{aligned}(11)\qquad &\{a_i,a_{i+1}\}^3_{\mathcal{T}} = 2a_ia_{i+1}b_{i+1}, && \{a_i,b_i\}^3_{\mathcal{T}} = -a_ib_i^2-a_i^2,\\ &\{a_i,b_{i+1}\}^3_{\mathcal{T}} = a_ib_{i+1}^2+a_i^2, && \{a_i,b_{i+2}\}^3_{\mathcal{T}} = a_ia_{i+1},\\ &\{a_{i+1},b_i\}^3_{\mathcal{T}} = -a_ia_{i+1}, && \{b_i,b_{i+1}\}^3_{\mathcal{T}} = a_i(b_i+b_{i+1}).\end{aligned}$$

Similarly, one can compute any higher order Poisson bracket.

The cubic bracket π_3 was found by Kupershmidt [19] via the infinite Toda lattice.

3.2. **The B_n-Toda system.** Generalizations of the Toda lattice for other root systems are well known and were first given in [4]. We consider here only the case of B_n.

The phase space for the B_n-Toda lattice is the affine variety of all Lax operators in $\mathfrak{sl}_{2n+1}$ of the form:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & & & & & \cdots & 0 & 0 \\ 1 & b_2 & a_2 & & & & & & & & 0 \\ \vdots & & \ddots & \ddots & & & & & & & \vdots \\ & & & b_{n-1} & a_{n-1} & & & & & & \\ & & & 1 & b_n & a_n & & & & & \\ & & & & 1 & 0 & -a_n & & & & \\ & & & & & -1 & -b_n & -a_{n-1} & & & \\ & & & & & & -1 & -b_{n-1} & & & \\ \vdots & & & & & & & \ddots & \ddots & & \vdots \\ 0 & & & & & & & & -1 & -b_2 & -a_1 \\ 0 & 0 & \cdots & & & & & & 0 & -1 & -b_1 \end{pmatrix} \tag{12}$$

This matrix actually lies in the real simple Lie algebra $\mathfrak{g} = \mathfrak{so}(n, n+1)$. We would like to have a multiple Hamiltonian formulation for the B_n-Toda systems

$$\frac{dL}{dt} = [L, (L^{k+1})_+]. \tag{13}$$

Note that, in this case, only the even powers of L give non-trivial integrals

$$H_{2k} = \frac{1}{2k}\,\mathrm{tr}(L^{2k}), \qquad (k = 1, 2, \ldots).$$

In order to write down a Hamiltonian formulation for these systems we recall that the B_n-Toda is a subsystem of the A_{2n}-system. This is well known and in fact, more is true: we can find a symmetry such that the B_n-Toda is obtained through a symmetry reduction from the A_{2n}-Toda. To see this, observe that conjugation by a diagonal matrix takes the Lax matrix (12) to the Lax matrix:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & & & & & \cdots & 0 & 0 \\ 1 & b_2 & a_2 & & & & & & & & 0 \\ \vdots & & \ddots & \ddots & & & & & & & \vdots \\ & & & b_{n-1} & a_{n-1} & & & & & & \\ & & & 1 & b_n & a_n & & & & & \\ & & & & 1 & 0 & a_n & & & & \\ & & & & & 1 & -b_n & a_{n-1} & & & \\ & & & & & & 1 & -b_{n-1} & & & \\ \vdots & & & & & & & \ddots & \ddots & & \vdots \\ 0 & & & & & & & & 1 & -b_2 & a_1 \\ 0 & 0 & \cdots & & & & & & 0 & 1 & -b_1 \end{pmatrix} \tag{14}$$

and this is just a matrix in $\mathcal{T}_{2n+1}$ satisfying $a_{2n+1-i} = a_i$ and $b_{2n+2-i} = -b_i$. Therefore, if we consider the involution $\phi : \mathcal{T}_{2n+1} \to \mathcal{T}_{2n+1}$ defined by

$$\phi(a_i, b_i) = (a_{2n+1-i}, -b_{2n+2-i}),$$

then the phase space of the B_n-Toda can be identified with the fixed point set of this involution. On the other hand the $\mathbb{Z}_2$-action generated by this involution is a Poisson action for the odd Poisson brackets. More precisely, we have:

