

# Invariants of Lie algebroids

Rui Loja Fernandes

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# PART 1

## Lie Algebroids: Basic Concepts

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## Basic Definitions

Lie algebroids are *geometric* vector bundles.

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- (ii) the **Leibniz identity** holds.

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A **morphism of Lie algebroids** is a bundle map  $\phi : A_1 \rightarrow A_2$  which preserves anchors and Lie brackets.

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## Basic Properties

The kernel and the image of the anchor give basic objects associated with any Lie algebroid:

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$$\mathfrak{g}_x \equiv \text{Ker } \#_x.$$

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$$x \mapsto \mathcal{D}_x \equiv \text{Im } \#_x.$$

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$$\mathfrak{g}_L = \bigcup_{x \in L} \mathfrak{g}_x \rightarrow L.$$

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The **short exact sequence of a leaf** is the short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \xrightarrow{\#} TL \longrightarrow 0$$

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EXAMPLES	$A$
Ordinary Geometry ( $M$ a manifold)	$TM$ $\downarrow$ $M$
Lie Theory ( $\mathfrak{g}$ a Lie algebra)	$\mathfrak{g}$ $\downarrow$ $\{*\}$
Foliation Theory ( $\mathcal{F}$ a regular foliation)	$T\mathcal{F}$ $\downarrow$ $M$
Equivariant Geometry ( $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g}$ $\downarrow$ $M$
Presymplectic Geometry ( $M$ presymplectic)	$TM \times \mathbb{R}$ $\downarrow$ $M$
Poisson Geometry ( $M$ Poisson)	$T^*M$ $\downarrow$ $M$

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# Lie Algebroid Cohomology

A first example of a global invariant of a Lie algebroid:

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## Lie Algebroid Cohomology

A first example of a global invariant of a Lie algebroid:

**A-differential forms:**  $\Omega^\bullet(A) = \Gamma(\wedge^\bullet A^*)$

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**A-differential:**  $d_A : \Omega^\bullet(A) \rightarrow \Omega^{\bullet+1}(A)$

$$d_A Q(\alpha_0, \dots, \alpha_r) \equiv \frac{1}{r+1} \sum_{k=0}^{r+1} (-1)^k \# \alpha_k (Q(\alpha_0, \dots, \hat{\alpha}_k, \dots, \alpha_r)) \\ + \frac{1}{r+1} \sum_{k < l} (-1)^{k+l+1} Q([\alpha_k, \alpha_l], \alpha_0, \dots, \hat{\alpha}_k, \dots, \hat{\alpha}_l, \dots, \alpha_r).$$

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**A-cohomology:**

$$H^\bullet(A) \equiv \frac{\text{Ker } d_A}{\text{Im } d_A}$$

In general, it is very hard to compute. . .

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## Examples

	$A$	$H^\bullet(A)$
Ordinary Geometry ( $M$ a manifold)	$TM$ $\downarrow$ $M$	de Rham cohomology
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Poisson Geometry ( $M$ Poisson)	$T^*M$ $\downarrow$ $M$	Poisson cohomology

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## Groupoids

A **groupoid** is a small category where every morphism is an isomorphism.

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$\mathcal{G} \equiv$  set of morphisms       $M \equiv$  set of objects.

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A **groupoid** is a small category where every morphism is an isomorphism.

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- **source** and **target** maps:

$$\begin{array}{ccc} & \xleftarrow{g} & \\ \bullet & & \bullet \\ t(g) & & s(g) \end{array} \qquad \mathcal{G} \begin{array}{c} \xrightarrow{t} \\ \xleftarrow{s} \end{array} M$$

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# Groupoids

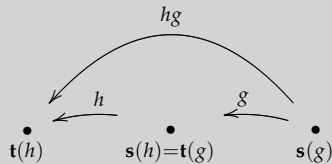
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- **product**:



$$\mathcal{G}^{(2)} = \{(h, g) \in \mathcal{G} \times \mathcal{G} : s(h) = t(g)\}$$

$$m : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$$

$$R_g : \mathbf{s}^{-1}(t(g)) \rightarrow \mathbf{s}^{-1}(s(g))$$

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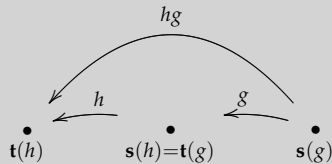
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- **identity**:  $\epsilon : M \hookrightarrow \mathcal{G}$



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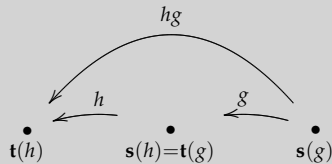
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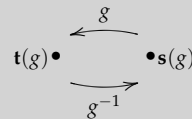
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- **identity**:  $\epsilon : M \hookrightarrow \mathcal{G}$



- **inverse**:  $\iota : \mathcal{G} \longrightarrow \mathcal{G}$



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## Lie Groupoids

A **Lie groupoid** is a groupoid where everything is  $C^\infty$ .

**Caution:**  $\mathcal{G}$  may not be Hausdorff, but all other manifolds ( $M$ ,  $\mathbf{s}$  and  $\mathbf{t}$ -fibers, . . .) are.

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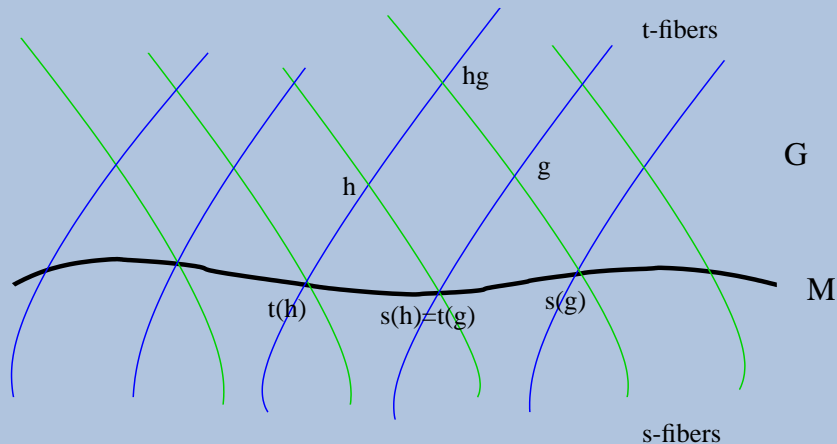
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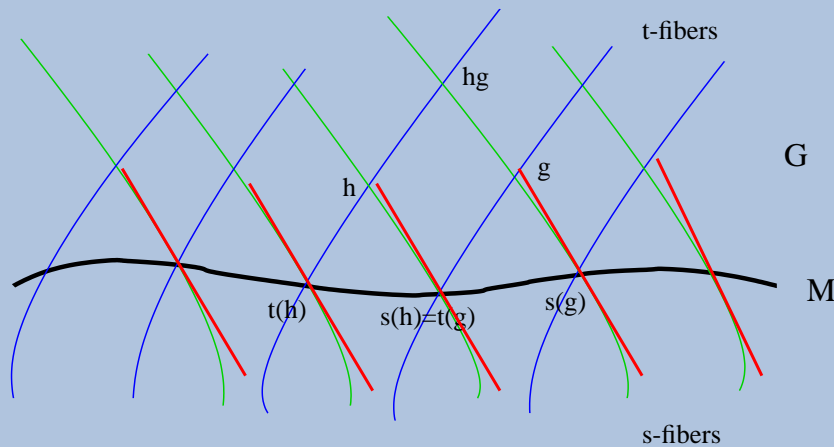
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$$A = \text{Ker } ds \Big|_M$$

