The Crossed Menagerie: an introduction to crossed gadgetry and cohomology in algebra and topology.

(Notes initially prepared for the XVI Encuentro Rioplatense de Álgebra y Geometría Algebraica, in Buenos Aires, 12-15 December 2006, extended for an MSc course (Summer 2007) at Ottawa. They form the first 7 chapters of a longer document that is still evolving!)

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October 9, 2008

Introduction

These notes were originally intended to supplement lectures given at the Buenos Aires meeting in December 2006, and have been extended to give a lot more background for a course in cohomology at Ottawa (Summer term 2007). They introduce some of the family of crossed algebraic gadgetry that have their origins in combinatorial group theory in the 1930s and '40s, then were pushed much further by Henry Whitehead in the papers on Combinatorial Homotopy, in particular, [113]. Since about 1970, more information and more examples have come to light, initially in the work of Ronnie Brown and Phil Higgins, (for which a useful central reference will be the forthcoming, [29]), in which crossed complexes were studied in depth. Explorations of crossed squares by Loday and Guin-Valery, [64, 81] and from about 1980 onwards indicated their relevance to many problems in algebra and algebraic geometry, as well as to algebraic topology have become clear. More recently in the guise of 2-groups, they have been appearing in parts of differential geometry, [10, 21] and have, via work of Breen and others, [17–20], been of central importance for *non-Abelian cohomology*. This connection between the crossed menagerie and non-Abelian cohomology is almost as old as the crossed gadgetry itself, dating back to Dedecker's work in the 1960s, [48]. Yet the basic message of what they are, why they work, how they relate to other structures, and how the crossed menagerie works, still need repeating, especially in that setting of non-Abelian cohomology in all its bewildering beauty.

The original notes have been augmented by additional material, since the link with non-Abelian cohomology was worth pursuing in much more detail. These notes thus contain an introduction to the way 'crossed gadgetry' interacts with non-Abelian cohomology and areas such as topological and homotopical quantum field theory. This entails the inclusion of a fairly detailed introduction to torsors, gerbes etc. This is based in part on Larry Breen's beautiful Minneapolis notes, [20].

If this is the first time you have met this sort of material, then some words of warning and welcome are in order.

There is much too much in these notes to digest in one go!

There is probably a lot more than you will need in your continuing research. For instance, the material on torsors, etc., is probably best taken at a later sitting and the chapter 'Beyond 2-types' is not directly used until a lot later, so can be glanced at.

I have concentrated on the group theoretic and geometric aspects of cohomology, since the non-Abelian theory is better developed there, but it is easy to attack other topics such as Lie algebra cohomology, once the basic ideas of the group case have been mastered and applications in differential geometry do need the torsors, etc. I have emphasised approaches using crossed modules (of groups). Analogues of these gadgets do exist in the other settings (Lie algebras, etc.), and most of the ideas go across without too much pain. If handling a non-group based problem (e.g. with monoids or categories), then the internal categorical aspect - crossed module as internal category in groups - would replace the direct method used here. Moreover the group based theory has the advantage of being central to both algebraic and geometric applications.

The aim of the notes is not to give an exhaustive treatment of cohomology. That would be impossible. If at the end of reading the relevant sections the reader feels that they have some intuition on the *meaning* and *interpretation* of cohomology classes in their own area, and that they can more easily attack other aspects of cohomological and homotopical algebra by themselves, then the notes will have succeeded for them.

Although not 'self contained', I have tried to introduce topics such as sheaf theory as and when necessary, so as to give a natural development of the ideas. Some readers will already have been introduced to these ideas and they need not read those sections in detail. Such sections are, I think, clearly indicated. They do not give all the details of those areas, of course. For a start, those details are not needed for the purposes of the notes, but the summaries do try to sketch in enough 'intuition' to make it reasonable clear, I hope, what the notes are talking about!

(This version is a shortened version of the notes. It does not contain the material on gerbes. It is still being revised. The full version will be made available later.)

Acknowledgements

These notes were started as extra backup for the lectures at the XVI Encuentro Rioplatense de Álgebra y Geometría Algebraica, in Buenos Aires, 12-15 December 2006. That meeting and thus my visit to Argentina was supported by several organisations there, CONICET, ANCPT, and the University of Buenos Aires, and in Uruguay, CSIC and PDT, and by a travel grant from the London Mathematical Society.

The visit would not have been possible without the assistance of Gabriel Minian and his colleagues and students, who provided an excellent environment for research discussions and, of course, the meeting itself.

The notes were continued for course MATH 5312 in the Spring of 2007 during a visit as a visiting professor to the Dept. of Mathematics and Statistics of the University of Ottawa. Thanks are due to Rick Blute, Pieter Hofstra, Phil Scott, Paul-Eugene Parent, Barry Jessup and Jonathan Scott for the warm welcome and the mathematical discussions on some of the material and the students of MATH 5312 for their interest and constructive comments.

Tim Porter, Bangor and Ottawa, Spring and Summer 2007

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Chapter 1

Preliminaries

1.1 Groups and Groupoids

Before launching into crossed modules, we need a word on groupoids. By a groupoid, we mean a small category in which all morphisms are isomorphisms. (If you have not formally met categories then do not worry, the idea will come through without that specific formal knowledge, although a quick glance at Wikipedia for the definition of a category might be a good idea at some time soon. You do not need category theory as such at this stage.) These groupoids typically arise in three situations (i) symmetry objects of a fibered structure, (ii) equivalence relations, and (iii) group actions. It is worth noting that several of the initial applications of groups were thought of, by their discoverers, as being more naturally this type of groupoid structure.

For the first, assume we have a family of sets $\{X_a : a \in A\}$. Typically we have a function $f : X \to A$ and $X_a = f^{-1}a$ for $a \in A$. We form the symmetry groupoid of the family by taking the index set, A, as the set of objects of the groupoid, $\mathcal{G}$, and, if $a, a' \in A$, then $\mathcal{G}(a, a')$, the set of arrows in our symmetry groupoid from a to a', is the set $Bijections(X_a, X_{a'})$. This $\mathcal{G}$ will contain all the individual symmetry groups / permutation groups of the various X_a, but will also record comparison information between different X_as.

Of course, any group is a groupoid with one object and if $\mathcal{G}$ is any groupoid, we have, for each object a of $\mathcal{G}$, a group $\mathcal{G}(a, a)$, of arrows that start and end at a. This is the 'automorphism group', $\mathsf{aut}_{\mathcal{G}}(a)$, of a within $\mathcal{G}$. It is also referred to as the vertex group of $\mathcal{G}$ at a, and denoted $\mathcal{G}(a)$. This later viewpoint and notation emphasise more the combinatorial, graph-like side of $\mathcal{G}$'s structure. Sometimes the notation $G[1]$ may be used for $\mathcal{G}$ as the process of regarding a group as a groupoid is a sort of 'suspension' or 'shift'. It is one aspect of 'categorification', cf. Baez and Dolan, [9].

That combinatorial side is strongly represented in the second situation, equivalence relations. Suppose that R is an equivalence relation on a set X. Going back to basics, R is a subset of $X \times X$ satisfying:

(a) if $a, b, c \in X$ and (a, b) and $(b, c) \in R$, then $(a, c) \in R$, i.e. R is transitive;

(b) for all $a \in X$, $(a, a) \in R$, alternatively the diagonal $\Delta \subseteq R$, i.e. R is reflexive;

(c) if $a, b \in X$ and $(a, b) \in R$, then $(b, a) \in R$, i.e. R is symmetric.

Two comments might be made here. The first is 'everyone knows that!', the second 'that is not the usual order to put them in! Why?'

It is a well known, but often forgotten, fact that from R, you get a groupoid (which we will denote by $\mathcal{R}$). The objects of $\mathcal{R}$ are the elements of X and $\mathcal{R}(a,b)$ is a singleton if $(a,b) \in \mathcal{R}$ and is empty otherwise. (There is really no need to label the single element of $\mathcal{R}(a,b)$, when this is non empty, but it is sometimes convenient to call it (a,b) at the risk of over using the ordered pair notation.) Now transitivity of R gives us a composition function: for $a,b,c \in X$,

$$\circ : \mathcal{R}(a,b) \times \mathcal{R}(b,c) \to \mathcal{R}(a,c).$$

(Remember that a product of a set with the empty set is itself always empty, and that for any set, there is a unique function with domain $\emptyset$ and codomain the set, so checking that this composition works nicely is slightly more subtle than you might at first think. This *is* important when handling the analogues of equivalence relations in other categories., then you cannot just write $(a,b)\circ(b,c) = (a,c)$, or similar, as 'elements' may not be obvious things to handle.) Of course this composition *is* associative, but if you have not seen the verification, it is important to think about it, looking for subtle points, especially concerning the empty set and empty function and how to do the proof without 'elements'.

This composition makes $\mathcal{R}$ into a category, since (a) gives the existence of identities for each object. ($Id_a = (a,a)$ in 'elementary' notation.) Finally (c) shows that each (a,b) is invertible, so $\mathcal{R}$ is a groupoid. (You now see why that order was the natural one for the axioms. You cannot prove that (a,a) is an identity until you have a composition, and similarly until you have identities, inverses do not make sense.) We may call $\mathcal{R}$, the groupoid of the equivalence relation R.

This shows how to think of R as a groupoid, $\mathcal{R}$. The automorphism groups, $\mathcal{R}(a)$, are all singletons as sets, so are trivial groups. Conversely any groupoid, $\mathcal{G}$, gives a diagram

$$\begin{tikzcd} Arr(\mathcal{G}) \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & Ob(\mathcal{G}) \arrow[l, shift left=2, "i"] \end{tikzcd}$$

with $s =$ 'source', $t =$ 'target'. It thus gives a function

$$Arr(\mathcal{G}) \xrightarrow{(s,t)} Ob(\mathcal{G}) \times Ob(\mathcal{G}) .$$

The image of this function is an equivalence relation as is easily checked. We will call this equivalence relation R for the moment. If $\mathcal{G}$ is a groupoid such that each $\mathcal{G}(a)$ is a trivial group, then each $\mathcal{G}(a,b)$ has at most one element (check it), so (s,t) is a one-one function and it is then trivial to note that $\mathcal{G}$ is isomorphic to the groupoid of the equivalence relation, R.

We have looked at this simple case in some detail as in applications of the basic ideas, especially in algebraic geometry, arguments using elements are quite tricky to give and the initial intuition coming from this set-based case can easily be forgotten.

The third situation, that of group actions, is also a common one in algebra and algebraic geometry. Equivalence relations often come from group actions. If G is a group and X is a G-set with (left) G-action

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ (g,x) & & g\cdot x \end{array} ,$$

(i.e. a function $act(g,x) = g\cdot x$, which must satisfy the rules $1\cdot x = x$ and for all $g_1, g_2 \in G$, $g_1\cdot(g_2\cdot x) = (g_1g_2)\cdot x$, a sort of associativity law,) then we get a groupoid $\mathcal{A}ct_G(X)$ as follows:

- the objects of $\mathcal{A}ct_G(X)$ are the elements of X;
- if $a, b, \in X$,
$$\mathcal{A}ct_G(X)(a,b) \cong \{g \mid g \cdot a = b\}.$$

An important word of caution is in order here. Logical complications can occur here if $\mathcal{A}ct_G(X)(a,b)$ is set equal to $\{g \mid g \cdot a = b\}$, since then a g can occur in several different 'hom-sets'. A good way to avoid this is to take
$$\mathcal{A}ct_G(X)(a,b) = \{(g,a) \mid g \cdot a = b\}.$$

This is a non-trivial change. It basically uses a disjoint union but although very simple it is fundamental in its implications. We could also do it by taking $Arr_{\mathcal{G}}(X) = G \times X$ with source and target maps $s(g,x) = x$, $t(g,x) = g \cdot x$. (It is **useful**, if you have not seen this before, to see how the various parts of the definition of an action match with parts of the structural rules of a groupoid. This is important as it indicates how, much later on, we will relax those rules in various ways.)

We have not discussed morphisms of groupoids. These are straightforward to define and to work with. Most of the concepts we will be handling in what follows exist in many-object, groupoid versions as well as single-object, group based ones. For simplicity we will often, but not always, give concepts in the group based form, and will leave the other many-object form 'to the reader'. The conversion is usually not that difficult.

For more details on the theory of groupoids, the best two sources are Ronnie Brown's book, [24] or Phil Higgins' monograph, now reprinted as [65].

1.2 A very brief introduction to cohomology

Partially as a case study, at least initially, we will be looking at various constructions that relate to group cohomology. Later we will explore a more general type of (non-Abelian) cohomology, including ideas about the non-Abelian cohomology of spaces, but that is for later. To start with we will look at a simple group theoretic problem that will be used for motivation at several places in what follows. Much of what is in books on group cohomology is the Abelian theory, whilst we will be looking more at the non-Abelian one. If you have not met cohomology at all, take a look at the Wikipedia entries for group cohomology. You may not understanding everything, but there are ideas there that will recur in what follows, and some terms that are described there or on linked entries, that will be needed later.

1.2.1 Extensions.

Given a group, G an extension of G by a group K is a group E with an epimorphism $p : E \to G$ whose kernel is isomorphic to K (i.e. a short exact sequence of groups
$$\mathcal{E} : 1 \to K \to E \xrightarrow{p} G \to 1.$$

As we asked that K is isomorphic to $Ker\, p$, we could have different groups E perhaps fitting into this, yet they would still be essentially the same extension. We say two extensions, $\mathcal{E}$ and $\mathcal{E}'$, are equivalent if there is an isomorphism between E and E' compatible with the other data. We can draw a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[d] & 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"'] & G \arrow[r] \arrow[d, "="] & 1 \\ \mathcal{E}' & 1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

A typical situation might be that you have an unknown group E' that you suspect is really E (i.e. is isomorphic to E). You find a known normal subgroup K of E is isomorphic to one in E' and that the two quotient groups are isomorphic,

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\cong"'] & E \arrow[r] \arrow[d, dashed, "?"'] & G \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & K' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 \end{tikzcd}$$

(But always remember, isomorphisms compare snap shots of the two structures and once chosen can make things more 'rigid' than perhaps they really 'naturally' are. For instance, we might have G a cyclic group of order 5 generated by an element a, and G' one generated by b. 'Naturally' we choose an isomorphism $\varphi : G \to G'$ to send a to b, but why? We could have sent a to any non-identity element of G' and need to be sure that this makes no difference. This is not just 'attention to detail'. It can be very important. It stresses the importance of $Aut(G)$, the group of automorphisms of G in this sort of situation.)

A simple case to illustrate that the extension problem is a valid one, is to consider $K = C_3 = \langle a \mid a^3\rangle$, $G = C_2 = \langle b \mid b^2\rangle$.

We could take $E = S_3$, the symmetric group on three symbols, or alternatively D_3 (also called D_6 to really confuse things, but being the symmetry group of the triangle). This has a presentation $\langle a, b \mid a^3, b^2, (ab)^2\rangle$. But what about $C_6 = \langle c \mid c^6\rangle$? This has a subgroup $\{1, c^2, c^4\}$ isomorphic to K and the quotient is isomorphic to G. Of course, S_3 is non-Abelian, whilst C_6 is. The presentation of C_6 needs adjusting to see just how similar the two situations are. This group also has a presentation $\langle a, b \mid a^3, b^2, aba^{-1}b\rangle$, since we can deduce $aba^{-1}b = 1$ from $[a, b] = 1$ and $b^2 = 1$ where in terms of the old generator c, $a = c^2$ and $b = c^3$. So there is a presentation of C_3 which just differs by a small 'twist' from that of S_3.

How could one be sure if S_3 and C_6 are the 'only' groups (up to isomorphism) that we could put in that central position? Can we classify all the extensions of G by K?

These extension problems were one of the impetuses for the development of a 'cohomological' approach to algebra, but they were not the only ones.

1.2.2 Invariants.

Another group theoretic input is via group representation theory and the theory of invariants. If G is a group of $n \times n$ invertible matrices then one can use the simple but powerful tools of linear algebra to get good information on the elements of G and often one can tie this information in to some geometric context, say, by identifying elements of G as leaving invariant some polytope or pattern, so G acts as a subgroup of the group of the symmetries of that pattern or object.

If therefore we use the group $Gl(n, \mathbb{K})$ of such invertible matrices over some field $\mathbb{K}$, then we could map an arbitrary G into it and attempt to glean information on elements of G from the corresponding matrices. We thus consider a group homomorphism

$$\rho : G \to Gl(n, \mathbb{K}),$$

then look for nice properties of the $\rho(g)$. of course, ρ need not be a monomorphism and then we will loose information in the process, but in any case such a morphism will make G act (linearly) on the vector space $\mathbb{K}^n$. We could, more generally, replace $\mathbb{K}$ by a general commutative ring R, in particular we could use the ring of integers, $\mathbb{Z}$, and then replace $\mathbb{K}^n$ by a general *module*, M, over R. If $R = \mathbb{Z}$, then this is just an Abelian group. (If you have not formally met modules look up a definition. The theory feels very like that of vector spaces to start with at least, but as elements in R need not have inverses, care needs to be taken - you cannot cancel or divide in general, so $rx = ry$ does not imply $x = y$! Having looked up a definition, for most of the time you can think of modules as being vector spaces or Abelian groups and you will not be far wrong. We will shortly but briefly mention modules over a group algebra $R[G]$ and that ring is not commutative, but again the complications that this does cause will not worry us at all.)

We can thus 'represent' G by mapping it into the automorphism group of M. This gives M the structure of a G-module. We look for invariants of the action of G on M - what are they? Suppose that G is some group of symmetries of some geometric figure or pattern, that we will call X, in $\mathbb{R}^n$, then for each $g \in G$, $gX = X$, since g acts by pushing the pattern around back onto itself. An invariant of G, considered as acting on M, or, to put it more neatly, of the G-module, M, is an element m in M such that $g.m = m$ for all $g \in G$. These form a submodule

$$M^G = \{m \mid gm = m \text{ for all } g \in G\}.$$

Clearly it will help in our understanding of the structure of G if we can calculate and analyse these modules of invariants. Now suppose we are looking at a submodule N of M, then N^G is a submodule of M^G and we can hope to start finding invariants, perhaps by looking at such submodules and the corresponding quotient modules, M/N. We have a short exact sequence

$$0 \to N \to M \to M/N \to 0,$$

but, although applying the (functorial) operation $(-)^G$ does yield

$$0 \to N^G \to M^G \to (M/N)^G,$$

the last map need not be onto so we may not get a short exact sequence and hence a nice simple way of finding invariants!

Example Try $G = C_2 = \{1, a\}$, $M = \mathbb{Z}$ the Abelian group of integers, with G action $a.n = -n$, and $N = 2\mathbb{Z}$, the subgroup of even integers, with the same G action. Now calculate the invariant modules M^G and N^G; they are both trivial, but $M/N \cong Z_2$, and ... what is $(M/N)^G$ for this example?

The way of studying this in general is to try to to continue the exact sequence further to the right in some universal and natural way (via the theory of derived functors). This is what cohomology does. We can get a long exact sequence,

$$0 \to N^G \to M^G \to (M/N)^G \to H^1(G, N) \to H^1(G, M) \to H^1(G, M/N) \to H^2(G, N) \to \dots.$$

But what are these $H^k(G, M)$ and how does one get at them for calculation and interpretation? In fact what is cohomology in general?

Its origins lie within Algebraic Topology as well as in Group Theory and that area provides some useful intuitions to get us started, before asking how to form group cohomology.

1.2.3 Homology and Cohomology of spaces.

Naively homology and cohomology give methods for measuring the holes in a space, holes of different dimensions yield generators in different (co)homology groups. The idea is easily seen for graphs and low dimensional simplicial complexes.

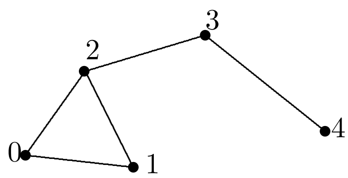
First we recall the definition of simplicial complex as we will need to be fairly precise about such objects and their role in relation to triangulations and related concepts.

Definition. A simplicial complex K is a set of objects, $V(K)$, called *vertices* and a set, $S(K)$, of finite non-empty subsets of $V(K)$, called *simplices*. The simplices satisfy the condition that if $\sigma \subset V(K)$ is a simplex and $\tau \subset \sigma$, $\tau \neq \emptyset$, then τ is also a simplex.

We say τ is a *face* of σ. If $\sigma \in S(K)$ has $p+1$ elements it is said to be a *p-simplex*. The set of p-simplices of K is denoted by K_p. The dimension of K is the largest p such that K_p is non-empty.

(We will sometimes use the notation $\mathcal{P}(X)$ for the power set of a set X, i.e. the set of subsets of X.)

When thinking about simplicial complexes, it is important to have a picture in our minds of a triangulated space (probably a surface or similar, a wireframe as in computer graphics). The simplices are the triangles, tetrahedra, etc., and are determined by their sets of vertices. Not every set of vertices need be a simplex, but if a set of vertices does correspond to a simplex then all its non-empty subsets do as well, as they give the faces of that simplex. Here is an example:



Here $V(K) = \{0,1,2,3,4\}$ and $S(K)$ consists of $\{0,1,2\}$, $\{2,3\}$, $\{3,4\}$ and all the non-empty subsets of these. Note the triangle $\{0,1,2\}$ is intended to be solid, (but I did not work out how to do it on the system I was using!)

Simplicial complexes are a natural combinatorial generalisation of (undirected) graphs. They not only have vertices and edges joining them, but also possible higher dimensional simplices relating paths in that low dimensional graph. It is often convenient to put a (total) order on the set $V(K)$ of vertices of a simplicial complex as this allows each simplex to be specified as a list $\sigma = \langle v_0, v_1, \ldots, v_n \rangle$ with $v_0 < v_1 < \ldots < v_n$, instead of as merely a set $\{v_0, v_1, \ldots, v_n\}$ of vertices. This, in turn, allows us to talk, unambiguously, of the k^{th} face of such a simplex, being the list with v_k omitted, so the zeroth face is $\langle v_1, \ldots, v_n \rangle$, the first is $\langle v_0, v_2, \ldots, v_n \rangle$ and so on.

Given two simplicial complexes K, L, then a function on the vertex sets, $f : V(K) \to V(L)$ is a *simplicial map* if it preserves simplices. (But that needs a bit of care to check out its exact meaning! ... for you to do. Look it up, or better try to see what the problem might be, try to resolve it your self and then look it up!)

1.2.4 Betti numbers and Homology

One of the first sorts of invariant considered in what was to become Algebraic Topology was the family of Betti numbers. Given a simple shape, the most obvious piece of information to note would be the number of 'pieces' it is made up of, or more precisely, the number of *components*. The idea is very well known, at least for graphs, and as simplicial complexes are closely related to graphs, we will briefly look at this case first.

For convenience we will assume the vertices $V = V(\Gamma)$ of a given finite graph, Γ, are ordered, so for each edge e of Γ, we can assign a source $s(e)$ and a target $t(e)$ amongst the vertices. Two vertices v and w are said to be in the same component of Γ if there is a sequence of edges $e_1, \ldots, e_k$ of Γ joining them[1]. There are, of course, several ways of thinking about this, for instance, define a relation $\sim$ on V by : for each e, $s(e) \sim t(e)$. Extend $\sim$ to an equivalence relation on V in the standard way, then $v \sim w$ if and only if they are in the same component. The zeroth Betti number, $\beta_0(\Gamma)$, is the number of components of Γ.

The first Betti number, $\beta_1(\Gamma)$, somewhat similarly, counts the number of cycles of Γ. We have ordered the vertices of Γ, so have effectively also directed its edges. If e is an edge, going from u to v, (so $u < v$ in the order on Γ_0), we write e also for the path going just along e and $-e$ for that going backwards along it, then extend our notation so $s(-e) = t(e) = v$, etc. Adding in these 'negative edges' corresponds to the formation of the symmetric closure of $\sim$. For the transitive closure we need to concatenate these simple one-edge paths: if e' is an edge or a 'negative edge' from v to w, we write $e + e'$ for the path going along e then e'. Playing algebraically with s and t and making them respect addition, we get a 'pseudo-calculation' for their difference $\partial = t - s$:

$$\partial(e + e') = t(e + e') - s(e + e') = t(e) + t(e') - s(e) - s(e') = t(e') - s(e) = u - w,$$

since $t(e) = v = s(e')$. In other words, defined in a suitable way, we would get that ∂, equal to 'target minus source', applies nicely to paths as well as edges, so that, for instance, two vertices would be related in the transitive closure of $\sim$ if there was a 'formal sum' of edges that mapped down to their 'difference'. We say 'formal sum' as this is just what it is. We will need 'negative vertices' as well as 'negative edges'.

We set this up more formally as follows: Let
$C_0(\Gamma)$ = the set of formal sums, $\sum_{v \in \Gamma_0} a_v v$ with $a_v \in \mathbb{Z}$, the additive group of integers, (an alternative form is to take $a_v \in \mathbb{R}$.;
$C_1(\Gamma)$ = the set of formal sums, $\sum_{e \in \Gamma_1} b_e e$ with $b_e \in \mathbb{Z}$,
where Γ_1 denotes the set of edges of Γ, and $\partial : C_1(\Gamma) \to C_0(\Gamma)$ defined by extending additively the mapping given on the edges by $\partial = t - s$.

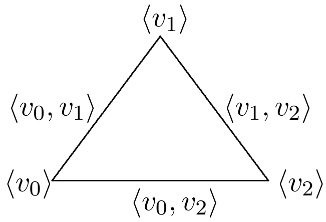
The task of determining components is thus reduced to calculating when integer vectors differ by the image of one in $C_1(\Gamma)$. The Betti number $\beta_0(\Gamma)$ is just the *rank* of the quotient $C_0(\Gamma)/Im(\partial)$, that is, the number of free generators of this commutative group. This would be exactly the dimension of this 'vector space' if we had allowed real coefficients in our formal sums not just integer ones.

Having reformulated components and $\sim$ in an algebraic way, we immediately get a pay-off in our determination of cycles. A cycle is a path which starts and ends at the same vertex; a path is

[1] In fact here, the ordering we have assumed on the vertices complicates the exposition a little but it is useful later on so will stick with it here.

being modelled by an element in $C_1(\Gamma)$, so a cycle is an element x in $C_1(\gamma)$ satisfying $\partial(x) = 0$. With this we have $\beta_1(\Gamma) = rank(Ker(\partial))$, a similar formulation to that for β_0. The similarity is even more striking if we replace the graph Γ by a simplicial complex K. We can then define in general and in any dimension p, $C_p(K)$ to be the commutative group of all formal sums $\sum_{\sigma \in K_p} a_\sigma \sigma$.

We next need to get an analogue of the $\partial = t - s$ formula. We want this to correspond to the boundary of the objects to which it is applied. For instance, if σ was the triangle / 2-simplex, $\langle v_0, v_1, v_2 \rangle$, we would want $\partial\sigma$ to be $\langle v_1, v_2 \rangle + \langle v_0, v_1 \rangle - \langle v_0, v_2 \rangle$, since going (clockwise) around the triangle, that cycle will be traced out:



If we write, in general, $d_i\sigma$ for the i^{th} face of a p-simplex $\sigma = \langle v_0, \ldots, v_p \rangle$, then in this 2-dimensional example $\partial\sigma = d_0\sigma - d_1\sigma + d_2\sigma$, changing the order for later convenience. This is the sum of the faces with weighting $(-1)^i$ given to $d_i\sigma$. This is consistent with $\partial = t - s$ in the lower dimension as $t = d_0$ and $s = d_1$. We can thus suggest that

$$\partial = \partial_p : C_p(K) \to C_{p-1}(K)$$

be defined on p-simplices by

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i d_i \sigma,$$

and then extended additively to all of $C_p(K)$.

As an example of what this does, look at a square K, with vertices v_0, v_1, v_2, v_3, edges $\langle v_i, v_{i+1} \rangle$ for $i = 0, 1, 2$ and $\langle v_0, v_2 \rangle$, and 2-simplices $\sigma_1 = \langle v_0, v_1, v_2 \rangle$ and $\sigma_2 = \langle v_0, v_2, v_3 \rangle$. As the square has these two 2-simplices, we can think of it as being represented by $\sigma_1 + \sigma_2$ in $C_2(K)$, then $\partial(\sigma_1 + \sigma_2) = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle - \langle v_0, v_3 \rangle$, as the two occurrences of the diagonal $\langle v_0, v_2 \rangle$ cancel out as they have opposite sign, and this is the path around the actual boundary of the square.

It is important to note that the boundary of a boundary is always trivial, that is, the composite mapping

$$C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} C_{p-2}(K)$$

is the mapping sending everything to $0 \in C_{p-1}(K)$.

The idea of the higher Betti numbers, $\beta_p(K)$ is that they measure the number of p-dimensional 'holes' in K. Imagine we has a tunnel-shaped hole through a space K, then we would have a cycle around the hole at one end of the tunnel and another around the hole at the other end. If we merely count cycles then we will get at least two such coming from this hole, but these cycles are linked as there is the cylindrical hole itself and that gives a 2 dimensional element with boundary the difference of the two cycles. In general a *p-cycle* will be an element x of $C_p(K)$ with trivial boundary, i.e., such that $\partial x = 0$, and we say that two p-cycles x and x' are *homologous* if there is an element y in $C_{p+1}(K)$ such that $\partial y = x - x'$. The 'holes' correspond to classes of homologous cycles as in our tunnel.

The number of 'independent' cycle classes in the various dimensions give the corresponding Betti number. Using some algebra this is easier to define rigorously, but at the same time the geometric insights from the vaguer description are important to try to retain. (They are not always put in a central enough position in textbooks!) This algebraic approach identifies $\beta_p(K)$ as the (torsion free) rank of a certain commutative group formed as follows: the p^{th} homology group of K is defined to be the quotient:

$$H_p(K) = \frac{Ker(\partial_p : C_p(K) \to C_{p-1}(K))}{Im(\partial_p : C_{p+1}(K) \to C_p(K))},$$

and then $\beta_p(K) = rank(H_p(K))$

Thus far we have from K built a sequence of modules $C(K)_n$, generated by the n-simplices of K and with homomorphisms $\partial_p : C_p(K) \to C_{p-1}(K)$ satisfying $\partial_{p-1}\partial_p = 0$.. (We abstract this structure calling it a *chain complex*. We will look at in more detail at several places later in these notes.)

Exercises: Try to investigate this homology in some very simple situations perhaps including some of the following:
(a) $V(K) = \{0, 1, 2, 3\}$, $S(K) = \mathcal{P}(V(K)) \setminus \{\emptyset, \{0, 1, 2, 3\}\}$. This is an empty tetrahedron so one expects one 3-dimensional hole., i.e. $\beta_3(K) = 1$ but the others are zero.
(b) $\Delta[2]$ is the (full) triangle and $\partial\Delta[2]$ its boundary, so is an empty triangle. Find the homology of $\partial\Delta[2] \times \partial\Delta[2]$, which is a triangulated torus.
(c) Find the homology of $\Delta[1] \times \partial\Delta[2]$, which is a cylinder.

Note , it is up to you to find the meaning of product in this context. Remember the discussion of the square, above, which is, of course $\Delta[1] \times \Delta[1]$.

Often cohomology is more use that homology. Starting with K and a module M work out $C^n(K, M) = Hom(C(K)_n, M)$. Now the boundary maps increase (upper) degree by one. The cohomology is $H^n(K, M) = Ker\, \partial^n / Im\, \partial^{n-1}$. Again this measures 'holes' detectable by M! What does that mean? The cohomology groups are better structured than the homology ones, but how are these invariants be interpreted?

A simplicial map $f : K \to L$ will induce a map on cohomology groups. Try it! We can equally well do this for chain or 'cochain complexes'. There is a notion of chain map between chain complexes, say, $\varphi : C \to D$ and such a map will induce maps on both homology ad cohomology. Of special interest is when the induced maps are isomorphisms. The chain map is then called a *quasi-isomorphism.*

1.2.5 Interpretation

The question of interpretation is a very crucial question, but, rather than answering it now, we will return to the cohomology of groups. The terminology may seem a bit strange. Here we have been talking about measuring holes in a space, so how does that relate to groups. The idea is that one builds a space from a group in such a way as the properties of the space reflect those of the group in some sense. The simplest case of this is an Eilenberg-MacLane space, $K(G, 1)$. The defining property of such a space is that its fundamental group is G whilst all other homotopy groups are trivial. Eilenberg and Maclane showed that however such a space was constructed its cohomology could be got just from G itself and that cohomology was related with the extension problem and the invariant module problem. Their method was to build a chain complex that would copy the structure of the chain complex on the $K(G, 1)$. This chain complex, *the bar resolution*, was very important because although in the group case there was an alternative route via the topological space $K(G, 1)$, for many other types of algebraic system (Lie algebras, associative algebras, commutative algebras, etc.), the analogous basic construction could be used, and in those contexts no space was available. Thus from G, we want to construct a nice chain complex directly. The construction is reasonably simple. It gives a natural way of getting a chain complex, but it does not exploit any particular features of the group so if the group is infinite, the modules will be infinitely generated, which will occupy us later, as we use insights from combinatorial group theory to construct smaller models for equivalent resolutions, and better still look at 'crossed' versions.

For the moment, we just need the definition (adapted from the account given in Wikipedia):

1.2.6 The bar resolution

In general, the basic idea of a *resolution* of an object is to replace the object of study, in our case a group, by an object that is in some category in which one can 'do' homotopy, such as that of chain complexes, so that the result is 'homotopy equivalent' to a thing corresponding to the group, that is, it models G well. This 'thing' usually plays a role a bit like the set of connected components of a space. Often the 'resolution' consists of free objects, but the terminology varies in different sources. This is a bit vague, so let us look at the bar resolution as an example.

The input data is a group G and a module M with a left G-action (i.e. a left G-module).

For $n \geq 0$, we let $C^n(G, M)$ be the group of all *functions* from the n-fold product G^n to M:

$$C^n(G, M) = \{\varphi : G^n \to M\}$$

This is an Abelian group; its elements are called the *n-cochains*. We further define group homomorphisms

$$\partial^n : C^n(G, M) \to C^{n+1}(G, M)$$

by

$$\begin{aligned} \partial^n(\varphi)(g_0, \ldots, g_n) \quad = \quad & g_0 \cdot \varphi(g_1, \ldots, g_n) \\ & + \sum_{i=0}^{n-1} (-1)^{i+1} \varphi(g_0, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ & + (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1}) \end{aligned}$$

These are known as the *coboundary homomorphisms.* The crucial thing to check here is $\partial^{n+1} \circ \partial^n = 0$, thus we have a chain complex and we can 'compute' its cohomology. For $n \geq 0$, define the group of n-cocycles as:

$$Z^n(G, M) = Ker\, \partial^n)$$

and the group of n-coboundaries as

$$\begin{cases} B^0(G, M) = 0 \\ B^n(G, M) = Im(\partial^{n-1}) \qquad n \geq 1 \end{cases}$$

and

$$H^n(G, M) = Z^n(G, M)/B^n(G, M).$$

Thinking about this topologically it is as if we had constructed a sort of space / simplicial complex K from G by taking $K_n = G^n$. We will see this idea several times later on. This cochain complex is often called the *bar resolution*. It exists in a normalised and a unnormalised form. This is the unnormalised one.

There are lots of properties that are easy to check here. Some will be suggested as exercises for you to do.

One further point is that this cohomology used a module, and so encodes 'commutative' or Abelian information. We will be also looking at the non-Abelian case.

Before we leave this introduction to cohomology, it should be mentioned that in the topological case, if we do not have a simplicial complex to start with, we either use the singular complex (see next section) which is a simplicial set and not a simplicial complex, but the theory extends easily enough, or we use open covers of the space to build a system of simplicial complexes approximating to the space. We will see this later as Čech cohomology. This is most powerful when the module M of coefficients is allowed to vary over the various points of the space. For this we will need the notion of sheaf, which will be discussed in some detail later.

1.3 Simplicial things in a category

1.3.1 Simplicial Sets

Simplicial objects are extremely useful. Simplicial sets extend ideas of simplicial complexes in a neat way. They combine a reasonably simple combinatorial definition with subtle algebraic properties. Their original construction was motivated in algebraic topology by the singular complex of a space.