Proposition 3.2. *The map* $\phi : \mathcal{T}_{2n+1} \to \mathcal{T}_{2n+1}$ *satisfies*

$$\phi_* \pi_k = (-1)^{k+1} \pi_k.$$

Proof. For lower order Poisson brackets one can check this relation by direct computation. For higher Poisson brackets, we note that expression (10) for the master symmetry Z_1, shows after some tedious computation that

$$\phi_* Z_1 = -Z_1 + X,$$

where X is a multiple of the first Hamiltonian vector field X_1. Now relation (9) gives

$$\mathcal{L}_{Z_1} \pi_k = (k-3)\pi_{k+1},$$

so it follows by induction that

$$\begin{aligned} \phi_* \pi_{k+1} &= \frac{1}{k-3} \phi_* \mathcal{L}_{Z_1} \pi_k \\ &= \frac{1}{k-3} \mathcal{L}_{\phi_* Z_1} \phi_* \pi_k \\ &= \frac{1}{k-3} \mathcal{L}_{-Z_1} (-1)^{k+1} \pi_k = (-1)^{k+2} \pi_{k+1}. \end{aligned}$$

□

Using Corollary 2.4, we conclude that the odd Poisson brackets for the A_{2n}-Toda induce Poisson brackets for the B_n-Toda lattices, which therefore possesses a multiple Hamiltonian formulation. In this way we have deduced the following result of [10]:

Corollary 3.3. *There exists a sequence of Poisson tensors* $\pi_1, \pi_3, \pi_5, \ldots$ *such that the* B_n*-Toda lattices (13) possess a multiple Hamiltonian formulation:*

$$\pi_{k+2} dH_{2l} = \pi_k dH_{2l+2}.$$

It should be noted that Theorem 2.1 gives an effective procedure to compute the Poisson brackets, as is illustrated by the following example.

Example 3.4. In this example we compute the cubic B_n-bracket. We denote by (a_i, b_j), where $i, j = 1, \ldots, n$, the coordinates on the phase space of the B_n-Toda lattice, and by $(\tilde{a}_i, \tilde{b}_j)$, where $i = 1, \ldots, 2n$, $j = 1, \ldots, 2n+1$, the coordinates on the phase space of the A_{2n}-Toda lattice.

Suppose we would like to compute the cubic B_n-bracket $\{a_i, b_i\}^3_{\mathcal{T}}$. First we extend the function a_i and b_i to invariant functions $\widehat{a}_i$ and $\widehat{b}_i$ on the phase space of the A_{2n}-Toda lattice. For example, we can take the functions

$$\widehat{a}_i = \frac{\tilde{a}_i + \tilde{a}_{2n+1-i}}{2}, \qquad \widehat{b}_i = \frac{\tilde{b}_i - \tilde{b}_{2n+2-i}}{2}.$$

Then we compute the A_{2n}-cubic bracket of these functions: we find using (11) that for $i < n$, we have

$$\left\{\widehat{a}_i, \widehat{b}_i\right\}^3_{\mathcal{T}} = -\frac{1}{4}\left(\tilde{a}_i \tilde{b}_i^2 + \tilde{a}_i^2 + \tilde{a}_{2n+1-i} \tilde{b}_{2n+2-i}^2 + \tilde{a}_{2n+1-i}^2\right),$$

while for $i = n$ we obtain

$$\left\{\widehat{a}_n, \widehat{b}_n\right\}^3_{\mathcal{T}} = -\frac{1}{4}\left(\tilde{a}_n\tilde{b}_n^2 + \tilde{a}_n^2 + \tilde{a}_{n+1}\tilde{b}_{n+2}^2 + \tilde{a}_{n+1}^2 + 2\tilde{a}_n\tilde{a}_{n+1}\right).$$

Finally, we restrict the brackets to the B_n-phase space, to obtain the cubic B_n-bracket

$$\{a_i, b_i\}^3_{\mathcal{T}} = -\frac{1}{2}\left(a_i b_i^2 + a_i^2\right), \qquad (i < n) \tag{15}$$

$$\{a_n, b_n\}^3_{\mathcal{T}} = -\frac{1}{2}\left(a_n b_n^2 + 2a_n^2\right). \tag{16}$$