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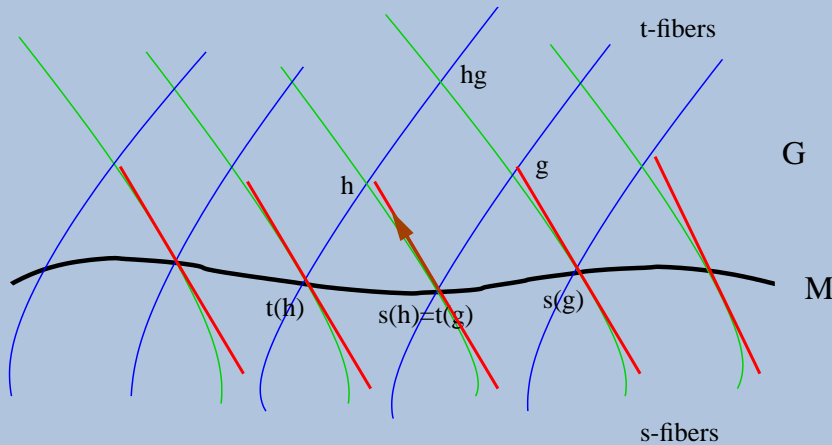
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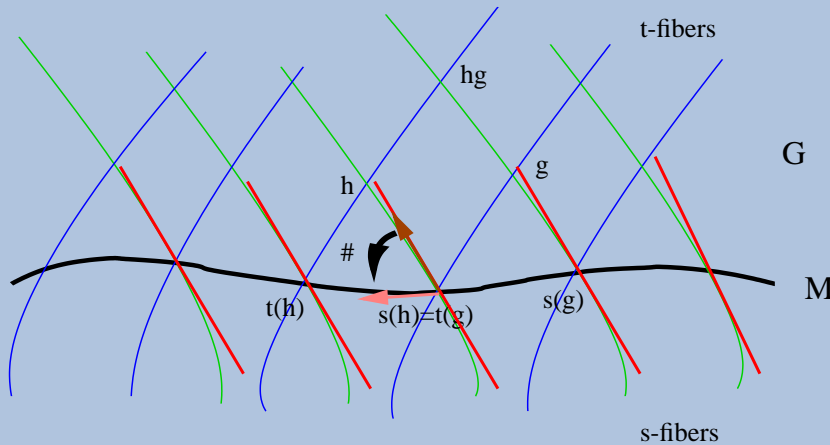
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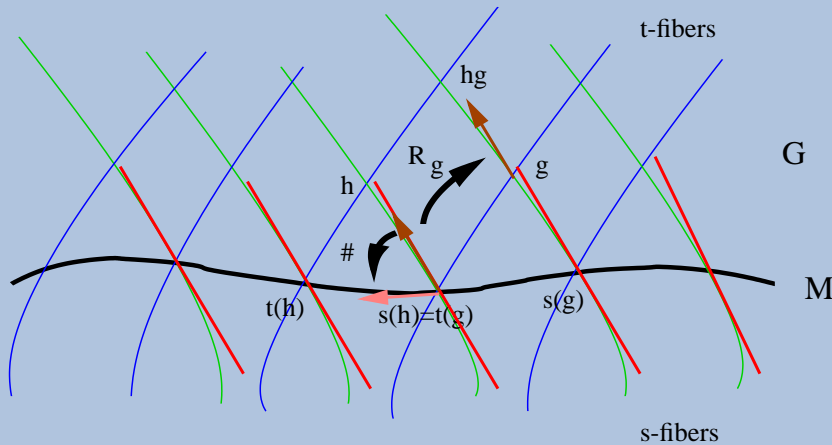
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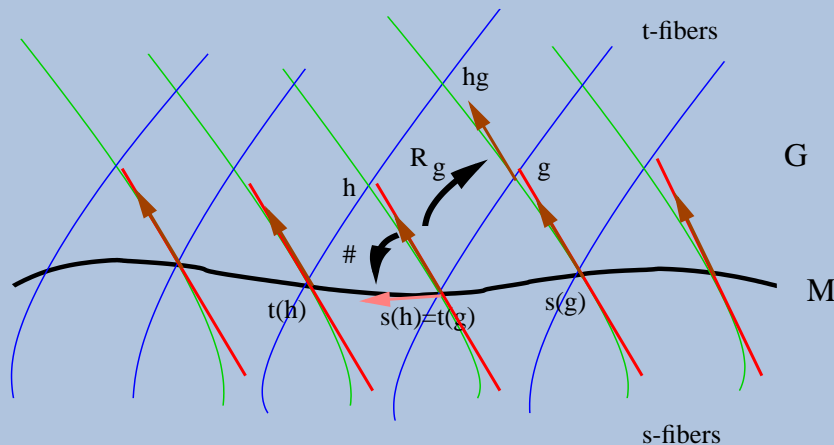
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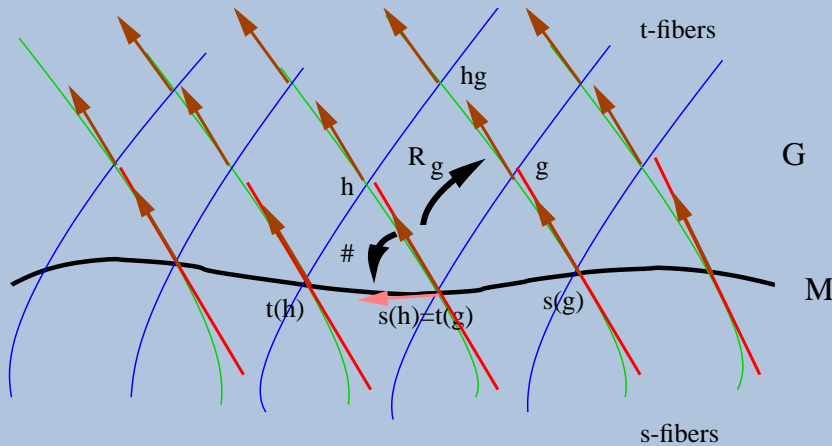
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$$A = \text{Ker } ds \Big|_M \quad \# = dt \Big|_A \quad [\alpha, \beta] = [X^\alpha, X^\beta]$$

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## Examples

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Equivariant Geometry ( $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g}$ $\downarrow$ $M$	gener. foliated cohomology	$G \times M$ $\Downarrow$ $M$
Poisson Geometry ( $M$ Poisson)	$T^*M$ $\downarrow$ $M$	Poisson cohomology	???