If X is a topological space, $Sing(X)$ denotes the collection of sets and mappings defined by

$$Sing(X)_n = Top(\Delta^n, X), \qquad n \in \mathbb{N},$$

where Δ^n is the usual topological n-simplex given, for example, by

$$\{\underline{x} \in \mathbb{R}^{n+1} \mid \sum x_i = 1; \text{ all } x_i \geq 0\}.$$

There are inclusion maps $\delta_i : \Delta^{n-1} \to \Delta^n$, mapping Δ^{n-1} to the face of Δ^n opposite the i^{th} vertex, and 'squashing' maps $\sigma_i : \Delta^{n+1} \to \Delta^n$ and these induce the face maps

$$d_i : Sing(X)_n \to Sing(X)_{n-1} \qquad 0 \leq i \leq n,$$

and degeneracy maps

$$s_i : Sing(X)_n \to Sing(X)_{n+1} \qquad 0 \leq i \leq n.$$

These satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i \qquad \text{if } i < j,$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ id & \text{if } i = j \text{ or } j+1, \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases}$$

$$s_i s_j = s_j s_{i-1} \qquad \text{if} \quad i > j.$$

Generally this structure is abstracted to give a family of sets, $\{K_n : n \geq 0\}$, face maps $d_i : K_n \to K_{n-1}$ and degeneracy maps, $s_i : K_n \to K_{n+1}$, satisfying these simplicial identities.

If $\mathcal{C}$ is any category, a simplicial object in $\mathcal{C}$ is given by a family of objects of $\mathcal{C}$, $\{K_n : n \geq 0\}$ and *morphisms* d_i and s_i as above. If $\mathbf{\Delta}$ denotes the category of finite ordinal sets, $[n] = \{0 < 1 < \ldots < n\}$ and order preserving functions between them, a special place being attributed to the coface functions, $\delta_i^n : [n-1] \to [n]$, and codegeneracies, $\sigma_i^n : [n] \to [n-1]$. The coface δ_i^n misses out the element i, $0 \leq i \leq n$, but is 'otherwise surjective', whilst σ_i^n maps both i and $i+1$ to i, but is otherwise injective. Given this, a simplicial object in $\mathcal{C}$ is simply a functor, $K : \mathbf{\Delta}^{op} \to \mathcal{C}$, with $K_n = K[n]$, $d_i^n = K(\delta_i^n)$ and $s_i^n = K(\sigma_i^n)$, so the obvious definition of a simplicial map will be a natural transformation of functors, $f : K \to L$. This translates as a family of morphisms, $f_n : K_n \to L_n$, compatible in the obvious way with the d_i and s_i.

We denote the category of simplicial objects in $\mathcal{C}$ by $Simp(\mathcal{C})$ or $Simp.\mathcal{C}$, but will shorten $Simp(Sets)$ to $\mathcal{S}$.

The category $\mathcal{S}$ models all homotopy types of spaces. It is a presheaf category, so is a topos and has a lot of nice structure including products, and mapping space objects $\underline{\mathcal{S}}(K, L)$, where

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

Here $\Delta[n] = \mathcal{S}(-, [n])$, the standard simplicial n-simplex. This has a special n-simplex, namely the element ι_n, in $\Delta[n]_n$ determined by the identity map.

The Yoneda lemma, from category theory, gives us an isomorphism $\mathcal{S}(\Delta[n], K) \cong K_n$, and so, for any n-simplex, x, gives us a simplicial map $\ulcorner x \urcorner : \Delta[n] \to K$, which is sometimes called the *name* of x From $\ulcorner x \urcorner$, you get x back by evaluating on $\ulcorner x \urcorner$ on ι_n.

Examples of simplicial sets.

(i) If $\mathcal{A}$ is a small category or a groupoid, we can form a simplicial set, $Ner(\mathcal{A})$, defined by $Ner(\mathcal{A})_n = Cat([n], \mathcal{A})$, with the obvious face and degeneracy maps induced by composition with the analogues of the δ_i and σ_i. The simplicial set, $Ner(\mathcal{A})$, is called the *nerve of the category* $\mathcal{A}$. An n-simplex in $Ner(\mathcal{A})$ is a sequence of n composable arrows in $\mathcal{A}$.

This is easier to understand in pictures:

$Ner(A)_0$ is the set of objects;

$Ner(A)_1$ is the set of arrows or morphisms;

$Ner(A)_2$ is the set of composable pairs of morphisms, so $\sigma \in Ner(\mathcal{A})_2$ will be of form $\sigma = (a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2)$. Visualising this as a triangle shows the faces more clearly:

$$\begin{tikzcd} & a_1 \arrow[dr, "\alpha_2"] & \\ a_0 \arrow[ur, "\alpha_1"] \arrow[rr, "\alpha_1\alpha_2"'] & & a_2 \end{tikzcd}$$

The case $Ner(A)_n$ for $n = 3$, etc. are left to you. This *is* worth doing if you have not seen it before.

Note that in these contexts, we will sometimes use composition in the 'left-to-right' order, but in general categorical settings will use gf being first do f then g. To stick exclusively to one or the other is usually awkward, so we use both as appropriate. This sometimes means we have to take extra care over the conventions we are using at a particular time.

If we have a group G, consider it as the one object groupoid $G[1]$ as before, then $Ner(G[1])$ is really the simplicial set corresponding to our construction of the bar resolution of G. It is called the *nerve of* G, and is a *classifying space* for G, an aspect that we will explore later in some detail.

(ii) Suppose we have a simplicial complex K, then it almost is a simplicial set. There are some problems, but they are easily resolved. If we, a bit naïvely, set K_n to be the set of n-simplices of K, then how are we to define the face maps, and if K has no simplices in dimensions greater than n say, K_{n+1} will be empty so degeneracies cause problems as you cannot map from a non-empty set to an empty one!

That was too naïve, so we pick a partial order on the vertices of K such that any simplex is totally ordered, (for instance, a total order on $V(K)$ does the job, but may not be convenient sometimes and so may be 'overkill'). Now reset K_n to be the set of all ordered strings $\sigma = \langle x_0, \ldots, x_n\rangle$ of vertices, for which the underlying (unordered) set is a simplex of K. The degeneracies now can be handled simply. For example, if $\sigma = \langle x_0, x_1\rangle$ is a 1-simplex in this simplicial set, then $s_0\sigma = \langle x_0, x_0, x_1\rangle$, whilst $s_1\sigma = \langle x_0, x_1, x_1\rangle$. (The details are left to you to complete. Note we did not specify how to define the face maps, so you need to do that as well and to verify that it all fits together neatly.)

There is sometimes a need to replace a simplicial object by another one that is closely related. This corresponds, in the case of a simplicial complex considered as a simplicial set, to reversing the total order used, and more generally, corresponds, at the level of the category, $\mathbf{\Delta}$, to a set function reversing the order of the finite ordinals. The effect of this on simplicial sets is what we will call *conjugation*.

Definition: Given a simplicial set, X, the *conjugate of* X, denoted $Conj\, X$, is defined by

$$(Conj\, X)_n = X_n,$$

$$d_i^n : (Conj\, X)_n \to (Conj\, X)_{n-1} = d_{n-i}^n : X_n \to X_{n-1}$$

and

$$s_i^n : (Conj\, X)_n \to (Conj\, X)_{n+1} = s_{n-i}^n : X_n \to X_{n+1}.$$

To check this is does give one a simplicial set is simple and so is **left to you**: it is also clear that $(Conj)^2X = X$. However, there is in general no isomorphism $X \to Conj\, X$. In fact, in general, the only morphisms between a simplicial set and its conjugate will be the trivial morphisms taking X to some 'point' in $Conj\, X$, where by a 'point' we mean the subsimplicial set generated by a single vertex. Clearly the two simplicial sets are geometrically equivalent in some sense, just as the category $\mathbf{\Delta}$, with the order inverted is 'essentially the same' category as $\mathbf{\Delta}$; however, there is no functor between them to reflect this fact.

If you want, or need, to learn more about simplicial set theory, as such, the old paper of Curtis, [43] and Peter May's monograph, [89], are very readable. There is a fairly well behaved notion of homotopy in $\mathcal{S}$, and simplicial homotopy theory is the subject of many good books. A chatty introduction to it can be found in Kamps and Porter, [76], which, of course, is highly recommended!

The homotopy theory of simplicial sets yields a notion of weak equivalence. (This is similar to 'quasi-isomorphism' in the homotopy theory of chain complexes.) There are homotopy groups and

$f : K \to L$ is a weak equivalence if f induces isomorphisms on all homotopy groups. We will not need the detailed definition yet.

We next look at some simplicial algebraic gadgets, especially simplicial groups and simplicially enriched groupoids. Initially we will concentrate on the first, but must mention the second for completeness and for later use.

1.3.2 Simplicial Objects in Categories other than *Sets*

If $\mathcal{A}$ is any category, we can form $Simp.\mathcal{A} = \mathcal{A}^{\mathbf{\Delta}^{op}}$. (Sometimes we will use a variant notation: $Simp(\mathcal{A})$, as occasionally the first notation may be ambiguous.)

These categories often have a good notion of homotopy as briefly mentioned above; see also the discussion of simplicially enriched categories in [76]. Of particular use are:

(i) *Simp.Ab*, the category of simplicial Abelian groups. This is equivalent to the category of chain complexes by the Dold-Kan theorem, which we will mention in more detail later.

(ii) *Simp.Grps*, the category of simplicial groups. This 'models' all connected homotopy types, by Kan, [77] (cf., Curtis, [43]). There are adjoint functors $G : \mathcal{S}_{conn} \to Simp.Grps$, $\overline{W} : Simp.Grps \to \mathcal{S}_{conn}$, with the two natural maps $G\overline{W} \to Id$ and $Id \to \overline{W}G$ being weak equivalences.

Results on simplicial groups by Carrasco, [36], generalise the Dold-Kan theorem to the non-Abelian case, (cf., Carrasco and Cegarra, [37]).

(iii) '*Simp.Grpds*': in 1984 Dwyer and Kan, [51], (and also Joyal and Tierney, and Duskin and van Osdol, cf., Nan Tie, [101, 102]) noted how to generalise the $(G, \overline{W})$ adjoint pair to handle all simplicial sets, not just the connected ones. (Beware there are several important printing errors in the paper [51].) For this they used a special type of simplicial groupoid. Although the term used in [51] was exactly that, 'simplicial groupoid', this is really a misnomer and may give the wrong impression, as not all simplicial objects in the category of groupoids are used. A probably better term would be 'simplicially enriched groupoid', although 'simplicial groupoid with discrete objects' is also used. We will denote this category by $\mathcal{S}-Grpds$.

This category 'models' all homotopy types using a mix of algebra and combinatorial structure. We will later describe both G and $\overline{W}$ in some detail, and will use simplicially enriched groupoids and simplicially enriched categories as well.

(iv) *Nerves of internal categories:* Suppose that $\mathcal{D}$ is a category with finite limits and C is an *internal categoryinternal category* in $\mathcal{D}$. What does that mean? In our earlier discussion on groupoids, we had the diagram that looked a bit like

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=2, "i"] \end{tikzcd} .$$

We complete this one stage to build in the set of composable pairs $C_2 = C_1 \times_{C_0} C_1$ and the multiplication/ composition map, which we denote here by m.

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "p_1"] \arrow[r, "m"] \arrow[r, shift right=2, "p_2"'] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & C_0 \arrow[l, shift left=3, "i"] \end{tikzcd} .$$

We did this previously within the category of sets, but could do it equally well in $\mathcal{D}$. We should also mention an object C_3 given by a 'triple pullback' which is useful when discussing the associativity of composition. This will give us the analogue of a small category, but in which the object of objects and the object of arrows are both themselves objects of $\mathcal{D}$ and the source target and composition maps are all morphisms in that category.

If one interprets this for $\mathcal{D} = Sets$, it becomes clear that this diagram that we seem to be building is part of the diagram specifying the nerve of the small category, C, with C_0 the set of objects, C_1 that of morphisms, C_2 that of composable pairs and so on. (We have not specified the two degeneracies from C_1 to C_2 in the diagram, but this is merely because we left the details of the rules governing identities out of our earlier discussion.) This builds a simplicial object in $\mathcal{D}$ as follows: take an n-fold pullback to get

$$C_n = \underbrace{C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} \ldots \times_{C_0} C_1}_{n},$$

define face and degeneracies by the same sort of rules as in the set based nerve, that is, in dimension n, d_0 and d_n each leave out an end, whilst the d_i use the composition in the category to get a composite of two adjacent 'arrows', and the degeneracies are 'insertion of identities'. (Working out how to do these morphisms in terms of diagrams is quite fun!) We thus get a simplicial object in $\mathcal{D}$ called the *nerve of the internal category, C*. We will use this in several situations later in a key way. In particular we will use the case $\mathcal{D} = Grps$.

(v) *Bisimplicial and multisimplicial objects:* A useful category in which we can take simplicial objects is $\mathcal{S}$ itself, and the same is true for other categories of form $Simp(\mathcal{A})$. For simplicity we will start by looking at simplicial objects in $\mathcal{S}$.

As a simplicial object in a category $\mathcal{A}$ is just a functor from $\mathbf{\Delta}^{op}$ to $\mathcal{A}$, a simplicial object in $\mathcal{S}$ is such a functor taking values that themselves are functors from $\mathbf{\Delta}^{op}$ to $Sets$. Another way to look at these is a 'functor of two variables' using a categorical version of the way that a function of two variables, $f : X \times Y \to Z$, can be thought of as a function $\tilde{f} : X \to Z^Y$ from X to the set of functions from Y to Z. Of course, $f(x, y) = \tilde{f}(x)(y)$ and similarly for the functors. We thus have a description of a simplicial object in $\mathcal{S}$ as corresponding to a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Sets$.

Definition: A *bisimplicial set* is a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to Sets$. . A *morphism of bisimplicial sets*, $f : X \to Y$ is a natural transformation between the corresponding functors. More generally a *bisimplicial object in a category* $\mathcal{A}$ is a functor $X : \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathcal{A}$, similarly for the corresponding morphisms. The corresponding categories will denoted $Bi\mathcal{S} := BiSimp(Sets)$ and in general $BiSimp(\mathcal{A})$.

A simplicial set can be specified by giving sets X_n and face and degeneracy 'operators' between them satisfying the simplicial idenities. A bisimplicial set is similarly specified by a bi-indexed family of sets $X_{p,q}$ and two families of simplicial operators. We may use the terms 'horizontal' and 'vertical' for these two families as that is how the corresponding diagrams are often drawn. For instance, the bottom part of a bisimplicial set will look a bit like the following:

$$\begin{tikzcd}
 & \vdots \arrow[d, shift left=2, "d_0^v"] \arrow[d] \arrow[d, shift right=2, "d_2^v"'] & \vdots \arrow[d, shift left=2, "d_0^v"] \arrow[d] \arrow[d, shift right=2, "d_2^v"'] \\
\cdots \arrow[r, shift left=2, "d_0^h"] \arrow[r] \arrow[r, shift right=2, "d_2^h"'] & X_{1,1} \arrow[r, shift left, "d_0^h"] \arrow[r, shift right, "d_1^h"'] \arrow[d, shift left, "d_0^v"] \arrow[d, shift right, "d_1^v"'] & X_{0,1} \arrow[d, shift left, "d_0^v"] \arrow[d, shift right, "d_1^v"'] \\
\cdots \arrow[r, shift left=2, "d_0^h"] \arrow[r] \arrow[r, shift right=2, "d_2^h"'] & X_{1,0} \arrow[r, shift left, "d_0^h"] \arrow[r, shift right, "d_1^h"'] & X_{0,0}
\end{tikzcd}$$

(As usual in such diagrams, there is not really room to show the degeneracy maps and so these are omitted from the picture.) In addition to the simplicial identities holding in each direction, each horizontal face or degeneracy has to be a simplicial map between the vertical simplicial sets. Practically this means that the diagram must commute.

We will later meet bisimplicial groups, and also briefly multisimplicial objects in which the number of variables is not limited to two. For instance, the nerve of a simplicial group is most naturally viewed as a bisimplicial set, and similarly the nerve of a bisimplicial group is a trisimplicial set, that is a functor from $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$ to *Sets*. There are ways of passing between such things as we will see later.

Perhaps the simplest example of bisimplicial sets is that of the bisingular complex of a space, X. This has in 'bidegree' (p, q) the set of continuous maps from $\Delta^p \times \Delta^q$ to X and with the two types of simplicial operators corresponding to composition in the first or second variable. We will see later that this type of construction is very useful when thinking about products and pairings in cohomology.

(vi) *Cosimplicial things:* At certain points in the development of cohomology and related areas we will have need to talk of cosimplicial sets.

Definition: A *cosimplicial set* is a functor $K : \mathbf{\Delta} \to Sets$, and a morphism of such is a natural transformation between the corresponding functors. The category of such will be denoted $CoSimp(Sets)$, and similarly for the obvious generalisations to other settings, namely *cosimplicial objects in a category* $\mathcal{A}$, being functors $K : \mathbf{\Delta} \to \mathcal{A}$ with the corresponding morphisms being natural transformations and forming a category $CoSimp(\mathcal{A})$.

This looks at one and the same time very similar and very different to simplicial objects. Certainly analysis of, say, simplicial groups is much easier than that of cosimplicial groups, but, as any functor, $K : \mathbf{\Delta} \to \mathcal{A}$, gives uniquely a functor, $K^{op} : \mathbf{\Delta}^{op} \to \mathcal{A}^{op}$, a cosimplicial object is also a simplicial object in the opposite category. The problem, thus, is that often the opposite category of a well known category, such as that of groups, is a lot less nice. Even the dual of *Sets* is not that 'well behaved'. None-the-less the identification of cosimplicial objects as simplicial objects in a dual category usually suggests how to construct what is needed in a context.

We will see explicit examples of cosimplicial objects later. Of course, we may also need to mix 'cosimplicial' and 'simplicial' in both simple and 'multi-' forms. Cosimplicial simplicial sets being the main use for such mixes that we will see.

1.3.3 The Moore complex and the homotopy groups of a simplicial group

Given a simplicial group G, *the Moore complex*, (NG, ∂), *of* G is the chain complex defined by

$$NG_n = \bigcap_{i=1}^{n} Ker\, d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from d_0^n by restriction. (Note there is no assumption that the NG_n are Abelian.

The n^{th} *homotopy group*, π_n(G), of G is the n^{th} homology of the Moore complex of G, i.e.,

$$\begin{aligned} \pi_n(G) &\cong H_n(NG, \partial), \\ &= \big(\bigcap_{i=0}^{n} Ker\, d_i^n\big)/d_{n+1}^{n+1}\big(\bigcap_{i=0}^{n} Ker\, d_i^{n+1}\big). \end{aligned}$$

(You should check that $\partial NG_{n+1} \lhd NG_n$.) The interpretation of NG and $\pi_n(G)$ is as follows:

for $n = 1$, $g \in NG_1$,

$$\partial g \bullet \xrightarrow{\ g\ } \bullet 1$$

and $g \in NG_2$ looks like

$$\begin{tikzcd} & \bullet \arrow[dr, "\partial g"] \arrow[d, phantom, "g"] & \\ \bullet \arrow[ur, "1"] \arrow[rr, "1"'] & & \bullet \end{tikzcd}$$

and so on.

We note that $g \in NG_2$ is in $Ker\, \partial$ if it looks like

$$\begin{tikzcd} & \bullet \arrow[dr, "1"] \arrow[d, phantom, "g"] & \\ \bullet \arrow[ur, "1"] \arrow[rr, "1"'] & & \bullet \end{tikzcd}$$

whilst it will give the trivial element of $\pi_2(G)$ if there is a 3-simplex x with g on its third face and all other faces identity.

This simple interpretation of the elements of NG and $\pi_n(G)$ will 'pay off' later by aiding interpretation of some of the elements in other situations. The homotopy groups we have introduced above have been defined purely algebraically as homology of a related complex. Any simplicial group gives us a base pointed simplicial set simply by forgetting the group structure and taking the identity element as the base point. Any pointed simplicial set gives homotopy groups in two different ways. There is an intrinsic way that is described in detail in, for instance, May's book, [89], but they can also be defined via a geometric realisation, which produces a space from the simplicial set. These two ways always give the same answer, and in the case that we are looking at of an underlying simplicial set of a simplicial group, this group coincides with that defined via the Moore complex. (This is easily found in the literature if you want to check up on it.)

n-equivalences and homotopy n-types Let $n \geq 0$. A morphism $f : G \to H$ of simplicial group(oid)s is an *n-equivalence* if the induced homomorphisms, $\pi_k(f) : \pi_k(G) \to \pi_k(H)$ are isomorphisms for all $k < n$.

Inverting the n-equivalences in *Simp.Grps* gives a category $Ho_n(Simp.Grps)$ and two simplicial groups have the same *n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Remark and warning: For a space or simplicial set K, $\pi_k(K) \cong \pi_{k-1}(\mathcal{G}(K))$, so these simplicial group n-types correspond to restrictions on $\pi_k(K)$ for $k \leq n$ in the spatial context.

To consider the application of this to homotopical and homological algebra, we will also need the following:

Definitions: (i) A simplicial group, G, is *augmented* by specifying a constant simplicial group $K(G_{-1}, 0)$ and a surjective group homomorphism, $f = d_0^0 : G_0 \to G_{-1}$ with $fd_0^1 = fd_1^1 : G_1 \to G_{-1}$. An *augmentation* of the simplicial group G is then a map

$$G \longrightarrow K(G_{-1}, 0),$$

where $K(G_{-1}, 0)$ is the constant simplicial group with value G_{-1}.

(ii) An augmented simplicial group, (G, f), is *acyclic* if the corresponding complex is acyclic, i.e., $H_n(NG) \cong 1$ for $n > 0$ and $H_0(NG) \cong G_{-1}$.

Remarks: (i) The above notions are just particular instances of the general notion of an *augmented simplicial object* in a category, and the corresponding idea of *acyclic* such things in settings where the definition makes sense.

(ii) When considering augmented simplicial objects, we sometimes use the notation d_0 or d_0^0 for the augmentation map as then the condition $fd_0^1 = fd_1^1$ becomes $d_0d_0 = d_0d_1$, which is a natural extension of the simplicial identities.

1.3.4 Kan complexes and Kan fibrations

Within the category of simplicial sets, there is an important subcategory determined by those objects that satisfy the Kan condition, that is the *Kan complexes*.

As before we set $\Delta[n] = \mathbf{\Delta}(-, [n]) \in \mathcal{S}$, then, for each i, $0 \leq i \leq n$, we can form, within $\Delta[n]$, a subsimplicial set, $\Lambda^i[n]$, called the *(n, i)-horn* or *(n, i)-box*, by discarding the top dimensional n-simplex (given by the identity map on $[n]$) and its i^{th} face. We must also discard all the degeneracies of those simplices.

By an *(n, i)-horn* or *box* in a simplicial set K, we mean a simplicial map $f : \Lambda^i[n] \to K$. Such a simplicial map corresponds intuitively to a family of n simplices of dimension $(n - 1)$, fitting together to form a 'funnel' or 'empty horn' shaped subcomplex within K. The family is thus a sequence, $(k_0, \ldots, k_{i-1}, -, k_{i+1}, \ldots, k_n)$, with each $k_\ell \in K_{n-1}$, satisfying $d_\ell k_j = d_{j-1}k_\ell$, for $\ell < j$, whenever both k_ℓ and k_j are in the sequence. The idea is that a Kan fibration of simplicial sets is a map in which the horns in the domain can be 'filled' if their images in the codomain can be. More formally:

Definition: A map $p : E \to B$ is a *Kan fibration* if, for any n, i as above, given any (n, i)-horn in E, specified by a map $f_1 : \Lambda^i[n] \to E$, together with an n-simplex, $f_0 : \Delta[n] \to B$, such that

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

commutes, then there is an $f : \Delta[n] \to E$ such that $pf = f_0$ and $f.inc = f_1$, i.e., f lifts f_0 and extends f_1.

We also say that p satisfies the *Kan lifting condition* if this is true.

Definition: A simplicial set, K, is a *Kan complex* if the unique map $K \to \Delta[0]$ is a Kan fibration. This is equivalent to saying that every horn in K has a filler, i.e., any $f_1 : \Lambda^i[n] \to Y$ extends to an $f : \Delta[n] \to Y$.

Singular complexes, $Sing(X)$, and the simplicial mapping spaces, $\underline{Top}(X, Y)$, are always Kan complexes.

Lemma 1 *The nerve of a category, C, is a Kan complex if and only if the category is a groupoid.* ∎

The proof is left to **the reader**.

This is very important as the filler structure involves compositions and inverses, so encodes the *algebraic* structure of C. Later we will use this many times, sometimes explicitly, but often it will be giving structure behind the scenes, for instance, internally within some other category.

If G is a simplicial group, then its underlying simplicial set is a Kan complex. Moreover, given a box in G, there is an algorithm for filling it using products of degeneracy elements. A form of this algorithm is given below. More generally if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration.

The following description of the algorithm is adapted from May's monograph, [89], page 67.

Proposition 1 *Let G be a simplicial group, then every box has a filler.*

Proof: Let $(y_0, \ldots, y_{k-1}, -, y_{k+1}, \ldots, y_n)$ give a horn in G_{n-1}, so the y_is are $(n-1)$ simplices that fit together as if they were all but one, the k^{th} one, of the faces of an n-simplex. There are three cases:

(i) $k = 0$: Let $w_n = s_{n-1}y_n$ and then $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$ for $i = n, \ldots, 1$, then w_1 satisfies $d_iw_1 = y_i$, $i \neq 0$;

(ii) $0 < k < n$: Let $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, k-1$, then take $w_n = w_{k-1}(s_{n-1}d_nw_{k-1})^{-1}s_{n-1}y_n$, and finally a downwards induction given by $w_i = w_{i+1}(s_{i-1}d_iw_{i+1})^{-1}s_{i-1}y_i$, for $i = n, \ldots, k+1$, then w_{k+1} gives $d_iw_{k+1} = y_i$ for $i \neq k$;

(iii) the third case, $k = n$ uses $w_0 = s_0y_0$ and $w_i = w_{i-1}(s_id_iw_{i-1})^{-1}s_iy_i$ for $i = 0, \ldots, n-1$, then w_{n-1} satisfies $d_iw_{n-1} = y_i$, $i \neq n$. ∎

Some discussion of how you can think of this algorithm can be found in [76].

(You could see if you can adapt the idea of this proof to prove the result mentioned immediately before the statement, namely: if $f : G \to H$ is an epimorphism of simplicial groups, then the underlying map of simplicial sets is a Kan fibration. What about the converse?)

Later on we will meet the simplicial mapping space, $\underline{\mathcal{S}}(K, L)$, of simplicial maps from K to L. It is defined by $\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L)$, with the obvious induced maps. It is easy to see that if L is a Kan complex, then so is $\underline{\mathcal{S}}(K, L)$, for any K. (Try to prove it, but then look at May, [89], to compare your attempt with his proof.)

1.3.5 T-complexes

There is quite a difference between the Kan complex structure of the nerve of a groupoid, G, and that of a singular complex. In the first, if we are given a (n, i)-horn, then there is *exactly one* n-simplex in $Ner(G)$, since the (n, i)-horn has a chain of n-composable arrows of G in it (at least unless $(n, i) = (2, 0)$ or $(2, 2)$, which cases are **left to you**) and that chain gives the required n-simplex. In other words, there is a 'canonical' filler for any horn. In $Sing(X)$, there will usually be many fillers. (Think about why this is true.)

One attempt to handle 'canonical fillers' interacts with a notion that we will encounter later on, namely that of crossed complexes, for which see section 3.1. The resulting notion of a simplicial T-complex is one sort of 'Kan complex with canonical fillers' and various of the intuitions and arguments that this introduces will recur frequently in the following chapters. It assumes there is always a unique special filler. There may be other non-special ones, but that is not controlled in the process, as we will see. Simplicial T-complexes were introduced by Dakin, [**?**]:

Definition: A *simplicial T-complex* consists of a pair (K, T), where K is a simplicial set and $T = (T_n)_{n\geq 1}$ is a graded subset of K with $T_n \subseteq K_n$. Elements of T are called *thin*. The thin structure satisfies the following axioms:

$T.1$ Every degenerate element is thin.

$T.2$ Every box in K has a unique thin filler.

$T.3$ A thin filler of a thin box also has its last face thin.

Example: The nerve of a groupoid has a T-complex structure in which each simplex of dimension greater than or equal to 2 is thin. Our earlier comments give the proof. Conversely, if (K, T) is a T-complex with $T_n = K_n$ for all $n \geq 2$, then K is the nerve of a groupoid with set of objects K_0 and set of arrows, K_1. (It is **left to you** to see how to compose arrows, to prove that it is an associative composition, and that there are identities at all objects.)

A box or horn is, of course, as in section 1.3.4, a collection of n-simplices that fits together like the collection of all but one faces of an $(n + 1)$-simplex. The collection of such n-boxes with given face missing can be formulated in terms of a pullback and hence axioms $T2$ and $T3$ can be encoded in a form suitable for adapting to other contexts. Similar ideas are used by Duskin, [49], and Nan-Tie, [101, 102], and we will have occasion to refer back to these later. We will need to adapt those ideas initially to T-complexes within the setting of groups (group T-complexes as below) but later we may need them in various other settings. Group T-complexes were briefly considered by

Ashley, [7], but their main theory has been clarified and extended by Carrasco, [36], and Cegarra and Carrasco, [37], using ideas that will be discussed briefly later.

1.3.6 Group T-complexes

Definition: A *group T-complex* is "a T-complex (G,T) in which G is a simplicial group and T is a graded subgroup of G", (Ashley, [7]).

Ashley proved a series of results that gave a neat alternative formulation of this concept. We note the following observations:

Lemma 2 *Let $D=(D_n)_{n\geq 1}$ be the graded subgroup of G generated by the images of the degeneracy maps, $s_i : G_n \to G_{n+1}$, for all i and n, then any box in G has a standard filler in D.*

Proof: In fact, the algorithmic formulae used when proving that any simplicial group is a Kan complex (cf., Proposition 1) give a filler defined as a product of degenerate copies of the faces of the box. ■

Proposition 2 *If (G,T) is a group T-complex then $T = D$.*

Proof: To see this, we note that axiom $T1$ implies that $D \subseteq T$. Conversely if $t \in T_n$, then it fills the box made up of $(\_, d_1 t, \ldots, d_n t)$. This, in turn, has a filler, d, in D, but, as this filler is also thin, it must be that $t = d$, since thin fillers are uniquely determined ($T2$). ■

This is neat since it says there is essentially at most one group T-complex structure on any given simplicial group. The next results says when such a structure does exist.

Theorem 1 *(Ashley, [7]) If G is a simplicial group, then (G,D) is a group T-complex if and only if $NG \cap D$ is the trivial graded subgroup.*

Proof: One way around, this is nearly trivial. If (G,D) is a group T-complex and $x \in NG_n$, then x fills a box $(\_,1,\ldots,1)$, so if $x \in NG_n \cap D_n$, x must itself be the thin filler, however 1 is also a thin filler for this box, so $x = 1$ as required.

Conversely if $NG \cap D = \{1\}$, then we must check $T2$ and $T3$, $T1$ being trivial. As any box has a standard filler in D, we only have to check uniqueness, but if x and y are in D_n, and both fill the same box (with the k^{th} face missing) then $z = xy^{-1}$ fills a box with 1s on all faces (and the k^{th} face missing).

If $k = 0$, then as $z \in NG_n \cap D_n$, we have $z = 1$ and x and y are equal. If $k > 0$, assume that if $\ell < k$ and $z \in D_n \cap \bigcap_{i\neq \ell} Ker\, d_i$ then $z = 1$, (i.e, that we have uniqueness up to at least the $(k-1)^{st}$ case). Consider $w = z s_{k-1} d_k z^{-1}$. This is still in D_n and $d_i w = 1$ unless $i = k-1$, hence by assumption $w = 1$. Of course, this implies that $z = s_{k-1} d_k z$, but then $d_{k-1} z = d_k z$. We know that $d_{k-1} z = 1$, so $d_k z = 1$ and $z = 1$, i.e., $x = y$ and we have uniqueness at the next stage.

To verify $T3$, assume that $x \in D_{n+1}$ and each $d_i x \in D_n$ for $i \neq k$, then we can assume that $k = 0$, since otherwise we can skew the situation around as before to get that to be true, verify it in that case and 'skew' it back again later. Suppose therefore that $d_i x \in D_n$ for all $0 < i < n$. As x must be the degenerate filler given by the standard method, we can calculate x as follows: let

$w_n = s_{n-1}d_n x$, $w_i = w_{i+1}(s_{i-1}d_i w_{i+1})^{-1}s_{i-1}y_i$ for $i = 1$, then $x = w_1$. We can therefore check that $d_0 x \in D_n$ as required. ■

Remark: Ashley, [7], in fact assumes a seemingly stronger conclusion, namely that $D_n \cap \bigcup_{\ell=0}^{n}(\bigcap_{i\neq\ell} Ker\, d_i) = 1$. The reduction to the single case is noted by Carrasco, [36].

Thus a group T-complex is a simplicial group in which the Moore complex contains no non-trivial product of degenerate elements.

It is often useful to have a 'dimensionwise' terminology in the following sense. We could say that a group T-complex satisfies the *thin filler condition* or simply, the *T-condition*, in all dimensions. That suggests that we extract that condition 'dimensionwise' as follows:

Definition: A simplicial group G satisfies the *thin filler condition in dimension n* if $NG_n \cap D_n$ is trivial. We may abbreviate that to *T-condition in dimension n*.

This terminology lends itself well to such variants as 'G satisfies the *thin filler condition in dimensions greater that k*' meaning that $NG_n \cap D_n$ is trivial for all $n > k$, and so on.

Chapter 2

Crossed modules - definitions, examples and applications

We will give these for groups, although there are analogues for many other algebraic settings.

2.1 Crossed modules

Definition: A *crossed module*, (C, G, δ), consists of groups C and G with a left action of G on C, written $(g, c) \to {}^{g}c$ for $g \in G$, $c \in C$, and a group homomorphism $\delta : C \to G$ satisfying the following conditions:
CM1) for all $c \in C$ and $g \in G$,

$$\delta({}^{g}c) = g\delta(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$,

$${}^{\delta(c_2)}c_1 = c_2c_1c_2^{-1}.$$

(CM2 is called the *Peiffer identity.*)

If (C, G, δ) and (C', G', δ') are crossed modules, a *morphism*, $(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$, *of crossed modules* consists of group homomorphisms $\mu : C \to C'$ and $\eta : G \to G'$ such that
(i) $\delta'\mu = \eta\delta$ and (ii) $\mu({}^{g}c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$, $g \in G$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted $CMod$.

Several well known situations give rise to crossed modules. The verification is left to you.

2.1.1 Algebraic examples of crossed modules

(i) Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion, then we will say (H, G, i) is a *normal subgroup pair.* In this case, of course, G acts on the left of H by conjugation and the inclusion homomorphism i makes (H, G, i) into a crossed module, an 'inclusion crossed modules'. Conversely it is an easy exercise to prove

Lemma 3 *If (C, G, ∂) is a crossed module, ∂C is a normal subgroup of G.* ■

(ii) Suppose G is a group and M is a left G-module; let $0 : M \to G$ be the trivial map sending everything in M to the identity element of G, then $(M, G, 0)$ is a crossed module.

Again conversely:

Lemma 4 *If (C, G, ∂) is a crossed module, $K = Ker\,\partial$ is central in C and inherits a natural G-module structure from the G-action on C. Moreover, $N = \partial C$ acts trivially on K, so K has a natural G/N-module structure.* ■

Again the proof is left as an exercise.

As these two examples suggest, general crossed modules lie between the two extremes of normal subgroups and modules, in some sense, just as groupoids lay between equivalence relations and G-sets. Their structure bears a certain resemblance to both - they are “external” normal subgroups but also are “twisted” modules.

(iii) Let G be a group, then, as usual, let $Aut(G)$, denote the group of automorphisms of G. Conjugation gives a homomorphism

$$\iota : G \to Aut(G).$$

Of course, $Aut(G)$ acts on G in the obvious way and ι is a crossed module. We will need this later so will give it its own name: $\mathsf{Aut}(G)$.

More generally if L is some type of algebra then $U(L) \to Aut(L)$ will be a crossed module, where $U(L)$ denotes the units of L and the morphism send a unit to the automorphism given by conjugation by it.

(iv) We suppose given a morphism

$$\theta : M \to N$$

of left G-modules and form the semi-direct product $N \rtimes G$. This group we make act on M via the projection from $N \rtimes G$ to G.