The other brackets are computed in a similar fashion and they give the following expressions

$$\begin{aligned}
&\{a_i, a_{i+1}\}^3_{\mathcal{T}} = a_i a_{i+1} b_{i+1}, && \{b_i, b_{i+1}\}^3_{\mathcal{T}} = a_i(b_i + b_{i+1}),\\
&\{a_i, b_{i+1}\}^3_{\mathcal{T}} = \frac{1}{2}\left(a_i b_{i+1}^2 + a_i^2\right), && \{a_i, b_{i+2}\}^3_{\mathcal{T}} = \frac{1}{2}a_i a_{i+1},\\
&\{a_{i+1}, b_i\}^3_{\mathcal{T}} = -\frac{1}{2}a_i a_{i+1}.
\end{aligned} \tag{17}$$

Remark 3.5. It is not hard to check that the (non-periodic) C_n-Toda system is a subsystem of the A_{2n-1}-Toda system. The reader can check that the multiple Hamiltonian structure for the C_n-Toda lattices found in [11], [7] can be obtained from the multiple Hamiltonian structure for the A_{2n-1}-Toda systems, as in the B_n-case. In fact, the involution $\phi : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$ defined by

$$\phi(a_i, b_i) = (a_{2n+1-i}, -b_{2n+2-i}).$$

satisfies $\phi_* \pi_k = (-1)^k \pi_k$, so the same method applies.

4. The Volterra Lattices

4.1. **The A_n-Volterra system.** We now turn to the (non-periodic) A_n-Volterra system. Its phase space $\mathcal{K}_n$ is the subspace of $\mathcal{T}_n$ consisting of all Lax operators (4) with zeros on the diagonal:

$$L = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & a_2 & & & 0 \\ 0 & 1 & & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & & & & 0 & a_{n-1} \\ 0 & 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}. \tag{18}$$

Note that $\mathcal{K}_n$ is not a Poisson subspace of $\mathcal{T}_n$. However, $\mathcal{K}_n$ is the fixed manifold of the involution $\psi : \mathcal{T}_n \to \mathcal{T}_n$ defined by

$$((a_1, a_2 \ldots, a_n), (b_1, b_2 \ldots, b_n)) \mapsto ((a_1, a_2 \ldots, a_n), (-b_1, -b_2 \ldots, -b_n)),$$

and we have (see [16]):

Proposition 4.1. *$\psi : \mathcal{T}_n \to \mathcal{T}_n$ is a Poisson automorphism of $(\mathcal{T}_n, \{\cdot,\cdot\}^k_{\mathcal{T}})$, if k even.*

Proof. Again, one checks by direct computation for $k < 4$ that

$$\psi_* \pi_k = (-1)^k \pi_k.$$

On the other hand, we have that $\psi_* Z_1 = -Z_1$, so by induction we see that

$$\begin{aligned}\psi_* \pi_{k+1} &= \frac{1}{k-3} \psi_* \mathcal{L}_{Z_1} \pi_k \\ &= \frac{1}{k-3} \mathcal{L}_{\psi_* Z_1} \psi_* \pi_k \\ &= \frac{1}{k-3} \mathcal{L}_{-Z_1} (-1)^k \pi_k = (-1)^{k+1} \pi_{k+1}.\end{aligned}$$

and the result follows. □

Therefore, by Corollary 2.4, $\mathcal{K}_n$ inherits a family of Poisson brackets $\pi_2, \pi_4, \dots$. For example, the quadratic bracket can be computed from formulas (7), in a form entirely analogous to Example 3.4, and is given by

$$\{a_i, a_j\}^2_{\mathcal{K}} = a_i a_j (\delta_{i,j+1} - \delta_{i+1,j}), \tag{19}$$

while the Poisson bracket π_4 is found to be given by the formulas

$$\begin{aligned}\{a_i, a_{i+1}\}^4_{\mathcal{K}} &= a_i a_{i+1} (a_i + a_{i+1}), && (i = 1, \dots, n-1) \\ \{a_i, a_{i+2}\}^4_{\mathcal{K}} &= a_i a_{i+1} a_{i+2}, && (i = 1, \dots, n-2).\end{aligned} \tag{20}$$

The formulas for the brackets π_2 and π_4 appeared in [9], and the existence of this hierarchy of Poisson brackets is proved in [16]. The first three Poisson structures of this system appeared first in [13] where the Volterra lattice is treated in detail, however the quadratic and cubic brackets do not coincide with the brackets in [9].