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## PART 2

# The Weinstein Groupoid and Integrability

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# *A-Homotopy*

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## A-Homotopy

**Proposition 2.1.** *For every Lie groupoid  $\mathcal{G}$  there exists a unique source simply-connected Lie groupoid  $\hat{\mathcal{G}}$  with the same associated Lie algebroid.*

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Construction is similar to Lie group case:

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- $P(\mathcal{G}) = \{g : I \rightarrow \mathcal{G} \mid \mathbf{s}(g(t)) = x, g(0) = 1_x\}$ ;

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The quotient gives the **monodromy groupoid**:

$$\tilde{\mathcal{G}} \equiv P(\mathcal{G}) / \sim \rightrightarrows M$$

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## A-Homotopy (cont.)

**Lemma 2.2.** The map  $D^R : P(\mathcal{G}) \rightarrow P(A)$  defined by

$$(D^R g)(t) \equiv \left. \frac{d}{ds} g(s)g^{-1}(t) \right|_{s=t}$$

is a homeomorphism onto

$$P(A) \equiv \left\{ a : I \rightarrow A \mid \frac{d}{dt} \pi(a(t)) = \#a(t) \right\} \quad (\mathbf{A}\text{-paths}).$$

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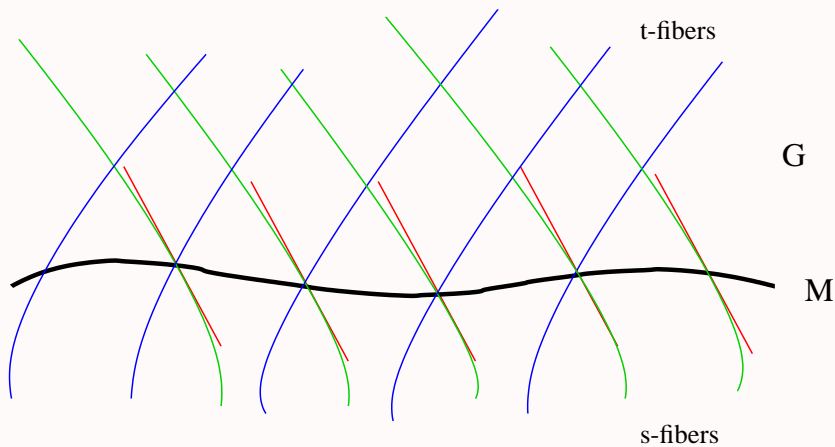
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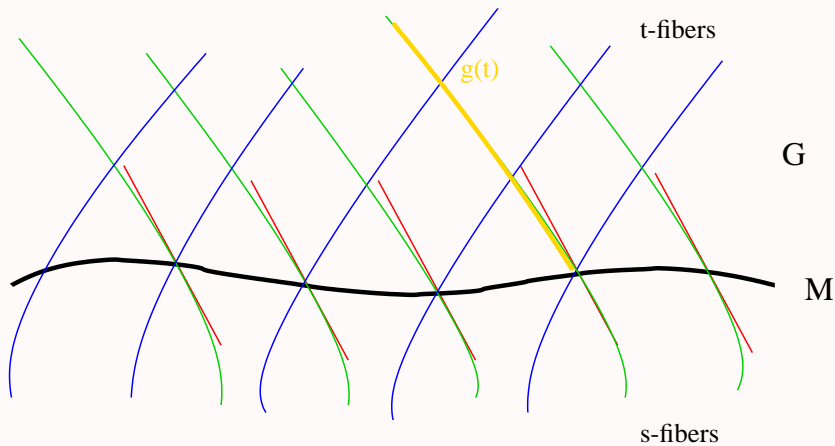
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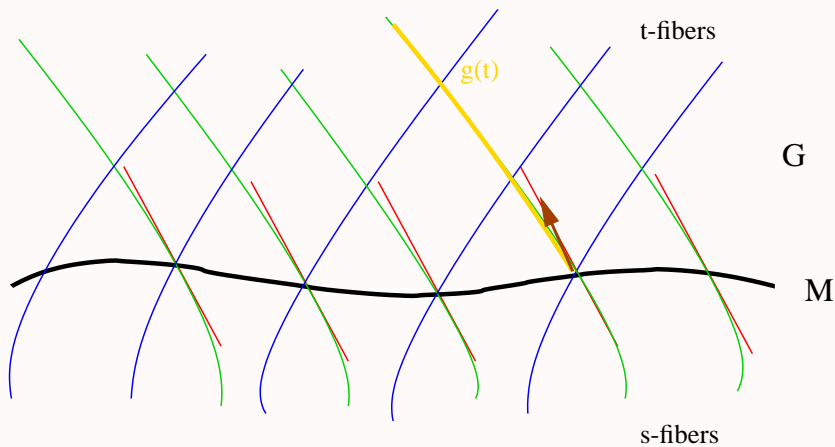
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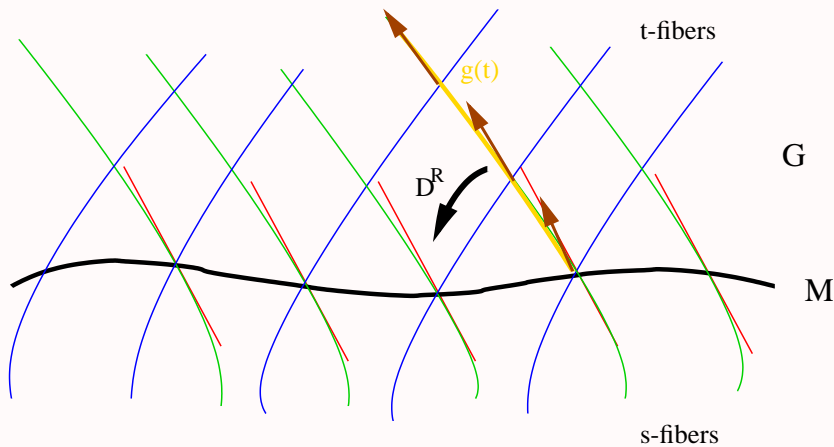
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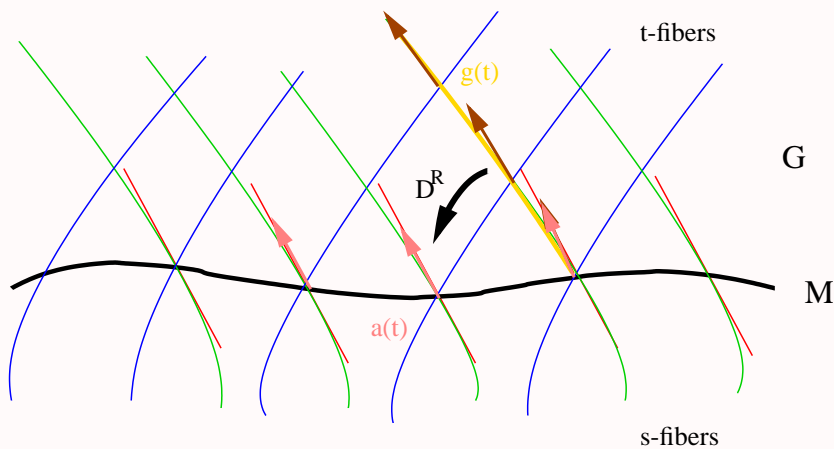
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## *A-Homotopy (cont.)*