We define a morphism

$$\partial : M \to N \rtimes G$$

by $\partial(m) = (\theta(m), 1)$, where 1 denotes the identity element of G, then $(M, N \rtimes G, \partial)$ is a crossed module. In particular, if A and B are Abelian groups, and B is considered to act trivially on A, then any homomorphism, $A \to B$ is a crossed module.

(v) As a last algebraic example for the moment, let

$$1 \to K \xrightarrow{a} E \xrightarrow{b} G \to 1$$

be an extension of groups with K a central subgroup of E, i.e. a *central extension* of G by K. For each $g \in G$, pick an element $s(g) \in b^{-1}(g) \subseteq E$. Define an action of G on E by: if $x \in E$, $g \in G$, then

$${}^{g}x = s(g)xs(g)^{-1}.$$

This is well defined, since if $s(g)$, $s'(g)$ are two choices, $s(g) = ks'(g)$ for some $k \in K$, and K is central. (This also shows that this *is* an action.) The structure (E, G, b) is a crossed module.

A particular important case is: for R a ring, let $E(R)$ be, as before, the group of elementary matrices of R, $E(R) \subseteq Gl(R)$ and $St(R)$, the corresponding Steinberg group with $b : St(R) \to E(R)$, the natural morphism, (see later or [92], for the definition). Then this gives a central extension

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

and thus a crossed module. In fact, more generally,

$$b : St(R) \to Gl(R)$$

is a crossed module. The group $Gl(R)/Im(b)$ is $K_1(R)$, the first algebraic K-group of the ring.

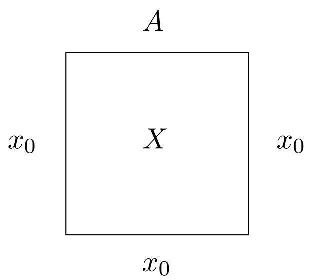
2.1.2 Topological Examples

In topology there are several examples that deserve looking at in detail as they do relate to aspects of the above algebraic cases. They require slightly more topological knowledge that has been assumed so far.

(vi) Let X be a pointed space, with $x_0 \in X$ as its base point, and A a subspace with $x_0 \in A$. Recall that the second relative homotopy group, $\pi_2(X, A, x_0)$, consists of relative homotopy classes of continuous maps

$$f : (I^2, \partial I^2, J) \to (X, A, x_0)$$

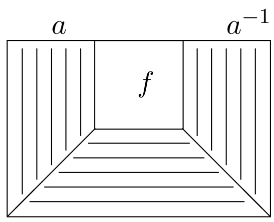
where ∂I^2 is the boundary of I^2, the square, $[0,1] \times [0,1]$, and $J = \{0,1\} \times [0,1] \cup [0,1] \times \{0\}$. Schematically f maps the square as:



so the top of the boundary goes to A, the rest to x_0 and the whole thing to X. The relative homotopies considered then deform the maps in such a way as to preserve such structure, so intermediate mappings also send J to x_0, etc. Restriction of such an f to the top of the boundary clearly gives a homomorphism

$$\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

to the fundamental group of A, based at x_0. There is also an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ given by rescaling the 'square' given by



where f is partially 'enveloped' in a region on which the mapping is behaving like a.

Of course, this gives a crossed module

$$\pi_2(X, A, x_0) \to \pi_1(A, x_0).$$

A direct proof is quite easy to give. One can be found in Hilton's book, [66] or in Brown-Higgins-Sivera, [29]. Alternatively one can use the argument in the next example.

(vii) Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence of pointed spaces. Thus p is a fibration, $F = p^{-1}(b_0)$, where b_0 is the basepoint of B. The fibre F is pointed at f_0, say, and f_0 is taken as the basepoint of E as well.

There is an induced map on fundamental groups

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

and if a is a loop in E based at f_0, and b a loop in F based at f_0, then the composite path corresponding to aba^{-1} is homotopic to one wholly within F. To see this, note that $p(aba^{-1})$ is null homotopic. Pick a homotopy in B between it and the constant map, then lift that homotopy back up to E to one starting at aba^{-1}. This homotopy is the required one and its other end gives a well defined element ${}^ab \in \pi_1(F)$ (abusing notation by confusing paths and their homotopy classes). With this action $(\pi_1(F), \pi(E), \pi_1(i))$ is a crossed module. This will not be proved here, but is not that difficult. Links with previous examples are strong.

If we are in the context of the above example, consider the inclusion map, f of a subspace A into a space X (both pointed at $x_0 \in A \subset X$). Form the corresponding fibration

$$i^f : M^f \to X$$

by forming the pullback

$$\begin{array}{ccc} M^f & \xrightarrow{\pi^f} & X^I \\ {\scriptstyle j^f}\downarrow & & \downarrow{\scriptstyle e_0} \\ A & \xrightarrow[f]{} & X \end{array}$$

so M^f consists of pairs (a, λ), where $a \in A$ and λ is a path from $f(a)$ to some point $\lambda(1)$. Set $i^f = e_1\pi^f$, so $i^f(a, \lambda) = \lambda(1)$. It is standard that i^f is a fibration and its fibre is the subspace $F_h(f) = \{(a, \lambda) \mid \lambda(1) = x_0\}$, often called the *homotopy fibre* of f. The base point of $F_h(f)$ is taken to be the constant path at x_0, (x_0, c_{x_0}).

If we note that

$$\pi_1(F_h(f)) \cong \pi_2(X, A, x_0)$$

$$\pi_1(M^f) \cong \pi_1(A, x_0)$$

(even down to the descriptions of the actions, etc.), the link with the previous example becomes clear, and thus furnishes another proof of the statement there.

(viii) The link between fibrations and crossed modules can also be seen in the category of simplicial groups. A morphism $f : G \to H$ of simplicial groups is a fibration if and only if each f_n is an epimorphism. This means that a fibration is determined by the fibre over the identity which is, of course, the kernel of f. The $(G, \overline{W})$-links between simplicial groups and simplicial sets mean that the analogue of π_1 is π_0. Thus the fibration f corresponds to

$$Ker\, f \stackrel{\triangleleft}{\to} G$$

and each level of this is a crossed module by our earlier observations. Taking π_0, it is easy to check that

$$\pi_0(Ker\, f) \to \pi_0(G)$$

is a crossed module. In fact any crossed module is isomorphic to one of this form. (Proof left to the reader.)

If $\mathsf{M} = (C, G, \partial)$ is a crossed module, then we sometimes write $\pi_0(\mathsf{M}) := G/\partial C$, $\pi_1(\mathsf{M}) := Ker\, \partial$, and then have a 4-term exact sequence:

$$0 \to \pi_1(\mathsf{M}) \to C \xrightarrow{\partial} G \to \pi_0(\mathsf{M}) \to 1.$$

In topological situations when M provides a model for (part of) the homotopy type of a space X or a pair (X, A), then typically $\pi_1(\mathsf{M}) \cong \pi_2(X)$, $\pi_0(\mathsf{MM}) \cong \pi_1(X)$.

MacLane and Whitehead, [88], showed that crossed modules give algebraic models for all homotopy 2-types of connected spaces. We will visit this result in more detail later, but loosely a 2-equivalence between spaces is a continuous map that induces isomorphisms on π_1 and π_2, the first two homotopy groups. Two spaces have the same 2-type if there is a zig-zag of 2-equivalences joining them.

We next turn to the use of crossed modules in combinatorial group theory.

2.2 Group presentations, identities and 2-syzyzgies

2.2.1 Presentations and Identities

(cf. Brown-Huebschmann, [30]) We consider a presentation $\mathcal{P} = (X : R)$ of a group G. The elements of X are called *generators* and those of R *relators*. We then have a short exact sequence,

$$1 \to N \to F \to G \to 1,$$

where $F = F(X)$, the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R. The group F acts on N by conjugation: ${}^u c = ucu^{-1}$, $c \in N, u \in F$ and the elements of N are words in the conjugates of the elements of R:

$$c = {}^{u_1}(r_1^{\varepsilon_1})^{u_2}(r_2^{\varepsilon_2}) \ldots {}^{u_n}(r_n^{\varepsilon_n})$$

where each ε_i is $+1$ or -1. One also says such elements are *consequences* of R. Heuristically an *identity among the relations* of $\mathcal{P}$ is such an element c which equals 1. The problem of what this means is analogous to that of working with a relation in R. For example, in the presentation

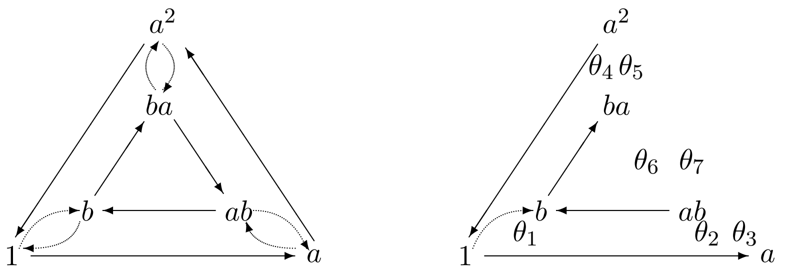
$(a : a^3)$ of C_3, the cyclic group of order 3, if a is thought of as being an element of C_3, then $a^3 = 1$, so why is this different from the situation with the 'presentation', $(a : a = 1)$? To get around that difficulty the free group on the generators $F(X)$ was introduced and, of course, in $F(\{a\})$, a^3 is not 1. A similar device, namely *free crossed modules* on the presentation will be introduced in a moment to handle the identities. Before that consider some examples which indicate that identities exist even in some quite common-or-garden cases.

Example 1: Suppose $r \in R$, but it is a power of some element $s \in F$, i.e. $r = s^m$. Of course, $rs = sr$ and

$$ {}^s r r^{-1} = 1 $$

so ${}^s r.r^{-1}$ is an identity. In fact, there will be a unique $z \in F$ with $r = z^q$, q maximal with this property. This z is called the *root of* r and if $q > 1$, r is called a *proper power*.

Example 2: Consider one of the standard presentations of S_3, $(a, b : a^3, b^2, (ab)^2)$. Write $r = a^3$, $s = b^2$, $t = (ab)^2$. Here the presentation leads to F, free of rank 2, but $N(R) \subset F$, so it must be free as well, by the Nielsen-Schreier theorem. Its rank will be 7, given by the Schreier index formula or, geometrically, it will be the fundamental group of the Cayley graph of the presentation. This group is free on generators corresponding to edges outside a maximal tree as in the following diagram:



The Cayley graph of S_3 and a maximal tree in it.

The set of normal generators of $N(R)$ has 3 elements; $N(R)$ is free on 7 elements (corresponding to the edges not in the tree), but is specified as consisting of products of conjugates of r, s and t, and there are infinitely many of these. Clearly there must be some slight redundancy, i.e., there must be some identities among the relations!

A path around the outer triangle corresponds to the relation r; each other region corresponds to a conjugate of one of r, s or t. (It may help in what follows to think of the graph being embedded on a 2-sphere, so 'outer' and 'outside' mean 'round the back face.) Consider a loop around a region. Pick a path to a start vertex of the loop, starting at 1. For instance the path that leaves 1 and goes along a, b and then goes around aaa before returning by $b^{-1}a^{-1}$ gives $abrb^{-1}a^{-1}$. Now the path around the outside can be written as a product of paths around the inner parts of the graph, e.g. $(abab)b^{-1}a^{-1}b^{-1}(bb)(b^{-1}a^{-1}b^{-1}a^{-1})\ldots$ and so on. Thus r can be written in a non-trivial way as a product of conjugates of r, s and t. (An explicit identity constructed like this is given in [30].)

Example 3: In a presentation of the free Abelian group on 3 generators, one would expect the commutators, $[x, y]$, $[x, z]$ and $[y, z]$. The well-known identity, usually called the Jacobi identity, expands out to give an identity among these relations (again see [30], p.154 or Loday, [82].)

2.2.2 Free crossed modules and identities

The idea that an identity is an equation in conjugates of relations leads one to consider formal conjugates of symbols that label relations. Abstracting this a bit, suppose G is a group and $f : Y \to G$, a function 'labelling' the elements of some subset of G. To form a conjugate, you need a thing being conjugated and an element 'doing' the conjugating, so form pairs (p, y), $p \in G$, $y \in Y$, to be thought of as ${}^p y$, the *formal conjugate* of y by p. Consequences are words in conjugates of relations, *formal consequences* are elements of $F(G \times Y)$. There is a function extending f from $G \times Y$ to G given by

$$\bar{f}(p, y) = pf(y)p^{-1},$$

converting a formal conjugate to an actual one and this extends further to a group homomorphism

$$\phi : F(G \times Y) \to G$$

defined to be $\bar{f}$ on the generators. The group G acts on the left on $G \times Y$ by multiplication: $p.(p', y) = (pp', y)$. This extends to a group action of G on $F(G \times Y)$. For this action, ϕ is G-equivariant if G is given its usual G-group structure by conjugations / inner automorphisms. Naively identities are the elements in the kernel of this, but there are some elements in that kernel that are there regardless of the form of function f. In particular, suppose that $g_1, g_2 \in G$ and $y_1, y_2 \in Y$ and look at

$$(g_1, y_1)(g_2, y_2)(g_1, y_1)^{-1}((g_1 f(y_1)g_1^{-1})g_2, y_2)^{-1}.$$

Such an element is always annihilated by ϕ. The normal subgroup generated by such elements is called the Peiffer subgroup. We divide out by it to obtain a quotient group. This is the construction of the free crossed module on the function f. If f is, as in our initial motivation, the inclusion of a set of relators into the free group on the generators we call the result the *free crossed module on the presentation* $\mathcal{P}$ and denote it by $C(\mathcal{P})$.

We can now formally define the module of identities of a presentation $\mathcal{P} = (X : R)$. We form the free crossed module on $R \to F(X)$, which we will denote by $\partial : C(\mathcal{P}) \to F(X)$. The *module of identities* of $\mathcal{P}$ is $Ker\,\partial$. By construction, the group presented by $\mathcal{P}$ is $G \cong F(X)/Im\,\partial$, where $Im\,\partial$ is just the normal closure of the set, R, of relations and we know that $Ker\,\partial$ is a G-module. We will usually denote the module of identities by $\pi_{\mathcal{P}}$.

We can get to $C(\mathcal{P})$ in another way. Construct a space from the combinatorial information in $C(\mathcal{P})$ as follows. Take a bunch of circles labelled by the elements of X; call it $K(\mathcal{P})_1$, it is the 1-skeleton of the space we want. We have $\pi_1(K(\mathcal{P})_1 \cong F(X)$. Each relator $r \in R$ is a word in X so gives us a loop in $K(\mathcal{P})_1$, following around the circles labelled by the various generators making up r. This loop gives a map $S^1 \xrightarrow{f_r} K(\mathcal{P})_1$. For each such r we use f_r to glue a 2-dimensional disc e^2_r to $K(\mathcal{P})_1$ yielding the space $K(\mathcal{P})$. The crossed module $C(\mathcal{P})$ is isomorphic to $\pi_2(K(\mathcal{P}), K(\mathcal{P})_1) \xrightarrow{\partial} \pi_1(K(\mathcal{P})_1$.

The main problem is how to calculate $\pi_{\mathcal{P}}$ or equivalently $\pi_2(K(\mathcal{P}))$. One approach is via an associated chain complex. This can be viewed as the chains on the universal cover of $K(\mathcal{P})$, but can also be defined purely algebraically, for which see Brown-Huebschmann, [30], or Loday, [82]. That algebraic - homological approach leads to 'homological syzygies'. For the moment we will concentrate on:

2.2.3 Homotopical syzygies:

There are both homological and homotopical syzygies. We will concentrate on the homotopical versions. For the homological form, have a look at Loday's article, [82] or Kapranov and Saito, [78] and later on in these notes.

We have built a complex, $K(\mathcal{P})$, from a presentation $\mathcal{P}$ of a group G. Any element in $\pi_2(K(\mathcal{P}))$ can, of course, be represented by a map from S^2 to $K(\mathcal{P})$ and by cellular approximation can be replaced, up to homotopy, by a cellular decomposition of S^2 and a cellular map $\phi : S^2 \to K(\mathcal{P})$. We will adopt the terminology of Kapranov and Saito, [78], and Loday, [82], in referring to a pair consisting of a cellular subdivision of S^2 together with a cellular map, as above, as a *homotopical 2-syzygy*. Of course, such an object corresponds to an identity among the relations of $\mathcal{P}$, but is a specific representative of such an identity. A family $\{\phi_\lambda\}_{\lambda\in\Lambda}$ of such homotopical 2-syzygies is then called *complete* when the homotopy classes $\{[\phi_\lambda]\}_{\lambda\in\Lambda}$ generate $\pi_2(K(\mathcal{P}))$.

In this case, we can use the ϕ_λ to form the next stage of the construction of an Eilenberg-MacLane space, $K(G,1)$, by killing this π_2. More exactly, rename $K(\mathcal{P})$ as $X(2)$ and form

$$X(3) := X(2) \cup \bigcup_{\lambda\in\Lambda} e^3_\lambda,$$

by, for each $\lambda \in \Lambda$, attaching a 3-cell, e^3_λ to $X(2)$ using ϕ_λ. Of course, we then have

$$\pi_1(X(3)) \cong G, \quad \pi_2(X(3)) = 0.$$

Again $\pi_3(X(3))$ may be non-trivial, so we consider homotopical 3-syzygies. Such an object, s, will consist of an oriented polytope decomposition of S^3 together with a continuous map, f_s from S^3 to $X(3)$, which sends the i-skeleton of that decomposition to $X(i)$, $i = 0, 1, 2$. At this stage we have $X(0) = K(\mathcal{P})_0$, a point, $X(1) = K(\mathcal{P})_1$, and $X(2) = K(\mathcal{P})_2$. One wants enough such 3-syzygies, s, identified algebraically and combinatorially, so that the corresponding homotopy classes, $\{[f_s]\}$ generate $\pi_3(X(3))$.

It is clear, by induction, that we get a notion of homotopical n-syzygy. We assume $X(n)$ has been built inductively by attaching cells of dimension $\leq n$ along homotopical k-syzygies for $k < n$, so that

$$\pi_1(X(n)) \cong G, \quad \pi_k(X(n)) = 0, \quad k = 2, \dots, n-1,$$

then a *homotopical n-syzygy*, s, is an oriented polytope decomposition of S^n and a continuous cellular map $f_s : S^n \to X(n)$. After a choice of a set $\mathcal{R}_n$ of n-syzygies, so that $\{[s_s] \mid s \in \mathcal{R}_n\}$ generates $\pi_n(X(n))$ as a G-module, we can form $X(n+1)$ by attaching $n+1$-dimensional cells e^{n+1}_s along these f_s for $s \in \mathcal{R}_n$.

If we can do this in a sensible way, for all n, we say the resulting system of syzygies is *complete* and the limit space $X(\infty) = \bigcup X(n)$ is then a cellular model for BG, the classifying space of the group G.

This construction is, of course, just a homotopical version of the construction of a free resolution of the trivial G-module, $\mathbb{Z}$. We could consider how to form simplicial resolutions 'step-by-step' (see here, starting page 71) as another combinatorial way to replace $K(\mathcal{P})$ and more generally $K(G,1)$. Alternatively there is a way of using this to get what is called a crossed resolution of G, but more on that later.

Remark: Some additional aspects of this can be found in Loday's paper [82], in particular the link with the 'pictures' of Igusa, [70, 71].

2.2.4 Examples of homotopical syzygies

Example and construction: Given any group G, we can find a presentation with $\{\langle g\rangle \mid g \neq 1, g \in G\}$ as set of generators and a relation $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle gg'\rangle^{-1}$ for each pair (g, g') of elements of G. (We write $\langle 1\rangle = 1$ for convenience.)

The relation $r_{g,g'}$ gives a triangle

$$\begin{tikzcd} & \cdot \arrow[dr, "g'"] & \\ \cdot \arrow[ur, "g"] \arrow[rr, "gg'"'] & & \cdot \end{tikzcd}$$

and, for each triple (g, g', g''), we get a homotopical 2-syzygy in the form of a tetrahedron.

Higher homotopical syzygies occur for any tuple, $(g_1, \ldots, g_n)$, of non-identity elements of G, by labelling a n-simplex. The limiting cellular space, $X(\infty)$, constructed from this context is just the usual model of the classifying space BG as geometric realisation of the nerve of G. The corresponding free resolution, $(C_*(G), d)$, is the classical *normalised bar resolution*. Using this bar resolution above dimension 2 together with the crossed module of the presentation at the base, one gets the standard free crossed resolution of the group, G, to which we will return later.

Example: Syzygies for the Steinberg group (cf. Kapranov and Saito, [78]) Let R be an associative ring with 1. The elementary matrices $\varepsilon_{ij}(a)$, over the ring R are the matrices having

$$\varepsilon_{ij}(a)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ a & \text{if } (k,l) = (i,j), a \in R \\ 0 & \text{otherwise,} \end{cases}$$

These satisfy some relations by virtue of their definition regardless of what R is. The *Steinberg group*, $St_n(R)$, has generators $x_{ij}(a)$, corresponding to these matrices and satisfying these generic relations. Explicitly it has relations,

St1 $\quad x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b)$;

St2 $\quad [x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} 1 & \text{if } i \neq \ell, j \neq k, \\ x_{i,\ell}(ab) & i \neq \ell, j = k. \end{cases}$

(These groups are nested so that $St_n(R) \subset St_{n+1}(R)$ and our earlier mention of the Steinberg group $St(R)$ corresponded to the direct limit of this nested sequence.)

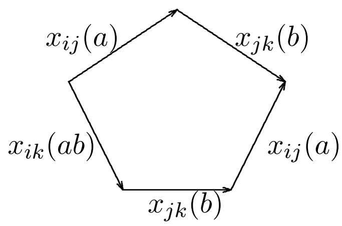
The identities / homotopical 2-syzygies are built from three types of polygon: a) a triangle, $T_{ij}(a, b)$ for each i, j, $i \neq j$, coming from St1;

b) a square,

$$\begin{tikzcd} \cdot \arrow[r, "x_{ij}(a)"] \arrow[d, "x_{kl}(b)"'] & \cdot \arrow[d, "x_{kl}(b)"] \\ \cdot \arrow[r, "x_{ij}(a)"'] & \cdot \end{tikzcd}$$

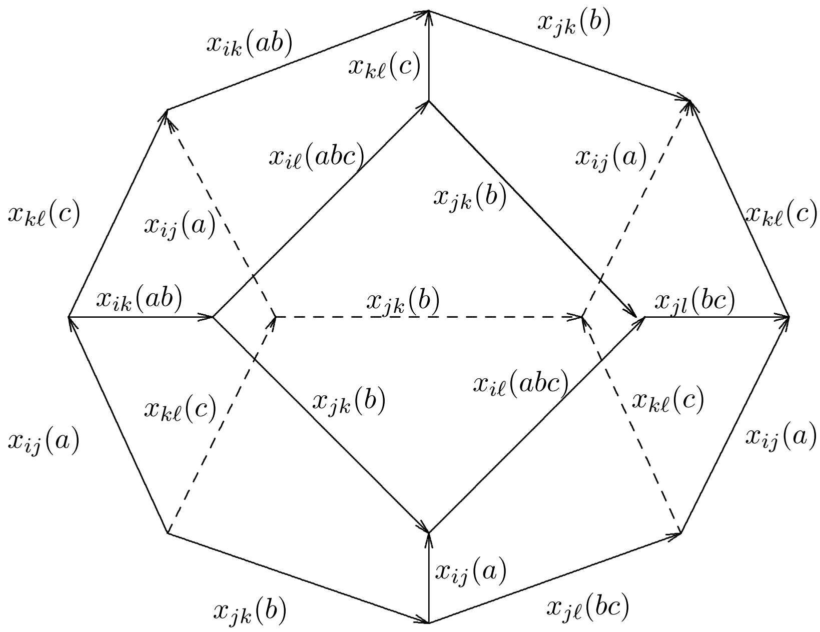
corresponding to the first case of St2 and

c) a pentagon, for the second:



Then for any pairs (i, j), (k, l), (m, p) with $x_{ij}(a)$, $x_{kl}(b)$, $x_{mp}(c)$, commuting by virtue of St2's first clause, we will have a homotopical syzygy in the form of a labelled cube.

There is also a homotopy 2-syzygy given by the associahedron labelled by generators as shown:



Remark: Kapranov and Saito, [78], have conjectured that the space $X(\infty)$ obtained by gluing labelled higher Stasheff polytopes together, is homotopically equivalent to the homotopy fibre of

$$f : BSt(R) \to BSt(R)^+,$$

where $(-)^+$ denotes Quillen's plus construction. The associahedron is a Stasheff polytope and, by encoding the data that goes to build the identities / syzygies schematically in a 'hieroglyph', Kapranov and Saito make a link between such hieroglyphs and certain polytopes.

2.3 Cohomology, crossed extensions and algebraic 2-types

2.3.1 Cohomology and extensions, continued

Suppose we have any group extension

$$\mathcal{E} : \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

with K Abelian, but not necessarily central. We can look at various possibilities.

If we can *split* p, by a homomorphism $s : G \to E$, with $ps = Id_G$, then, of course, $E \cong K \rtimes G$ by the isomorphisms,

$$e \longrightarrow (esp(e)^{-1}, p(e)),$$

$$ks(g) \longleftarrow (k, g),$$

which are compatible with the projections etc., so there is an equivalence of extensions

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & K \rtimes G \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

Our convention for multiplication in $K \rtimes G$ will be

$$(k, g)(k', g') = (k^g k', gg').$$

We say that the extension is a *split extension* and call s a *splitting* both of the extension and of the epimorphism. Both expressions are used.

But what if p does not split. We can build a (small) category of extensions $\mathcal{E}xt(G, K)$ with objects such as $\mathcal{E}$ above and in which a morphism from $\mathcal{E}$ to $\mathcal{E}'$ is a diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

By the 5-lemma, α will be an isomorphism, so $\mathcal{E}xt(G, K)$ is a groupoid.

In $\mathcal{E}$, the epimorphism p is usually not splittable, but as a function between sets, it is onto so we can pick an element in each $p^{-1}(g)$ to get a transversal (or set of coset representatives), $s : G \to E$. We get a comparison pairing / obstruction map or 'factor set' :

$$f : G \times G \to E$$

$$f(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1},$$

which will be trivial, (i.e. $f(g_1, g_2) = 1$ for all $g_1, g_2 \in G$ exactly if s splits p, i.e. is a homomorphism). This construction assumes that we know the multiplication in E, otherwise we cannot form this product! On the other hand given this 'f', we can work out the multiplication. As a set, E will be the product $K \times G$, identified with it by the same formulae as in the split case, noting that $pf(g_1, g_2) = 1$, we have

$$(k_1, g_1)(k_2, g_2) = ({k_1}^{s(g_1)} k_2 f(g_1, g_2), g_1g_2).$$

The product is *twisted* by the pairing f. Of course, we need this multiplication to be associative and, to ensure that, f must satisfy a cocycle condition:

$${}^{s(g_1)}f(g_2, g_3)f(g_1, g_2g_3) = f(g_1, g_2)f(g_1g_2, g_3).$$

This is a well known formula from group cohomology, more so if written additively:

$$s(g_1)f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0.$$

Here we actually have various parts of the nerve of G involved in the formula. The group G 'is' a small category (groupoid with one object), which we will, for the moment, denote $\mathcal{G}$. The triple $\sigma = (g_1, g_2, g_3)$ is a 3-simplex in $Ner(\mathcal{G})$ and its faces are

$$\begin{aligned} d_0\sigma &= (g_2, g_3), \\ d_1\sigma &= (g_1g_2, g_3), \\ d_2\sigma &= (g_1, g_2g_3), \\ d_3\sigma &= (g_1, g_2). \end{aligned}$$

This is all very classical. We can use it in the usual way to link $\pi_0(\mathcal{E}xt(G,K))$ with $H^2(G,K)$ and so is the 'modern' version of Schreier's theory of group extensions, at least in the case that K is Abelian.

For a long time there was no obvious way to look at the elements of $H^3(G,K)$ in a similar way. In MacLane's homology book, [85], you can find a discussion from the classical viewpoint. In Brown's [22], the link with crossed modules is sketched although no references for the details are given, for which see MacLane's [87].

If we have a crossed module $C \xrightarrow{\partial} P$, then we saw that $Ker\,\partial$ is central in C and is a $P/\partial C$-module. We thus have a 'crossed 2-fold extension':

$$K \xrightarrow{i} C \xrightarrow{\partial} P \xrightarrow{p} G,$$

where $K = Ker\,\partial$ and $G = P/\partial C$. (We will write $N = \partial C$.)

Repeat the same process as before for the extension

$$N \to P \to G,$$

but take extra care as N is usually not Abelian. Pick a transversal $s : G \to P$ giving $f : G\times G \to N$ as before (even with the same formula). Next look at

$$K \xrightarrow{i} C \to N,$$

and lift f to C via a choice of $F(g_1, g_2) \in C$ with image $f(g_1, g_2)$ in N.

The pairing f satisfied the cocycle condition, but we have no means of ensuring that F will do so, i.e. there will be, for each triple (g_1, g_2, g_3), an element $c(g_1, g_2, g_3) \in C$ such that

$$^{s(g_1)}F(g_2, g_3)F(g_1, g_2g_3) = i(c(g_1, g_2, g_3))F(g_1, g_2)F(g_1g_2, g_3),$$

and some of these $c(g_1, g_2, g_3)$ may be non-trivial. The $c(g_1, g_2, g_3)$ will satisfy a cocycle condition correspond to a 4-simplex in $Ner(\mathcal{G})$, and one can reconstruct the crossed 2-fold extension up to equivalence from F and c. Here 'equivalence' is generated by maps of 'crossed' exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & C' \arrow[r] & P' \arrow[r] & G \arrow[r] & 1, \end{tikzcd}$$

but these morphisms need not be isomorphisms. Of course, this identifies $H^3(G,K)$ with π_0 of the resulting category.

What about $H^4(G,K)$? Yes, something similar works, but we do not have the machinery to do it here, yet.

2.3.2 Not really an aside!

Suppose we start with a crossed module $\mathsf{C} = (C, P, \partial)$. We can build an internal category $\mathcal{X}(\mathsf{C})$ in *Grps* from it. The group of objects of $\mathcal{X}(\mathsf{C})$ will be P and the group of arrows $C \rtimes P$. The source map

$$s : C \rtimes P \to P \quad \text{is} \quad s(c, p) = p,$$

the target

$$t : C \rtimes P \to P \quad \text{is} \quad t(c, p) = \partial c.p.$$

(That looks a bit strange. That sort of construction usually does not work, multiplying two homomorphisms together is a recipe for trouble! - but it does work here:

$$\begin{aligned} t((c_1, p_1).(c_2, p_2)) &= t(c_1{}^{p_1}c_2, p_1p_2) \\ &= \partial(c_1{}^{p_1}c_2).p_1p_2, \end{aligned}$$

whilst $t(c_1, p_1).t(c_2, p_2) = \partial c_1.p_1.\partial c_2.p_2$, but remember $\partial(c_1{}^{p_1}c_2) = \partial c_1.p_1.\partial c_2.p_1^{-1}$, so they are equal.)

The identity morphism is $i(p) = (1, p)$, but what about the composition. Here it helps to draw a diagram. Suppose $(c_1, p_1) \in C \rtimes P$, then it is an arrow

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

and we can only compose it with (c_2, p_2) if $p_2 = \partial c_1.p_1$. This gives

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1 \xrightarrow{(c_2,\partial c_1.p_1)} \partial c_2 \partial c_1.p_1.$$

The obvious candidate for the composite arrow is (c_2c_1, p_1) and it works!

In fact, $\mathcal{X}(\mathsf{C})$ is an *internal groupoid* as $(c_1^{-1}, \partial c_1.p_1)$ is an inverse for (c_1, p_1).

Now if we started with an internal category

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & G_0 \arrow[l, shift left=2, "i"] \end{tikzcd},$$

etc., then set $P = G_0$ and $C = Ker\, s$ with $\partial = t\,|_C$ to get a crossed module.

Theorem 2 *(Brown-Spencer,[34]) The category of crossed modules is equivalent to that of internal categories in Grps.* ∎

You have, almost, seen the proof. As beginning students of algebra, you learnt that equivalence relations on groups need to be congruence relations for quotients to work well and that congruence relation 'are the same as' normal subgroups. That is the essence of the proof needed here, but we have groupoids rather than equivalence relations and crossed modules rather than normal subgroups.

This is a good place to mention 2-groups. The notion of 2-category is one that should be fairly clear even if you have not met it before. For instance the category of small categories, functors and natural transformations is a 2-category. Between each pair of objects we have not just a set of functors as morphisms but a small category of them with the natural transformations between them as the arrows in this second level of structure. The notion of 2-category is abstracted from this. We will not give a formal definition here (but suggest that you look one up if you have not met the idea before). A 2-category thus has objects, arrows or morphisms (or sometimes '1-cells') between them and then some 2-cells between them. A *2-groupoid* is a 2-category in which all 1-cells and 2-cells are invertible. If the 2-groupoid has just one object then we call it a *2-group*. Internal categories in *Grps* are really exactly the same as 2-groups. The Brown-Spencer theorem thus constructs the *associated 2-group of a crossed module.*

2.3.3 Perhaps a bit more of an aside ... for the moment!

This is quite a good place to mention the groupoid based theory of all this. The resulting objects look like abstract 2-categories and are 2-groupoids. We have a set of objects K_0, a set of arrows K_1, depicted $x \xrightarrow{p} y$, and a set of two cells

$$\begin{tikzcd} x \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c.p"'] \arrow[r, phantom, "\Downarrow(c,p)"] & y \end{tikzcd} .$$

In our previous diagrams, as all the elements of P started and ended at the same single object, we could shift dimension down one step; our old objects are now arrows and our old arrows are 2-cells. We will return to this later.

The important idea to note here is that a 'higher dimensional category' has a link with an algebraic object. The 2-group(oid) provides a useful way of interpreting the structure of the crossed module *and* indicates possible ways towards similar applications and interpretations elsewhere. For instance, a presentation of a monoid leads more naturally to a 2-category than to any analogue of a crossed module, since kernels are less easy to handle than congruences in Mon.

There are other important interpretations of this. Categories such as that of vector spaces, Abelian groups or modules over a ring, have an additional structure coming from the tensor product, $A \otimes B$. They are monoidal categories. One can 'multiply' objects together and this is linked to a related multiplication on morphisms between the objects. In many of the important examples the multiplication is not strictly associative, so for instant, if A, B, C are objects there is an isomorphism between $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, but this isomorphism is most definitely not the identity as the two objects are constructed in different ways. A similar effect happens in the category of sets with ordinary Cartesian product. The isomorphism is there because of universal properties but it is again not the identity. It satisfies some coherence conditions, (a cocycle condition in disguise), relating to associativity of four fold tensors and the associahedron that we gave earlier, is a corresponding diagram for the five fold tensors. (Yes, there is a strong link but that is not for these notes!) Our 2-group(oid) is the 'suspension' or 'categorification' of a similar structure. We can multiply objects and 'arrows' and the result is a strict 'gr-groupoid', i.e. a strict monoidal category with inverses. This is vague here, but will gradually be explored later on. If you want to explore the ideas further now, look at Baez and Dolan, [9].

Just as associativity in a monoid is replaced by a 'lax' associativity 'up to coherent isomorphisms' in the above, gr-groupoids are 'lax' forms of internal categories in groups and thus indicate the presence of a crossed module-like structure, albeit in a weakened or 'laxified' form. Later we will see naturally occurring gr-groupoid structures associated with some constructions in non-Abelian cohomology. There is also a sense in which the link between fibrations and crossed modules given earlier here, indicates that fibrations are like a related form of lax crossed modules. In the notion of fibred category and the related Grothendieck construction, this intuition begins to be 'solidified' into a clearer strong relationship.