It follows that the restriction of the integrals H_{2k} to $\mathcal{K}_n$ gives a set of commuting integrals, with respect to these Poisson brackets. Also, for i odd, the Lax equations (6) lead to Lax equations for the corresponding flows, merely by putting all b_i equal to zero. Taking $i = 1$, we recover the system

$$\dot{a}_i = a_i (a_{i-1} - a_{i+1}), \qquad i = 1, \dots, n, \tag{21}$$

which we called the A_n-Volterra lattice in the introduction. More generally, taking i odd we find a family of commuting Hamiltonian vector fields on $\mathcal{K}_n$ which are restrictions of the Toda vector fields, while for i even the Toda vector fields X_i are not tangent to $\mathcal{K}_n$. Hence, we obtain a family of integrable systems admitting a multiple Hamiltonian formulation:

$$X_{2(k+l)-1} = \pi_{2k+2} dH_{2l} = \pi_{2k} dH_{2l+2}, \qquad (k = 1, 2, \dots).$$

This system was shown to be a completely integrable system in [20], while the periodic version of this system was proved to be an a.c.i. system in [15, 16].

4.2. **Generalized Volterra systems.** We now describe the construction of the generalized Volterra systems of Bogoyavlensky (see [5]).

Let $\mathfrak{g}$ be a simple Lie algebra, with rank $\mathfrak{g} = n$, and let $\Pi = \{\omega_1, \omega_2, \dots, \omega_n\}$ be a Cartan-Weyl basis for the simple roots in $\mathfrak{g}$. There exist unique positive integers k_i such that

$$k_0 \omega_0 + k_1 \omega_1 + \cdots + k_n \omega_n = 0,$$

where $k_0 = 1$ and ω_0 is the minimal negative root. We consider the Lax pair:

$$\dot{L} = [B, L],$$

where

$$L(t) = \sum_{i=1}^{n} b_i(t) e_{\omega_i} + e_{\omega_0} + \sum_{1 \le i < j \le n} [e_{\omega_i}, e_{\omega_j}],$$

$$B(t) = \sum_{i=1}^{n} \frac{k_i}{b_i(t)} e_{-\omega_i} + e_{-\omega_0}.$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra. For every root $\omega_a \in \mathfrak{h}^*$ there is a unique $H_{\omega_a} \in \mathfrak{h}$ such that $\omega(h) = \beta(H_{\omega_a}, h)$, for all $h \in \mathfrak{h}$, where β denotes the Killing form. Also, β induces an inner product on $\mathfrak{h}^*$ by setting $\langle \omega_a, \omega_b \rangle = \beta(H_{\omega_a}, H_{\omega_b})$, and we define

$$c_{ij} = \begin{cases} 1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i < j \\ 0 & \text{if } \langle \omega_i, \omega_j \rangle = 0 \text{ or } i = j \\ -1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i > j \end{cases}$$

With these choices, the Lax pair above is equivalent to the system of o.d.e.'s

$$\dot{b}_i = -\sum_{j=1}^{n} \frac{k_j c_{ij}}{b_j}. \tag{22}$$

To obtain a Lotka-Volterra type system one introduces a new set of variables by

$$\begin{aligned} x_{ij} &= c_{ij} b_i^{-1} b_j^{-1}, \\ x_{ji} &= -x_{ij}, \\ x_{jj} &= 0. \end{aligned}$$

Note that $x_{ij} \neq 0$ iff there exists an edge in the Dynkin diagram for the Lie algebra $\mathfrak{g}$ connecting the vertices ω_i and ω_j. System (22), in the variables x_{ij}, takes the form

$$\dot{x}_{ij} = x_{ij} \sum_{s=1}^{n} k_s \left(x_{is} + x_{js} \right), \tag{23}$$

which is a Lotka-Volterra type system. We call (23) the Bogoyavlensky-Volterra system associated with $\mathfrak{g}$ (or the $\mathfrak{g}$-Volterra system for short).