Can transport “ $\sim$ ” and “ $\cdot$ ” to  $P(A)$ :

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## ***A*-Homotopy (cont.)**

Can transport “ $\sim$ ” and “ $\cdot$ ” to  $P(A)$ :

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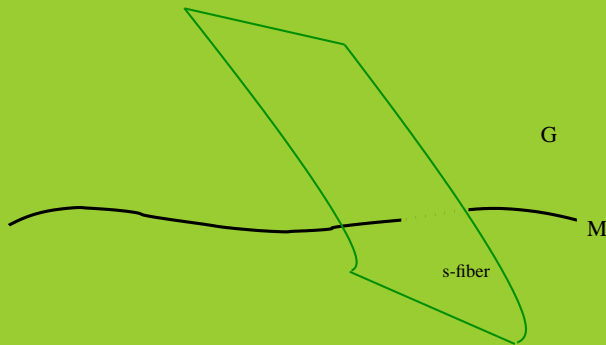
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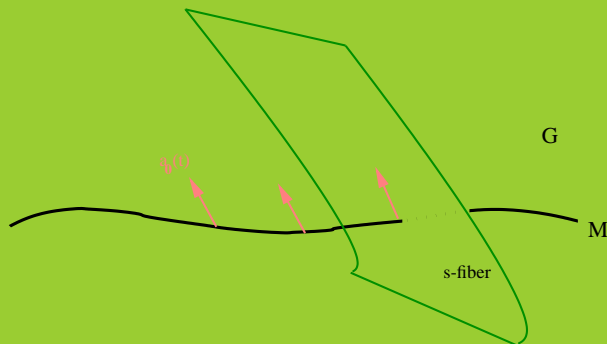
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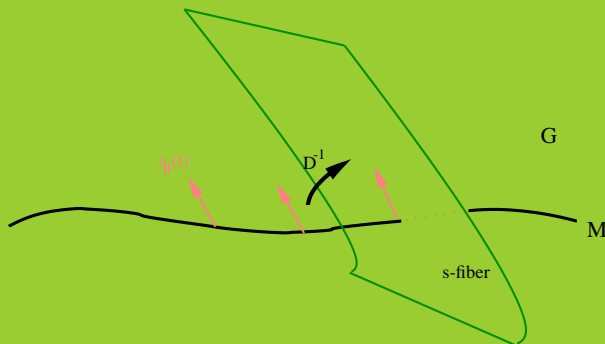
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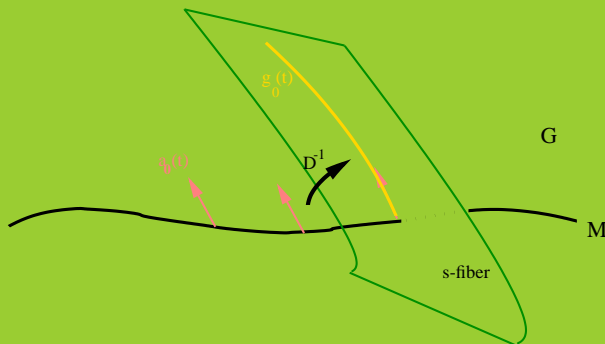
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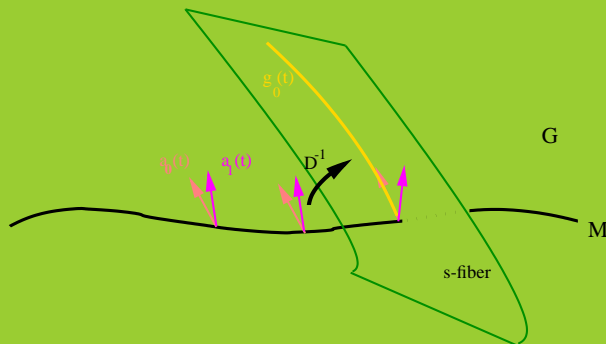
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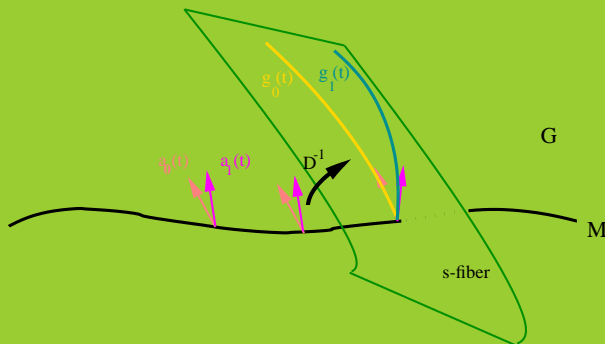
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# The Weinstein Groupoid

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## The Weinstein Groupoid

Observe that:

- An  $A$ -path is a Lie algebroid map  $TI \rightarrow A$ ;

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## The Weinstein Groupoid

Observe that:

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For *any* Lie algebroid  $A$ , the **Weinstein Groupoid** of  $A$  is:

$$\mathcal{G}(A) = P(A)/\sim \text{ where } \left\{ \begin{array}{l} \mathbf{s} : \mathcal{G}(A) \rightarrow M, \quad [a] \mapsto \pi(a(0)) \\ \mathbf{t} : \mathcal{G}(A) \rightarrow M, \quad [a] \mapsto \pi(a(1)) \\ M \hookrightarrow \mathcal{G}(A), \quad x \mapsto [0_x] \end{array} \right.$$

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- $\mathcal{G}(A)$  is a *topological* groupoid with source simply-connected fibers;

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## Examples

	$A$	$H^\bullet(A)$	$\mathcal{G}$	$\mathcal{G}(A)$
Ordinary Geometry ( $M$ a manifold)	$TM$ ↓ $M$	de Rham cohomology	$M \times M$ ↓↓ $M$	$\pi_1(M)$ ↓↓ $M$
Lie Theory ( $\mathfrak{g}$ a Lie algebra)	$\mathfrak{g}$ ↓ $\{*\}$	Lie algebra cohomology	$G$ ↓↓ $\{*\}$	Duistermaat-Kolk construction of $G$
Foliation Theory ( $\mathcal{F}$ a regular foliation)	$T\mathcal{F}$ ↓ $M$	foliated cohomology	$\text{Hol}$ ↓↓ $M$	$\pi_1(\mathcal{F})$ ↓↓ $M$
Equivariant Geometry ( $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g}$ ↓ $M$	gener. foliated cohomology	$G \times M$ ↓↓ $M$	$\mathcal{G}(\mathfrak{g}) \times M$ ↓↓ $M$
Poisson Geometry ( $M$ Poisson)	$T^*M$ ↓ $M$	Poisson cohomology	???	Poisson $\sigma$ -model (Cattaneo & Felder)

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## Integrability of Lie Algebroids

A Lie algebroid  $A$  is **integrable** if there exists a Lie groupoid  $\mathcal{G}$  with  $A$  as associated Lie algebroid.

