WHAT IS POISSON GEOMETRY?

RUI LOJA FERNANDES

1. Motivation and Origins

In Classical Mechanics one learns how to describe a mechaninal system with n degrees of freedom evolving with time. Briefly, the state of the system at time t is described by a point $(q(t), p(t))$ in **phase space** $\mathbb{R}^{2n}$. Here the $(q^1(t), \dots, q^n(t))$ are the configuration coordinates and the $(p_1(t), \dots, p_n(t))$ are the momentum coordinates of the system. The evolution of the system in time is determined by a function $h : \mathbb{R}^{2n} \to \mathbb{R}$, called the **hamiltonian**: if $(q(0), p(0))$ is the initial state of the system, then the state at time t is obtained by solving **Hamilton's equations**:

$$\begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i}, \\ \\ \dot{p}_i = -\frac{\partial H}{\partial q^i}, \end{cases} \qquad (i = 1, \dots, n). \tag{1}$$

This description of mechanics is the departing point for Poisson geometry.

First, one starts by defining a new product $\{f_1, f_2\}$ between any two smooth functions f_1 and f_2, called the **Poisson bracket**, by setting:

$$\{f_1, f_2\} := \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q^i} - \frac{\partial f_1}{\partial q^i} \frac{\partial f_2}{\partial p_i} \right). \tag{2}$$

Note that this product is *not associative*. In fact, one has instead the very important **Jacobi identity**:

$$\{f_1, \{f_2, f_3\}\} + \{f_1, \{f_2, f_3\}\} + \{f_1, \{f_2, f_3\}\} = 0,$$

valid for any smooth functions f_1, f_2 and f_3. It is also **skew-symmetric**:

$$\{f_1, f_2\} = -\{f_2, f_1\},$$

and **bilinear** over $\mathbb{R}$:

$$\{f, af_1 + bf_2, f\} = a\{f, f_1\} + b\{f, f_2\}, \quad (a, b \in \mathbb{R}).$$

You can find all these properties in the definition of a *Lie algebra*, and we will pursue this connection later. There is a fourth property which relates this new product with the usual product of two functions. This property is the **Leibniz identity**:

$$\{f, f_1 \cdot f_2\} = f_1 \cdot \{f, f_2\} + \{f, f_1\} \cdot f_2.$$

All this is quite algebraic, so let us turn back to the dynamics. We now observe that, once a function h has been fixed, Hamilton's equations (1) can be written in the form:

$$\dot{x}^i = \{h, x^i\}, \quad (i = 1, \dots, n)$$

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where x^i is any of the coordinate functions (q^i, p_i). Actually, let us observe that we can define a vector field X_h in $\mathbb{R}^{2n}$ by setting:

$$X_h(f) := \{h, f\}.$$

Note that, by the Leibniz identity, X_h is indeed a derivation so defines a vector field which in the coordinates (q^i, p_i) is given by:

$$X_h = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p_i} \right).$$

Then Hamilton's equations are just the equations for the integral curves of this vector field, i.e., the equations:

$$\dot{x}(t) = X_h(x(t)).$$

In this way, we have geometrized mechanics, and we are now ready to proceed to the next abstraction level.

2. Poisson brackets

Let me quote Michael Spivak (the author of some very nice textbooks in Calculus and Differential Geometry):

> *"There are all good reasons why definitions should be hard and theorems should be easy."*

So here is our main definition:

Definition 2.1. A **Poisson bracket** on a manifold M is a Lie bracket $\{\ ,\ \}$ on its space of smooth functions $C^\infty(M)$ which satisfies the Leibniz identity:

$$\{f, f_1 \cdot f_2\} = f_1 \cdot \{f, f_2\} + \{f, f_1\} \cdot f_2. \tag{3}$$

A **Poisson manifold** $(M, \{\ ,\ \})$ is a manifold M together with a choice of a Poisson bracket on it.