2.3.4 Back to 2-types

From our crossed module, $\mathsf{C} = (C, P, \partial)$, we build the internal groupoid $\mathcal{X}(\mathsf{C})$ as above, then apply the nerve construction internally to the internal groupoid structure to get a simplicial group, $K(\mathsf{C})$. We need this in some detail in low dimensions.

$$\begin{array}{rcll} K(\mathsf{C})_0 & = & P & \\ K(\mathsf{C})_1 & = & C \rtimes P & \qquad d_0 = t, d_1 = s \\ K(\mathsf{C})_2 & = & C \rtimes (C \rtimes P), & \end{array}$$

where $d_0(c_2, c_1, p) = (c_2, \partial c_2.p)$, $d_1(c_2, c_1, p) = (c_2.c_1, p)$ and $d_2(c_2, c_1, p) = (c_1, p)$. The pattern continues with $K(\mathsf{C})_n = C \rtimes (\ldots \rtimes (C \rtimes P) \ldots)$, having n-copies of C. The d_i, $0 < i < n$ are given by multiplication in C, d_0 is induced from t and d_n is a projection. The s_i are insertions of identities. (We will examine this in more detail later.) We say $K(\mathsf{C})$ is the *nerve of the crossed module*, C. The simplicial set $\overline{W}(K(\mathsf{C}))$ or its geometric realisation, would be called the *classifying space* of C and we will look at this in much more detail later on. (A word of caution: for G a group considered as a crossed module, this 'nerve' is not the nerve of G in the sense used earlier. It is just the constant simplicial group corresponding to G. What is often called the nerve of G is what here has been called its classifying space. One way to view this is to note that $\mathcal{X}(\mathsf{C})$ has two independent structures, one a group, the other a category, and *this* nerve is of the category structure. The group G considered as a crossed module is like a set considered as a (discrete) category, having only identity arrows.)

The Moore complex of $K(\mathsf{C})$ is easy to calculate and is just $NK(\mathsf{C})_i = 1$ if $i \geq 2$; $NK(\mathsf{C})_1 \cong C$; $NK(\mathsf{C})_0 \cong P$ with the $\partial : NK(\mathsf{C})_1 \to NK(\mathsf{C})_0$ being exactly the given ∂ of C. (This is left as an exercise. It is a useful one to do in detail.)

Proposition 3 *(Loday,[81]) The category $CMod$ of crossed modules is equivalent to the subcategory of Simp.Grps, consisting of those simplicial groups, G, having Moore complexes of length 1, i.e. $NG_i = 1$ if $i \geq 2$.* ∎

This raises the interesting question as to whether it is possible to find alternative algebraic descriptions of the structures corresponding to Moore complexes of length n.

Is there any way of going directly from simplicial groups to crossed modules? Yes. The last two terms of the Moore complex will give us:

$$\partial : NG_1 \to NG_0 = G_0$$

and G_0 acts on NG_1 by conjugation via s_0, i.e. if $g \in G_0$ and $x \in NG_1$, then $s_0(g)xs_0(g)^{-1}$ is also in NG_1. (Of course, we could use multiple degeneracies to make g act on an $x \in NG_n$ just as easily.) As $\partial = d_0$, it respects the G_0 action, so CM1 is satisfied. In general, CM2 will not be satisfied. Suppose $g_1, g_2 \in NG_1$ and examine ${}^{\partial g_1}g_2 = s_0d_0g_1.g_2.s_0d_0g_1^{-1}$. This is rarely equal to $g_1g_2g_1^{-1}$. We write $\langle g_1, g_2\rangle = [g_1, g_2][g_2, s_0d_0g_1] = g_1g_2g_1^{-1}.({}^{\partial g_1}g_2)^{-1}$, so it measures the obstruction to CM2 for this pair g_1, g_2. This is often called the *Peiffer commutator* of g_1 and g_2. Noting that $s_0d_0 = d_0s_1$, we have an element

$$\{g_1, g_2\} = [s_0g_1, s_0g_2][s_0g_2, s_1g_1] \in NG_2$$

and $\partial\{g_1, g_2\} = \langle g_1, g_2\rangle$. This second pairing is called the *Peiffer lifting* (of the Peiffer commutator). Of course, if $NG_2 = 1$, then CM2 is satisfied (as for $K(\mathsf{C})$, above).

We could work with what we will call $M(G, 1)$, namely

$$\overline{\partial} : \frac{NG_1}{\partial NG_2} \to NG_0,$$

with the induced morphism and action. (As $d_0d_0 = d_0d_1$, the morphism is well defined.) This is a crossed module, but we could have divided out by less if we had wanted to. We note that $\{g_1, g_2\}$ is a product of degenerate elements, so we form, in general, the subgroup $D_n \subseteq NG_n$, generated by all degenerate elements.

Lemma 5

$$\overline{\partial} : \frac{NG_1}{\partial(NG_2 \cap D_2)} \to NG_0$$

is a crossed module. ■

This is, in fact, $M(sk_1G, 1)$, where sk_1G is the 1-skeleton of G, i.e., the subsimplicial group generated by the k-simplices for $k = 0, 1$.

The kernel of $M(G, 1)$ is $\pi_1(G)$ and the cokernel $\pi_0(G)$ and

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

represents a class $k(G) \in H^3(\pi_0(G), \pi_1(G))$. Up to a notion of 2-equivalence, $M(G, 1)$ represents the 2-type of G completely. This is an algebraic version of the result of MacLane and Whitehead we mentioned earlier. Once we have a bit more on cohomology, we will examine it in detail.

This use of $NG_2 \cap D_2$ and our noting that $\{g_1, g_2\}$ is a product of degenerate elements may remind you of group T-complexes and thin elements. Suppose that G is a group T-complex in the sense of our discussion at the end of the previous chapter (page 29). In a general simplicial group the subgroups, $NG_n \cap D_n$, will not be trivial. They give measure of the extent to which homotopical information in dimension n on G depends on 'stuff' from lower dimensions., i.e., comparing G with its $(n-1)$-skeleton. (Remember that in homotopy theory, invariants such as the homotopy groups do not necessarily vanish above the dimension of the space, just recall the sphere S^2 and the subtle structure of its higher homotopy groups.)

The construction here of $M(sk_1G, 1)$ involves 'killing' the images of our possible multiple 'D-fillers' for horns, forcing uniqueness. We will see this again later.

Chapter 3

Crossed complexes and (Abelian) Cohomology

Accurate encoding of homotopy types is tricky. Chain complexes, even of G-modules, can only record certain, more or less Abelian, information. Simplicial groups, at the opposite extreme, can encode all connected homotopy types, but at the expense of such a large repetition of the essential information that makes calculation, at best, tedious and, at worst, virtually impossible. Complete information on truncated homotopy types can be stored in the cat^n-groups of Loday, [81]. We will look at these later. An intermediate model due to Blakers and Whitehead, [113], is that of a crossed complex. The algebraic and homotopy theoretic aspects of the theory of crossed complexes have been developed by Brown and Higgins, (cf. [26, 27], etc., in the bibliography and the forthcoming monograph by Brown, Higgins and Sivera, [29]) and by Baues, [12–14].

3.1 Crossed complexes: the Definition

We will initially look at reduced crossed complexes, i.e. the group rather than the groupoid based case.)

Definition: A *crossed complex*, which will be denoted C, consists of a sequence of groups and morphisms

$$\mathsf{C} : \ldots \to C_n \stackrel{\delta_n}{\to} C_{n-1} \stackrel{\delta_{n-1}}{\to} \ldots \to C_3 \stackrel{\delta_3}{\to} C_2 \stackrel{\delta_2}{\to} C_1$$

satisfying the following:
CC1) $\delta_2 : C_2 \to C_1$ is a crossed module;
CC2) each C_n, $(n > 2)$, is a left $C_1/\delta_1 C_2$-module and each δ_n, $(n > 2)$ is a morphism of left $C_1/\delta_2 C_2$-modules, (for $n = 3$, this means that δ_3 commutes with the action of C_1 and that $\delta_3(C_3) \subset C_2$ must be a $C_1/\delta_2 C_2$-module);
CC3) $\delta\delta = 0$.

The notion of a morphism of crossed complexes is clear. It is a graded collection of morphisms preserving the various structures. We thus get a category, Crs_{red} of reduced crossed complexes.

As we have that a crossed complex is a particular type of chain complex (of non-Abelian groups near the bottom), it is natural to define its homology groups in the obvious way.

Definition: If C is a crossed complex, its n^{th} *homology group* is

$$H_n(\mathsf{C}) = \frac{Ker\,\delta_n}{Im\,\delta_{n+1}}.$$

These homology groups are, of course, functors from Crs_{red} to the category of Abelian groups.

Definition: A morphism $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$ is called a *weak equivalence* if it induces isomorphisms on all homology groups.

There are good reasons for considering the homology groups of a crossed complex as being its homotopy groups. For example, if the crossed complex comes from a simplicial group then the homotopy groups of the simplicial group are the same as the homology groups of the given crossed complex (possibly shifted in dimension, depending on the notational conventions you are using).

The non-reduced version of the concept is only a bit more difficult to write down. It has C_1 as a groupoid on a set of objects C_0 with each C_k, a family of groups indexed by the elements of C_0. The axioms are very similar; see [29] for instance or many of the papers by Brown and Higgins listed in the bibliography. This gives a category, Crs, of (unrestricted) crossed complexes and morphisms between them. This category is very rich in structure. It has a tensor product structure, denoted $\mathsf{C} \otimes \mathsf{D}$ and a corresponding mapping complex construction, $\mathsf{Crs}(\mathsf{C}, \mathsf{D})$, making it into a monoidal closed category. The details are to be found in the papers and book listed above.

3.1.1 Examples: crossed resolutions

As we mentioned earlier, a resolution of a group (or other object) is a model for the homotopy type represented by the group, but which usually is required to have some nice freeness properties. With crossed complexes we have some notion of homotopy around, just as with chain complexes, so we can apply that vague notion of resolution in this context as well. This will give us some neat examples of crossed complexes that are 'tuned' for use in cohomology.

A *crossed resolution* of a group G is a crossed complex, C, such that for each $n > 1$, $Im\,\delta_n = Ker\,\delta_{n-1}$ and there is an isomorphism, $C_1/\delta_2 C_2 \cong G$.

A crossed resolution can be constructed from a presentation $\mathcal{P} = (X : R)$ as follows:

Let $C(P) \to F(X)$ be the free crossed module associated with $\mathcal{P}$. We set $C_2 = C(\mathcal{P})$, $C_1 = F(X)$, $\delta_1 = \partial$. Let $\kappa(\mathcal{P}) = Ker(\partial : C(\mathcal{P}) \to F(X))$. This is the module of identities of the presentation and is a left G-module. As the category G-*Mod* has enough projectives, we can form a free resolution $\mathbb{P}$ of $\kappa(\mathcal{P})$. To obtain a crossed resolution of G, we join $\mathbb{P}$ to the crossed module by setting $C_n = P_{n-2}$ for $n > 3$, $\delta_n = d_{n-2}$ for $n > 3$ and the composite from P_0 to $C(P)$ for $n = 3$.

3.1.2 The standard crossed resolution

We next look at a particular case of the above, namely the standard crossed resolution of G. In this, which we will denote by $\mathsf{C}G$, we have

(i) $C_1G =$ the free group on the underlying set of G. The element corresponding to $u \in G$ will be denoted by $[u]$.

(ii) C_2G is the free crossed module over C_0G on generators, written $[u,v]$, considered as elements of the set $G \times G$, in which the map δ_1 is defined on generators by

$$\delta[u,v] = [uv]^{-1}[u][v].$$

(iii) For $n > 3$, C_nG is the free left G-module on the set G^{n+1}, but in which one has equated to zero any generator $[u_1, \ldots, u_{n+1}]$ in which some u_i is the identity element of G.

If $n > 2$, $\delta : C_{n+1}G \to C_nG$ is given by the usual formula

$$\begin{aligned}\delta[u_1, \ldots, u_{n+1}] &= {}^{[u_1]}[u_2, \ldots, u_{n+1}] \\ &\quad + \sum_{i=1}^{n}(-1)^i[u_1, \ldots, u_iu_{i+1}, \ldots, u_{n+1}] + (-1)^{n+1}[u_1, \ldots, u_n].\end{aligned}$$

For $n = 2$, $\delta : C_3G \to C_2G$ is given by

$$\delta[u,v,w] = {}^{[u]}[v,w].[u,v]^{-1}.[uv,w]^{-1}[u,vw].$$

This is the crossed analogue of the inhomogeneous bar resolution, $\mathsf{B}G$, of the group G. A groupoid version can be found in Brown-Higgins, [25], and the abstract group version in Huebschmann, [68]. In the first of these two references, it is pointed out that $\mathsf{C}G$, as constructed, is isomorphic to the crossed complex, $\underline{\pi}(BG)$, of the classifying space of G considered with its skeletal filtration.

For any filtered space $\underline{X} = (X_n)_{n\in\mathbb{N}}$, the fundamental crossed complex, $\underline{\pi}(\underline{X})$, is, in general, a non-reduced crossed complex. It is defined to have

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a\in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1X_1X_0$, and $\underline{\pi}(\underline{X})_2$, the family, $(\pi_2(X_2, X_1, a))_{a\in X_0}$.

There are two useful, but conflicting, conventions as to indexation in crossed complexes. In the topologically inspired one, the bottom group is C_1, in the simplicial and algebraic one, it is C_0. Both get used and both have good motivation. The natural indexation for the standard crossed resolution would seem to be with C_n being generated by n-tuples, i.e. the topological one. (I am not sure that all instances of the other have been avoided, so please be careful!)

G-augmented crossed complexes. Crossed resolutions of G are examples of G-augmented crossed complexes. A *G-augmented crossed complex* consists of a pair (C, ϕ) where C is a crossed complex and where $\phi : C_1 \to G$ is a group homomorphism satisfying

(i) $\phi\delta_1$ is the trivial homomorphism;

(ii) $Ker\,\phi$ acts trivially on C_i for $i \geq 3$ and also on C_2^{Ab}.

A *morphism*

$$(\alpha, Id_G) : (\mathsf{C}, \phi) \to (\mathsf{C}', \phi')$$

of G-augmented crossed complexes consists of a morphism

$$\alpha : \mathsf{C} \to \mathsf{C}'$$

of crossed complexes such that $\phi'\alpha_0 = \phi$.

This gives a category, Crs_G, which behaves nicely with respect to change of groups, i.e. if $\varphi : G \to H$, then there are induced functors between the corresponding categories.

3.2 Crossed complexes and chain complexes: I

(Some of the proofs here are given in more detail as they are less routine and are not that available elsewhere.)

We have introduced crossed complexes where normally chain complexes of modules would have been used. We have seen earlier the bar resolution and now we have the standard crossed resolution. What is the connection between them? The answer is approximately that chain complexes form a category equivalent to a reflective subcategory of Crs. In other words, there is a canonical way of building a chain complex from a crossed one akin to the process of Abelianising a group. The resulting reflection functor sends the standard crossed resolution of a group to the bar resolution. The details involve some interesting ideas.

In chapter 2, we saw that, given a morphism $\theta : M \to N$ of modules over a group G, $\partial : M \to N \rtimes G$, given by $\partial(m) = (\theta(m), 1_G)$ is a crossed module, where $N \rtimes G$ acts on M via the projection to G. That example easily extends to a functorial construction which, from a positive chain complex, D, of G-modules, gives us a crossed complex $\Delta_G(\mathsf{D})$ with $\Delta_G(\mathsf{D})_n = D_n$ if $n > 1$ and equal to $D_1 \rtimes G$ for $n = 1$.

Lemma 6 $\Delta_G : Ch(G\text{−}Mod) \to Crs_G$ *is an embedding.*

Proof: That Δ_G is a functor is easy to see. It is also easy to check that it is full and faithful, that is it induces bijections,

$$Ch(G\text{−}Mod)(\mathsf{A}, \mathsf{B}) \to Crs_G(\Delta_G(\mathsf{A}), \Delta_G(\mathsf{B})).$$

The augmentation of $\Delta_G(A)$ is given by the projection of $A_1 \rtimes G$ onto G. ■

We can thus turn a positive chain complex into a crossed complex. Does this functor have a left adjoint? i.e. is there a functor $\xi_G : Crs_G \to Ch(G\text{−}Mod)$ such that

$$Ch(G\text{−}Mod)(\xi_G(\mathsf{C}), \mathsf{D}) \to Crs_G(\mathsf{C}, \Delta_G(\mathsf{D}))?$$

If so it would suggest that chain complexes of G-modules are like G-augmented crossed complexes that satisfy some additional equational axioms. As an example of a similar situation think of 'Abelian groups' within 'groups' for which the inclusion has a left adjoint, namely Abelianisation $(G)^{Ab} = G/[G, G]$. Abelian groups are of course groups that satisfy the additional rule $[x, y] = 1$. Other examples of such situations are nilpotent groups of a given finite rank c. The subcategories of this general form are called *varieties* and, for instance, the study of varieties of groups is a very interesting area of group theory. Incidentally, it is possible to define various forms of cohomology modulo a variety in some sense. We will not explore that here.

We thus need to look at morphisms of crossed complexes from a crossed complex C to one of form $\Delta_G(\mathsf{D})$, and we need therefore to look at morphisms into a semidirect product. These are useful for other things, so are worth looking at in detail.

3.2.1 Semi-direct product and derivations.

Suppose that we have a diagram

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & K \rtimes G \arrow[dl, "proj"] \\ & G & \end{tikzcd}$$

where K is a G-module (written additively, so we write $g.k$ not ${}^g k$ for the action). This is like the very bottom of the situation for a morphism $f : \mathsf{C} \to \Delta_G(\mathsf{D})$.

As the codomain of f is a semidirect product, we can decompose f, as a function, in the form

$$f(h) = (f_1(h), \alpha(h)),$$

identifying its second component using the diagram. The mapping f_1 is not a homomorphism. As f is one, however, we have

$$(f_1(h_1h_2), \alpha(h_1h_2)) = f(h_1)f(h_2) = (f_1(h_1) + \alpha(h_1)f_1(h_2), \alpha(h_1h_2)),$$

i.e. f_1 satisfies

$$f_1(h_1h_2) = f_1(h_1) + \alpha(h_1)f_1(h_2)$$

for all $h_1, h_2 \in H$.

3.2.2 Derivations and derived modules.

We will use the identification of G-modules for a group G with modules over the group ring $\mathbb{Z}[G]$ of G. Recall that this ring is obtained from the free Abelian group on the set G by defining a multiplication extending linearly that of G itself. (Formally if, for the moment, we denote by e_g, the generator corresponding to $g \in G$, then an arbitrary element of $\mathbb{Z}[G]$ can be written as $\sum_{g\in G} n_g e_g$ where the n_g are integers and only finitely many of them are non-zero. The multiplication is by 'convolution' product, that is,

$$\Big(\sum_{g\in G} n_g e_g\Big)\Big(\sum_{g\in G} m_g e_g\Big) = \sum_{g\in G}\Big(\sum_{g_1\in G} n_{g_1} m_{g_1^{-1}g} e_g\Big).$$

We will also need the augmentation $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$, given by $\varepsilon(\sum_{g\in G} n_g e_g) = \sum_{g\in G} n_g$ and its kernel $I(G)$, known as the augmentation ideal.

Definitions: Let $\phi : G \to H$ be a homomorphism of groups. A *ϕ-derivation*

$$\partial : G \to M$$

from G to a left $\mathbb{Z}[H]$-module, M, is a mapping from G to M, which satisfies the equation

$$\partial(g_1g_2) = \partial(g_1) + \phi(g_1)\partial(g_2)$$

for all g_1, $g_2 \in G$.

Such φ-derivations are really all derived from a universal one.

A *derived module* for ϕ consists of a left $\mathbb{Z}[H]$-module, D_ϕ, and a ϕ-derivation, $\partial_\phi : G \to D_\phi$ with the following universal property:

Given any left $\mathbb{Z}[H]$-module, M, and a ϕ-derivation $\partial : G \to M$, there is a unique morphism

$$\beta : D_\phi \to M$$

of $\mathbb{Z}[H]$-modules such that $\beta\partial_\phi = \partial$.

The set of all ϕ-derivations from G to M has a natural Abelian group structure. We denote this set by $Der_\phi(G, M)$. This gives a functor from *H-Mod* to *Ab*, the category of Abelian groups. If (D_ϕ, ∂_ϕ) exists, then it sets up a natural isomorphism

$$Der_\phi(G, M) \cong H\!-\!Mod(D_\phi, M),$$

i.e., (D_ϕ, ∂_ϕ) represents the ϕ-derivation functor.

3.2.3 Existence

The treatment of derived modules that is found in Crowell's paper, [41], provides a basis for what follows. In particular it indicates how to prove the existence of (D_ϕ, ∂_ϕ) for any ϕ.

Form a $\mathbb{Z}[H]$-module, D, by taking the free left $\mathbb{Z}[H]$-module, $\mathbb{Z}[H]^{(X)}$, on a set of generators, $X = \{\partial g : g \in G\}$. Within $\mathbb{Z}[H]^{(X)}$ form the submodule, Y, generated by the elements

$$\partial(g_1 g_2) - \partial(g_1) - \phi(g_1)\partial(g_2).$$

Let $D = \mathbb{Z}[H]^{(X)}/Y$ and define $d : G \to D$ to be the composite:

$$G \xrightarrow{\eta} \mathbb{Z}[H]^{(X)} \xrightarrow{quotient} D,$$

where η is "inclusion of the generators", $\eta(g) = \partial g$. Thus d by construction, will be a ϕ-derivation. The universal property is easily checked and hence (D_ϕ, ∂_ϕ) exists.

We will later on construct (D_ϕ, ∂_ϕ) in a different way which provides a more amenable description of D_ϕ, namely as a tensor product. As a first step towards this description, we shall give a simple description of D_G, that is, the derived module of the identity morphism of G. More precisely we shall identify (D_G, ∂_G) as being $(I(G), \partial)$, where, as above, $I(G)$ is the augmentation ideal of $\mathbb{Z}[G]$ and $\partial : G \to I(G)$ is the usual map, $\partial(g) = g - 1$.

Our earlier observations give us the following useful result:

Lemma 7 *If G is a group and M is a G-module, then there is an isomorphism*

$$Der_G(G, M) \to Hom/G(G, M \rtimes G)$$

where $Hom/G(G, M \rtimes G)$ is the set of homomorphisms from G to $M \rtimes G$ over G, i.e., $\theta : G \to M \rtimes G$ such that for each $g \in G$, $\theta(g) = (g, \theta'(g))$ for some $\theta'(g) \in M$. ■

3.2.4 Derivation modules and augmentation ideals

Proposition 4 *The derivation module D_G is isomorphic to $I(G) = Ker(\mathbb{Z}[G] \to \mathbb{Z})$. The universal derivation is*

$$d_G : G \to I(G)$$

given by $d_G(g) = g - 1$.

Proof:

We introduce the notation $f_\delta : I(G) \to M$ for the $\mathbb{Z}[G]$-module morphism corresponding to a derivation

$$\delta : G \to M.$$

The factorisation $f_\delta d_G = \delta$ implies that f_δ must be defined by $f_\delta(g-1) = \delta(g)$. That this works follows from the fact that $I(G)$, as an Abelian group, is free on the set $\{g - 1 : g \in G\}$ and that the relations in $I(G)$ are generated by those of the form

$$g_1(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1).$$

■

We note a result on the augmentation ideal construction that is not commonly found in the literature.

The proof is easy and so will be omitted.

Lemma 8 *Given groups G and H in $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\phi"] \\ H \arrow[r, "\delta'"'] & N \end{tikzcd} \qquad (*)$$

where δ, δ' are derivations, M is a left $\mathbb{Z}[G]$-module, N is a left $\mathbb{Z}[H]$-module and ϕ is a module map over ψ, i.e., $\phi(g.m) = \psi(g)\phi(m)$ for $g \in G$, $m \in M$. Then the corresponding diagram

$$\begin{tikzcd} I(G) \arrow[r, "f_\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\phi"] \\ I(H) \arrow[r, "f_{\delta'}"'] & N \end{tikzcd} \qquad (**)$$

is commutative. ■

The earlier proposition has the following corollaries:

Corollary 1 *The subset $Im\, d_G = \{g - 1 : g \in G\} \subset I(G)$ generates $I(G)$ as a $\mathbb{Z}[G]$-module. Moreover the relations between these generators are generated by those of the form*

$$(g_1 g_2 - 1) - (g_1 - 1) - g_1(g_2 - 1).$$

■

It is useful to have also the following reformulation of the above results stated explicitly.

Corollary 2 *There is a natural isomorphism*

$$Der_G(G, M) \cong G{-}Mod(I(G), M).$$

■

3.2.5 Generation of $I(G)$.

The first of these two corollaries raises the question as to whether, if $X \subset G$ generates G, does the set $G_X = \{x - 1 : x \in X\}$ generate $I(G)$ as a $\mathbb{Z}[G]$-module.

Proposition 5 *If X generates G, then G_X generates $I(G)$.*

Proof: We know $I(G)$ is generated by the $g - 1$s for $g \in G$. If g is expressible as a word of length n in the generators X then we can write $g - 1$ as a $\mathbb{Z}[G]$-linear combination of terms of the form $x - 1$ in an obvious way. (If $g = w.x$ with w of lesser length than that of g, $g - 1 = w - 1 + w(x - 1)$, so use induction on the length of the expression for g in terms of the generators.) ■

When G is free: If G is free, say, $G \cong F(X)$, i.e., is free on the set X, we can say more.

Proposition 6 *If $G \cong F(X)$ is the free group on the set X, then the set $\{x - 1 : x \in X\}$ freely generates $I(G)$ as a $\mathbb{Z}[G]$-module.*

Proof: (We will write F for $F(X)$.) The easiest proof would seem to be to check the universal property of derived modules for the function $\delta : F \to \mathbb{Z}[G]^{(X)}$, given on generators by

$$\delta(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

then extended using the derivation rule to all of F using induction. This uses essentially that each element of F has a *unique* expression as a reduced word in the generators, X.

Suppose then that we have a derivation $\partial : F \to M$, define $\overline{\partial} : \mathbb{Z}[G]^{(X)} \to M$ by $\overline{\partial}(e_x) = \partial(x)$, extending linearly. Since by construction $\overline{\partial}\delta = \partial$ and is the unique such homomorphism, we are home. ■

Note: In both these proofs we are thinking of the elements of the free module on X as being functions from X to the group ring, these functions being of 'finite support', i.e. being non-zero on only a finite number of elements of X. This can cause some complications if X is infinite or has some topology as it will in some contexts. The *idea* of the proof will usually go across to that situation but details have to change. (A situation in which this happens is in profinite group theory where the derivations have to be continuous for the profinite topology on the group, see [106].)

3.2.6 (D_ϕ, d_ϕ), the general case.

We can now return to the identification of (D_ϕ, d_ϕ) in the general case.

Proposition 7 *If $\phi : G \to H$ is a homomorphism of groups, then $D_\phi \cong \mathbb{Z}[H] \otimes_G I(G)$, the tensor product of $\mathbb{Z}[H]$ and $I(G)$ over G.*

Proof: If M is a $\mathbb{Z}[H]$-module, we will write $\phi^*(M)$ for the restricted $\mathbb{Z}[G]$-module, i.e. M with G-action given by $g.m := \phi(g).m$. Recall that the functor ϕ^* has a left adjoint given by sending a G-module, N to $\mathbb{Z}[H] \otimes_G N$, i.e. take the tensor of Abelian groups, $\mathbb{Z}[H] \otimes N$ and divide out by $x \otimes g.n \equiv x\phi(g) \otimes n$.

With this notation we have a chain of natural isomorphisms,

$$\begin{aligned} Der_\phi(G, M) &\cong Der_G(G, \phi^*(M)) \\ &\cong G{-}Mod(I(G), \phi^*(M)) \\ &\cong H{-}Mod(\mathbb{Z}[H] \otimes_G I(G), M), \end{aligned}$$

so by universality,

$$D_\phi \cong \mathbb{Z}[H] \otimes_G I(G),$$

as required. ■

3.2.7 D_ϕ for $\phi : F(X) \to G$.

The above will be particularly useful when ϕ is the "co-unit" map, $F(X) \to G$, for X a set that generates G. We could, for instance, take $X = G$ as a set, and ϕ to be the usual natural epimorphism.

In fact we have the following:

Corollary 3 *Let $\phi : F(X) \to G$ be an epimorphism of groups, then there is an isomorphism*

$$D_\phi \cong \mathbb{Z}[G]^{(X)}$$

of $\mathbb{Z}[G]$-modules. In this isomorphism, the generator ∂_x, of D_ϕ corresponding to $x \in X$, satisfies

$$d_\phi(x) = \partial_x$$

for all $x \in X$. ■

(You should check that you see how this follows from our earlier results.)

3.3 Associated module sequences

3.3.1 Homological background

Given an exact sequence

$$1 \to K \to L \to Q \to 1$$

of abstract groups, then it is a standard result from homological algebra that there is an associated exact sequence of modules,

$$0 \to K^{Ab} \to \mathbb{Z}[Q] \otimes_L I(L) \to I(Q) \to 0.$$

There are several different proofs of this. Homological proofs give this as a simple consequence of the Tor^L-sequence corresponding to the exact sequence

$$0 \to I(L) \to \mathbb{Z}[L] \to \mathbb{Z} \to 0$$

together with a calculation of $Tor_1^L(\mathbb{Z}[Q], \mathbb{Z})$, but we are not assuming that much knowledge of standard homological algebra. That homological proof also, to some extent, hides what is happening at the 'elementary' level, in both the sense of 'simple' and also that of'what happens to the 'elements' of the groups and modules concerned.

The second type of proof is more directly algebraic and has the advantage that it accentuates various universal properties of the sequence. The most thorough treatment of this would seem to be by Crowell, [41], for the discrete case. We outline it below.

3.3.2 The exact sequence.

Before we start on the discussion of the exact sequence, it will be useful to have at our disposal some elementary results on Abelianisation of the groups in a crossed module. Here we actually only need them for normal subgroups but we will need it shortly anyway in the more general form. Suppose that (C, P, ∂) is a crossed module, and we will set $A = Ker\partial$ with its module structure that we looked at before, and $N = \partial C$, so A is a P/N-module.

Lemma 9 *The Abelianisation of C has a natural $\mathbb{Z}[P/N]$-module structure on it.*

Proof: First we should point out that by "Abelianisation" we mean $C^{Ab} = C/[C, C]$, which is, of course, Abelian and it suffices to prove that N acts trivially on C^{Ab}, since P already acts in a natural way. However, if $n \in N$, and $\partial c = n$, then for any $c' \in C$, we have that ${}^n c' = {}^{\partial c} c' = cc'c^{-1}$, hence ${}^n c'(c')^{-1} \in [C, C]$ or equivalently

$${}^n(c'[C, C]) = c'[C, C],$$

so N does indeed act trivially on C^{Ab}. ∎

Of course N^{Ab} also has the structure of a $\mathbb{Z}[P/N]$-module and thus a crossed module gives one three P/N-modules. These three are linked as shown by the following proposition.

Proposition 8 *Let (C, P, ∂) be a crossed module. Then the induced morphisms*

$$A \to C^{Ab} \to N^{Ab} \to 0$$

form an exact sequence of $\mathbb{Z}[P/N]$-modules.

Proof: It is clear that the sequence

$$1 \to A \to C \to N \to 1$$

is exact and that the induced homomorphism from C^{Ab} to N^{Ab} is an epimorphism. Since the composite homomorphism from A to N is trivial, A is mapped into $Ker(C^{Ab} \to N^{Ab})$ by the composite $A \to C \to C^{Ab}$. It is easily checked that this is onto and hence the sequence is exact as claimed. ∎

Now for the main exact sequence result here:

Proposition 9 *Let*

$$1 \to K \xrightarrow{\phi} L \xrightarrow{\psi} Q \to 1$$

be an exact sequence of groups and homomorphisms. Then there is an exact sequence

$$0 \to K^{Ab} \xrightarrow{\tilde{\phi}} \mathbb{Z}[Q]\hat{\otimes}_L I(L) \xrightarrow{\tilde{\psi}} I(Q) \to 0$$

of $\mathbb{Z}[Q]$*-modules.*

Proof: By the universal property of D_ψ, there is a unique morphism

$$\tilde{\psi} : D_\psi \to I(Q)$$

such that $\tilde{\psi}\partial_\psi = I(\psi)\partial_L$.

Let $\delta : K \to K^{Ab} = K/[K, K]$ be the canonical Abelianising morphism. We note that $\partial_\psi\phi : K \to D_\psi$ is a homomorphism (since

$$\begin{aligned}
\partial_\psi\phi(k_1k_2) &= \partial_\psi\phi(k_1) + \psi\phi(k_1)\partial_\psi\phi(k_2) \\
&= \partial_\psi\phi(k_1) + \partial_\psi\phi(k_2),)
\end{aligned}$$

so let $\tilde{\phi} : K^{Ab} \to D_\psi$ be the unique morphism satisfying $\tilde{\phi}\delta = \partial_\psi\phi$ with K^{Ab} having its natural $\mathbb{Z}[Q]$-module structure.

That the composite $\tilde{\psi}\tilde{\phi} = 0$ follows easily from $\psi\phi = 0$. Since D_ψ is generated by symbols $d\ell$ and $\tilde{\psi}(d\ell) = \psi(\ell) - 1$, it follows that $\tilde{\psi}$ is onto. We next turn to "$Ker\ \tilde{\psi} \subseteq Im\ \tilde{\phi}$".

If we can prove $\alpha : D_\psi \to I(Q)$ is the cokernel of $\tilde{\phi}$, then we will have checked this inclusion and incidentally will have reproved that $\tilde{\psi}$ is onto.

Now let $D_\psi \to C$ be any morphism such that $\alpha\tilde{\phi} = 0$. Consider the diagram

$$\begin{tikzcd}
K \arrow[r, "\phi"] \arrow[d, "\delta"'] & L \arrow[r, "\psi"] \arrow[d, "\partial_\psi"] & Q \arrow[d, "\partial_Q"] \\
K^{Ab} \arrow[r, "\tilde{\phi}"] & D_\psi \arrow[r, "\tilde{\psi}"] \arrow[dr, "\alpha"'] & \mathcal{C}(Q) \\
& & C
\end{tikzcd}$$

The composite $\alpha\partial_\psi$ vanishes on the image of ϕ since $\alpha\partial_\psi\phi = \alpha\tilde{\phi}\delta$ and $\alpha\tilde{\phi}$ is assumed zero. Define $d : Q \to C$ by $d(q) = \alpha\partial_\psi(\ell)$ for $\ell \in L$ such that $\psi(\ell) = q$. As $\alpha\partial_\psi$ vanishes on $Im\ \phi$, this is well defined and

$$\begin{aligned}
d(q_1q_2) &= \alpha\partial_\psi(\ell_1\ell_2) \\
&= \alpha\partial_\psi(\ell_1) + \alpha(\psi(\ell_1)\partial_\psi(\ell_2)) \\
&= d(q_1) + q_1 d(q_2)
\end{aligned}$$

so d factors as $\bar{\alpha}\partial_Q$ in a unique way with $\bar{\alpha} : I(Q) \to C$. It remains to prove that $\alpha = \tilde{\psi}$, but

$$\begin{aligned}
\tilde{\psi}\partial_\psi &= I_C(\psi)\partial_L \\
&= \partial_Q\psi
\end{aligned}$$

by the naturality of ∂. Now finally note that $\bar{\alpha}\partial_Q = d$ and $d\psi = \alpha\partial_\psi$ to conclude that $\tilde{\psi}\partial_\psi$ and $\alpha\partial_\psi$ are equal. Equality of α and $\bar{\alpha}\tilde{\psi}$ then follows by the uniqueness clause of the universal property of (D_ψ, ∂_ψ).

Next we need to check that $K^{Ab} \to D_\psi$ is a monomorphism. To do this we use the fact that there is a transversal, $s : Q \to L$, satisfying $s(1) = 1$. This means that, following Crowell, [41] p. 224, we can for each $\ell \in L$, $q \in Q$, find an element $q \times \ell$ uniquely determined by the equation

$$\phi(q \times \ell)) = s(q)\ell s(q\psi(\ell))^{-1},$$

which, of course, defines a function from $Q \times L$ to K. Crowell's lemma 4.5 then shows

$$q \times \ell_1\ell_2 = (q \times \ell_1)(q\psi(\ell_1) \times \ell_2) \quad \text{for} \quad \ell_1, \ell_2 \in L.$$

Now let $M = \mathbb{Z}[Q]^{(X)}$, with $X = \{\partial\ell : \ell \in L\}$, so that there is an exact sequence

$$M \to D_\psi \to 0.$$

The underlying group of $\mathbb{Z}[Q]$ is the free Abelian group on the underlying set of Q. Similarly M, above, has, as underlying group, the free Abelian group on the set $Q \times X$.