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## Integrability of Lie Algebroids

A Lie algebroid  $A$  is **integrable** if there exists a Lie groupoid  $\mathcal{G}$  with  $A$  as associated Lie algebroid.

**Lemma 2.3.**  *$A$  is integrable iff  $\mathcal{G}(A)$  is a Lie groupoid.*

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The **monodromy group** at  $x$  is

$$N_x(A) \equiv \text{Im } \partial \subset Z(\mathfrak{g}_L).$$

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To measure the discreteness of  $N_x(A)$  we set:

$$r(x) \equiv d(N_x - \{0\}, \{0\}) \quad (\text{with } d(\emptyset, \{0\}) = +\infty).$$

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# Obstructions to Integrability

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## Obstructions to Integrability

**Theorem 2.4 (Crainic and RLF, 2001).** *A Lie algebroid is integrable iff both the following conditions hold:*

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## Obstructions to Integrability

**Theorem 2.4 (Crainic and RLF, 2001).** *A Lie algebroid is integrable iff both the following conditions hold:*

(i) *Each monodromy group is discrete, i.e.,  $r(x) > 0$ ,*

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This gives previous known criteria:

Lie (1890's), Chevalley (1930's), Van Est (1940's), Palais (1957), Douady & Lazard (1966), Phillips (1980), Almeida & Molino (1985), Mackenzie (1987), Weinstein (1989), Dazord & Hector (1991), Alcade Cuesta & Hector (1995), Debord (2000), Mackenzie & Xu (2000), Nistor (2000).

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**Corollary 2.5.** *A Lie algebroid is integrable if, for all leaves  $L \in \mathcal{F}$ , either of the following conditions holds:*

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**Corollary 2.5.** *A Lie algebroid is integrable if, for all leaves  $L \in \mathcal{F}$ , either of the following conditions holds:*

- (i)  $\pi_2(L)$  is finite (e.g.,  $L$  is 2-connected);

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## Computing the Obstructions

In many examples it is possible to compute the monodromy groups:

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## Computing the Obstructions

In many examples it is possible to compute the monodromy groups:

**Proposition 2.6.** *Assume there exists a splitting:*

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \xrightarrow{\#} TL \longrightarrow 0$$

$\longleftarrow \sigma$

*with center-valued curvature 2-form*

$$\Omega_\sigma(X, Y) = \sigma([X, Y]) - [\sigma(X), \sigma(Y)] \in Z(\mathfrak{g}_L), \quad \forall X, Y \in \mathfrak{X}(L)$$

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For the obvious splitting, the curvature is  $\Omega_\sigma = \omega$ . We obtain:

$$N_x = \left\{ \int_\gamma \omega : [\gamma] \in \pi_2(L, x) \right\}.$$

*Conclusion:*  $A = TM \times \mathbb{R}$  is integrable iff the group of spherical periods of  $\omega$  is discrete.

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## Example: Regular Poisson Manifolds.

Let  $(M, \{ , \})$  be a regular Poisson manifold. Fix a symplectic leaf  $L \subset M$  and  $x \in L$ .