Remember that a Lie bracket is just a bilinear operation, which is skew-symmetric and satisfies the Jacobi identity. We can summarize part of what we said before by saying that $\mathbb{R}^{2n}$ with the bracket defined by (2) is a Poisson manifold.

Let $(M, \{\ ,\ \})$ be a Poisson manifold. If $h \in C^\infty(M)$ is a any smooth function, we associate to it a vector field X_h, called the **Hamiltonian vector field** of h, by setting for any smooth function $f \in C^\infty(M)$:

$$X_h(f) := \{h, f\}.$$

Therefore, on a a Poisson manifold a function determines dynamics.

Just to give you the flavour of the theory, let us look quickly at first integrals. Remember that a function f is a first integral of a vector field X if for any integral curve $x(t)$ of X we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)) = 0.$$

Note that this happens if, and only if, $X(f) = 0$. Therefore, for the hamiltonian vector field X_h the first integrals are precisely those functions f such that $\{h, f\} = 0$. This yields immediately the well-known fact that the hamiltonian is a conserved quantity:

Theorem 2.2. *The hamiltonian function h is a first integral of the hamiltonian vector field X_h.*

Proof. Due to the skew-symmetry of the bracket, we have:

$$X_h(h) = \{h, h\} = 0.$$

□

Actually, the Poisson bracket allows us to do even better:

Theorem 2.3. *If f_1 and f_2 are first integrals of X_h then so is $\{f_1, f_2\}$.*

Proof. Using the Jacobi identity we find:

$$X_h(\{f_1, f_2\}) = \{h, \{f_1, f_2\}\} = \{\{h, f_1\}, f_2\} + \{f_1, \{h, f_2\}\} = 0.$$

□

All this is quite elementary and familiar. I only wanted to stress the point that the Poisson bracket is the essential ingredient in any study of Hamiltonian dynamics.

So far we have been quite algebraic. So where is the geometry? It turns out that there is both *local* and *global* geometry underlying Poisson brackets. I will explain this briefly in the next two sections.

3. Local Poisson geometry

If $(M, \{\ ,\ \})$ is a Poisson manifold and we choose a local coordinate system $(U, x^1, \dots, x^m)$, the Poisson bracket takes the local form:

$$\{f_1, f_2\} = \sum_{i<j}^{m} C^{ij} \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j},$$

where the coefficients are the *structure functions* $C^{ij} := \{x^i, x^j\}$. Note that by changing coordinates these structures functions can get simplified or more complex.

The local study of Poisson brackets is based upon the following theorem of Alan Weinstein:

Theorem 3.1 (Darboux-Weinstein)**.** *Let $(M, \{\ ,\ \})$ be a Poisson manifold and $x_0 \in M$. There is an even number $r = 2n$ and local coordinates $(U, (q^1, \dots, q^n, p_1, \dots, p_n, y^1, \dots, y^s))$ centered at x_0 such that:*

$$\{f_1, f_2\} = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q^i} - \frac{\partial f_1}{\partial q^i} \frac{\partial f_2}{\partial p_i} \right) + \sum_{i<j}^{s} \phi^{ij} \frac{\partial f_1}{\partial y^i} \frac{\partial f_2}{\partial y^j},$$

where $\phi^{ij} = \phi^{ij}(y)$ are functions that depend only on the (y^i) coordinates and vanish at x_0.

You can find an elegant proof of this result in the original Weinstein's paper [6]. By the way, this is a very nice paper, which lays down the foundations of local Poisson geometry, and which I recommend as a good starting point to learn Poisson geometry.

The number $r = 2n$ is called the **rank** of the Poisson bracket at x_0. Locally a Poisson bracket splits into a product of the standard Poisson bracket on $\mathbb{R}^{2n}$ and a singular Poisson bracket, vanishing at x_0.