Define a map $\tau : M \to K^{Ab}$ of Abelian groups by

$$\tau(a, \partial\ell) = \delta(q \times \ell).$$

We check that if $p(m) = 0$, then $\tau(m) = 0$. Since $Ker\, p$ is generated as a $\mathbb{Z}[Q]$-module by elements of the form

$$\partial(\ell_1\ell_2) - \partial\ell_1 - \psi(\ell_1)\partial\ell_2,$$

it follows that as an Abelian group, $Ker\, p$ is generated by the elements

$$(q, \partial(\ell_1\ell_2)) - (q, \partial\ell_1) - (q\psi(\ell_1), \partial\ell_2).$$

We claim that τ is zero on these elements; in fact

$$\begin{aligned}\tau(q, \partial(\ell_1\ell_2)) &= \delta(q \times (\ell_1\ell_2)) \\ &= \delta(q \times \ell_1) + \delta(q\psi(\ell_1) \times \ell_2) \\ &= \tau(q, \ell_1) + \tau(q\psi(\ell_1), \ell_2).\end{aligned}$$

Thus τ induces a map $\eta : D_\psi \to K^{Ab}$ of Abelian groups.

Finally we check $\eta\tilde{\phi} =$ identity, so that $\tilde{\phi}$ is a monomorphism: let $b \in K^{Ab}$, $k \in K$ be such that $\delta(k) = b$, then

$$\begin{aligned}\eta\tilde{\phi}(b) &= \eta\tilde{\phi}\delta(k) \\ &= \eta\partial_\psi(k) \\ &= \delta(1 \times \phi(k)),\end{aligned}$$

but $1 \times \phi(k)$ is uniquely determined by

$$\phi(1 \times \phi(k)) = s(1)\phi(k)s(1\psi\phi(k))^{-1} = \phi(k),$$

since $s(1) = 1$, hence $1 \times \phi(k) = k$ and $\eta\tilde{\phi}(b) = \delta(k) = b$ as required. ■

A discussion of the way in which this result interacts with the theory of covering spaces can be found in Crowell's paper already cited. We will very shortly see the connection of this module sequence with the Jacobian matrix of a group presentation and the Fox free differential calculus. It is this latter connection which suggests that we need more or less explicit formulae for the maps $\tilde{\phi}$ and $\tilde{\psi}$ and hence requires that Crowell's detailed proof be used, not the slicker homological proof.

3.3.3 Reidemeister-Fox derivatives and Jacobian matrices

At various points we will refer to Reidemeister-Fox derivatives as developed by Fox in a series of articles, see [59], and also summarised in Crowell and Fox, [42]. We will call these derivatives *Fox derivatives*.

Suppose G is a group and M a G-module and let $\delta : G \to M$ be a derivation, (so $\delta(g_1g_2) = \delta(g_1) + g_1\delta(g_2)$ for all $g_1, g_2 \in G$), then, for calculations, the following lemma is very valuable, although very simple to prove.

Lemma 10 *If $\delta : G \to M$ is a derivation, then*
(i) $\delta(1_G) = 0$;
(ii) $\delta(g^{-1}) = -g^{-1}\delta(g)$ for all $g \in G$;
(iii) for any $g \in G$ and $n \geq 1$,

$$\delta(g^n) = (\sum_{k=0}^{n-1} g^k)\delta(g).$$

Proof: As was said, these are easy to prove.
$\delta(g) = \delta(1g) + 1\delta((g)$, so $\delta(1) = 0$, and hence (i); then

$$\delta(1) = \delta(g^{-1}g) = \delta(g^{-1}) + g^{-1}\delta(g)$$

to get (ii), and finally induction to get (iii). ■

The Fox derivatives are derivations taking values in the group ring as a left module over itself. They are defined for $G = F(X)$, the free group on a set X. (We usually write F for $F(X)$ in what follows.)

Definition: For each $x \in X$, let

$$\frac{\partial}{\partial x} : F \to \mathbb{Z}F$$

be defined by
(i) for $y \in X$,

$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

(ii) for any words, $w_1, w_2 \in F$,

$$\frac{\partial}{\partial x}(w_1w_2) = \frac{\partial}{\partial x}w_1 + w_1\frac{\partial}{\partial x}w_2.$$

Of course, a routine proof shows that the derivation property in (ii) defines $\frac{\partial w}{\partial x}$ for any $w \in F$.
This derivation, $\frac{\partial}{\partial x}$, will be called the *Fox derivative with respect to the generator x.*

Example: Let $X = \{u, v\}$, with $r \equiv uvuv^{-1}u^{-1}v^{-1} \in F = F(u, v)$, then

$$\begin{aligned} \frac{\partial r}{\partial u} &= 1 + uv - uvuv^{-1}u^{-1}, \\ \frac{\partial r}{\partial v} &= u - uvuv^{-1} - uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

This relation is the typical braid group relation, here in Br_3, and we will come back to these simple calculations later.

It is often useful to extend a derivation $\delta : G \to M$ to a linear map from $\mathbb{Z}G$ to M by the simple rule that $\delta(g+h) = \delta(g) + \delta(h)$.

We have
$$Der(F, \mathbb{Z}F) \cong F{-}Mod(IF, \mathbb{Z}F),$$
and that
$$IF \cong \mathbb{Z}F^{(X)},$$
with the isomorphism matching each generating $x-1$ with e_x, the basis element labelled by $x \in X$. (The universal derivation then sends x to e_x.)

For each given x, we thus obtain a morphism of F-modules:
$$d_x : \mathbb{Z}F^{(X)} \to \mathbb{Z}F$$
with
$$\begin{aligned} d_x(e_y) &= 1 \quad \text{if } y = x \\ d_x(e_y) &= 0 \quad \text{if } y \neq x, \end{aligned}$$
i.e., the 'projection onto the x^{th}-factor' or 'evaluation at $x \in X$' depending on the viewpoint taken of the elements of the free module, $\mathbb{Z}F^{(X)}$.

Suppose now that we have a group presentation, $\mathcal{P} = (X : R)$, of a group, G. Then we have a short exact sequence of groups
$$1 \to N \xrightarrow{\phi} F \xrightarrow{\gamma} G \to 1,$$
where $N = N(R)$, $F = F(X)$, i.e., N is the normal closure of R in the free group F. We also have a free crossed module,
$$C \xrightarrow{\partial} F,$$
constructed from the presentation and hence, two short exact sequences of G-modules with $\kappa(\mathcal{P}) = Ker\,\partial$, the module of identities of $\mathcal{P}$,
$$0 \to \kappa(\mathcal{P}) \to C^{Ab} \to N^{Ab} \to 0,$$
and also
$$0 \to N^{Ab} \xrightarrow{\tilde{\phi}} IF \otimes_F \mathbb{Z}G \to IG \to 0.$$
We note that the first of these is exact because N is a free group, further
$$C^{Ab} \cong \mathbb{Z}G^{(R)},$$
(the proof is left to you to manufacture from earlier results), and the map from this to N^{Ab} in the first sequence sends the generator e_r to $r[N, N]$.

We next revisit the derivation of the associated exact sequence (Proposition 9, page 57) in some detail to see what $\tilde{\phi}$ does to $r[N, N]$. We have $\tilde{\phi}(r[N, N]) = \partial_\gamma \phi(r) = \partial_\gamma(r)$, considering r now as an element of F, and by Corollary 3, on identifying D_γ with $\mathbb{Z}G^{(X)}$ using the isomorphism between IF and $\mathbb{Z}F^{(X)}$, we can identify $\partial_\gamma(x) = e_x$. We are thus left to determine $\partial_\gamma(r)$ in terms of the $\partial_\gamma(x)$, i.e., the e_x. The following lemma does the job for us.

Lemma 11 *Let $\delta : F \to M$ be a derivation and $w \in F$, then*

$$\delta w = \sum_{x \in X} \frac{\partial w}{\partial x} \delta x.$$

Proof: By induction on the length of w. ■

In particular we thus can calculate

$$\partial_\gamma(r) = \sum \frac{\partial r}{\partial x} e_x.$$

Tensoring with $\mathbb{Z}G$, we get

$$\tilde{\phi}(r[N,N]) = \sum \frac{\partial r}{\partial x} e_x \otimes 1.$$

There is one final step to get this into a usable form:

From the quotient map $\gamma : F \to G$, we, of course, get an induced ring homomorphism, $\gamma : \mathbb{Z}F \to \mathbb{Z}G$, and hence we have elements $\gamma(\frac{\partial r}{\partial x}) \in \mathbb{Z}G$. Of course,

$$\frac{\partial r}{\partial x} e_x \otimes 1 = e_x \otimes \gamma(\frac{\partial r}{\partial x}),$$

so we have, on tidying up notation just a little:

Proposition 10 *The composite map*

$$\mathbb{Z}G^{(R)} \to N^{Ab} \to \mathbb{Z}G^{(X)}$$

sends e_r to $\sum \gamma(\frac{\partial r}{\partial x}) e_x$ and so has a matrix representation given by $J_{\mathcal{P}} = \left(\gamma(\frac{\partial r_i}{\partial x_j})\right)$. ■

Definition: The *Jacobian matrix* of a group presentation, $\mathcal{P} = (X : R)$ of a group G is

$$J_{\mathcal{P}} = \left(\gamma(\frac{\partial r_i}{\partial x_j})\right),$$

in the above notation.

The application of γ to the matrix of Fox derivatives simplifies expressions considerable in the matrix. The usual case of this is if a relator has the form rs^{-1}, then we get

$$\frac{\partial rs^{-1}}{\partial x} = \frac{\partial r}{\partial x} - rs^{-1}\frac{\partial s}{\partial x}$$

and if r or s is quite long this looks moderately horrible to work out! However applying γ to the answer, the term rs^{-1} in the second of the two terms becomes 1. We can actually think of this as replacing rs^{-1} by $r - s$ when working out the Jacobian matrix.

Example: Br_3 **revisited.** We have $r \equiv uvuv^{-1}u^{-1}v^{-1}$, which has the form $(uvu)(vuv)^{-1}$. This then gives

$$\gamma(\frac{\partial r}{\partial u}) = 1 + uv - v \quad \text{and} \quad \gamma(\frac{\partial r}{\partial v}) = u - 1 - vu,$$

abusing notation to ignore the difference between u, v in $F(u, v)$ and the generating u, v in Br_3.

Homological 2-syzygies: In general we obtain a truncated chain complex:

$$\mathbb{Z}G^{(R)} \xrightarrow{d_2} \mathbb{Z}G^{(X)} \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{d_0} \mathbb{Z} \to 0,$$

with d_2 given by the Jacobian matrix of the presentation, and d_1 sending generator e^1_x to $1 - x$, so $Im\, d_1$ is the augmentation ideal of $\mathbb{Z}G$.

Definition: A *homological 2-syzygy* is an element in $Ker\, d_2$..

A homological 2-syzygy is thus an element to be killed when building the third level of a resolution of G. What are the links between homotopical and homological syzygies? Brown and Huebschmann, [30], show they are isomorphic, as $Ker\, d_2$ is isomorphic to the module of identities. We will examine this result in more detail shortly.

Extended example: Homological Syzygies for the braid group presentations: The Artin braid group, Br_{n+1}, defined using $n + 1$ strands is given by

- generators: y_i, $i = 1, \ldots, n$;
- relations: $r_{ij} \equiv y_i y_j y_i^{-1} y_j^{-1}$ for $i + 1 < j$;
  $r_{ii+1} \equiv y_i y_{i+1} y_i y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$ for $1 \leq i < n$.

We will look at such groups only for small values of n.

By default, Br_2 has one generator and no relations, so is infinite cyclic.

The group Br_3: (We will simplify notation writing $u = y_1$, $v = y_2$.)

This then has presentation $\mathcal{P} = (u, v : r \equiv uvuv^{-1}u^{-1}v^{-1})$. It is also the 'trefoil group', i.e., the fundamental group of the complement of a trefoil knot. If we construct $X(2) = K(\mathcal{P})$, this is already a $K(Br_3, 1)$ space, having a trivial π_2. There are no higher syzygies.

We have all the calculation for working with homological syzygies here. The key part of the complex is the Jacobian matrix as that determines d_2:

$$d_2 = \begin{pmatrix} 1 + uv - v & u - 1 - vu \end{pmatrix}.$$

This has trivial kernel, but, in fact, that comes most easily from the identification with homotopical syzygies.

The group Br_4: simplifying notation as before, we have generators u, v, w and relations

$$\begin{aligned} r_u &\equiv vwvw^{-1}v^{-1}w^{-1}, \\ r_v &\equiv uwu^{-1}w^{-1}, \\ r_w &\equiv uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

The 1-syzygies are made up of hexagons for r_u and r_w and a square for r_v. There is a fairly obvious way of fitting together squares and hexagons, namely as a permutohedron, and there is a labelling of such that gives a homotopical 2-syzygy.

The presentation yields a truncated chain complex with d_2

$$\mathbb{Z}G^{(r_u,r_v,r_w)} \xrightarrow{d_2} \mathbb{Z}G^{(u,v,w)}$$

with

$$d_2 = \begin{pmatrix} 0 & 1+vw-w & v-1-wv \\ 1-w & 0 & u-1 \\ 1+uv-v & u-1-vu & 0 \end{pmatrix}$$

and Loday, [82], has calculated that for the permutohedral 2-syzygy, s, one gets another term of the resolution, $\mathbb{Z}G^{(s)}$, and a $d_3 : \mathbb{Z}G^{(s)} \to \mathbb{Z}G^{(r_u,r_v,r_w)}$ given by

$$d_3 = \begin{pmatrix} 1+vu-u-wuv & v-vwu-1-uv-vuwv & 1+vw-w-uvw \end{pmatrix}.$$

For more on methods of working with these syzygies, have a look at Loday's paper, [82], and some of the references that you will find there.

3.4 Crossed complexes and chain complexes: II

3.4.1 The reflection from Crs to chain complexes

It is now time to return to the construction of a left adjoint for Δ_G.

Proposition 11 *The functor Δ_G has a left adjoint.*

Proof: We construct the left adjoint explicitly as follows:

Let $f. : (\mathsf{C},\phi) \to \Delta_G(M.)$ be a morphism in Crs_G, then we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\phi"] \arrow[d, "f_0"] & G \arrow[d, "Id_G"] \\
\cdots \arrow[r] & M_2 \arrow[r, "\delta_2"] & M_1 \arrow[r, "\delta_1"] & M_0 \rtimes G \arrow[r, "pr_G"] & G
\end{tikzcd}$$

Since the right hand square commutes, f_0 is given by some formula

$$f_0(c) = (\partial(c), \phi(c)),$$

where $\partial : C_0 \to M_0$ is a ϕ-derivation. Thus $\partial = \tilde{f}_0\partial_\phi$ for a unique G-module morphism, $\tilde{f}_0 : D_\phi \to M_0$, and f_0 factors as

$$C_0 \xrightarrow{\bar{\phi}} D_\phi \rtimes G \xrightarrow{\tilde{f}_0 \rtimes G} M_0 \rtimes G,$$

where $\bar{\phi}(c) = (\partial_\phi(c), \phi(c))$.

The map $\partial_\phi\delta_1 : C_1 \to D_\phi$ is a homomorphism since

$$\begin{aligned} \partial_\phi\delta_1(c_1c_2) &= \partial_\phi\partial_1(c_1) + \phi\partial_1(c_1)\partial_\phi\partial_1(c_2) \\ &= \partial_\phi\partial_1(c_1) + \partial_\phi\partial_1(c_2), \end{aligned}$$

$\phi\partial_1$ being trivial (because (C,ϕ) is G-augmented). We thus obtain a map $d : C_1^{Ab} \to D_\phi$ given by $d(c[C,C]) = \partial_\phi\partial_1(c)$ for $c \in C_1$. As we observed earlier the Abelian group C_1^{Ab} has a natural $\mathbb{Z}[G]$-module structure making d a G-module morphism.

Similarly there is a unique G-module morphism,

$$\tilde{f}_1 : C_1^{Ab} \to M_1,$$

satisfying

$$\tilde{f}_1(c[C,C]) = f_1(c).$$

Since for $c \in C_1$,

$$(d_1\tilde{f}_1(c), 1) = f_0(\delta_1 c) = (\tilde{f}_0\partial_\phi(\delta_1 c_1), 1),$$

we have that the diagram

$$\begin{tikzcd} C_1^{Ab} \arrow[r, "\tilde{f}_1"] \arrow[d, "d"'] & M_1 \arrow[d, "d_1"] \\ D_\phi \arrow[r, "\tilde{f}_0"] & M_0 \end{tikzcd}$$

commutes.

We also note that since $\delta_2 : C_2 \to C_1$ maps into $Ker\ \delta_1$, the composite

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\text{can}} C_1^{Ab} \xrightarrow{d} D_\phi,$$

being given by $d(\delta_2(c)[C,C] = \partial_\phi\delta_1\delta_2(c)$, is trivial and that $\tilde{f}_1\delta_2(c[C,C]) = f_1\delta_2(c) = d_2f_2(c)$, thus we can define $\xi = \xi_G(\mathsf{C}, \phi)$ by

$$\begin{aligned} \xi_n &= C_n \text{ if } n \geq 2 \\ \xi_1 &= C_1^{Ab}, \\ \xi_0 &= D_\phi, \end{aligned}$$

the differentials being as constructed. We note that as $Ker\ \phi$ acts trivially on all C_n for $n \geq 2$, all the C_n have $\mathbb{Z}[G]$-module structures.

That ξ_G gives a functor

$$Crs \to Ch(G{-}Mod)$$

is now easy to check using the uniqueness clauses in the universal properties of D_ϕ and Abelianisation. Again uniqueness guarantees that the process "f goes to $\tilde{f}$" gives a natural isomorphism

$$Ch(G - Mod)(\xi_G(\mathsf{C}, \phi), \mathsf{M}) \cong Crs_G((\mathsf{C}, \phi), \Delta_G(\mathsf{M}))$$

as required. ■

It is relatively easy to extend the above natural isomorphism to handle morphisms of crossed complexes over different groups. For a detailed treatment one needs a discussion of the way that the change of groups functors work with crossed modules or crossed complexes, that is, if we have a morphism of groups $\theta : G \to H$ then we would expect to get functors between Crs_G and Crs_H induced by θ. These do exist and are very nicely behaved, but they will not be discussed here, see [106] for a full treatment in the more general context of profinite groups.

3.4.2 Crossed resolutions and chain resolutions

One of our motivations for introducing crossed complexes was that they enable us to model more of the sort of information encoded in a $K(G,1)$ than does the usual standard algebraic models, e.g. a chain complex such as the bar resolution. In particular, whilst the bar resolution is very good for cohomology with Abelian coefficients for non-Abelian cohomology the crossed version can allow us to push things further, but then comparison on the Abelian theory is very necessary! It is therefore of importance to see how this $K(G,1)$ information that we have encoded changes under the functor $\xi : Crs \to Ch(G{-}Mod)$.

We start with a crossed resolution determined in low dimensions by a presentation $\mathcal{P} = (X : R)$ of a group, G. Thus, in this case, $C_0 = F(X)$ with $\phi : F(X) \to G$, the 'usual' epimorphism, and $C_1 \to C_0$ is $C \to F(X)$, the free crossed module on $R \to F(X)$. It is not too hard to show that $C_1^{Ab} \cong \mathbb{Z}[G]^{(R)}$, the free $\mathbb{Z}[G]$-module on R. (The proof is left as an exercise.) This maps down onto $N(R)^{Ab}$, the Abelianisation of the normal closure of R in $F(X)$ via a map

$$\partial_* : \mathbb{Z}[G]^{(R)} \to N(R)^{Ab},$$

given by $\partial_*(e_r) = r[N(R), N(R)]$, where e_r is the generator of $\mathbb{Z}[G]$ corresponding to $r \in R$.

There is also a short exact sequence

$$1 \to N(R) \xrightarrow{i} F(X) \xrightarrow{\phi} G \to 1$$

and hence by Proposition 9, a short exact sequence

$$0 \to N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G] \otimes_F I(F) \xrightarrow{\tilde{\phi}} I(G) \to 0$$

(where we have written $F = F(X)$).

By the Corollary to Proposition 7, we have

$$\mathbb{Z}[G] \otimes_F I(F) \cong \mathbb{Z}[G]^{(X)}.$$

The required map $C_1^{Ab} \to D_\phi$ is the composite

$$\mathbb{Z}[G]^{(R)} \xrightarrow{\partial_*} N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G]^{(X)}.$$

We have given an explicit description of ∂_* above, so to complete the description of d, it remains to describe $\tilde{i}$, but $\tilde{i}$ satisfies $\tilde{i}\delta = \partial_\phi i$, where $\delta : N(R) \to N(R)^{Ab}$, so $\tilde{i}(r[N(R), N(R)]) = d_\phi(r)$. Thus if r is a relator, i.e., if it is in the image of the subgroup generated by the elements of R, then $\partial(e_r)$ can be written as a finite sum of the form $\sum_x a_x e_x$ and the elements $a_x \in \mathbb{Z}[G]$ are the images of the Fox derivatives.

This operator can best be viewed as the Alexander matrix of a presentation of a group, further study of this operator depends on studying transformations between free modules over group rings, and we will not attempt to study those here.

The rest of the crossed resolution does not change and so, on replacing $I(G)$ by $\mathbb{Z}[G] \to \mathbb{Z}$, we obtain a free pseudocompact $\mathbb{Z}[G]$-resolution of the trivial module $\mathbb{Z}$,

$$\ldots \to \mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}$$

built up from the presentation. This is the complex of chains on the universal cover, $\widetilde{K(G,1)}$, where $K(G,1)$ is constructed starting from a presentation $\mathcal{P}$.

3.4.3 Standard crossed resolutions and bar resolutions

We next turn to the special case of the standard crossed resolution of G discussed briefly earlier. Of course this is a special case of the previous one, but it pays to examine it in detail.

Clearly in $\xi = \xi(\mathsf{C}G, \phi)$, we have:
ξ_0 = the free $\mathbb{Z}[G]$-module on the underlying set of G, individual generators being written $[u]$, for $u \in G$;
ξ_1 = the free $\mathbb{Z}[G]$ -module on $G \times G$, generators being written $[u, v]$;
$\xi_n = C_n G$, the free $\mathbb{Z}[G]$ -module on G^{n+1}, etc.

The map $d_2 : \xi_2 \to \xi_1$ induced from δ_2 is given by

$$d_2[u, v, w] = u[v, w] - [u, v] - [uv, w] + [u, vw],$$

and the map $d_1 : \xi_1 \to \xi_0$ by

$$\begin{aligned} d_1([u, v]) &= d_\phi([uv]^{-1}[u][v]) \\ &= v^{-1}u^{-1}(-[uv] + [u] + u[v]), \end{aligned}$$

a unit times the usual bar resolution formula. Thus, as claimed earlier, the standard crossed resolution is the crossed analogue of the bar resolution.

3.4.4 The intersection $A \cap [C, C]$.

We next turn to a comparison of homological and homotopical syszygies. We have almost all the preliminary work already. The next ingredient is a result that will identify the intersection of the kernel of a crossed module, $A = Ker(C \xrightarrow{\partial} P)$ and the commutator subgroup of C.

The kernel of the homomorphism from A to C^{Ab} is, of course, $A \cap [C, C]$ and this need not be trivial. In fact, Brown and Huebschmann ([30], p.160) note that in examples of type $(G, Aut(G), \partial)$, the kernel of ∂ is, of course, the centre ZG of G and $ZG \cap [G, G]$ can be non-trivial, for instance, if G is dicyclic or dihedral.

We will adopt the same notation as previously with $N = \partial P$ etc.

Proposition 12 *If in the exact sequence of groups*

$$1 \to A \to C \xrightarrow{p} N \to 1,$$

the epimorphism from C to N is split (the splitting need not respect G-action), then $A \cap [C, C]$ is trivial.

Proof: Given a splitting $s : N \to C$, (so ps is the identity on N), then the group C can be written as $A \rtimes s(N)$. The commutators in C, therefore, all lie in $s(N)$ since A is Abelian, but then, of course, $A \cap [C, C]$ cannot contain any non-trivial elements. ■

Brown and Huebschmann, [30], p. 168, prove that for an group G with presentation $\mathcal{P}$, the module of identities for $\mathcal{P}$ is naturally isomorphic to the second homology group, $H_2(\tilde{K}(\mathcal{P}))$, of the universal cover of $K(\mathcal{P})$, the 2-complex of the presentation. We can approach this via the algebraic constructions we have.

Given a presentation $\mathcal{P} = \langle X : R\rangle$ of a group G, the algebraic analogue of $K(\mathcal{P})$, we have argued above, is the free crossed module $C(\mathcal{P}) \xrightarrow{d} F(X)$ and the chains on the universal cover of $K(\mathcal{P})$ will be given by ξ_G of this, i.e., by the chain complex

$$\mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)}.$$

In general there will be a short exact sequence

$$0 \to \kappa(\mathcal{P}) \cap [C(\mathcal{P}), C(\mathcal{P})] \to \kappa(\mathcal{P}) \to H_2(\xi(C(\mathcal{P})) \to 0.$$

This short exact sequence yields the Brown-Huebschmann result as $N(R)$ will a free group so the epimorphism onto $N(R)$ splits and we can use the above Proposition 12. We thus get

Proposition 13 *If $\mathcal{P} = \langle X : R\rangle$ is a free presentation of G, then there is an isomorphism*

$$\kappa \xrightarrow{\cong} H_2(\xi(C_{\mathcal{C}}(\mathcal{P})) = Ker(d : \mathbb{Z}[G]^R \to \mathbb{Z}[G]^X).$$

■

Note: Here we are using something that will not be true in all algebraic settings. A subgroup of a free group is always free, but the analogous statement for free algebras of other types is not true.

3.5 Simplicial groups and crossed complexes

3.5.1 From simplicial groups to crossed complexes

Given any simplicial group G, the formula,

$$\mathsf{C}(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions with at its 'bottom end' the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0$$

gives a crossed complex with ∂ induced from the boundary in the Moore complex. The detailed proof is too long to indicate here. It just checks the axioms one by one.

We should have a glance at this formula from various viewpoints, some of which will be revisited later. Once again there is a clear link with the non-uniqueness of fillers for horns in a simplicial group if it is not a group T-complex. We have all those $(NG_n \cap D_n)$ terms involved!

Suppose that we had our simplicial group G and wanted to construct a quotient of it that was a group T-complex. We could do this in a silly way since the trivial simplicial group is clearly a group T-complex, but let us keep the quotient as large as possible. This problem is related to the question of whether the category of group T-complexes forms a reflexive subcategory of *Simp.Grps*. The condition $NG \cap D = 1$ looks like some sort of 'equational specification'. Our question can thus really be posed as follows: Suppose we have a simplicial group morphism $f : G \to H$ and H is a group T-complex. Remember that in group T-complexes, as against the non-algebraic ones, the thin structure is not an added bit of structure. The thin elements are determined by the degeneracies, so whether or not H is or is not a group T-complex is somehow its own affair, and nothing to do with any external factors! Does f factor universally through some 'group T-complexification' of G? Something like

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "proj"'] & & H \\ & G/T(G) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

with $G/T(G)$ a group T-complex and $\hat{f}$ uniquely determined by the diagram.

One sensible way to look at such a question is to assume, provisionally, that such a factorisation exists and to see what $T(G)$ would have to be. In general, if $f : G \to H$ is any simplicial group morphism (with no restriction on H for the moment), then with a hopefully obvious notation,

$$f_n(NG_n \cap D(G)_n) \subseteq NH_n \cap D(H)_n,$$

since f sends degenerate elements to degenerate elements and preserves products! Back in our situation in which H is a group T-complex, then $f_n(NG_n \cap D(G)_n) = 1$, for the simple reason that the right hand side of that displayed formula is trivial by assumption. We thus have that if some such $T(G)$ exists, then we must have $NG_n \cap D(G)_n \subseteq T(G)_n$ and our first attempt might be to look at the possibility that they should be equal. This is wrong and for fairly trivial reasons. The subgroup $T(G)_n$ of G_n has to be normal if we are to form the quotient by it, and there is no reason why $NG_n \cap D(G)_n$ should be a normal subgroup in general.

We might then be tempted to take the normal subgroup generated by $NG_n \cap D(G)_n$, but that is 'defeatist' in this situation. We might hope to do detailed calculations with the subgroup and if it is specified as a normal closure, we will lose some of our ability to do that, at least without considerable more effort. (Let's be lazy and see if we can get around that difficulty.) If we look again, we find another thing that 'goes wrong' with any attempt to use $NG_n \cap D(G)_n$ as it is. This subgroup would be within NG_n, of course, and we want to induce a map from the Moore complex of G to that of $G/T(G)$. For that to work, we would need not only $NG_n \cap D(G)_n \subseteq T(G)_n$, but the image of $NG_n \cap D(G)_n$ under d_0 to be in $T(G)_{n-1}$. Going up a dimension, we thus need not only $NG_n \cap D(G)_n$, but $d_0(NG_{n+1} \cap D(G)_{n+1}) \subseteq T(G)_n$. We thus need the product subgroup $(NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$ to be in $T(G)_n$. This looks a bit complicated. Do we need to go any further up the Moore complex? No, because $d_0 d_0$ is trivial. We might thus try

$$T(G)_n = (NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$$

You might now think that this is a bit silly because we would still need this product subgroup to be normal in order to form the quotient ... , but it is! The lack of normality of our earlier attempt is absorbed by the image of the next level up. (That is pretty!)

Of course, there are very good reasons why this works. These involve what are sometimes called *Peiffer pairings*. We will see some of these later.

As a consequence of the above discussion, we more or less have:

Proposition 14 *If G is a group T-complex, then NG is a crossed complex.* ■

We certainly have a sketch of

Proposition 15 *The full subcategory of Simp.Grps determined by the group T-complexes is a reflective subcategory.* ■

Of course, the details of the proofs of both of these are left for you to write down. Nearly all of the reasoning for the second result is there for you, but some of the detailed calculations for the first are quite tricky.

The close link between group T-complexes and crossed complexes is evident from these results. You might guess that they form equivalent categories. They do. We will look at the way back from crossed complexes (of groups) to simplicial groups later on, but we need to get back to cohomology.

3.5.2 Simplicial resolutions, a bit of background

We need some such means of going from simplicial groups to crossed complexes so because we can also use simplicial resolutions to 'resolve' a group (and in many other situations). We first sketch in some historical background.

In the 1960s, the connection between simplicial groups and cohomology was examined in detail. The basic idea was that given the adjoint "free-forget" pair of functors between *Groups* and *Sets*, one could generate a free resolution of a group, G, using the resulting comonad (or cotriple) (cf. MacLane, [85]). This resolution was not, however, by a chain complex but by a free simplicial group, F, say. It was then shown (Barr and Beck, [11]) that given any G-module, M, and working in the category of groups over G, one could form the cosimplicial G-module, $Hom_{Gps/G}(F, M)$, and hence, by a dual form of the Dold-Kan theorem, a cochain complex $C(G, M)$, whose homotopy type, and hence whose homology, was independent of the choice of F. This homology was the usual Eilenberg-MacLane cohomology of G with coefficients in M, but with a shift in dimension (cf. Barr and Beck, [11]).

Other theories of cohomology were developed at about the same time by Grothendieck and Verdier, [5], André, [3, 4], and Quillen, [107, 108]. The first of these was designed for use with "sites", that is, categories together with a Grothendieck topology.

André and Quillen developed, independently, a method of defining cohomology using simplicial resolutions. Their work is best known in commutative algebra, but their methods work in greater generality. Unlike the theory of Barr and Beck (monadic cohomology), they only assume there is enough structure to construct free resolutions; a (co)monad is just one way of doing this. In particular, André, [3, 4], describes a step-by-step, almost combinatorial, process for constructing such resolutions. This ties in well with our earlier comments about using a presentation of a group to construct a crossed resolution and the important link with syzygies. André's method is the simplicial analogue of this.

We will assume for the moment that we have a simplicial resolution, F, of our group, G. Both André and Quillen then consider applying a derived module construction dimensionwise to F, obtaining a simplicial G-module. They then use "Dold-Kan" to give a chain complex of G-modules, which they call the "cotangent complex", denoted L_G or $LAb(G)$, of G (at least in the case of commutative algebras). The homotopy type of $LAb(G)$ does not depend on the choice of resolution and so is a useful invariant of G. We will need to look at this construction in more detail, but will consider a slightly more general situation to start with.

3.5.3 Free simplicial resolutions

Standard theory (cf. Duskin, [49]) shows that if F and F' are free simplicial resolutions of groups, G and H, say, and $f : G \to H$ is a morphism, then f can be lifted to $f' : F \to F'$. The method is the simplicial analogue of lifting a homomorphism of modules to a map of resolutions of those modules, which you should look at first as it is technically simpler. Any two such lifts are homotopic (by a simplicial homotopy).

Of course, f will also lift to a morphism of crossed complexes, $f : \mathsf{C}(F) \to \mathsf{C}(F')$, and any two such lifts will be *homotopic* as crossed complex morphisms. Thus whatever simplicial lift, $f' : F \to F'$, we choose, $\mathsf{C}(f')$ will be a lift in the "crossed" case, and although we do not know at this stage of our discussion of the theory if a homotopy between two simplicial lifts is transferred to a homotopy between the images under C, this does not matter as the *relation* of homotopy is preserved at least in this case of resolutions.

Any group has a free simplicial resolution. There is the obvious adjoint pair of functors

$$\begin{aligned} U &: Groups \to Sets \\ F &: Sets \to Groups \end{aligned}$$

Writing $\eta : Id \to UF$ and $\varepsilon : FU \to Id$ for the unit and counit of this adjunction (cf. MacLane, [85, 86]), we have a comonad (or cotriple) on *Groups*, the free group comonad, $(FU, \varepsilon, F\eta U)$. We write $L = FU$, $\delta = F\eta U$, so that

$$\varepsilon : L \to I$$

is the counit of the comonad whilst

$$\delta : L \to L^2$$

is the comultiplication. (For the reader who has not met monads or comonads before, (L, η, δ) behaves as if it was a monoid in the dual of the category of "endofunctors" on *Groups*, see MacLane, [86] Chapter VI. We will explore them briefly in section **??**, starting on page **??**.)

Now suppose G is a group and set $F(G)_i = L^{i+1}(G)$, so that $F(G)_0$ is the free group on the underlying set of G and so on. The counit (which is just the augmentation morphism from $FU(G)$ to G) gives, in each dimension, face morphisms

$$d_i = L^{n-i}\varepsilon L^i(G) : L^{n+1}(G) \to L^n(G),$$

for $i = 0, \ldots, n$, whilst the comultiplication gives degeneracies

$$\begin{aligned} s_i &: L^n(G) \to L^{n+1}(G) \\ s_i &= L^{n-1-i}\delta L^i, \end{aligned}$$

for $i = 0, \ldots, n-1$, satisfying the simplicial identities.

Remark: Here we follow the conventions used by Duskin, in his Memoir, [49]. Later we will also need to look at similar resolutions where the labelling of the faces and degeneracies are reversed.

This simplicial group, $F(G)$, satisfies $\pi_0(F(G)) \cong G$ (the isomorphism being induced by $\varepsilon(G) : F_0(G) \to G$) and $\pi_n(F(G))$ is trivial if $n \geq 1$. The reason for this is simple. If we apply U once more to $F(G)$, we get a simplicial set and the unit of the adjunction

$$\eta : 1 \to UF$$

allows one to define for each n

$$\eta U(FU)^n : UL^n \to UL^{n+1},$$

which gives a natural contraction of the augmented simplicial set, $UF(G) \to U(G)$, (cf. Duskin, [49]). We will look at this in detail in our later treatment of augmentations, etc. For the moment, it suffices to accept the fact that we do get a resolution, as we do not need to know the details of why this construction works, at least not yet.

If we denote the constant simplicial group on G by $K(G,0)$, the augmentation defines a simplical homomorphism

$$\overline{\varepsilon} : F(G) \to K(G,0)$$

satisfying $U\overline{\varepsilon}.inc = Id$, where $inc : UK(G,0) \to UF(G)$ is the 'inclusion' of simplicial sets given by η, and then these extra maps, $(UF)^n\eta U$, in fact, give a homotopy between $inc.U\overline{\varepsilon}$ and the identity map on $UF(G)$, i.e., $\overline{\varepsilon}$ is a weak homotopy equivalence of simplicial groups. Thus $F(G)$ is a free simplicial resolution of G. It is called the *comonadic free simplicial resolution* of G.

This simplicial resolution has the advantage of being functorial, but the disadvantage of being very big. We turn next to a 'step-by-step' method of constructing a simplicial resolution using ideas pioneered by André, [4], although most of his work was directed more towards commutative algebras, cf. [3].