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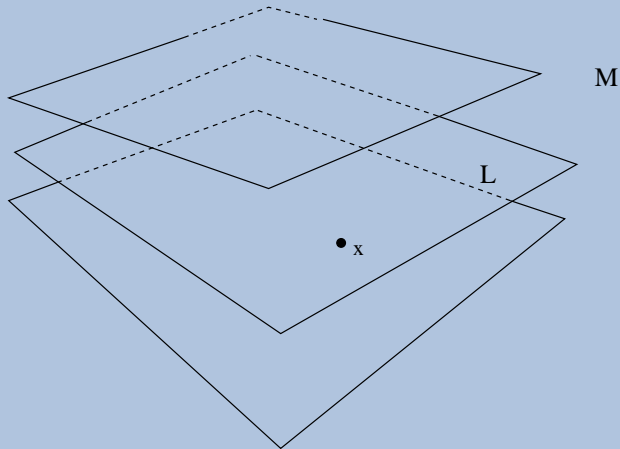
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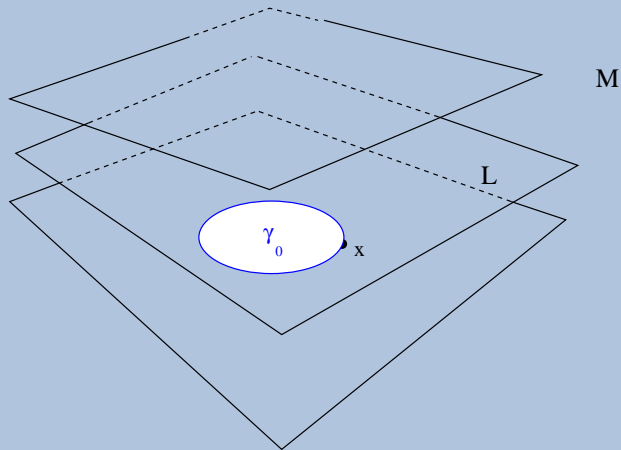
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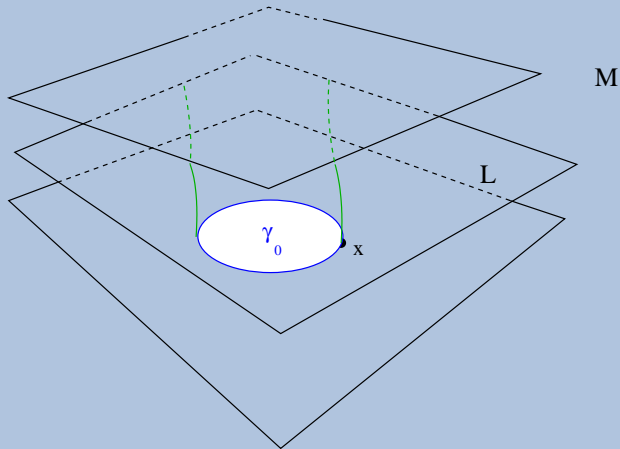
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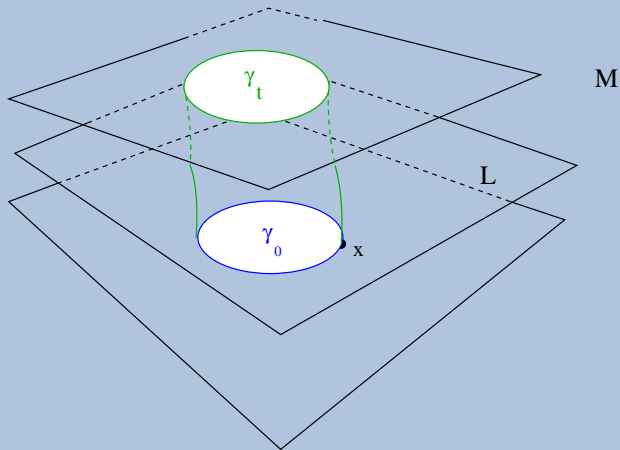
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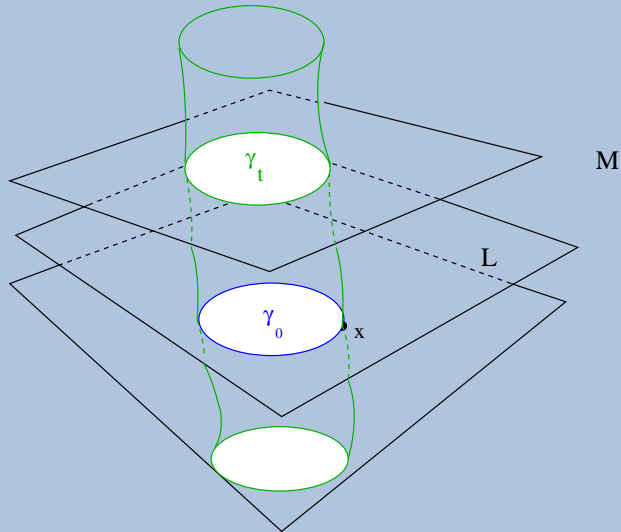
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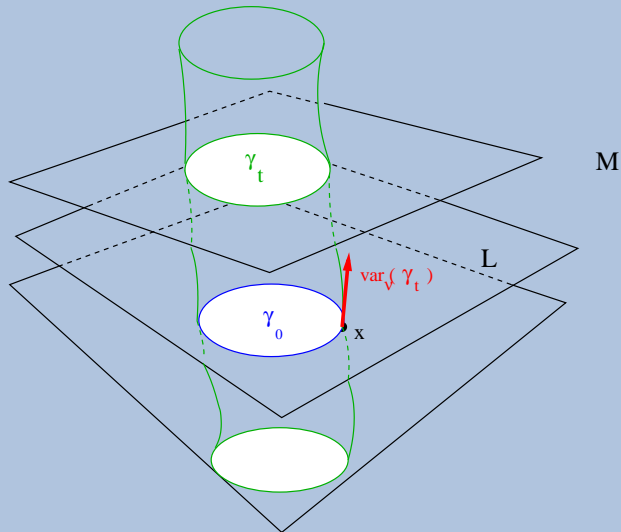
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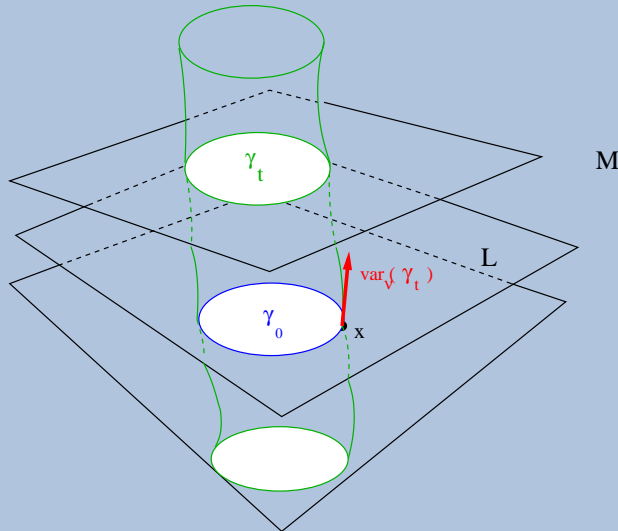
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Let  $(M, \{, \})$  be a regular Poisson manifold. Fix a symplectic leaf  $L \subset M$  and  $x \in L$ .



**Proposition 2.7.** For a foliated family  $\gamma_t : \mathbb{S}^2 \rightarrow M$ , the derivative of the symplectic areas

$$\left. \frac{d}{dt} A(\gamma_t) \right|_{x=0},$$

depends only on the class  $[\gamma_0] \in \pi_2(L, x)$  and  $\text{var}_{\nu}(\gamma_t) = [d\gamma_t/dt|_{t=0}] \in \nu(L)_x$ .

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## Example: Regular Poisson Manifolds.

Define the **variation of symplectic variations**  $A'(\gamma_0) \in \mathcal{V}_x^*(L)$  by

$$\langle A'(\gamma_0), \text{var}_v(\gamma_t) \rangle = \left. \frac{d}{dt} A(\gamma_t) \right|_{t=0}$$

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we have:

$$N_x = \{A'(\gamma) : [\gamma] \in \pi_2(L, x)\} \subset \mathfrak{v}_x^*(L).$$

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*Some consequences:*

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*Some consequences:*

- Every two dimensional Poisson manifold is integrable;

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$$N_x = \{A'(\gamma) : [\gamma] \in \pi_2(L, x)\} \subset \mathfrak{v}_x^*(L).$$

*Some consequences:*

- Every two dimensional Poisson manifold is integrable;
- A Poisson structure in  $M = \mathbb{R}^3 - \{0\}$  with leaves the spheres  $x^2 + y^2 + z^2 = \text{const.}$  is integrable iff the symplectic areas of the spheres have no critical points.

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## Example: Regular Poisson Manifolds.

Define the **variation of symplectic variations**  $A'(\gamma_0) \in \mathfrak{v}_x^*(L)$  by

$$\langle A'(\gamma_0), \text{var}_v(\gamma_t) \rangle = \left. \frac{d}{dt} A(\gamma_t) \right|_{t=0}$$

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## PART 3

# Other Invariants: Holonomy, Characteristic Classes and K-Theory

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# Lie Algebroid Connections

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## Lie Algebroid Connections

An  $A$ -connection is a bundle map  $h : p^*A \rightarrow TP$  s.t.:

$$\begin{array}{ccc} & P & \curvearrowright G \\ & \downarrow p & \\ A & \xrightarrow{\pi} & M \end{array}$$

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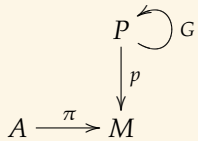
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## Lie Algebroid Connections

An  $A$ -connection is a bundle map  $h : p^*A \rightarrow TP$  s.t.:

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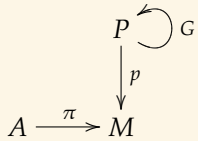
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where  $u \in P, a \in A_x$  with  $x = p(u)$ , and  $g \in G$ .