Example 4.2. Let us take $\mathfrak{g} = A_n$. Then we have the Dynkin diagram

with $k_i = 1$, $i = 0, \ldots, n$. If we label the edges of this diagram by $a_1, \ldots, a_n$, then system (23) takes precisely the form of the A_n-Volterra lattice.

We now turn our attention to the B_n-case.

4.3. **The B_n-Volterra system.** For $\mathfrak{g} = B_n$, the Dynkin diagram has the form

and for the obvious labeling (and after a linear change of variables) we obtain the B_n-Volterra system

$$\begin{cases} \dot{a}_1 = -a_1 a_2 \\ \dot{a}_i = a_i(a_{i-1} - a_{i+1}), \qquad (i = 2, \ldots, n-1) \\ \dot{a}_n = a_n \left(a_{n-1} + a_n \right). \end{cases} \tag{24}$$

The multiple Hamiltonian structure for the B_n-Volterra system can be obtained from the A_{2n}-Volterra system in the same fashion as the multiple Hamiltonian formulation for the B_n-Toda system was obtained from the A_{2n}-Toda system (cf. Section 3.2). In fact, we notice that equations (24) for the B_n-Volterra system can be obtained from equations (21) for the A_{2n}-Volterra system by setting $a_{2n+i-i} = -a_i$ (this defines an invariant submanifold). Hence, we take as phase space of the B_n-Volterra system the Lax operators of the form (18) satisfying these additional restrictions:

$$L = \begin{pmatrix} 0 & a_1 & 0 & \cdots & & & & & \cdots & 0 & 0 \\ 1 & 0 & a_2 & & & & & & & & 0 \\ \vdots & & \ddots & \ddots & & & & & & & \vdots \\ & & & 0 & a_{n-1} & & & & & & \\ & & & 1 & 0 & a_n & & & & & \\ & & & & 1 & 0 & -a_n & & & & \\ & & & & & 1 & 0 & -a_{n-1} & & & \\ & & & & & & 1 & 0 & & & \\ \vdots & & & & & & & \ddots & \ddots & & \vdots \\ 0 & & & & & & & & 1 & 0 & -a_1 \\ 0 & 0 & \cdots & & & & & & 0 & 1 & 0 \end{pmatrix}, \tag{25}$$

Notice that this subspace appears as the fixed point set of the involution $\varphi : \mathcal{K}_{2n} \to \mathcal{K}_{2n}$ defined by

$$\varphi(a_i) = -a_{2n+1-i}.$$

We will see below that this involution can be used to obtain the multiple Hamiltonian structure of the B_n-Volterra system.

Remark 4.3. There is a diagonal matrix D which conjugates the Lax matrix given above for the B_n system, to the following Lax matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & \cdots & & & & & \cdots & 0 & 0 \\ 1 & 0 & a_2 & & & & & & & & 0 \\ \vdots & & \ddots & \ddots & & & & & & & \vdots \\ & & & 0 & a_{n-1} & & & & & & \\ & & & 1 & 0 & a_n & & & & & \\ & & & & 1 & 0 & a_n & & & & \\ & & & & & -1 & 0 & a_{n-1} & & & \\ & & & & & & -1 & 0 & & & \\ \vdots & & & & & & & \ddots & \ddots & & \vdots \\ 0 & & & & & & & & -1 & 0 & a_1 \\ 0 & 0 & \cdots & & & & & & 0 & -1 & 0 \end{pmatrix}. \tag{26}$$

Notice that these Lax matrices *are not* obtained as Lax matrices for the B_n-Volterra system (matrices of the form (12)) with zeros on the diagonal.

We now have the following proposition analogous to Propositions 3.2 and 4.1. The proof is similar and will be omitted.