3.5.4 Step-by-Step Constructions

This section is a brief résumé of how to construct simplicial resolutions by hand rather than functorially. This allows a better interpretation of the generators in each level of the resolution. These are the simplicial analogues of higher syzygies. The work depends heavily on a variety of sources, mainly [3], [80] and [95]. André only treats commutative algebras in detail, but Keune [80] does discuss the general case quite clearly. The treatment here is adapted from the paper by Mutlu and Porter, [98].

Recall of notation: We first recall some notation and terminology, which will be used in the construction of a simplicial resolution. Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. Define the following maps: the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(k) = \begin{cases} k & \text{if} \quad k < i, \\ k+1 & \text{if} \quad k \geq i, \end{cases}$$

for $0 \leq i \leq n \neq 0$. The increasing surjective monotone map $\alpha_i^n : [n+1] \to [n]$ is given by

$$\alpha_i^n(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i, \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m,n\}$ the set of increasing surjective maps $[m] \to [n]$.

3.5.5 Killing Elements in Homotopy Groups

Let G be a simplicial group and let $k \geq 1$ be fixed. Suppose we are given a set, Ω, of elements: $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathsf{G})$, then we can choose a corresponding set of elements $\theta_\lambda \in NG_{k-1}$ so that $x_\lambda = \theta_\lambda\ \partial_k(NG_k)$. (If $k = 1$, then as $NG_0 = G_0$, the condition that $\theta_\lambda \in NG_0$ is immediate.) We want to 'kill' the elements in Ω.

We form a new simplicial group F_n where
1) F_n is the free G_n-group, (i.e., group with G_n-action)

$$F_n = \coprod_{\lambda,t} G_n\{y_{\lambda,t}\} \text{ with } \lambda \in \Lambda \text{ and } \ t \in \{n, k\},$$

where $G_n\{y\} = G_n * < y >$, the co-product of G_n and a free group generated by y.

2) For $0 \leq i \leq n$, the group homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : G_n \to G_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\alpha_i^n, \quad t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the group homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : G_n \to G_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective,} \\ t'(\theta_\lambda) & \text{if} \qquad\quad u = \delta_k^k t', \\ 1 & \text{if} \qquad\quad u = \delta_j^k t' \quad \text{with} \ \ j \neq k, \end{cases}$$

by extending multiplicatively.

We sometimes denote the F, so constructed by $\mathsf{G}(\Omega)$.

Remark: In a 'step-by-step' construction of a simplicial resolution, (see below), there will thus be the following properties: i) $F_n = G_n$ for $n < k$, ii) $F_k =$ a free G_k-group over a set of non-degenerate indeterminates, all of whose faces are the identity except the k^{th}, and iii) F_n is a free G_n-group on some degenerate elements for $n > k$.

We have immediately the following result, as expected.

Proposition 16 *The inclusion of simplicial groups $G \hookrightarrow F$, where $F = G(\Omega)$, induces a homomorphism*

$$\pi_n(G) \longrightarrow \pi_n(F)$$

for each n, which for $n < k - 1$ is an isomorphism,

$$\pi_n(G) \cong \pi_n(F)$$

and for $n = k-1$, is an epimorphism with kernel generated by elements of the form $\bar{\theta}_\lambda = \theta_\lambda \partial_k NG_k$, where $\Omega = \{x_\lambda : \lambda \in \Lambda\}$. ■

3.5.6 Constructing Simplicial Resolutions

The following result is essentially due to André, [3].

Theorem 3 *If G is a group, then it has a free simplicial resolution $\mathbb{F}$.*

Proof: The repetition of the above construction will give us the simplicial resolution of a group. Although 'well known', we sketch the construction so as to establish some notation and terminology.

Let G be a group. The zero step of the construction consists of a choice of a free group F and a surjection $g : F \to G$ which gives an isomorphism $F/Ker\, g \cong G$ as groups. Then we form the constant simplicial group, $F^{(0)}$, for which in every degree n, $F_n = F$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j.

Thus $F^{(0)} = K(F, 0)$ and $\pi_0(F^{(0)}) = F$. Now choose a set, Ω^0, of normal generators of the closed normal subgroup $N = Ker\,(F \xrightarrow{g} G)$, and obtain the simplicial group in which $F_1^{(1)} = F(\Omega^0)$ and for $n > 1$, $F_n^{(1)}$ is a free F_n-group over the degenerate elements as above. This simplicial group will be denoted by $F^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of the group G*.

The subsequent steps depend on the choice of sets, Ω^0, $\Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$. Let $F^{(k)}$ be the simplicial group constructed after k steps, that is, *the k-skeleton of the resolution*. The set Ω^k is formed by elements a of $F_k^{(k)}$ with $d_i^k(a) = 1$ for $0 \leq i \leq k$ and whose images $\bar{a}$ in $\pi_k(F^{(k)})$ generate that module over $F_k^{(k)}$ and $F^{(k+1)}$.

Finally we have inclusions of simplicial groups

$$F^{(0)} \subseteq F^{(1)} \subseteq \cdots \subseteq F^{(k-1)} \subseteq F^{(k)} \subseteq \cdots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial group F with $F_n = F_n^{(k)}$ if $n \leq k$. This F, or, more exactly, (F, g), is thus a simplicial resolution of the group G.

The proof of theorem is completed. ∎

Remark: A variant of the 'step-by-step' construction gives: *if G is a simplicial group, then there exists a free simplicial group F and a continuous epimorphism $F \longrightarrow G$ which induces isomorphisms on all homotopy groups.* The details are omitted as they should be reasonably clear.

The key observation, which follows from the universal property of the construction, is a freeness statement:

Proposition 17 *Let $F^{(k)}$ be a k-skeleton of a simplicial resolution of G and $(\Omega^k, g^{(k)})$ k-dimension construction data for $F^{(k+1)}$. Suppose given a simplicial group morphism $\Theta : F^{(k)} \longrightarrow G$ such that $\Theta_*(g^{(k)}) = 0$, then Θ extends over $F^{(k+1)}$.*

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k g^{(k)}$ to NG_{k+1}, a lift that must exist since $\Theta_*(\pi_k(F^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, this proposition is as often as close to 'left adjointness' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We have not talked here about the homotopy of simplicial group morphisms, and so will not discuss homotopy invariance of this construction for which one adapts the description given by André, [3], or Keune, [80]. Of course, the resolution one builds by any means would be homotopicallly equivalent to any other so, for cohomological purposes, it makes no difference how the resolution is built.

Of course, from any simplicial resolution F of G, you can get an augmented crossed complex $C(F)$ over G using the formula given earlier and this is a crossed resolution.

3.6 Cohomology and crossed extensions

3.6.1 Cochains

Consider a G-module, M, and a non-negative integer n. We can form the chain complex, $K(M, n)$, having M in dimension n and zeroes elsewhere. We can also form a crossed complex, $\mathsf{K}(M, n)$, that plays the role of the n^{th} Eilenberg-MacLane space of M in this setting. We may call it the n^{th} *Eilenberg-MacLane crossed complex* of M:

If $n = 0$, $\mathsf{K}(M,n)_0 = G \ltimes M$, $\mathsf{K}(M,n)_i = 0$, $i > 0$.

If $n \geq 1$, $\mathsf{K}(M,n)_0 = G$, $\mathsf{K}(M,n)_n = M$, $\mathsf{K}(M,n)_i = 0$, $i \neq 0$ or n.

One way to view cochains is as chain complex morphisms. Thus on looking at $Ch(\mathsf{B}G, K(M,n))$, one finds exactly $Z^{n+1}(G,M)$, the $(n+1)$-cocycles of the cochain complex $C(G,M)$. We can also view $Z^{n+1}(G,M)$ as $Crs_G(\mathsf{C}G, \mathsf{K}(M,n))$.

In the category of chain complexes, one has that a homotopy from $\mathsf{B}G$ to $K(M,n)$ between 0 and f, say, is merely a coboundary, so that $H^{n+1}(G,M) \cong [\mathsf{B}G, K(M,n)]$, adopting the usual homotopical notation for the group of homotopy classes of maps from the bar resolution $\mathsf{B}G$ to $K(M,n)$. This description has its analogue in the crossed complex case as we shall see.

3.6.2 Homotopies

Let C, C' be two crossed complexes with Q and Q' respectively as the cokernels of their bottom morphism. Suppose λ, $\mu : \mathsf{C} \to \mathsf{C}'$ are two morphisms inducing the same map $\varphi : Q \to Q'$.

A homotopy from λ to μ is a family, $h = \{h_k : k \geq 1\}$, of maps $h_k : C_k \to C'_{k+1}$ satisfying the following conditions:

H1) $h_0 : C_1 \to C'_2$ is a derivation along μ_0 (i.e. for $x, y \in C_0$,

$$h_0(xy) = h_0(x)({}^{\mu_0}h_0(y)),)$$

such that

$$\delta_1 h_0(x) = \lambda_0(x)\mu_0(x)^{-1}, \quad x \in C_0.$$

H2) $h_1 : C_1 \to C'_2$ is a C_0-homomorphism with C_0 acting on C'_2 via λ_0 (or via μ_0, it makes no difference) such that

$$\delta_2 h_1(x) = \mu_1(x)^{-1}(h_0\delta_1(x)^{-1}\lambda_1(x)) \text{ for } x \in C_1.$$

H3) for $k \geq 2$, h_k is a Q-homomorphism (with Q acting on the C'_k via the induced map $\varphi : Q \to Q'$) such that

$$\delta_{k+1}h_k + h_{k-1}\delta_k = \lambda_k - \mu_k.$$

We note that the condition that λ and μ induce the same map, $\varphi : Q \to Q'$, is, in fact, superfluous as this is implied by $H1$.

The properties of homotopies and the relation of homotopy are as one would expect. One finds $H^{n+1}(G,M) \cong [\mathsf{C}G, \mathsf{K}(M,n)]$. Given that in higher dimensions, this is the same set exactly as $[\mathsf{B}G, K(M,n)]$ means that there is not much to check and so the proof has been omitted.

3.6.3 Huebschmann's description of cohomology classes

The transition from this position to obtaining Huebschmann's descriptions of cohomology classes, [68], is now more or less formal. We will, therefore, only sketch the main points.

If G is a group, M is a G-module and $n \geq 1$, a *crossed n-fold extension* is an exact augmented crossed complex,

$$0 \to M \to C_n \to \ldots \to C_2 \to C_1 \to G \to 1.$$

The notion of similarity of such extensions is analogous to that of n-fold extensions in the Abelian Yoneda theory, (cf. MacLane, [85]), as is the definition of a Baer sum. We leave the details to you. This yields an Abelian group, $Opext^n(G,M)$, of similarity classes of crossed n-fold extensions of G by M.

Given a cohomology class in $H^{n+1}(G, M)$ realisable as a homotopy class of maps, $f : \mathsf{C}G \to \mathsf{K}(M, n)$, one uses f to form an induced crossed complex, much as in the Abelian Yoneda theory:

$$\begin{tikzcd} & J_n(G) \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{pushout}"] & C_n \arrow[r] \arrow[d] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d] & G \arrow[d, equal] \\ 0 \arrow[r] & M \arrow[r] & M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & G \end{tikzcd}$$

where $J_n(G)$ is $Ker(C_nG \to C_{n-1}G)$. (Thus J_nG is also $Im(C_{n+1}G \to C_nG)$ and as the map f satisfies $f\delta = 0$, it is zero on the subgroup $\delta(C_{n+2}G)$ (i.e. is constant on the cosets) and hence passes to $Im(C_{n+1}G \to C_nG)$ in a well defined way.) Arguments using lifting of maps and homotopies show that the assignment of this element of $Opext^n(G, M)$ to $cls(f) \in H^{n+1}(G, M)$ establishes an isomorphism between these groups.

3.6.4 Abstract Kernels.

The importance of having such a description of classes in $H^n(G, M)$ probably resides in low dimensions. To describe classes in $H^3(G, M)$, one has, as before, crossed 2-fold extensions

$$0 \to M \to C_2 \xrightarrow{\partial} C_1 \to G \to 1,$$

where ∂ is a crossed module. One has for any group G, a crossed 2-fold extension

$$0 \to Z(G) \to G \xrightarrow{\partial_G} Aut(G) \to Out(G) \to 1$$

where ∂_G sends $g \in G$ to the corresponding inner automorphism of G. An *abstract kernel* (in the sense of Eilenberg-MacLane, [54]) is a homomorphism $\psi : Q \to Out(G)$ and hence provides, by pulling back, a 2-fold extension of Q by the centre $Z(G)$ of G.

3.7 2-types and cohomology

In classifying homotopy types and in obstruction theory, one frequently has invariants that are elements in cohomology groups of the form $H^m(X, \pi)$, where typically π is the n^{th} homotopy group of some space. When dealing with homotopy types, π will be a group, usually Abelian with a π_1-action, i.e., we are exactly in the situation described earlier, except that X is a homotopy type not a group. Of course, provided that X is connected, we can replace X by a simplicial group, bringing us even nearer to the situation of this section. We shall work within the category of simplicial groups.

3.7.1 2-types

A morphism

$$f : G \to H$$

of simplicial groups is called a *2-equivalence* if it induces isomorphisms

$$\pi_0(f) : \pi_0(G) \to \pi_0(H,)$$

and

$$\pi_1(f) : \pi_1(G) \to \pi_1(H).$$

We can form a quotient category, $Ho_2(Simp.Grps)$, of *Simp.Grps* by formally inverting the 2-equivalences, then we say two simplicial groups, G and H, have the same *2-type*, (or, more exactly, *homotopy 2-type*), if they are isomorphic in $Ho_2(Simp.Grps)$.

This is, of course, just a special case of the general notion of n-type in which "n-equivalences" are inverted, thus forming the quotient category $Ho_n(Simp.Grps)$.

We recall the following from earlier:

Definition: An *n-equivalence* is a morphism, f, of simplicial groups (or groupoids) inducing isomorphisms, $\pi_i(f)$, for $i = 0, 1, \ldots, n-1$.

Definition: Two simplicial groups, G and H, have the same *n-type* (or, more exactly, *homotopy n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Sometimes it is convenient to say that a simplicial group, G, *is* an n-type. This is taken to mean that it represents an n-equivalence class *and* has zero homotopy groups above dimension $n-1$.

3.7.2 Example: 1-types

Before examining 2-types in detail, it will pay to think about 1-types. A morphism f as above is a 1-equivalence if it induces an isomorphism on π_0, i.e., $\pi_0(f)$ is an isomorphism. Given any group G, there is a simplicial group, $K(G,0)$ consisting of G in each dimension with face and degeneracy maps all being identities. Given a simplicial group, H, having $G \cong \pi_0(H)$, the natural quotient map

$$H_0 \to \pi_0(H) \cong G,$$

extends to a natural 1-equivalence between H and $K(\pi_0(H), 0)$.

It is fairly routine to check that

$$\pi_0 : Simp.Grps \to Grps$$

has $K(-,0)$ as an adjoint and that, as the unit is a natural 1-equivalence, and the counit an isomorphism, this adjoint pair induces an equivalence between the category $Ho_1(Simp.Grps)$ of 1-types and the category, *Grps*, of groups. In other words,

groups are algebraic models for 1-types.

3.7.3 Algebraic models for n-types?

So much for 1-types. Can one provide algebraic models for 2-types or, in general, n-types? We touched on this earlier. The criteria that any such "models" might satisfy are debatable. Perhaps ideally, or even unrealistically, there should be an isomorphism class of algebraic "gadgets" for each 2-type. An alternative weaker solution is to ask that a notion of equivalence between the models is possible, and that only equivalence classes, not isomorphism classes, correspond to 2-types, but, in addition, the notion of equivalence is algebraically defined. It is this weaker possibility that corresponds to our aim here.

3.7.4 Algebraic models for 2-types.

If G is a simplicial group, then we can form a crossed module

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0,$$

where the action of G_0 is via the degeneracy, $s_0 : G_0 \to G_1$, and ∂ is induced by d_0. (As before we will denote this crossed module by $M(G,1)$.) The kernel of ∂ is

$$\frac{Ker\, d_0 \cap Ker\, d_1}{d_0(NG_2)} \cong \pi_1(G),$$

whilst its cokernel is

$$\frac{G_0}{d_0(NG_1)} \cong \pi_0(G),$$

and so we have a crossed 2-fold extension

$$0 \to \pi_1(G) \to \frac{NG_1}{d_0(NG_2)} \to G_0 \to \pi_0(G) \to 1$$

and hence a cohomology class $k(G) \in H^3(\pi_0(G), \pi_1(G))$.

Suppose now that $f : G \to H$ is a morphism of simplicial groups, then one obtains a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "\pi_1(f)"'] & \frac{NG_1}{d_0(NG_2)} \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, "f_0"] & \pi_0(G) \arrow[r] \arrow[d, "\pi_0(f)"] & 1 \\
0 \arrow[r] & \pi_1(H) \arrow[r] & \frac{NH_1}{d_0(NH_2)} \arrow[r] & H_0 \arrow[r] & \pi_0(H) \arrow[r] & 1
\end{tikzcd}$$

If, therefore, f is a 2-equivalence, $\pi_0(f)$ and $\pi_1(f)$ will be isomorphisms and the diagram shows that, modulo these isomorphisms, $k(G)$ and $k(H)$ are the same cohomology class, i.e. the 2-type of G determines π_0, π_1 and this cohomology class, k in $H^3(\pi_0, \pi_1)$.

Conversely, suppose we are given a group π, a π-module, M, and a cohomology class $k \in H^3(\pi, M)$, then we can realise k by a 2-fold extension

$$0 \to M \to C \xrightarrow{\partial} G \to \pi \to 1$$

as above.

The crossed module, $\mathsf{C} = (C, G, \partial)$, determines a simplicial group $K(\mathsf{C})$ as follows:

Suppose $\mathsf{C} = (C, P, \partial)$ is any crossed module, we construct a simplicial group, $K(\mathsf{C})$, by

$$K(\mathsf{C})_0 = P, \qquad K(\mathsf{C})_1 = C \rtimes P,$$

$$s_0(p) = (1, p),\ d_0^1(c, p) = \partial c.p,\ d_1^1(c, p) = p.$$

Assuming $K(\mathsf{C})_n$ is defined and that it acts on C via the unique composed face map to $K(\mathsf{C})_0 = P$ followed by the given action of P on C, we set

$$\begin{aligned} K(\mathsf{C})_{n+1} &= C \rtimes K(\mathsf{C})_n; \\ d_0^{n+1}(c_{n+1}, \ldots, c_1, p) &= (c_{n+1}, ..., c_2, \partial c_1.p); \\ d_i^{n+1}(c_{n+1}, \ldots, c_{i+1}, c_i, \ldots, c_1, p) &= (c_{n+1}, \ldots, c_{i+1}c_i, \ldots c_1, p) \\ & \qquad \text{for } 0 < i < n+1; \\ d_{n+1}^{n+1}(c_{n+1}, \ldots, c_1, p) &= (c_n, \ldots, c_1, p); \\ s_i^n(c_n, \ldots, c_1, p) &= (c_n, \ldots, 1, \ldots, c_1, p), \end{aligned}$$

where the 1 is placed in the i^{th} position.

Clearly $Ker\, d_1^1 = \{(c, p) : p = 1\} \cong C$, whilst $Ker\, d_1^2 \cap Ker\, d_2^2 = \{(c_2, c_1, p) : (c_1, p) = (1, 1) \text{ and } (c_2c_1, p) = (1, 1)\} \cong \{1\}$, hence the "top term" of $M(K(\mathsf{C}), 1)$ is isomorphic to C itself, whilst $K(\mathsf{C})_0$ is P itself. The boundary map ∂ in this interpretation is the original ∂, since it maps $(c, 1)$ to $d_0(c)$, i.e., we have

Lemma 12 *There is a natural isomorphism*

$$\mathsf{C} \cong M(K(\mathsf{C}), 1).$$

■

This construction is the internal nerve of the corresponding internal category in *Grps*, as we noted earlier. All the ideas that go into defining the nerve of a category adapt to handling internal categories, and they produce simplicial objects in the corresponding ambient category. As we have a simplicial group $K(\mathsf{C})$, we might check if it is a group T-complex, but this is more or less immediate as $NK(\mathsf{C})_n = 1$ for $n \geq 2$, whilst $NK(\mathsf{C})_1$ is $\{(c, p) : p = 1\}$ and $s_0(K(\mathsf{C})_0 = \{(c, p) : c = 1\}$.

Suppose now that we had chosen an equivalent 2-fold extension

$$0 \to M \to C' \xrightarrow{d'} G' \to \pi \to 1$$

The equivalence guarantees that there is a zig-zag of maps of 2-fold extensions joining it to that considered earlier. We need only look at the case of a direct basic equivalence:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & C \arrow[r, "\partial"] \arrow[d] & G \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, "="] & 1 \\ 0 \arrow[r] & M \arrow[r] & C' \arrow[r, "\partial'"] & G' \arrow[r] & \pi \arrow[r] & 1 \end{tikzcd}$$

giving a map of crossed modules, $\varphi : \mathsf{C} \to \mathsf{C}'$, where $\mathsf{C}' = (C', G', \partial')$. This induces a morphism of simplicial groups,

$$K(\varphi) : K(\mathsf{C}) \to K(\mathsf{C}'),$$

that is, of course, a 2-equivalence. If there is a longer zig-zag between C and C', then the intermediate crossed modules give intermediate simplicial groups and a zig-zag of 2-equivalences so that $K(\mathsf{C})$ and $K(\mathsf{C}')$ are isomorphic in $Ho_2(Simp.Grps)$, i.e. they have the same 2-type. This argument can, of course, be reversed.

If G and H have the same 2-type, they are isomorphic within the category $Ho_2(Simp.Grps)$, so they are linked in $Simp.Grps$ by a zig-zag of 2-equivalences, hence the corresponding cohomology classes in $H^3(\pi_0(G), \pi_1(G))$ are the same up to identification of $H^3(\pi_0(G), \pi_1(G))$ and $H^3(\pi_0(H), \pi_1(H))$. This proves the simplicial group analogue of the result of MacLane and Whitehead, [88], that we mentioned earlier, giving an algebraic model for 2-types of connected CW-complexes.

Theorem 4 *(MacLane and Whitehead, [88]) 2-types are classified by a group π_0, a π_0-module, π_1 and a class in $H^3(\pi_0, \pi_1)$.* ■

We have handled this in such a way so as to derive an equivalence of categories:

Proposition 18 *There is an equivalence of categories,*

$$Ho_2(Simp.Grps) \cong Ho(CMod),$$

where $Ho(CMod)$ is formed from $CMod$ by formally inverting those maps of crossed modules that induce isomorphisms on both the kernels and the cokernels. ■

3.8 Re-examining group cohomology with Abelian coefficients

3.8.1 Interpreting group cohomology

We have had

- A definition of group cohomology via the bar resolution: for a group G and a G-module, M:

  $$H^n(G, M) = H^n(C(G, M))$$

  together with an identification of $C(G, M)$ with maps from the classifying space / nerve BG of G to M, up to shifts in dimension;

- Interpretations

$$\begin{array}{rcl} H^0(G, M) & \cong & M^G, \text{ the module of invariants} \\ H^1(G, M) & \cong & Der(G, M)/Pder(G, M) \\ & - & \text{by inspection, where } Pder(G, M) \text{ is the submodule of} \\ & & \qquad \text{principal derivations;} \\ H^2(G, M) & \cong & Opext(G, M), \text{ i.e. classes of extensions} \\ & & \qquad 0 \to M \to H \to G \to 1 \end{array}$$

  and we also have

$$\begin{array}{rcl} H^n(G, M) & \cong & Opext^n(G, M), n \geq 2, \text{ via crossed resolutions} \\ & \cong & [\mathsf{C}(G), \mathsf{K}(M, n)] \end{array}$$

Another interpretation, which will be looked at shortly is as $Ext^n(\mathbb{Z}, M)$, where $\mathbb{Z}$ is given the trivial G-module structure. This leads to

$$H^n(G, M) \cong Ext^{n-1}(I(G), M),$$

via the long exact sequence coming from

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

3.8.2 The *Ext* long exact sequences

There are several different ways of examining the long exact sequence that we need. We will use fairly elementary methods rather than more 'homologically intensive' one. These latter ones are very elegant and very powerful, but do need a certain amount of development before being used. The more elementary ones have, though, a hidden advantage. The intuitions that they exploit are often related to ones that extend, at least partially, to the non-Abelian case and also to the geometric situations that will be studied later in the notes.

The idea is to explore what happens to an exact sequence of modules

$$\mathcal{E}: \quad 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

over some given ring (we need it for G-modules so there the ring is $\mathbb{Z}[G]$, the group ring of G), when we apply the functor $Hom(-, M)$ for M another module. Of course one gets a sequence

$$Hom(\mathcal{E}, M): 0 \to Hom(C, M) \xrightarrow{\beta^*} Hom(B, M) \xrightarrow{\alpha^*} Hom(A, M)$$

and it is easy to check that this is exact, but there is no reason why α^* should be onto since a morphism $f : A \to M$ may or may not extend to some g defined over the bigger module B. For instance, if $M = A$, and f is the identity morphism, then f extends if and only if the sequence splits (so $B \cong A \oplus C$). We examine this more closely.

We have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

and can form a new diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "\overline{f}"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M \arrow[r, "\overline{\alpha}"] & N \arrow[r, "\overline{\beta}"] & C \arrow[r] & 0 \end{tikzcd}$$

where the left hand square is a pushout. You should check that you see why there is an induced morphism $\overline{\beta} : N \to C$ 'emphusing the universal property of pushouts. (This is important as sometimes one wants this sort of construction, or argument, for sheaves of modules and there working with elements causes some slight difficulties.) The existence of this map is guaranteed by the universal property and does not depend on a particular construction of N. Of course this means that the bottom line is defined only up to isomorphism although we can give a very natural explicit model for N, namely it can be represented as the quotient of $B \oplus M$ by the submodule L of elements of the form $(\alpha(a), -f(a))$ for $a \in A$. Then we have $\overline{\beta}(b, m) = \beta(b)$. (Check it is well defined.) It is also useful to have the corresponding formulae for $\overline{\alpha}(m) = (0, m) + L$ and for $\overline{f}(b) = (b, 0) + L$. This gives an extension of modules

$$f^*(\mathcal{E}): \quad 0 \to M \xrightarrow{\overline{\alpha}} N \xrightarrow{\overline{\beta}} C \to 0.$$

If f extends over B to give g, so $g\alpha = f$, then we have a morphism $g' : N \to M$ given by $g'((m, b) + L) = m + g(b)$. (Check that g' is well defined.)

Lemma 13 *f extends over B if and only if $f^*(\mathcal{E})$ is a split extension.*

Proof: We have done the 'only if'. If $f^*(\mathcal{E})$ is split, there is a projection $g' : N \to M$ such that $g'\overline{\alpha}(m) = m$ for all m. Define $g = g'\overline{f}$ to get the extension. ■

We thus get a map

$$Hom(A, M) \xrightarrow{\delta} Ext^1(C, M)$$

$$\delta(f) = [f^*(\mathcal{E})]$$

which extends the exact sequence one step to the right.

Here it is convenient to define $Ext^1(C, M)$ to be the set (actually Abelian group) of extensions of form

$$0 \to M \to ? \to C \to 0$$

modulo equivalence (isomorphism of middle terms with the ends fixed). The Abelian group structure is given by Baer sum (see entry in Wikipedia, or many standard texts on homological algebra).

Important aside: 'Recall' the '*snake lemma*: given a commutative diagram of modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\mu"'] & N \arrow[r] \arrow[d, "\nu"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \\ 0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0 \end{tikzcd}$$

there is an exact sequence

$$0 \to Ker\,\mu \to Ker\,\nu \to Ker\,\psi \xrightarrow{\delta} Coker\,\mu \to Coker\,\nu \to Coker\,\psi \to 0$$

This has as a corollary that if μ and ψ are isomorphisms then so is ν. (Do check that you can construct δ and prove exactness, i.e. using a simple diagram chase.)

Back to extensions: It is fairly easy to show that $Hom(\mathcal{E}, M)$ extends even further to 6 terms with

$$\ldots \xrightarrow{\beta^*} Ext^1(B, M) \xrightarrow{\alpha^*} Ext^1(A, M)$$

Here is how α^* is constructed. Suppose $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ gives an element of $Ext^1(B, M)$, then we can form a diagram

$$\begin{tikzcd} \alpha^*(\mathcal{E}_1): & 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & \alpha^{-1}(N) \arrow[r, "p'"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ \mathcal{E}_1: & 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

by restricting $\mathcal{E}_1$ along α using a pull back in the right hand square. We can give $\alpha^{-1}(N)$ explicitly in the form that the usual construction of pullbacks in categories of modules gives it to us

$$\alpha^{-1}(N) \cong \{(a, n) \mid \alpha(a) = p(n)\}$$

and p' and α' are projections. The construction of β^* is done similarly using pullback along β. It is then easy to check that the obvious extension to $Hom(\mathcal{E}, M)$, mentioned above, is exact, but that there is again no reason why α^* should be onto. (Of course, knowledge of the purely homological way of getting these exact sequence will suggest that there is an $Ext^2(C, M)$ term to come.)

We examine an obstruction to it being so. Suppose given $\mathcal{E}' : 0 \to M \to N_1 \xrightarrow{p'} A \to 0$, giving us an element of $Ext'(A, M)$. If α^* were onto, we would need a $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ such that $\alpha^{-1}(N) \cong N_1$ leaving M fixed and relating to α as above by a pullback. We can splice $\mathcal{E}'$ and $\mathcal{E}_1$ together to get a suitable looking diagram

$$\mathcal{E}' * \mathcal{E}_1 : \quad \begin{tikzcd} 0 \arrow[r] & M \arrow[r] & N' \arrow[rr] \arrow[dr, "p'"'] & & B \arrow[r] & C \arrow[r] & 0 \\ & & & A \arrow[ur, "\alpha"'] & & & \end{tikzcd}$$

and the row is exact. If we change $\mathcal{E}'$ by an isomorphism than clearly this spliced sequence would react accordingly. If you check up, as suggested, on the Baer sum structure if $Ext^1(A, M)$ and $Ext^2(C, M)$ then you can again check that the above splicing construction yields a homomorphism from the first group to the second. Moreover there is no reason not to extend the splicing construction to a pairing operation on the whole graded family of Ext-groups. This is given in detail in quite a few of the standard books on Homological Algebra, so will not be gone into here.

Two facts we do need to have available are about the structure of $Ext^2(C, M)$. Let $\mathcal{E}xt^2(C, M)$ be the category of 4-term exact sequences

$$0 \to M \to N \to P \to C \to 0$$

and morphisms which are commuting diagrams

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\ 0 \arrow[r] & M \arrow[r] & N' \arrow[r] & P' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

then $Ext^2(C, M)$ is the set of connected components of this category. The important thing to note is that the morphisms are not isomorphisms in general, so two 4-term sequences give the same element in $Ext^2(C, M)$ if they are linked by a zig-zag of intermediate terms of this form. The second fact is that the zero for the Baer sum addition is the class of the 4-term extension

$$0 \to M \to M \xrightarrow{0} C \to C \to 0$$

with ‘equals’ on the unmarked maps.

Suppose now that the top row in

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\overline{p}"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

is obtained by restriction along α from the bottom row. We now form the spliced sequence

$$0 \to M \to N_1 \xrightarrow{\alpha\overline{p}} B \to C \to 0.$$

We would hope that this 4-term sequence was trivial, i.e. equivalence to the zero one. We clearly must use the given element in $Ext^1(B, M)$ in a constructive way in the proof that it is trivial, so we form the pushout of $\alpha\overline{p}$ along α' getting us a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0\, , \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

with the middle square a pushout. It is now almost immediate that the morphism from B to B' is split, since we can form a commutative square

$$\begin{tikzcd} N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[d, "="] \\ N \arrow[r, "p"'] & B \end{tikzcd}$$

giving us the required splitting from B' to B. It is now a simple use of the snake lemma, to show that the complementary summand of B in B' is isomorphic to C. We thus have that the bottom row of the diagram above is of the form

$$0 \to M \to N \to B \oplus C \to C.$$

This looks hopeful but to finish off the argument we just produce the morphism:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & M \arrow[r, "0"] \arrow[d] & C \arrow[r] \arrow[d, "incl_2"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r, "incl_1 p"'] & B \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and we have a sequence of maps joining our spliced sequence to the trivial one. (A similar argument goes through in higher dimensions.) Now you should try to prove that if a spliced sequence is linked to a trivial one then it does come from an induced one. That is quite tricky, so look it up in a standard text. An alternative approach is to use the homological algebra to get the trivialising element (coboundary or homotopy, depending on your viewpoint) and then to construct the extension from that. Another thing to do is to consider how the Ext-groups, $Ext^k(A, M)$, vary in M rather than with A. This will be left to you.

3.8.3 From Ext to group cohomology

If we look briefly at the classical homological algebraic method of defining $Ext^K(A, M)$, we would take a projective resolution $\mathsf{P}.$ of A, apply the functor $Hom(-, M)$, to get a cochain complex $Hom(\mathsf{P}., M)$, then take its (co)homology, with $H^n(Hom(\mathsf{P}., M))$ being isomorphic to $Ext^n(A, M)$, or, if you prefer, $Ext^n(A, M)$ being defined to be $H^n(Hom(\mathsf{P}., M))$. This method can be studied in most books on homological algebra (we cite for instance, MacLane, [85], Hilton and Stammbach, [67] and Weibel, [112]), so is easily accessible to the reader - and we will not devote much space to it here as a result. We will however summarise some points, notation, definitions of terms etc., some of which you probably know.

First the notion of projective module: A module P is projective if given any epimorphism $f : B \to C$ the induced map $Hom(P, f) : Hom(P, B) \to Hom(P, C)$ is onto. In other words any map from P to C can be lifted to one from P to B.

Any free module is projective.

Of the properties of projectives that we will use, we will note that $Ext^n(P, M) = 0$ for P projective and for any M. To see this recall that any n-fold extension of P by M will end with an epimorphism to P, but such things split as their codomain is projective. It is now relatively easy to use this splitting to show the extension is equivalent to the trivial one.

A resolution of a module A is an augmented chain complex

$$\mathsf{P}. : \ldots \to P_1 \to P_0 \to M$$

which is exact, i.e. it has zero homology in all dimensions. This means that the augmentation induces an isomorphism between $P_0/\partial P_1$ and M. The resolution is projective if each P_n is a projective module.

If $\mathsf{P}.$ and $\mathsf{Q}.$ are both projective resolutions of A, then the cochain complexes $Hom(\mathsf{P}., M)$ and $Hom(\mathsf{Q}., M)$ always have the same homology. (Once again this is standard material from homological algebra so is left to the reader to find in the usual sources.)

An example of a projective resolution is given by the bar resolution, $\mathsf{B}G.$, and the construction $C^n(G, M)$ in the first chaper is exactly $Hom(\mathsf{B}G., M)$. This reolution ends with $BG_0 = \mathbb{Z}(G)$ and the resolution resolves the Abelian group $\mathbb{Z}$ with trivial G-module structure. (This can be seen from our discussion of homological syzygies where we had

$$\mathbb{Z}(G)^{(R)} \to \mathbb{Z}(G)^{(X)} \to \mathbb{Z}(G) \to \mathbb{Z}.$$

In fact we have

$$H^n(G, M) \cong Ext^n(\mathbb{Z}, M)$$

by the fact that $\mathsf{B}G.$ is a projective resolution of $\mathbb{Z}$ and then we can get more information using the short exact sequence

$$0 \to I(G) \to \mathbb{Z}(G) \to \mathbb{Z} \to 0.$$

As $\mathbb{Z}(G)$ is a free G-module, it is projective and the long exact sequence for $Ext(-, M)$ thus has every third term trivial (at least for $n > 0$), so

$$Ext^n(\mathbb{Z}, M) \cong Ext^{n-1}(I(G), M)$$

giving another useful interpretation of $H^n(G, M)$.

3.8.4 Exact sequences in Cohomology

Of course, the identification of $H^n(G, M)$ as $Ext^n(\mathbb{Z}, M)$ means that, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of G-modules, we will get a long exact sequence in $H^n(G, -)$, just by looking at the long exact sequence for $Ext^n(\mathbb{Z}, -)$.