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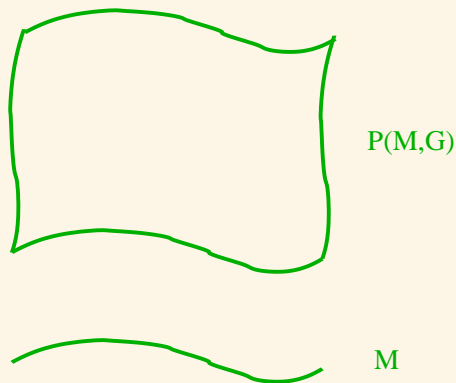
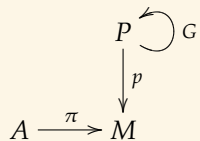
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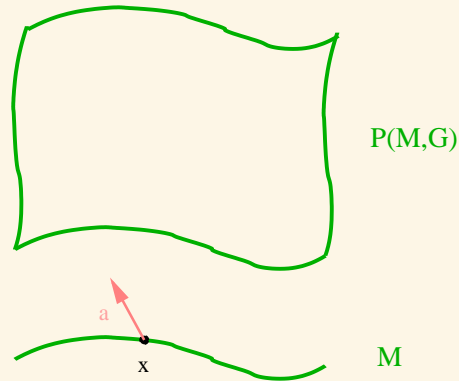
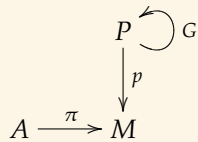
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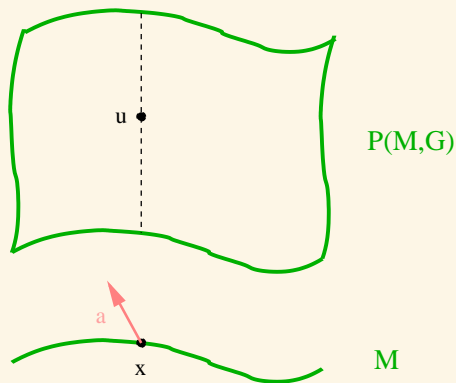
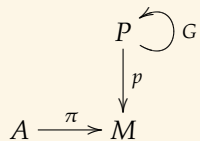
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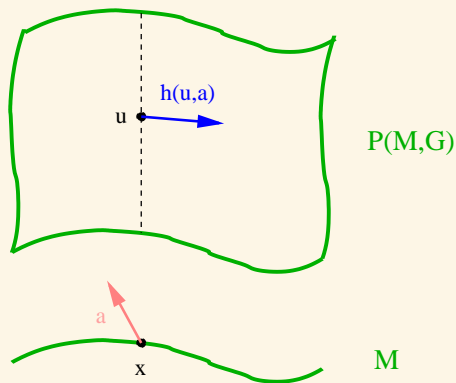
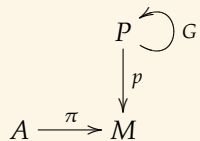
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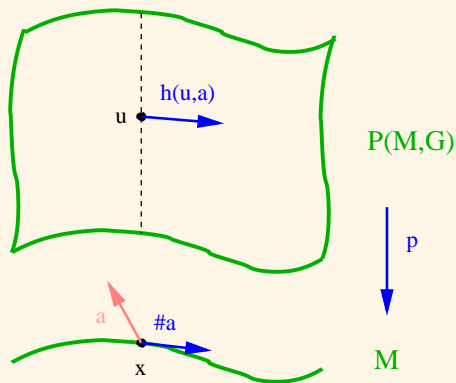
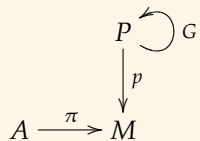
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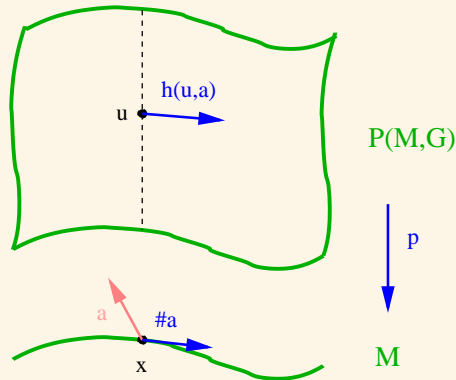
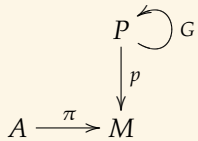
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Similar to usual connections (case  $A = TM$ ) but:

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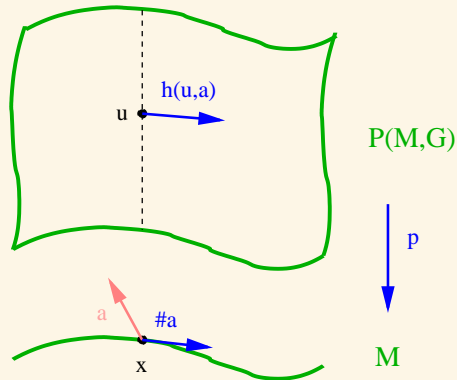
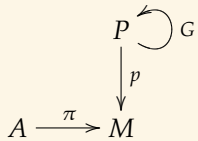
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Similar to usual connections (case  $A = TM$ ) but:

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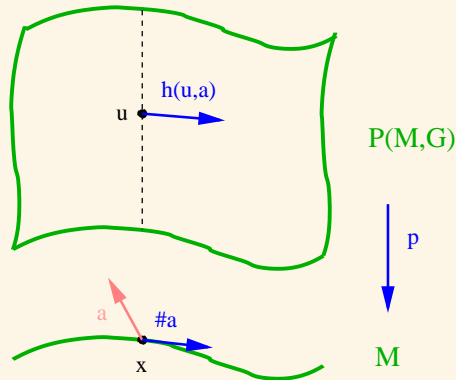
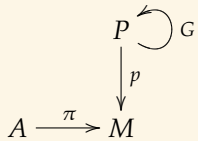
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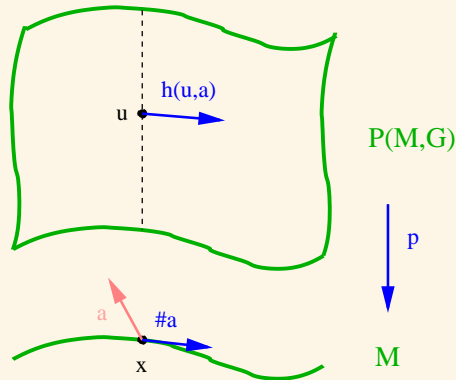
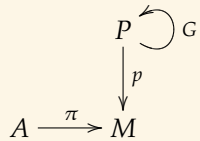
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Similar to usual connections (case  $A = TM$ ) but:

- $A$ -connections are not determined by distribution  $\text{Im } h$ ;
- $\parallel$ -transport can be defined only along  $A$ -paths;
- Flat connections may have non-discrete holonomy;

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# *A-Holonomy*

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## ***A*-Holonomy**

Using non-linear connections one obtains the ***A*-holonomy** homomorphism:

$$\text{Hol} : \mathcal{G}(A)_x \rightarrow \text{Out}(A_L^\perp),$$

where:

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**Theorem 3.1 (RLF, 2001).** *Let  $L$  be a compact, transversely stable leaf of  $A$ , with finite holonomy. Then:*