Proposition 4.4. *Let π_k ($k = 2, 4, \dots$) be the Poisson tensors of the A_{2n}-Volterra system. Then*

$$\varphi_* \pi_k = (-1)^{k/2} \pi_k.$$

In this way, by Corollary 2.4, the phase space of the B_n-Volterra system carries Poisson brackets $\pi_4, \pi_8, \dots$. For example, from (20) we can compute, using the same technique as in Example 3.4, the following explicit formulas for the bracket π_4:

$$
\begin{aligned}
\{a_i, a_{i+1}\}^4_{\mathcal{K}} &= \frac{1}{2} a_i a_{i+1}(a_i + a_{i+1}), \\
\{a_{n-1}, a_n\}^4_{\mathcal{K}} &= \frac{1}{2} a_{n-1} a_n (a_{n-1} + 2a_n), \qquad (i = 1, \dots, n-2) \\
\{a_i, a_{i+2}\}^4_{\mathcal{K}} &= \frac{1}{2} a_i a_{i+1} a_{i+2}.
\end{aligned} \tag{27}
$$

The restriction of the invariant functions

$$H_{4k} = \frac{1}{4k} \operatorname{tr}(L^{4k})$$

to this space, defines a hierarchy of systems, possessing a multiple Hamiltonian formulation:

$$X_{4(k+l)-1} = \pi_{4k+4} dH_{4l} = \pi_{4k} dH_{4l+4}, \qquad (k = 1, 2, \dots).$$

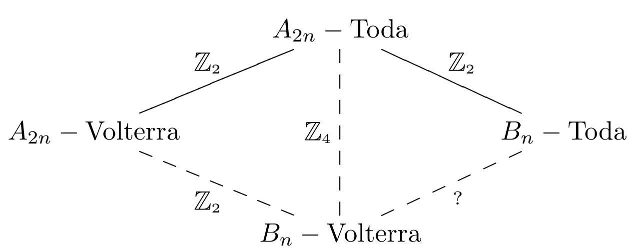
The function $I_4 = \frac{1}{4} \sum_{i=1}^{n-1} (2a_i^2 + a_i a_{i+1})$ gives the B_n-Volterra system (24). The brackets above, as well as the master symmetries for these systems (including the C_n and D_n cases) appear in a recent preprint of Kouzaris ([18]). However, the Lax matrices above lead to Lax pairs which are *different* from the Lax pairs given by Kouzaris.

Finally we remark that the C_n-Volterra system can be identified with the B_n-Volterra systems (see [18]), and hence admits the same description.

5. Epilogue: From Toda to Volterra and back

In this concluding section we would like to explain what we know so far about the connection between the Toda and Volterra systems, including the results obtained above. The relation between the two systems relies on special symmetries of the phase spaces.

We have shown above that the phase space of the A_n-Volterra system appears as the fixed point set of a Poisson involution of the A_n-Toda phase space. On the other hand, we have also shown that the multiple Hamiltonian structure of the B_n-Toda system can be obtained from the A_{2n}-Toda system. Hence, to get to the B_n-Volterra system, there are *a priori* two distinct ways to proceed, as explained by the following diagram:



We have seen that the correct way to proceed is to choose the *left* side of this diagram. In fact, we saw above that we can get from A_{2n}-Toda to A_{2n}-Volterra by a Poisson involution $\psi : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$ (i.e., a $\mathbb{Z}_2$-symmetry). Also, we can get from

A_{2n}-Volterra to B_n-Volterra using another Poisson involution $\varphi : \mathcal{K}_{2n} \to \mathcal{K}_{2n}$ (and, hence, again a $\mathbb{Z}_2$-symmetry). Note also that we can go straight from A_{2n}-Toda to B_n-Volterra using a $\mathbb{Z}_4$-symmetry: if one defines the map $\varphi : \mathcal{T}_{2n} \to \mathcal{T}_{2n}$ by

$$\widetilde{\varphi}(a_i, b_i) = (-a_{2n+1-i}, \sqrt{-1}\, b_{2n+2-i}),$$

then one checks that the group

$$G = \left\{I, \widetilde{\varphi}, \widetilde{\varphi}^2, \widetilde{\varphi}^3\right\}$$

acts by Poisson automorphisms on $(\mathcal{T}_{2n}, \pi_{4k})$, $k = 1, 2, \ldots$. The fixed point set (i.e., the phase space for B_n-Volterra) inherits Poisson structures π_{4k} and the even flows reduce to this space. Notice also that $\widetilde{\varphi}|_{\mathcal{K}_{2n}} = \varphi$ and $\widetilde{\varphi}^2 = \psi$, so this reduction in one stage (the middle line in the diagram) coincides with the reduction in two stages (the left part of the diagram).