What is more interesting - but much more difficult - is to study the way that $H^n(G, M)$ varies as G changes. For a start it is not completely clear what this means! If we change the group in a short exact sequence,t

$$1 \to G \to H \to K \to 1$$

say, then what type of modules should be used fro the ‘coefficients’, that is to say a G-modules or one over H or K. This problem is, of course, related to the change of groups along an arbitrary homomorphism, so we will look at an group homomorphism $\varphi : G \to H$, with no assumptions as to monomorphism, or normal inclusion, at least to start with.

Suppose given such a φ, then the ‘restriction functor’ is

$$\varphi^* : H{-}Mod \to G{-}Mod,$$

where, if N is in $H{-}Mod$, $\varphi^*(N)$ has the same underlying Abelian group structure as N, but is a G-module via the action, $g.n := \varphi(g).n$. We have already used that φ^* has a left adjoint φ_* given by $\varphi_*(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M$. Now we also need a right adjoint for φ^*.

To construct such an adjoint, we use the old device of assuming that it exists, studying it and then extracting a construction from that study. We have M in $G{-}Mod$ and N in $H{-}Mod$, and we assume a natural isomorphism

$$G{-}Mod(\varphi^*(N), M) \cong H{-}Mod(N, \varphi_\sharp(M)).$$

If we take $N = \mathbb{Z}H$, then, as $H{-}Mod(\mathbb{Z}H, \varphi_\sharp(M)) \cong \varphi_\sharp(M)$, we have a construction of $\varphi_\sharp(M)$, at least as an Abelian group. In fact this gives

$$\varphi_\sharp(M) \cong G{-}Mod(\varphi^*(\mathbb{Z}H), M)$$

and as $\mathbb{Z}H$ is also a right G-module, via $h.g := h.\varphi(g)$, we have a left G-module structure of $\varphi_\sharp(M)$ as expected. In fact, this is immediate from the naturality of the adjunction isomorphism using the left hand position of $G{-}Mod(\varphi^*(\mathbb{Z}H), M)$, as for fixed M, the functor converts the right G-action of $\mathbb{Z}$ to a left one on $\varphi_\sharp(M)$. This allows us to get an explicit elementwise formula for this action as follows: let $m^* : \mathbb{Z}H \to M$ be a left G-module mrphsim This can be specified by what it does to the natural basis of $\mathbb{Z}H$ (as Abelian group), and so is often written $m^* : H \to M$, where the function m^* must satisfy a G-equivariance property: $m^*(\varphi(g).h) = g.m^*(h)$. Any such function can, of course, be extended linearly to a G-module morphism of the earlier form. If $g \in G$, we get a morphism

$$-.\varphi(g) : \varphi^*(\mathbb{Z}H) \to \varphi^*(\mathbb{Z}H)$$

given by ‘h goes to $h\varphi(g)$’. This *is* a G-module morphism as the G-module structure is by *left* multiplication, which is independent of this right multiplication. Applying $G{-}Mod(-, M)$, we get $g.m^*$ is given by

$$g.m^*(h) - m^*(h.\varphi(g).$$

This is a *left* G-module structure, although at first that may seem strange. That it is linear is easy to check. What take a little bit of work is to check $(g_1g_2).m^* = g_1(g_2.m^*)$: applying both sides to an element $h \in H$ gives

$$(g_1g_2).m^*(h) = m^*(h\varphi(g_1)\varphi(g_2)),$$

whilst

$$g_1(g_2.m^*)(h) = (g_2.m^*)(h.\varphi(g_1)) = m^*(h\varphi(g_1)\varphi(g_2)).$$

(The checking that $g_1.m^*$ does satisfy the G-equivariance property is left to the reader.)

Remark: There are great similarities between the above calculations and those needed later when examining bitorsors. This is almost certainly not coincidental.

We built $\varphi_\sharp(M)$ in such a way that it is obviously functorial in M and gives a right adjoint to φ^*. This implies that there is a natural morphism

$$i : N \to \varphi_\sharp\varphi^*(N).$$

We denote this second module by N^*, when the context removes any ambiguity, and especially when φ is the inclusion of a subgroup. The morphism sends n to $n^* : H \to N$, where $n^*(h) = h.n$. (Check that $n^*(\varphi(g).h) = g.n^*(h)$. This reminds us that the codomain of n^* is infact just the *set* N underlying both the H-module N and the G-module $\varphi^*(N)$.)

We examine the cohomology groups $H^n(H, N^*)$. These are the (co)homology groups of the cochain complex $Hom(\mathsf{P}., N^*)$, where $\mathsf{P}.$ is a projective H-module resolution of $\mathbb{Z}$. The adjunction shows that this is isomorphic to $Hom(\varphi^*(\mathsf{P}.), \varphi^*(N))$. If $\varphi^*(\mathsf{P}.)$ is a projective G-module resolution of the trivial G-module $\mathbb{Z}$ then the cohomology of this complex will be $H^n(G, N)$, where N has the structure $\varphi^*(N)$.

The condition that free or projective H modules restrict to free or projective G-modules is satisfied in one important case, namely when G is a subgroup of H, since $\mathbb{Z}H$ is a free Abelian group on the *set* H and H is a disjoint union of right G-cosets, so $\mathbb{Z}H$ splits as a G-module into a direct sum of copies of $\mathbb{Z}G$. This provides part of the proof of Shapiro's lemma

Proposition 19 *If $\varphi : G \to H$ is an inclusion, then for a H-module N, there is a natural isomorphism*

$$H^n(H, N^*) \cong H^n(G, N).$$

■

Corollary 4 *The morphism $i : N \to N^*$ and the above isomorphism yield the restriction morphism*

$$H^n(H, N) \to H^n(G, N).$$

■

This suggest other results. Suppose we have an extension

$$1 \to N \to G \to Q \to 1$$

(so here we replace H by G with N in the old role of G, but in addition, being normal in G).

If we look at $\mathsf{B}N$ and $\mathsf{B}G$ in dimension n, these are free modules over the sets N^n and G^n respectively, with the inclusion between them; G is a disjoint union of N-cosets, indexed by elements of Q, so can we use this to derive properties of the cokernel of $\mathbb{Z}G \otimes_{\mathbb{Z}N} \mathsf{B}N \to \mathsf{B}G$, and to tie them into some resolution of Q, or perhaps, of $\mathbb{Z}$ as a trivial Q-module. The answer must clearly be positive, perhaps with some restrictions such as finiteness, but there are several possible ways of getting to an answer having slightly different results. (You have in the (φ_*, φ^*) and $(\varphi^*, \varphi_\sharp)$ adjunctions, enough of the tools needed to read detailed accounts in the literature, so we will not give them here.)

This also leads to relative cohomology groups and their relationship with the cohomology of the quotient Q. We can also consider the crossed resolutions of the various groups in the extension and work, say, with the induced maps

$$\mathsf{C}(N) \to \mathsf{C}(C)$$

looking at its cokernel or better what should be called its homotopy cokernel.

Another possibility is to examine $\mathsf{C}(N)$ and $\mathsf{C}(Q)$ and the cocycle information needed to specify the extension, and to use all this to try to construct a crossed resolution of G. (We will see something related to this in our examination of non-Abelian cohomology a little later.) A simple case of this is when the extension is split, $G \cong N \rtimes Q$ and using a twisted tensor product for crossed complexes, one can produce a suitable $\mathsf{C}(N) \otimes_\tau \mathsf{C}(Q)$ resolving G, (see Tonks, [111]).

Chapter 4

Beyond 2-types

4.1 Crossed squares

The title of this chapter promises to go beyond 2-types and we have so far only done this with the crossed complexes. These do give all the homotopy groups of a simplicial group, but the homotopy types they represent are of a fairly simple type as they have vanishing Whitehead products.

We will return to crossed complexes later on, but will now go to 3-types and crossed squares.

4.1.1 An introduction to crossed squares

We saw earlier that crossed modules were like normal subgroups except that the inclusion map is replaced by a homomorphism that need not be a monomorphism. We even noted that all crossed modules are, up to isomorphism, obtainable by applying π_0 to a simplicial "inclusion crossed module".

Given a pair of normal subgroups M, N of a group G, we can form a square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

in which each morphism is an inclusion crossed module and there is a commutator map

$$h : M \times N \to M \cap N$$

$$h(m, n) = [m, n].$$

This forms a *crossed square* of groups. We will be dealing with crossed squares as crossed n-cubes, for $n = 2$, later. Here we will give an interim definition of crossed squares. The notion is due to Guin-Walery and Loday, [64], and this slightly shortened form of the definition is adapted from Brown-Loday, [32].

4.1.2 Crossed squares, definition and examples

A *crossed square* (more correctly *crossed square of groups*) is a commutative square of groups and homomorphisms

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"] & P \end{tikzcd}$$

together with actions of the group P on L, M and N (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h : M \times N \to L$. This structure is to satisfy the following axioms:
(i) the maps λ , λ' preserve the actions of P, furthermore with the given actions, the maps μ, ν and $\kappa = \mu\lambda = \mu'\lambda'$ are crossed modules;
(ii) $\lambda h(m,n) = m^n m^{-1}$, $\lambda' h(m,n) = {}^m n n^{-1}$;
(iii) $h(\lambda \ell, n) = \ell^n \ell^{-1}$, $h(m, \lambda' \ell) = {}^m \ell \ell^{-1}$;
(iv) $h(mm', n) = {}^m h(m', n) h(m, n)$, $h(m, nn') = h(m,n)^n h(m, n')$;
(v) $h({}^p m, {}^p n) = {}^p h(m, n)$;
for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

There is an evident notion of morphism of crossed squares, just preserve all the structure, and we obtain a category Crs^2, the *category of crossed squares.*

Examples
(a) Given any simplicial group, G, and two simplicial normal subgroups, M and N, the square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

with inclusions and with $h = [\ ,\ ] : M \times N \to G$ is a simplicial "inclusion crossed square" of simplicial groups. Applying π_0 to the diagram gives a crossed square and, in fact, all crossed squares arise in this way (up to isomorphism).
b) Any simplicial group, G, yields a crossed square, $M(G, 2)$, defined by

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3)} \arrow[r] \arrow[d] & Ker\, d_1 \arrow[d] \\ Ker d_2 \arrow[r] & G_1 \end{tikzcd}$$

for suitable maps. This is, in fact, part of the construction that shows that all connected 3-types are modelled by crossed squares.

Another way of encoding 3-types is using the truncated simplicial group and Conduché's notion of 2-crossed module.

4.2 2-crossed modules and related ideas

4.2.1 Truncations.

Definition: Given a chain complex, (X, ∂), and an integer n, the *truncation of X at level n* is the complex $t_{n]}X$ defined by

$$(t_{n]}X)_i = \begin{cases} 0 & \text{for } i > n \\ X_n/Im\,\partial_n & \\ X_i & \text{for } i < n. \end{cases}$$

For $i < n$, the differential of $t_{n]}X$ is the same as that of X, whilst the n^{th}-differential is induced by ∂.

(For more on truncations see Illusie [72, 73]). Truncation is, of course, functorial.

This construction will work for chain complexes of groups provided each $Im\,\partial$ is a *normal* subgroup of the corresponding X, i.e., provided X is a *normal chain complex of groups.*

Proposition 20 *There is a truncation functor $t_{n]} : Simp.Grps \to Simp.Grps$ such that there is a natural isomorphism*

$$t_{n]}NG \cong Nt_{n]}G,$$

where N is the Moore complex functor from Simp.Grps to the category of normal chain complexes of groups.

Proof: We first note that $d_0(NG_{n+1})$ is contained in G_n as a normal subgroup and that all face maps of G vanish on it. We can thus take

$$\begin{aligned} (t_{n]}G)_i &= G_i \text{ for all } i < n \\ (t_{n]}G)_n &= G_n/d_0(NG_{n+1}) \end{aligned}$$

and for $i > n$, we take the semidirect decomposition of G_i, which we will see shortly, given by Proposition 33, delete all occurrences of NG_k for $k > n$ and replace any NG_n by $NG_n/d_0(NG_{n+1})$. The definition of face and degeneracy is easy as is the verification that $t_{n]}N$ and $Nt_{n]}$ are the same and that the various actions are compatible. ■

This truncation functor has nice properties. (In the chain complex case, these are discussed in Illusie, [72].)

Proposition 21 *Let $T_{n]}$ be the full subcategory of Simp.Grps defined by the simplicial groups whose Moore complex is trivial in dimensions greater than n and let $i_n : T_{n]} \to Simp.Grps$ be the inclusion functor.*

a) The functor $t_{n]}$ is left adjoint to i_n. (We will usually drop the i_n and so also write $t_{n]}$ for the composite functor.)

b) The natural transformation, η, co-unit of the adjunction, is a natural epimorphism which induces an isomorphism on π_i for $i \leq n$. The unit of the adjunction is isomorphic to the identity transformation, so $T_{n]}$ is a reflective subcategory of Simp.Grps.

c) For any simplicial group G, $\pi_i(t_{n]}G) = 0$ if $i > n$.

d) To the inclusion, $T_{n]} \to T_{n+1]}$, there corresponds a natural epimorphism η_n from $t_{n+1]}$ to $t_{n]}$. If G is a simplicial group, the kernel of $\eta_n(G)$ is a $K(\pi_{n+1}(G), n+1)$, i.e., has a single non-zero homotopy group in dimension $n+1$, that being $\pi_{n+1}(G)$, i.e., is an 'Eilenberg-MacLane space' of type $(\pi_{n+1}(G), n+1)$. ■

As each statement is readily verified using the Moore complex and the semidirect product decomposition, the proof of the above will be left to you, however you will need Proposition 33, page 123.

Definition: We will say that a simplicial group, G, is *n-truncated* if $NG_k = 1$ for all $k > n$.

Of course, $T_{n]}$ is the category of n-truncated simplicial groups.

A comparison of these properties with those of the *coskeleta functors* (cf. Artin and Mazur, [**?**]) is worth making. We will not look at this in detail here, but will just summarise the results. We will meet them again later on; see page 219.

Given any integer $k \geq 0$, there is a functor, $cosk_k$, defined on the category of simplicial sets, which is the composite of a truncation functor (differently defined) and its right adjoint. The n-simplices of $cosk_k\, X$ are given by $Hom(sk_k\Delta[n], X)$, the set of simplicial maps from the k-skeleton of the n-simplex, $\Delta[n]$, to the simplicial set, X. There is a canonical map from X to $cosk_k\, X$, whose homotopy fibre is $(k-1)$-connected. The canonical map from $cosk_k\, X$ to $cosk_{k-1}\, X$ thus has homotopy fibre an Eilenberg-MacLane 'space' of type $(\pi_k(X), k)$.

This k-coskeleton is constructed using finite limits and there is an analogue in any category of simplicial objects in a category $\mathcal{D}$ provided only that $\mathcal{D}$ has finite limits, thus in particular in $Simp.Grps$. Conduché, [39], has calculated the Moore complex of $cosk_{k+1}\, G$ for a simplicial group, G, using a construction described in Duskin's Memoir, [49]. His result gives

$$\begin{aligned} N(cosk_{k+1}G)_r &= 0 \qquad \text{if } r > k+2 \\ N(cosk_{k+1}G)_{k+2} &= Ker(\partial_{k+1} : NG_{k+1} \to NG_k), \end{aligned}$$

and

$$N(cosk_{k+1}G)_r = NG_r \qquad \text{if } r \leq k+1.$$

There is an epimorphism from $cosk_{n+1}G$ to $t_{n]}G$, which, on passing to Moore complexes, gives

$$\begin{tikzcd} 0 \arrow[r] & Ker\,\partial_{k+1} \arrow[r] \arrow[d] & NG_{k+1} \arrow[r] \arrow[d] & NG_k \arrow[r, "\partial_{k+1}"] \arrow[d] & NG_{k-1} \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & NG_k/Im/\,\partial_{m+1} \arrow[r] & NG_{k-1} \end{tikzcd}$$

This epimorphism of chain complexes thus has a kernel with trivial homology. The epimorphism therefore induces an isomorphism on all homotopy groups and hence is a weak homotopy equivalence. We may thus use either $t_{n]}G$ or $cosk_{n+1}G$ as a model of the n-type of G.

4.2.2 Truncated simplicial groups and the Brown-Loday lemma

The theory of crossed n-cubes that we have hinted at above is not the only way of encoding higher n-types. Another method would be to use these truncated simplicial groups as suggested above. A detailed study of this is complicated in high dimension, but feasible for 3-types and, in fact, reveals some interesting insights into crossed squares in the process.

As a first step to understanding truncated simplicial groups a bit more, we will give a variant of an argument that we have already seen. We will look at a 1-truncated simplicial group. The analysis is really a simple use of the sort of insights given by the Brown-Loday lemma.

Proposition 22 *(The Brown-Loday lemma) Let N_2 be the (closed) normal subgroup of G_2 generated by elements of the form*

$$F_{(1),(0)}(x,y) = [s_1x, s_0y][s_0y, s_0x]$$

for $x, y \in NG_1 = Ker\, d_1$. Then $NG_2 \cap D_2 = N_2$ and consequently

$$\partial(NG_2 \cap D_2) = [Ker\, d_0, Ker\, d_1].$$

■

Note the link with group T-complex type conditions through the intersection, $NG_2 \cap D_2$.

The form of this element, $F_{(1),(0)}(x,y)$, is obtained by taking the two elements, x and y, of degree 1 in the Moore complex of a simplicial group, G, mapping them up to degree 2 by complementary degeneracies, and then looking at the component of the result that is in the Moore complex term, NG_2. (It is easy to show that G_2 is a semidirect product of NG_2 and degenerate copies of lower degree Moore complex terms.) The idea behind this pairing can be extended to higher dimensions. It gives the *Peiffer pairings*

$$F_{\alpha,\beta} : NG_p \times NG_q \to NG_{p+q}.$$

In general, these take $x \in NG_p$ and $y \in NG_q$ and (α, β) a complimentary pair of index strings (of suitable lengths), and sends (x, y) to the component in NG_{p+q} of $[s_\alpha x, s_\beta y]$; see the series of papers [96–100]. This again uses the Conduché decomposition lemma, [39], that we will see later on, cf. page 123. It is also worth noting that the Peiffer pairing ends up in $NG_{p+q} \cap D_{p+q}$, so would all be zero in a group T-complex.

A very closely related notion is that of hypercrossed complex as in Carrasco and Cegarra, [36, 37]. There one uses the component of $s_\alpha x.s_\beta y$ in NG_{p+q} to give a pairing and adds cohomological information to the result to get a reconstruction technique for G from NG, i.e., an *ultimate Dold-Kan theorem*, thus hypercrossed complexes generalise 2-crossed modules and 2-crossed complexes to all dimensions.

4.2.3 1- and 2-truncated simplicial groups

Suppose that G is a simplicial group and that $NG_i = 1$ for $i \geq 2$. This leaves us just with

$$\partial : NG_1 \to NG_0.$$

We make $NG_0 = G_0$ act on NG_1 by conjugation as before

$$^g c = s_0(g)cs_0(g)^{-1} \text{ for } g \in G_0, c \in NG_1,$$

and, of course, $\partial(^g c) = g.\partial c.g^{-1}$. Thus the first crossed module axiom is satisfied. For the other one, we note that $F_{(1),(0)}(c_1, c_2) \in NG_2$, which is trivial, so

$$\begin{aligned} 1 &= d_0([s_1c_1, s_0c_2][s_0c_2, s_0c_1]) \\ &= [s_0d_0c_1, c_2][c_2, c_1] = (^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1}, \end{aligned}$$

so the Peiffer identity holds as well. Thus $\partial : NG_1 \to NG_0$ is a crossed module. As we have already seen that the functor $\mathcal{G}$ provides a way to construct a simplicial group from a crossed module and that the result has Moore complex of length 1, we have the following slight reformulation of earlier results:

Proposition 23 *The category of crossed modules is equivalent to the subcategory $T_{1]}$ of 1-truncated simplicial groups.* ■

The main reason for restating and proving this result in this form is that we can glean more information from the proof for examining the next level, 2-truncated simplicial groups.

If we replace our 1-truncated simplicial group by an arbitrary one, then we have already introduced the idea of a Peiffer commutator of two elements, and there we used the term 'Peiffer lifting' without specifying what particular interest the construction had. We recall that here: Given a simplicial group, G, and two elements $c_1, c_2 \in NG_1$ as above, then the *Peiffer commutator* of c_1 and c_2 is defined by

$$\langle c_1, c_2\rangle = ({}^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1}.$$

We met earlier, $F_{(1),(0)}$, which gives the *Peiffer lifting* denoted

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

where

$$\{c_1, c_2\} = [s_1c_1, s_0c_2][s_0c_2, s_0c_1]$$

and we noted

$$\partial\{c_1, c_2\} = \langle c_1, c_2\rangle.$$

These structures come into their own for a 2-truncated simplicial group. Suppose that G is now a simplicial group, which is 2-truncated, so its Moore complex looks like:

$$\ldots 1 \to NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0.$$

For the moment, we will concentrate our attention on the morphism ∂_2.

The group NG_1 acts on NG_2 via conjugation using s_0 or s_1. We will use s_0 for the moment, so that if $g \in NG_1$ and $c \in NG_2$,

$${}^{g}c = s_0(g)cs_0(g)^{-1}.$$

It is once again clear that $\partial_2({}^{g}c) = g.\partial_2(c).g^{-1}$ and, as before, we consider, for $c_1, c_2 \in NG_2$ this time, the Peiffer pairing given by

$$[s_1c_1, s_0c_2][s_0c_2, s_0c_1],$$

which is, this time, the component of $[s_1c_1, s_0c_2]$ in NG_3. However that latter group is trivial, so this element is trivial, and hence, so is its image in NG_2. The same calculation as before shows that, with this s_0-based action of NG_1 on NG_2, (NG_2, NG_1, ∂_2) is a crossed module.

We also know that there is a Peiffer lifting

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

which measures the obstruction to $NG_1 \to NG_0$ being a crossed module, since $\partial\{-,-\}$ is the Peiffer commutator, whose vanishing is equivalent to $NG_1 \to NG_0$ being a crossed module. We do not have yet in our investigation a detailed knowledge of how the two structures interact, nor any other distinguishing properties of $\{-,-\}$. We will not give such a detailed derivation here, but from it we can obtain the following:

Proposition 24 *Let G be a 2-truncated simplicial group. The Peiffer lifting*

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

has the following properties:
(i) it is a map such that if $m_0, m_1 \in NG_1$,

$$\partial\{m_0, m_1\} = {}^{\partial m_0}m_1.(m_0m_1m_0^{-1})^{-1};$$

(ii) if $\ell_0, \ell_1 \in NG_2$,

$$\{\partial\ell_0, \partial\ell_1\} = [\ell_0, \ell_1];$$

(iii) if $\ell \in NG_2$ and $m \in NG_1$, then

$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(iv) if $m_0, m_1, m_2 \in NG_1$, then
a) $\{m_0, m_1m_2\} = \{m_0, m_1\}^{(m_0m_1m_0^{-1})}\{m_0, m_2\}$,
b) $\{m_0m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1m_2m_1^{-1}\}$;
(v) if $n \in NG_0$ and $m_0, m_1 \in NG_1$, then

$$^{n}\{m_0, m_1\} = \{{}^{n}m_0, {}^{n}m_1\}.$$

■

The above can be encoded in the definition of a 2-crossed module.

4.2.4 2-crossed modules, the definition

Definition: A *2-crossed module* is a normal complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

together with an action of N on all three groups and a mapping

$$\{-,-\} : M \times M \to L$$

such that

(i) the action of N on itself is by conjugation, and ∂_2 and ∂_1 are N-equivariant;

(ii) for all $m_0, m_1 \in M$,
$$\partial_2\{m_0, m_1\} = {}^{\partial_1 m_0}m_1.m_0m_1^{-1}m_0^{-1};$$

(iii) if $\ell_0, \ell_0 \in L$, then
$$\{\partial_2\ell_0, \partial_2\ell\} = [\ell_1, \ell_0];$$

(iv) if $\ell \in L$ and $m \in M$, then
$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(v) for all $m_0, m_1, m_2 \in M$,

(a) $\{m_0, m_1m_2\} = \{m_0, m_1\}\{\partial\{m_0, m_2\}, (m_0m_1m_0^{-1})\}\{m_0, m_2\}$;

(b) $\{m_0m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1m_2m_1^{-1}\}$;

(vi) if $n \in N$ and $m_0, m_1 \in M$, then

$$^n\{m_0, m_1\} = \{^nm_0, {}^nm_1\}.$$

The pairing $\{-,-\} : M \times M \to L$ is often called the *Peiffer lifting* of the 2-crossed module.

The only one of these axioms that looks 'daunting' is (v)a). Note that we have not specified that M acts on L. We could have done that as follows: if $m \in M$ and $\ell \in L$, define

$$^m\ell = \{\partial\ell, m\}\ell.$$

Now (v)a) simplifies to the expression

$$\{m_0, m_1m_2\} = \{m_0, m_1\}\,^{(m_0m_1m_0^{-1})}\{m_0, m_2\}.$$

We denote such a 2-crossed module by $\{L, M, N, \partial_2, \partial_1\}$, or similar, only adding in notation for the actions and the pairing if explicitly needed for the context. A *morphism of 2-crossed modules* is, fairly obviously, given by a diagram

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & N \arrow[d, "f_0"'] \\ L' \arrow[r, "\partial_2'"'] & M' \arrow[r, "\partial_1'"'] & N' \end{tikzcd},$$

where $f_0\partial_1 = \partial_1'f_1, f_1\partial_2 = \partial_2'f_2$,

$$f_1(^nm) = {}^{f_0(n)}f_1(m), \quad f_2(^n\ell) = {}^{f_0(n)}f_2(\ell),$$

and

$$\{-,-\}(f_1 \times f_1) = f_2\{-,-\},$$

for all $\ell \in L$, $m \in M$, $n \in N$.

These compose in an obvious way giving a category which we will denote by $2-CMod$.

The following should be clear.

Theorem 5 *The Moore complex of a 2-truncated simplicial group is a 2-crossed module. The assignment is functorial.* ∎

We will denote this functor by $\mathsf{C}^{(2)} : T_{2]} \to 2-CMod$. It is an equivalence of categories.

4.2.5 Examples of 2-crossed modules

Of course, the construction of 2-crossed modules from simplicial groups gives a generic family of examples, but we can do better than that and show how these new crossed gadgets link in with others that we have met earlier.

Example 1: Any crossed module gives a 2-crossed module, since if (M, N, ∂) is a crossed module, we need only add a trivial $L = 1$, and the resulting sequence

$$L \to M \to N$$

with the 'obvious actions' is a 2-crossed module! This is, of course, functorial and $CMod$ can be considered to be a full subcategory of $2-CMod$ in this way. It is a reflective subcategory since there is a reflection functor obtained as follows:

If

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

is a 2-crossed module, then $Im\,\partial_2$ is a normal subgroup of M and we have (with a small abuse of notation):

Proposition 25 *If $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ is a 2-crossed module then there is an induced crossed module structure on*

$$\partial_1 : \frac{M}{Im\,\partial_2} \to N.$$

■

But we can do better than this:

Example 2: Any crossed complex of length 2, that is one of form

$$\ldots \to 1 \to 1 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

gives us a 2-crossed complex on taking $L = C_2$, $M = C_1$ and $N = C_0$, with $\{m, m'\} = 1$ for all $m, m' \in M$. We will check this in a moment, but note that this gives a functor from $Crs_{2]}$ to $2-CMod$ extending the one we gave in Example 1.

Of course, (i) crossed complexes of length 2 are the same as 2-truncated crossed complexes.

4.2.6 Exploration of trivial Peiffer lifting

Suppose we have a 2-crossed module

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

with the extra condition that $\{m_0, m_1\} = 1$ for all $m_0, m_1 \in M$. The obvious thing to do is to see what each of the defining properties of a 2-crossed module give in this case.

(i) There is an action of N on L and M and the ∂s are N-equivariant. (This gives nothing new in our special case.)

(ii) $\{-, -\}$ is a lifting of the Peiffer commutator - so if $\{m_0, m_1\} = 1$, the Peiffer identity holds for (M, N, ∂_1), i.e. that is a crossed module;

(iii) if $\ell_0, \ell_1 \in L$, then $1 = \{\partial_2 \ell_0, \partial_2 \ell_1\} = [\ell_1, \ell_0]$, so L is Abelian
and,

(iv) as $\{-, -\}$ is trivial ${}^{\partial m}\ell = \ell$, so ∂M has trivial action on L.

Axioms (v) and (vi) vanish.

We leave the reader, if they so wish, to structure this into a formal proof that the 2-crossed module is precisely a 2-truncated crossed complex.

Our earlier discussion should suggest:

Proposition 26 *The category $Crs_{2]}$ of crossed complexes of length 2 is equivalent to the full sub-category of $2{-}CMod$ given by those 2-crossed complexes with trivial Peiffer lifting.* ■

We leave the proof of this to the reader.

A final comment is that in a 2-truncated simplicial group, G, one obviously has that it satisfies the thin filler condition (cf. page 30) in dimensions greater than 2, since $NG_k = 1$ for all $k > 2$ and if the Peiffer lifting is trivial in the corresponding 2-crossed module, G satisfies it in dimensions 2 as well. (As D_1 is $s_0(G_0)$, any simplicial group satisfies the thin filler condition in dimension 1.)

In the next section we will give other examples of 2-crossed modules, those coming from crossed squares.

4.2.7 2-crossed modules and crossed squares

We now have several 'competing' models for homotopy 3-types. Since we can go from simplicial groups to both crossed square and 2-crossed modules, there should be some link between the latter two situations. In his work on homotopy n-types, Loday gave a construction of what he called a 'mapping cone' for a crossed square. Conduché later noticed that this naturally had the structure of a 2-crossed module. This is looked at in detail in a paper by Conduché, [40].

Suppose that

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

is a crossed square, then its mapping cone complex is

$$L \stackrel{\partial_2}{\to} M \rtimes N \stackrel{\partial_1}{\to} P,$$

where $\partial_2\ell = (\lambda\ell^{-1}, \lambda'\ell)$ and $\partial_1(m,n) = \mu(m)\nu(n)$.

We first note that the semi-direct product $M \rtimes N$ is formed by making N act on M *via* P, i.e.

$${}^{n}m = {}^{\nu(n)}m,$$

where the P-action is the given one. The fact that (λ^{-1}, λ') and $\mu\nu$ are homomorphisms is an interesting and instructive, but easy, exercise:

i) $(m,n)(m',n') = (m^{\nu(n)}m', nn')$, so

$$\begin{aligned} \partial_1((m,n)(m',n')) &= \mu(m^{\nu(n)}m').\nu(nn') \\ &= \mu(m)\nu(n)\mu(m')\nu(n)^{-1}\nu(n)\nu(n') \\ &= (\mu(m)\nu(n))(\mu(m')\nu(n')); \end{aligned}$$

(ii) if $\ell, \ell' \in L$, then, of course,

$$\begin{aligned} \partial_1(\ell\ell') &= (\lambda(\ell\ell')^{-1}, \lambda'(\ell\ell')) \\ &= (\lambda(\ell')^{-1}\lambda(\ell)^{-1}, \lambda'(\ell)\lambda'(\ell')). \end{aligned}$$

whilst

$$\begin{aligned}\partial_1(\ell)\partial_1(\ell') &= (\lambda(\ell)^{-1}, \lambda'(\ell))(\lambda(\ell')^{-1}, \lambda'(\ell'))\\ &= (\lambda(\ell)^{-1}.{}^{\nu\lambda'(\ell^{-1})}\lambda(\ell')^{-1}, \lambda'(\ell\ell')),\end{aligned}$$

thus the second coordinates are the same, but, as $\nu\lambda' = \mu\lambda$, the first coordinates are also equal.

These elementary calculations are useful as they pave the way for the calculation of the Peiffer commutator of $x = (m, n)$ and $y = (c, a)$ in the above complex:

$$\begin{aligned}\langle x, y\rangle &= {}^{\partial x}y.xy^{-1}x^{-1}\\ &= {}^{\mu m.\nu n}(c, a).(m, n)({}^{a^{-1}}c^{-1}, a^{-1})({}^{n^{-1}}m^{-1}, n^{-1})\\ &= ({}^{\mu m\nu n}c, {}^{\mu m\nu n}a)(m^{\nu(na^{-1})}c^{-1}.{}^{\nu(na^{-1}n^{-1})}m^{-1}, na^{-1}n^{-1}),\end{aligned}$$

which on multiplying out and simplifying is

$$({}^{\nu(na^{-1}n^{-1})}m.m^{-1}, {}^{\mu m}(nan^{-1}).(na^{-1}n^{-1})).$$

(Note that any dependence on c vanishes!)

Conduché defined the Peiffer lifting in this situation by

$$\{x, y\} = h(m, nan^{-1}).$$

It is immediate to check that this works

$$\begin{aligned}\partial_2\{x, y\} &= (\lambda h(m, nan^{-1}), \lambda' h(m, nan^{-1}))\\ &= ({}^{\nu(na^{-1}n^{-1})}m.m^{-1}, {}^{\mu m}(nan^{-1}).(na^{-1}n^{-1}),\end{aligned}$$

by the axioms of a crossed square.

We will not check all the axioms for a 2-crossed module for this structure, but will note the proofs for one or two of them as they illustrate the connection between the properties of the h-map and those of the Peiffer lifting.

2CM(iii) : $\{\partial\ell_0, \partial\ell_1\} = [\ell_1, \ell_0]$. As $\partial\ell = (\lambda\ell^{-1}, \lambda'\ell)$, this needs the calculation of

$$h(\lambda\ell_0^{-1}, \lambda'(\ell_0\ell_1\ell_0^{-1})),$$

but the crossed square axiom :

$$h(\lambda\ell, n) = \ell.{}^{n}\ell^{-1}, \text{ and } h(m, \lambda'\ell) = {}^{m}\ell.\ell^{-1},$$

together with the fact that the map $\lambda : L \to M$ is a crossed module, give

$$\begin{aligned}h(\lambda\ell_0^{-1}, \lambda'(\ell_0\ell_1\ell_0^{-1})) &= {}^{\mu\lambda(\ell_0^{-1}}(\ell_0\ell_1\ell_0^{-1}).\ell_0\ell_1^{-1}\ell_0^{-1})\\ &= [\ell_1, \ell_0].\end{aligned}$$

We need $\{(m, n), (\lambda\ell^{-1}, \lambda'\ell)\}\{(\lambda\ell^{-1}, \lambda'\ell), (m, n)\}$ to equal ${}^{\mu(m)\nu(n)}\ell.\ell^{-1}$, but evaluating the initial expression gives

$$\begin{aligned}h(m, n.\lambda'\ell.n^{-1})h(\lambda\ell^{-1}, \lambda'\ell.n.\lambda'\ell^{-1}) &= h(m, \lambda'({}^{n}\ell))h(\lambda\ell^{-1}, \lambda'\ell.n.\lambda'\ell^{-1})\\ &= {}^{\mu(m)\nu(n)}\ell.{}^{\nu(n)}\ell^{-1}.\ell^{-1}.{}^{\nu\lambda'(\ell).\nu(n).\nu\lambda'\ell^{-1}}\ell,\end{aligned}$$

and this does simplify as expected to give the correct results.

We thus have two ways of going from a simplicial group, G, to a 2-crossed module:
(a) directly to get

$$\frac{NG_2}{\partial NG_3} \to NG_1 \to NG_0;$$

(b) indirectly via $M(G,2)$ and then by the above construction to get

$$\frac{NG_2}{\partial NG_3} \to Ker\, d_0 \rtimes Ker\, d_1 \to G_1$$

and they clearly give the same homotopy type. More precisely G_1 decomposes as $Ker\, d_0 \rtimes s_0G_0$ and the $Ker\, d_0$ factor in the middle term of (b) maps down to that in this decomposition by the identity map, thus d_0 induces a quotient map from (b) to (a) with kernel isomorphic to

$$1 \to Ker\, d_0 \xrightarrow{=} Ker\, d_0,$$

which is acyclic/contractible.

4.2.8 2-crossed complexes

(These were not discussed in the lectures in Buenos Aires due to lack of time.)