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- each leaf near  $L$  is a bundle over  $L$  whose fiber is a finite union of leaves of the transverse Lie algebroid structure.*

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# *A-derivatives*

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## *A*-derivatives

An *A*-connection on  $P = P(M, G)$  induces on any associated vector bundle  $E \rightarrow M$  an ***A*-derivative** operator:

$$\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E).$$

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Axioms for an *A*-derivative:

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For any  $\alpha \in \Gamma(A)$ ,  $s \in \Gamma(E)$ ,  $f \in C^\infty(M)$ :

(i)  $\nabla_{f\alpha}s = f\nabla_\alpha s$ ;

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$A$ -derivatives work like the usual covariant derivatives (case  $A = TM$ ):

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- The **torsion** of  $\nabla : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$  is

$$T_\nabla(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta].$$

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## Characteristic Classes

$A$ -connections lead to:

- A **Chern-Weil** theory for Lie algebroids [Vaisman, 1991; Kubarski, 1996; RLF, 2000];

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- **Characteristic classes of representations** of a Lie algebroid [Crainic, 2001].

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# K-theory

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# K-theory

**Flat  $A$ -connections  $\Leftrightarrow$  Representations of  $A$**

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## K-theory

**Flat  $A$ -connections  $\Leftrightarrow$  Representations of  $A$**

Axioms for a representation of  $A$ :

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## K-theory

**Flat  $A$ -connections  $\Leftrightarrow$  Representations of  $A$**

Axioms for a representation of  $A$ :

$E \rightarrow M$  is a vector bundle and there exists a product  $\Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$  such that:

(i)  $(f\alpha) \cdot s = f(\alpha \cdot s)$ ;

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- The appropriate equivalence relation(s) were introduced by [Ginzburg, 2001];
- Representations lead to **Morita equivalence** in the context of Lie algebroids [Ginzburg, 2001; Crainic & RLF, 2002].

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## THE LEIBNIZ IDENTITY.

For any sections  $\alpha, \beta \in \Gamma(A)$  and function  $f \in C^\infty(M)$ :

$$[\alpha, f\beta] = f[\alpha, \beta] + \# \alpha(f)\beta.$$

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## THE TANGENT LIE ALGEBROID.

$M$  - a manifold

- bundle:  $A = TM$ ;
- anchor:  $\# : TM \rightarrow TM, \# = \text{id}$ ;
- Lie bracket:  $[ , ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ ,  
usual Lie bracket of vector fields;
- characteristic foliation:  $\mathcal{F} = \{M\}$ .

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## THE LIE ALGEBROID OF A LIE ALGEBRA.

$\mathfrak{g}$  - a Lie algebra

- bundle:  $A = \mathfrak{g} \rightarrow \{*\}$ ;
- anchor:  $\# = 0$ ;
- Lie bracket:  $[ , ] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ ,  
given Lie bracket;
- characteristic foliation:  $\mathcal{F} = \{*\}$ .

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## THE LIE ALGEBROID OF A FOLIATION.

$\mathcal{F}$  - a regular foliation

- bundle:  $A = T\mathcal{F} \rightarrow M$ ;
- anchor:  $\# : T\mathcal{F} \hookrightarrow TM$ , inclusion;
- Lie bracket:  $[ , ] : \mathfrak{X}(\mathcal{F}) \times \mathfrak{X}(\mathcal{F}) \rightarrow \mathfrak{X}(\mathcal{F})$ ,  
usual Lie bracket restricted to vector fields tangent to  $\mathcal{F}$ ;
- characteristic foliation:  $\mathcal{F}$ .

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## THE ACTION LIE ALGEBROID.

$\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$  - an infinitesimal action of a Lie algebra

- bundle:  $A = M \times \mathfrak{g} \rightarrow M$ ;
- anchor:  $\# : A \rightarrow TM, \#(x, v) = \rho(v)|_x$ ;
- Lie bracket:  $[\cdot, \cdot] : C^\infty(M, \mathfrak{g}) \times C^\infty(M, \mathfrak{g}) \rightarrow C^\infty(M, \mathfrak{g})$

$$[v, w](x) = [v(x), w(x)] + (\rho(v(x)) \cdot w)|_x - (\rho(w(x)) \cdot v)|_x;$$

- characteristic foliation: orbit foliation.

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## THE LIE ALGEBROID OF A PRESYMPLECTIC MANIFOLD.

$M$  - an presymplectic manifold with closed 2-form  $\omega$

- bundle:  $A = TM \times \mathbb{R} \rightarrow M$ ;
- anchor:  $\# : A \rightarrow TM, \#(v, \lambda) = v$ ;
- Lie bracket:  $\Gamma(A) = \mathfrak{X}(M) \times C^\infty(M)$

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f) - \omega(X, Y));$$

- characteristic foliation:  $\mathcal{F} = \{M\}$ .

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## THE COTANGENT LIE ALGEBROID.

$M$  - a Poisson manifold with Poisson tensor  $\pi$

- bundle:  $A = T^*M$ ;
- anchor:  $\# : TM^* \rightarrow TM, \# \alpha = i_\pi \alpha$ ;
- Lie bracket:  $[ , ] : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^1(M)$ ,

Kozul Lie bracket:

$$[\alpha, \beta] = \mathcal{L}_{\# \alpha} \beta - \mathcal{L}_{\# \beta} \alpha - d\pi(\alpha, \beta);$$

- characteristic foliation: the symplectic foliation.

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## THE PAIR GROUPOID.

$M$  - a manifold

- **arrows:**  $\mathcal{G} = M \times M$ ;
- **objects:**  $M$ ;
- **target and source:**  $\mathbf{s}(x, y) = x, \mathbf{t}(x, y) = y$ ;
- **product:**  $(x, y) \cdot (y, z) = (x, z)$ ;

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## THE LIE GROUPOID OF A LIE GROUP.

$G$  - a Lie group

- **arrows:**  $\mathcal{G} = G$ ;
- **objects:**  $M = \{*\}$ ;
- **target and source:**  $\mathbf{s}(x) = \mathbf{t}(x) = *$ ;
- **product:**  $g \cdot h = gh$ ;

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## THE HOLONOMY GROUPOID.

$\mathcal{F}$  - a regular foliation in  $M$

- **arrows:**  $\mathcal{G} = \{[\gamma] : \text{holonomy equivalence classes}\};$
- **objects:**  $M;$
- **target and source:**  $\mathbf{s}([\gamma]) = \gamma(0), \mathbf{t}([\gamma]) = \gamma(1);$
- **product:**  $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma'];$

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## THE ACTION GROUPOID.

$G \times M \rightarrow M$  - an action of a Lie group on  $M$

- **arrows:**  $\mathcal{G} = G \times M$ ;
- **objects:**  $M$ ;
- **target and source:**  $\mathbf{s}(g, x) = x, \mathbf{t}(g, x) = gx$ ;
- **product:**  $(h, y) \cdot (g, x) = (hg, x)$ ;

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