If you write down Hamilton's equations in Darboux-Weinstein coordinates, you will see that a point in U can be connected to x_0 by an integral curve lying in U of some Hamiltonian vector field if, and only if, the integral curve lies in the level set $y^i = 0$ ($i = 1, \ldots, s$). This shows that points lying in the same integral curve of some Hamiltonian vector field will have the same rank.

Now we can get a more geometric picture as follows. Let us define an equivalence relation on M by declaring two points x and y to be equivalent if they can be connected by a piece-wise continuous curve made of integral curves of Hamiltonian vector fields. Using the Darboux-Weinstein splitting theorem, we can see that the equivalence classes are immersed submanifolds and that they form a (singular) foliation of M. Note that the dimension of each leaf is precisely the rank of the points of that leaf.

Let us focus our attention now on the singular part. By the Weinstein splitting theorem, we can assume that we are at a point x_0 where the rank is zero (otherwise, we pick a small transverse manifold to the leaf and we restrict attention to it). Now $(x^1, \ldots, x^m)$ are local coordinates centered at x_0, the Taylor expansion of the function $\{x^i, x^j\}$ around x_0 is

$$\{x^i, x^j\}(x) = \sum_{k=1}^{m} c_k^{ij} x^k + o(2),$$

where the c_k^{ij} are some constants, and $o(2)$ denotes terms that vanish to first order. The *linearization problem* asks:

- Is there a set of local coordinates where the higher order terms vanish so that the Poisson bracket becomes linear?

When this is possible, one says that the Poisson structure is **linearizable** around x_0. This problem is just an instance of a normal form problem. In these kind of problems one looks for local invariants that allow one to solve it. Let us explain one such local invariant.

Let us observe that the vector space $T^*_{x_0}M$ carries a natural Lie algebra structure: in terms of the base $\{\mathrm{d}_{x_0}x^1, \ldots, \mathrm{d}_{x_0}x^m\}$ the bracket is given by:

$$[\mathrm{d}_{x_0}x^i, \mathrm{d}_{x_0}x^j] = \sum_{k=1}^{m} c_k^{ij} \mathrm{d}_{x_0}x^k.$$

You can check, by changing variables, that this is independent of the choice of coordinates. This Lie algebra, called the **isotropy Lie algebra** at x_0, is a local invariant of our Poisson manifold. Its Killing form K, which in terms of the base above is given by:

$$K(q) = c_l^{im} c_m^{jl} q_i q_j,$$

is also an invariant. We have the following deep theorem:

Theorem 3.2 (Conn [3]). *If K is negative definite, then $\{\ ,\ \}$ is linearizable around x_0.*

Note that K is negative definite if, and only if, every Lie group integrating the isotropy Lie algebra is compact.

I hope this has given you a flavour of what (local) Poisson geometry. It is now time to turn into global Poisson geometry.

4. Global Poisson geometry

As we have explained in the previous section, a Poisson manifold has a (singular) foliation.

*** TO BE CONTINUED ***

5. Lie Groupoids and Lie algebroids

*** TO BE CONTINUED ***

6. Integrable Systems

*** TO BE CONTINUED ***

References

1. M. Adler, P. van Moerbeke and P. Vanhaecke, *Algebraic Integrable Systems*, Springer-Verlag, 2004.
2. A. Cannas da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, Berkeley Mathematics Lectures, vol. **10**, American Math. Soc. , Providence, 1999.
3. J. Conn, Normal forms for smooth Poisson structures, *Annals of Math.* **121** (1985), 565–593.
4. M. Crainic and R. L. Fernandes, Integrability of Lie brackets, *Ann. of Math.* **157** (2003).
5. M. Crainic and R. L. Fernandes, *Lectures on Integrability of Lie Brackets*, To be published by Mathematical Sciences Publishers, 2007. Available in www.math.ist.utl.pt/∼rfern/publications
6. A. Weinstein, The local structure of Poisson manifolds, *J. Differential Geometry* **18** (1983), 523–557.

Depart. de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, PORTUGAL

E-mail address: rfern@math.ist.utl.pt