On the other hand, there seems to be no such symmetry reduction from the B_n-Toda lattice to the B_n-Volterra lattice: the various Poisson structures and Hamiltonian functions we have for the B_n-Toda systems *do not* restrict to the phase space of the B_n-Volterra systems. We could however still find a connection between these two systems, which goes in the opposite direction.

To explain this connection, we perform the change of variable, $a_i = -2x_i^2$, and we consider the following equivalent (i.e., conjugate) form of the Lax matrix (25) (or, equivalently, (26)):

$$L = \begin{pmatrix}
0 & x_1 & 0 & \cdots & & & & \cdots & 0 & 0 \\
x1 & 0 & x_2 & & & & & & & 0 \\
\vdots & & \ddots & \ddots & & & & & & \vdots \\
 & & & 0 & x_n & & & & & \\
 & & & x_n & 0 & \sqrt{-1}x_n & & & & \\
 & & & & \sqrt{-1}x_n & 0 & & & & \\
\vdots & & & & & \ddots & \ddots & & & \vdots \\
0 & & & & & & & & 0 & \sqrt{-1}x_1 \\
0 & 0 & \cdots & & & & & 0 & \sqrt{-1}x_1 & 0
\end{pmatrix}.$$

In these new variables the equations for the B_n-Volterra lattice (24) become:

$$\begin{cases}
\dot{x_1} = x_1 x_2^2, & \\
\dot{x_i} = x_i(x_{i-1}^2 - x_{i+1}^2), & \qquad (i = 2, \ldots, n-1). \\
\dot{x_n} = -x_n(x_n^2 + x_{n-1}^2), &
\end{cases}$$

The relation between the B_n-Volterra system and the corresponding Toda system of types B_n and C_n is similar to the relation observed by Moser (see [20]) between the A_n-Volterra and the A_n-Toda lattices. One starts by taking the square of the Lax matrix above and notices that L^2 leaves certain subspaces invariant. Moreover, L^2 reduces on each of these invariant spaces to a symmetric Jacobi matrix. More precisely, assume that L^2 is a $N \times N$ matrix. Then there are two distinct cases:

- $N = 4n+1$: Removing all odd columns and all odd rows we end–up with an $2n \times 2n$ matrix and a Toda system of type C_n. On the other hand, removing all even columns and all even rows we end–up with an $2n+1 \times 2n+1$ matrix and a Toda system of type B_n.
- $N = 4n+3$: Removing all odd columns and all odd rows we end–up with an $2n+1 \times 2n+1$ matrix and a Toda system of type B_n. On the other

hand, removing all even columns and all even rows we end–up with an $2(n+1) \times 2(n+1)$ matrix and a Toda system of type C_{n+1}.

In this way, we have a procedure which takes us from a Volterra systems of type B_n (or C_n) to either a Toda system of type B_n or a Toda system of type C_n.

Example 5.1. Take $N = 9$ and $n = 2$. Omitting all even rows and all even columns of L^2 we obtain the 5×5 matrix

$$\begin{pmatrix} x_1^2 & x_1x_2 & 0 & 0 & 0 \\ x_1x_2 & x_2^2 + x_3^2 & x_3x_4 & 0 & 0 \\ 0 & x_3x_4 & 0 & -x_3x_4 & 0 \\ 0 & 0 & -x_3x_4 & -x_2^2 - x_3^2 & -x_1x_2 \\ 0 & 0 & 0 & -x_1x_2 & -x_1^2 \end{pmatrix}.$$

We identify this matrix with a symmetric Jacobi matrix of type B_2, i.e. we let

$$B_1 = x_1^2, B_2 = x_2^2 + x_3^2, A_1 = x_1x_2, A_2 = x_3x_4,$$

and the equations satisfied by A_1, A_2, B_1, and B_2 are

$$\begin{cases} \dot{A}_1 = A_1\left(B_2 - B_1\right), \\ \dot{A}_2 = -A_2B_2, \\ \dot{B}_1 = 2A_1^2, \\ \dot{B}_2 = 2A_2^2 - 2A_1^2. \end{cases}$$

These are precisely the Toda equations of type B_2.

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