Crossed complexes are a useful extension of crossed modules allowing not only the encoding of an algebraic model for the 2-type, but also information on the 'chains on the universal cover', e.g. if G is a simplicial group, we had $\mathsf{C}(G)$, the crossed complex constructed from the Moore complex of G, given by

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions and having at its 'bottom end' the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0.$$

For a crossed complex, $\pi(X)$, coming from a CW-complex (as a filtered space, filtered by its skeleta), these groups in dimensions ≥ 3 coincide with the corresponding groups of the complex of chains on the universal cover of X. In general, the analogue of that chain complex can be extracted functorially from a general crossed complex; see [29] or [106]. The tail on a crossed complex allows extra dimensions, not available just with crossed modules, in which homotopies can be constructed. The category Crs is very much better structured than is $CMod$ itself and so 'adding a tail' would seem to be a 'good thing to do', so with 2-crossed modules, we can try and do something similar, adding a similar 'tail'.

We have an obvious normal chain complex of groups that ends

$$\ldots \to C(G)_3 \to \frac{NG_2}{d_0(NG_3 \cap D_3)} \to NG_1 \to NG_0.$$

Here there are more of the structural Peiffer pairings of the Moore complex NG that survive to the quotient, but it should be clear that, as they take values in the $NG_n \cap D_n$, in general these will again be almost all trivial if the receiving dimension, n, is greater than 2. For $n \leq 2$, these pairings are those that we have been using earlier in this chapter. The one exceptional case that is important here, as in the crossed complex case, is that which gives the action of NG_0 on $C_n(G)$ for $n \geq 3$, which, just as before, gives $C_n(G)$ the structure of a $\pi_0 G$-module. Abstracting from this gives the definition of a 2-crossed complex.

Definition: A *2-crossed complex* is a normal complex of groups

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_0,$$

together with a 2-crossed module structure given on $C_2 \to C_1 \to C_0$ by a Peiffer lifting function $\{-,-\} : C_1 \times C_1 \to C_2$, such that, on writing $\pi = Coker(C_1 \to C_0)$,

(i) each C_n, $n \geq 3$ and $Ker\,\partial_2$ are π-modules and the ∂_n for $n \geq 4$, together with the codomain restriction of ∂_3, are π-module homomorphisms;

(ii) the π-module structure on $Ker\,\partial_2$ is the action induced from the C_0-action on C_2 for which the action of $\partial_1 C_1$ is trivial.

A 2-crossed complex morphism is defined in the obvious way, being compatible with all the actions, the pairings and Peiffer liftings. We will denote by $2-Crs$, the corresponding category.

Proposition 27 *The construction above defines a functor* $\mathsf{C}^{(2)}$ *from Simp.Grps to* $2-Crs$. ■

There are no prizes for guessing that the simplicial groups whose homotopy types are accurately encoded in $2-Crs$ by this functor are those that satisfy the thin condition in dimensions greater than 3. In fact, the construction of the functor $\mathsf{C}^{(2)}$ explicitly kills off the intersection $NG_k \cap D_k$ for $k \geq 3$.

We have noted above that any 2-crossed module,

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

gives us a short crossed complex by dividing L by the subgroup $\{M, M\}$, the image of the Peiffer lifting. (We do not need this, but $\{M, M\}$ is easily checked to be a normal subgroup of L.) We also discussed those 2-crossed complexes that had trivial Peiffer lifting. They were just the length 2 crossed complexes. This allows one to show that crossed complexes form a reflexive subcategory of $2-Crs$ and to give a simple description of the reflector:

Proposition 28 *There is an embedding*

$$Crs \to 2{-}Crs,$$

which has a left adjoint, L *say, compatible with the functors defined from Simp.Grps to* $2-Crs$ *and to* Crs, *i.e.* $\mathsf{C}(G) \cong \mathsf{L}\mathsf{C}^{(2)}(G)$. ■

4.3 Cat^n-groups and crossed n-cubes

4.3.1 Cat^2-groups and crossed squares

In the simplest examples of crossed squares, μ and μ' are normal subgroup inclusions and $L = M \cap N$, with h being the conjugation map. Moreover this type of example is almost 'generic' since, if

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & G \end{tikzcd}$$

is a simplicial crossed square constructed from a simplicial group, G, and two simplicial normal subgroups, M and N, then applying π_0, the square gives a crossed square and, up to isomorphism, all crossed squares arise in this way.

Although when first defined by D. Guin-Walery and J.-L. Loday, [64], the notion of crossed squares was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (see [81] and below).

Definition: A *cat*^{1}*-group* is a triple, (G, s, t), where G is a group and s, t are endomorphisms of G satisfying conditions
(i) $st = t$ and $ts = s$.
(ii) $[Ker\, s,\ Ker\, t] = 1$.

A cat^1-group is a reformulation of an internal groupoid in *Grps*. (The interchange law is given by the $[Ker, Ker]$ condition; left for you to check) As these latter objects are equivalent to crossed modules, we expect to be able to go between cat^1-groups and crossed modules without hindrance, and we can:

Setting $M = Ker\, s$, $N = Im\, s$ and $\partial = t|M$, then the action of N on M by conjugation within G makes $\partial : M \to N$ into a crossed module. Conversely if $\partial : M \to N$ is a crossed module, then setting $G = M \rtimes N$ and letting s, t be defined by

$$s(m, n) = (1, n)$$

and

$$t(m, n) = (1, \partial(m)n)$$

for $m \in M$, $n \in N$, we have that (G, s, t) is a cat^1-group. Again this is one of those simple, but key calculations that are well worth doing yourself.

For a cat^2-group, we again have a group, G, but this time with two independent cat^1-group structures on it. Explicitly:

Definition: A *cat*^{2}*-group* is a 5-tuple (G, s_1, t_1, s_2, t_2), where (G, s_i, t_i), $i = 1, 2$, are cat^1-groups and

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i, j = 1, 2, \quad i \neq j$.

There is an obvious notion of morphism between cat^2-groups and with this we obtain a category, $Cat^2(Grps)$.

Theorem 6 *[81] There is an equivalence of categories between the category of cat^2-groups and that of crossed squares.*

Proof: The cat^1-group (G, s_1, t_1) will give us a crossed module with $M = Ker\, s_1$, $N = Im\, s_1$, and $\partial = t|M$, but, as the two cat^1-group structures are independent, (G, s_2, t_2) restricts to give cat^1-group structures on both M and N and makes ∂ a morphism of cat^1-groups as is easily checked. We thus get a morphism of crossed modules

$$\begin{tikzcd} Ker\, s_1 \cap Ker\, s_2 \arrow[r] \arrow[d] & Im\, s_1 \cap Ker\, s_2 \arrow[d] \\ Ker\, s_2 \cap Im\, s_1 \arrow[r] & Im\, s_1 \cap Im\, s_2, \end{tikzcd}$$

where each morphism is a crossed module for the natural action, i.e., conjugation in G. It remains to produce an h-map, but this is given by the commutator within G, since, if $x \in Ker\, s_2 \cap Im\, s_1$ and $y \in Im\, s_2 \cap Ker\, s_1$, then $[x, y] \in Ker\, s_1 \cap Ker\, s_2$. It is easy to check the axioms for a crossed square. The converse is left as an exercise. ■

4.3.2 Cat^n-groups and crossed n-cubes, the general case

Of the two notions named in the title of this section, the first is easier to define.

Definition: A *cat^n-group* is a group G together with $2n$ endomorphisms $s_i, t_i, (1 \leq i \leq n)$ such that

$$s_i t_i = t_i, \text{ and } t_i s_i = s_i \text{ for all } i,$$

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \text{ for } i \neq j$$

and, for all i,

$$[Ker\, s_i, Ker\, t_i] = 1.$$

A cat^n-group is thus a group with n independent cat^1-group structures on it.

As a cat^1-group can also be reformulated as an internal groupoid in the category of groups, a cat^n-group, not surprisingly, leads to an internal n-fold groupoid in the same setting.

The definition of crossed n-cube as an n-fold crossed module was initially suggested by Ellis in his thesis. The only problem was to determine the sense in which one crossed module should act on another. Since the number of axioms controlling the structure increased from crossed modules to crossed squares, one might fear that the number and complexity of the axioms would increase drastically in passing to higher 'dimensions'. The formulation that resulted from the joint work, [56], of Ellis and Steiner showed how that could be avoided by encoding the actions and the h-maps in the same structure.

We write $\langle n \rangle$ for the set $\{1, \ldots, n\}$.

Definition: A *crossed n-cube*, M, is a family of groups, $\{M_A : A \subseteq \langle n \rangle\}$, together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, for $i \in \langle n \rangle, A \subseteq \langle n \rangle$, and functions, $h : M_A \times M_B \to M_{A \cup B}$, for $A, B \subseteq \langle n \rangle$, such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \langle n \rangle$, the following axioms hold:

(1) $\mu_i a = a$ if $a \notin A$

(2) $\mu_i\mu_j a = \mu_j\mu_i a$
(3) $\mu_i h(a,b) = h(\mu_i a, \mu_i b)$
(4) $h(a,b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
(5) $h(a,a') = [a,a']$
(6) $h(a,b) = h(b,a)^{-1}$
(7) $h(a,b) = 1$ if $a = 1$ or $b = 1$
(8) $h(aa',b) = {}^a h(a',b)h(a,b)$
(9) $h(a,bb') = h(a,b)^b h(a,b')$
(10) ${}^a h(h(a^{-1},b),c)^c h(h(c^{-1},a),b)^b h(h(b^{-1},c),a) = 1$
(11) ${}^a h(b,c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

A morphism of crossed n-cubes

$$\{M_A\} \to \{M'_A\}$$

is a family of homomorphisms, $\{f_A : M_A \to M'_A \mid A \subseteq \langle n\rangle\}$, which commute with the maps, μ_i, and the functions, h. This gives us a category, Crs^n, equivalent to that of cat^n-groups.

Remarks: 1. In the correspondence between cat^n-groups and crossed n-cubes (see Ellis and Steiner, [56]), the cat^n-group corresponding to a crossed n-cube, (M_A), is constructed as a repeated semidirect product of the various M_A. Within the resulting "big group", the h-functions interpret as being commutators. This partially explains the structure of the h-function axioms.

2. For $n = 1$, these eleven axioms reduce to the usual crossed module axioms. For $n = 2$, they give a crossed square:

$$\begin{tikzcd} M_{\langle 2\rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & M_{\{1\}} \arrow[d, "\mu_1"] \\ M_{\{2\}} \arrow[r, "\mu_2"'] & M_\emptyset \end{tikzcd},$$

with the h-map that was previously specified being $h : M_{\{1\}} \times M_{\{2\}} \to M_{\langle 2\rangle}$. The other h-maps in the above definition correspond to the various actions as explained in the definition itself.

Theorem 7 *[56] There are equivalences of categories*

$$Crs^n \simeq Cat^n(Grps),$$

■

4.4 Loday's Theorem and its extensions

In 1982, Loday proved a generalisation of the MacLane - Whitehead result that stated that connected homotopy 2-types (they called them 3-types) were modelled by crossed modules. The extension used cat^n-groups, and, as cat^1-groups 'are' crossed modules, we should expect cat^n-groups to model connected $(n+1)$-types (if the MacLane-Whitehead result is to be the $n = 1$ case, see page 90).

We have mentioned that 'simplicial groupoids' model all homotopy types and had a construction of both a crossed module $M(G,1)$ and a crossed square, $M(G,2)$ from a simplicial group, G. These are the $n = 1$ and $n = 2$ cases of a general construction of a crossed n-cube from G that we will give in a moment First we note a rather neat result.

We saw early on in these notes, (Lemma 3, page 31), that if $\partial : C \to P$ was a crossed module, then $\partial C \lhd P$, i.e. is a normal subgroup of P. A crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

can be thought of as a (horizontal or vertical,) crossed module of crossed modules:

$$\begin{array}{c} L \\ \downarrow \\ N \end{array} \longrightarrow \begin{array}{c} M \\ \downarrow \\ P \end{array}$$

(λ, ν) gives such a crossed module with domain (L, N, λ') and codomain (M, P, μ) and so on. (Working out the precise meaning of 'crossed module of crossed modules' and, in particular, what it should mean to have an action of one crossed module on another, is a very useful exercise; try it!) The image of (λ, ν) is a normal sub-crossed module of (M, P, μ), so we can form a quotient

$$\overline{\mu} : M/\lambda L \to P/\nu N,$$

and this is a crossed module. (This is not hard to check. There are lots of different ways of checking it, but perhaps the best way is just to show how $P/\nu N$ acts on $M/\lambda L$, in an obvious way, and then to check the induced map, $\overline{\mu}$, has the right properties - just by checking them. This gives one a feeling for how the various parts of the definition of a crossed square are used here.)

Another result from near the start of these notes, (Lemma 4), is that $Ker\,\partial$ is a central subgroup of C and ∂C acts trivially on it, so $Ker\,\partial$ has a natural $P/\partial C$-module structure. Is there an analogue of this for a crossed square? Of course, referring again to our crossed square, above, the kernel of (λ, ν) would be $\lambda' : Ker\,\lambda \to Ker\,\nu$ (omitting any indication of restriction of λ' for convenience). Both $Ker\,\lambda$ and $Ker\,\nu$ are Abelian, as they themselves are kernels of crossed modules, so $Ker\,\lambda$ is a $M/\lambda L$-module and $Ker\,\nu$ is a $P/\nu N$-module. (It is left to the diligent reader to work out the detailed structure here and to explore crossed modules that are modules over other ones.)

We had, for a given simplicial group, G, the crossed square

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3)} \arrow[r] \arrow[d] & Ker\,d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

which was denoted $M(G, 2)$. (The top horizontal and left vertical maps are induced by d_0.) Let us examine the horizontal quotient and kernel.

First the quotient, this has NG_1/d_0NG_2 as its 'top' group and $G_1/Ker\,d_0 \cong G_0$, as its bottom one. Checking all the induced maps shows quite quickly that the quotient crossed module is $M(G, 1)$, up to isomorphism.

What about the kernel? Well, the bottom horizontal map is an inclusion, so has trivial kernel, whilst the top is induced by d_0, and so the kernel here can be calculated to be $Ker\,d_0 \cap NG_2$, divided by $d_0(NG_3)$, but that is $Ker\,\partial/Im\,\partial$ in the Moore complex, so is $H_2(NG)$ and thus is $\pi_2(G)$. We thus have, from previous calculations, that for $M(G,1)$, there is a crossed 2-fold extension

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

and for $M(G,2)$, a similar object, a crossed 2-fold extension of crossed modules:

$$\begin{tikzcd}
1 \arrow[r] & \pi_2(G) \arrow[r] \arrow[d] & Ker\,d_1 \arrow[r] \arrow[d] & NG_2/d_0(NG_3) \arrow[r] \arrow[d] & Ker\,d_1 NG_1/d_0(NG_2) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 \arrow[r] & Ker\,d_0 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & 1
\end{tikzcd}$$

'Obviously' this should give an element of '$H^3(M(G,2),(\pi_2(G)\to 1))$', but we have not given any description of what that cohomology group should be. It can be done, but we will not go in that direction for the moment. Rather we will use the route via simplicial groups.

4.4.1 Simplicial groups and crossed n-cubes, the main ideas

We have that simplicial groups yield crossed squares by the $M(G,2)$ construction, and that, from $M(G,2)$, we can calculate $\pi_0(G)$, $\pi_1(G)$, and $\pi_2(G)$. If G represents a 3-type of a space (or the 2-type of a simplicial group), then we would expect these homotopy groups to be the only non-trivial ones. (Any simplicial group can be truncated to give one with these π_i as the only non-trivial ones.) This suggests that going from 3-types to crossed squares in a nice way should be just a question of combining the functorial constructions

$$\mathsf{Spaces} \xrightarrow{Sing} \mathsf{Simplicial\ Sets}$$

$$\mathsf{Simplicial\ Sets} \xrightarrow{G(\ )} \mathcal{S}\text{−}\mathsf{Groupoids}$$

$$\mathcal{S}\text{−}\mathsf{Groupoids} \xrightarrow{M(\ ,2)} \mathsf{Crossed\ squares}.$$

Of course, we would need to see if, for $f : X \to Y$ a 3-equivalence (so f induces isomorphisms on π_i for $i = 0,1,2,3$), what would be the relationship between the corresponding crossed squares. We would also need to know that each crossed square was in sense 'equivalent' to one of the form $M(G,2)$ for some G constructed from it, in other words to reverse, in part, the last construction. (The other constructions have well known inverses at the homotopy level.)

We will use a 'multinerve' construction, generalising the nerve that we have already met. We will denote this by $E^{(n)}(\mathsf{M})$ for M a crossed n-cube.

For $n = 1$, $E^{(1)}$ is just the nerve of the crossed module, so if $\mathsf{M} = (C,P,\partial)$, we have $E^{(1)}(\mathsf{M}) = K(\mathsf{M})$ as given already on page 45.

For $n = 2$, i.e., for a crossed square, M, we form the 'double nerve ' of the associated cat^2-group of M. From M, we first form the 'crossed module of cat^1-groups'

$$L \rtimes N \xrightarrow{(\lambda,\nu)} M \rtimes P,$$

where, for instance, in $M \rtimes P$ the source endomorphism is $s(m,p) = (1,p)$ and the target is $t(m,p) = (1,\partial m.p)$. (We could repeat in the horizontal direction to form $(L \rtimes N) \rtimes (M \rtimes P)$, which is the 'big group' of the cat^2-group associated to M, but, in fact, will not do this except implicitly, as it is easier to form a simplicial crossed module in this situation. This,

$$E^{(1)}(L \xrightarrow{\lambda'} N) \longrightarrow E^{(1)}(M \xrightarrow{\mu} P),$$

is obtained by applying the $E^{(1)}$ construction to the vertical crossed modules. The two parts are linked by a morphism of simplicial groups induced from (λ, ν) and which is compatible with the action of the right hand simplicial group on the left hand one. (This action is not that obvious to write down - unless you have already done the previously suggested 'exercises'. It uses the h-maps from $M \times N$ to L, etc. in an essential way, and is, in some ways, best viewed within $(L \rtimes N) \rtimes (M \rtimes P)$ as being derived from conjugation. Details are, for instance, in Porter, [106] or [104] as well as in the discussion of the equivalence between cat^n-groups and crossed n-cubes in the original, [56].)

With this simplicial crossed module, we apply the nerve in the second horizontal direction to get a bisimplicial group, $\mathcal{E}^{(2)}(\mathsf{M})$. (Of course, if we started with a crossed n-cube, we could repeat the application of the nerve functor n-times, one in each direction to get an n-simplicial group $\mathcal{E}^{(n)}(\mathsf{M})$.)

There are two ways of getting from a bisimplicial set or group to a simplicial one. One is the diagonal, so if $\{G_{p,q}\}$ is a bisimplicial group, $\mathsf{diag}(G_{\bullet,\bullet})_n = G_{n,n}$ with fairly obvious face and degeneracy maps. The other is the *codiagonal* (also sometimes called the 'bar construction'). This was introduced by Artin and Mazur, [6]. It picks up related terms in the various $G_{p,q}$ for $p+q = n$. (An example is for any simplicial group, G, on taking the nerve in each dimension. You get a bisimplicial set whose codiagonal is $\overline{W}(G)$, with the formula given later in these notes.) We will consider the codiagonal in some detail later on, (starting on page **??**). The two constructions give homotopically equivalent simplicial groups. Proofs of this can be found in several places in the literature, for instance, in the paper by Cegarra and Remedios, [**?**]. Here we will set $E^{(n)}(\mathsf{M}) = \mathsf{diag}\mathcal{E}^{(n)}(\mathsf{M})$.

At this stage, for the reader trying to understand what is going on here, it is worth calculating the Moore complex of these simplicial groups. This is technically quite tricky as it is easy to make a slip, but it is not hard to see that they are 'closely related' to the 2-crossed module / mapping cone complex:

$$L \to M \rtimes N \to P$$

that we met earlier, (page 98), that is due to Loday and Conduché, see [40]. Of course, such detailed calculations are much harder to generalise to crossed n-cubes and other techniques are used, see [104] or the alternative version based on the technology of cat^n-groups due to Bullejos, Cegarra and Duskin, [35].

In any of these approaches from a crossed n-cube or cat^n-group, you either extract a n-simplicial group and then a simplicial group, by diagonal or codiagonal, or going one stage further applying the nerve functor to the n-simplicial group to get a $(n+1)$-simplicial set, which is then 'attacked' using the diagonal or codiagonal functors to get out a simplicial set. This end result is the simplicial model for the crossed n-cube and has the same homotopy groups as M. Using the simplicial group approach, one applies the $M(-, n)$-functor, that we have so far seen only for $n = 1$ and 2, to get back a new crossed n-cube. This is not M itself in general, but is 'quasi-isomorphic' to it.

A morphism $f : \mathsf{M} \to \mathcal{N}$ of crossed n-cubes will be called a *trivial epimorphism* if $\mathcal{E}^{(n)}(f) : \mathcal{E}^{(n)}(\mathsf{M}) \to \mathcal{E}^{(n)}(\mathcal{N})$ is an epimorphism (and thus a fibration of simplicial groups having contractible kernel. Starting with the category, Crs^n, of crossed n-cubes, inverting the trivial epimorphisms gives a category, $Ho(Crs^n)$, and f will be called a *quasi-isomorphism* if it gives an isomorphism in this category. We can now state Loday's result in the form given in [104]:

Theorem 8 *The functor*

$$M(-,n) : Simp.Grps \to Crs^n$$

induces an equivalence of categories

$$Ho_n(Simp.Grps) \xrightarrow{\simeq} Ho(Crs^n).$$

■

As yet we have not actually given the definition of $M(G,n)$ for $n > 2$ so here it is:

Definition Given a simplicial group, G, the crossed n-cube, $M(G,n)$, is given by:
(a) for $A \subseteq \langle n\rangle$,

$$M(G,n)_A = \frac{\bigcap\{Ker\, d_j^n \,:\, j \in A\}}{d_0(Ker\, d_1^{n+1} \cap \bigcap\{Ker\, d_{j+1}^{n+1} \,:\, j \in A\})};$$

(b) if $i \in \langle n\rangle$, the homomorphism $\mu_i : M(G,n)_A \to M(G,n)_{A\setminus\{i\}}$ is induced from the inclusion of $\bigcap\{Ker\, d_j^n \,:\, j \in A\}$ into $\bigcap\{Ker\, d_j^n \,:\, j \in A \setminus \{i\}\}$;
(c) representing an element in $M(G,n)_A$ by $\overline{x}$, where $x \in \bigcap\{Ker\, d_j^n \,:\, j \in A\}$, (so the overbar denotes a coset), and, for $A, B \subseteq \langle n\rangle$, $\overline{x} \in M(G,n)_A$, $\overline{y} \in M(G,n)_B$,

$$h(\overline{x},\overline{y}) = \overline{[x,y]} \in M(G,n)_{A\cup B}.$$

Where this definition 'comes from' and why it works is a bit to lengthy to include here, so we refer the interested reader to [106]. From its many properties, we will mention just the following one, linking $M(G,n)$ with $M(G,n-1)$ in a similar way to that we have examined for $n = 2$.

We will use the following notation: $M(G,n)_1$ will denote the crossed $(n-1)$-cube obtained by restricting to those $A \subseteq \langle n\rangle$ with $1 \in A$ and $M(G,n)_0$ that obtained from the terms with $A \subseteq \langle n\rangle$ with $1 \notin A$.

Proposition 29 *Given a simplicial group G and $n \geq 1$, there is an exact sequence of crossed $(n-1)$-cubes:*

$$1 \to K \to M(G,n)_1 \xrightarrow{\mu_1} M(G,n)_0 \to M(G,n-1) \to 1,$$

where, if $B \subseteq \langle n-1\rangle$ and $B \neq \langle n-1\rangle$, then $K_B = \{1\}$, whilst $K_{\langle n-1\rangle} \cong \pi_n(G)$. ■

4.4.2 Squared complexes

We have met crossed squares and 2-crossed modules and the different ways they encode the homotopy 3-type. We have extended 2-crossed modules to 2-crossed complexes, so it is natural curiosity to try to extend crossed squares to a 'cube' formulation. We will see this is just the start of another hierarchy which is in some ways simpler than that suggested by the hypercrossed complexes, and their variants, etc. The first step is the following which was introduced by Ellis, [55].

Definition: A *squared complex* consists of a diagram of group homomorphisms

$$\begin{tikzcd} & & & & N \arrow[dr, "\mu"] & \\ \cdots \arrow[r] & C_4 \arrow[r, "\partial_4"] & C_3 \arrow[r, "\partial_3"] & L \arrow[ur, "\lambda'"] \arrow[dr, "\lambda"'] & & P \\ & & & & M \arrow[ur, "\mu'"'] & \end{tikzcd}$$

together with actions of P on L, N, M and C_i for $i \geq 3$, and a function $h : M \times N \longrightarrow L$. The following axioms need to be satisfied.

(i) The square $\left(\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M & \xrightarrow[\mu']{} & P \end{array}\right)$ is a crossed square;

(ii) The group C_n is Abelian for $n \geq 3$

(iii) The boundary homomorphisms satisfy $\partial_n\partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection $Ker\,\lambda \cap Ker\,\lambda'$;

(iv) The action of P on C_n for $n \geq 3$ is such that μM and $\mu' N$ act trivially. Thus each C_n is a π_0-module with $\pi_0 = P/\mu M \mu' N$.

(v) The homomorphisms ∂_n are π_0-module homomorphisms for $n \geq 3$.

This last condition does make sense since the axioms for crossed squares imply that $Ker\,\mu' \cap Ker\mu$ is a π_0-module.

Definition: A *morphism of squared complexes*,

$$\Phi : \left(C_*, \left(\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M & \xrightarrow[\mu']{} & P \end{array}\right)\right) \longrightarrow \left(C'_*, \left(\begin{array}{ccc} L' & \xrightarrow{\lambda} & N' \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M' & \xrightarrow[\mu']{} & P' \end{array}\right)\right)$$

consists of a morphism of crossed squares $(\Phi_L, \Phi_N, \Phi_M, \Phi_P)$, together with a family of equivariant homomorphisms Φ_n for $n \geq 3$ satisfying $\Phi_L\partial_3 = \partial'_3\Phi_L$ and $\Phi_{n-1}\partial_n = \partial'_n\Phi_n$ for $n \geq 4$. There is clearly a category $SqComp$ of squared complexes.

A squared complex is thus a crossed square with a 'tail' attached.

Any simplicial group will give us such a gadget by taking the crossed square to be $M(sk_2G, 2)$, that is,

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3 \cap D_3)} \arrow[r] \arrow[d] & Ker\,d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

and then, for $n \geq 3$,

$$C_n(G) = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}.$$

The above complex contains not only the information for the crossed square $M(G,2)$ that represents the 3-type, but also the whole of $\mathsf{C}^{(2)}(G)$, the 2-crossed complex of G and thus the crossed complex and the 'chains on the universal cover' of G.

The advantage of working with crossed squares or squared complexes rather than the more linearly displayed models is that they can more easily encode 'non-symmetric' information. We will show this in low dimensions here but will later indicate how to extend it to higher ones. For instance, one gets a building process for homotopy types that reflects more the algebra. In examples, given two profinite crossed modules, $\mu : M \to P$ and $\nu : N \to P$, there is a universal crossed square defining a 'tensor product' of the two crossed modules. We have

$$\begin{tikzcd} M\otimes N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square and hence represents a 3-type. It is universal with regard to crossed squares having the same right-hand and bottom crossed modules, (see [31, 32] for the original theory and [106] for its connections with other material).

Equivalently we could represent its 3-type as a 2-crossed module

$$M\otimes N \longrightarrow M \rtimes N \xrightarrow{\mu\nu} P$$

or

$$M\otimes N \longrightarrow \frac{(M \rtimes N)}{\sim} \longrightarrow \frac{P}{\mu M},$$

where $\sim$ corresponds to dividing out by the μM action. However, of these, the profinite crossed square lays out the information in a clearer format and so can often have some advantages.

4.5 Crossed $\mathbb{N}$-cubes

4.5.1 Just replace n by $\mathbb{N}$?

We have already suggested (page 93) how one might model all homotopy types using hypercrossed complexes, i.e. by adding more of the potential structure to the Moore complex of a simplicial group. We also saw how crossed modules (which are, from this viewpoint, 1-truncated hypercrossed complexes) generalised to crossed complexes, which have a better structured homotopical and homological algebra. We have seen earlier the transition from 2-crossed modules (= 2-truncated hypercrossed complexes) to 2-crossed complexes and briefly in the previous section, how crossed squares generalised to give squared complexes.

We will end this progression by looking at an elegant theoretical treatment of a generalisation of both crossed complexes and squared complexes. These gadgets are related to the "Moore chain complexes of order $(n+1)$ of a simplicial group", as briefly studied by Baues in [13], but have some of the advantages of crossed squares over 2-crossed modules, namely they can be 'non-symmetric', and hence are easily specified by, say, an 'inclusion crossed n-cube' consisting of a simplicial group and n simplicial normal subgroups. This allows for extra freedom in constructions. Also the axioms are very much simpler!

The definition of a crossed n-cube involves the set $\langle n\rangle = \{1, 2, \dots, n\}$. One obvious way to extend this, eliminating dependence on n, is to try replacing $\langle n\rangle$ by $\mathbb{N} = \{1, 2, \dots\}$ and taking the subsets A, B, C, in that definition to be finite, a condition previously automatic. This gives the notion of a crossed $\mathbb{N}$-cube:

Definition: A *crossed* $\mathbb{N}$-*cube*, M, is a family of groups,

$$\{M_A \mid A \subset \mathbb{N}, A \text{ finite}\},$$

together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, $(i \in \mathbb{N},\ A \subset_{fin} \mathbb{N})$, and functions, $h : M_A \times M_B \to M_{A\cup B}$, $(A, B \subset_{fin} \mathbb{N})$, such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, b, $b' \in M_B$, $c \in M_C$ and $i, j \in \mathbb{N}$, the following axioms hold:

(1) $\mu_i a = a$ if $a \notin A$
(2) $\mu_i\mu_j a = \mu_j\mu_i a$
(3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
(4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
(5) $h(a, a') = [a, a']$
(6) $h(a, b) = h(b, a)^{-1}$
(7) $h(a, b) = 1$ if $a = 1$ or $b = 1$
(8) $h(aa', b) = {}^a h(a', b)h(a, b)$
(9) $h(a, bb') = h(a, b)^b h(a, b')$
(10) ${}^a h(h(a^{-1}, b), c)^c h(h(c^{-1}, a), b)^b h(h(b^{-1}, c), a) = 1$
(11) ${}^a h(b, c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

(We have written $A \subset_{fin} \mathbb{N}$ as a shorthand for $A \subset \mathbb{N}$ with A finite.) Of course, these are formally identical to those given previously except in as much as there is no bound on the size of the finite sets A, B, C involved.

Examples: The first example is somewhat obvious, the second slightly surprising.

(i) As, for any n, $\langle n\rangle \subset \mathbb{N}$, if M is a crossed n-cube, then we can extend it trivially to an crossed $\mathbb{N}$-cube by defining $M_A = M_A$ if $A \subseteq \langle n\rangle$, and $M_A = 1$ otherwise. The h-maps $M_A \times M_B \to M_{A\cup B}$ are then clearly determined by those of the original crossed n-cube.

(ii) Suppose $\mathsf{M} = \{M_A, \mu_i, h\}$ is a crossed $\mathbb{N}$-cube, which is such that M_A is trivial unless A is of form $\langle n\rangle$ for some n, (where we interpret $\emptyset$ as being $\langle 0\rangle$, and so $M_\emptyset$ is not required to be trivial). We will write $C_n = M_{\langle n\rangle}$ and $\partial_n : C_n \to C_{n-1}$ for the morphism $\mu_n : M_{\langle n\rangle} \to M_{\langle n-1\rangle}$.

We note that $\partial_{n-1}\partial_n$ is trivial as it factorises via the trivial group:

$$\begin{tikzcd} M_{\langle n\rangle} \arrow[r] \arrow[d] & M_{\langle n-1\rangle} \arrow[d] \\ M_A \arrow[r] & M_{\langle n-2\rangle} \end{tikzcd}$$

where $A = \langle n\rangle - \{n-1\}$, so $M_A = 1$. We thus have that (C_n, ∂_n) is a complex of groups.

There is a pairing

$$C_0 \times C_n \to C_n$$

given by $h : M_\emptyset \times M_{\langle n\rangle} \to M_{\langle n\rangle}$, and thus an action

$$^a b = h(a,b)b,$$

whilst $\partial(^a b) = {}^a\partial b$, since $\mu_n h(a,b) = h(\mu_n a, \mu_n b)$, which is $h(a, \mu_n b)$, since $n \notin \emptyset$!

The map $\partial_1 : C_1 \to C_0$ is a crossed module by exactly the proof that a crossed 1-cube is a crossed module.

If $a = \partial_1 b$, then for $c \in C_n$, $n \geq 2$,

$$\begin{aligned} ac &= h(\partial_1 b, c)c \\ &= h(b, \mu_1 c)c, \end{aligned}$$

since $1 \in \langle 1\rangle \cap \langle n\rangle$, but $\mu_1 c \in M_{\langle n\rangle - \{1\}}$, the trivial group so

$$^a c = c.$$

We will not systematically check *all* the axioms, but clearly (C_n, ∂) is a crossed complex. (The detailed checking *is* best left to the reader.) Conversely any crossed complex gives a crossed $\mathbb{N}$-cube.

These examples show that both crossed n-cubes, for all n, and crossed complexes are examples of crossed $\mathbb{N}$-cubes. The obvious question, given our previous discussion, is to try to put Ellis' squared complex in the same framework. There is an obvious method to try out, and it works! One takes $M_A = 1$ unless $A = \langle n\rangle$ for some $n \in \mathbb{N}$ or if $A \subseteq \langle 2\rangle$. This does it, but it also indicates an effective way of encoding higher dimensional analogues of these squared complexes.

To do this, given $n \geq 1$, we have a subcategory of the category of crossed $\mathbb{N}$-cubes specified by the crossed n-cube complexes, that is, by $M_A = 1$ unless $A = \langle m\rangle$ for some $m \in \mathbb{N}$ or if $A \subseteq \langle n\rangle$ for the given n.

As we are going to explore these gadgets in a bit of detail, we introduce some notation.

$Crs^{\mathbb{N}}$ will denote the category of crossed $\mathbb{N}$-cubes of groups; $Crs^n.Comp$ will denote the subcategory of $Crs^{\mathbb{N}}$ determined by the crossed n-cube complexes. Thus, for instance, $Crs^1.Comp$ becomes an alternative notation for the category of crossed complexes.

4.5.2 From simplicial groups to crossed n-cube complexes

To show how these gadgets relate to ordinary 'bog-standard' models of homotopy types, we will show how to obtain a crossed n-cube complex from a simplicial group G.

To obtain a crossed n-cube complex from a simplicial group G, one analyses the constructions giving crossed complexes and crossed square complexes. For crossed complexes, one used the relative homotopy groups of G, so that the base crossed module is

$$\frac{NG_1}{(NG_1 \cap D_1)d_0(NG_2 \cap D_2)} \to G_0,$$

but $NG_1 \cap D_1 = 1$ since D_1 is generated by the $s_0(g)$ with $g \in G_0$.

For an arbitrary simplicial group, H, the crossed module $M(H,1)$ was given by

$$\frac{NH_1}{d_0(NH_2)} \to H_0,$$

so the earlier crossed module was $M(sk_1G, 1)$, as $N(sk_1G)_2 = NG_2 \cap D_2$.

Similarly for the crossed square complex associated to G, we explicitly took the 'base' crossed square to be $M(sk_2G, 2)$.

Proposition 30 *Let G be a simplicial group and $n \in \mathbb{N}$. Define a family M_A, $A \subset \mathbb{N}$, A finite, by*
(i) if $A = \langle m \rangle$ and $m > n$, then

$$M_A = \frac{NG_m}{(NG_m \cap D_m)d_0(NG_{m+1} \cap D_{m+1})};$$

(ii) if $A \subseteq \langle n \rangle$,

$$\begin{aligned} M_A &= M(sk_n G, n)_A \\ &= \frac{\bigcap\{Ker\, d_j^n : j \in A\}}{d_0(Ker\, d_1^{n+1} \cap \bigcap\{Ker\, d_{j+1}^{n+1} : j \in A\} \cap D_{n+1})} : \end{aligned}$$

(iii) if A is otherwise, then M_A is trivial.
Further define $\mu_i : M_A \to M_{A-\{i\}}$ by
(iv) if $i \in A$, then μ_i is the identity morphism;
(v) if $A = \langle m \rangle$, with $m > n$ and $i = m$, then μ_m is induced by d_0, and is trivial if $i \neq m$;
(vi) if $A \subseteq \langle n \rangle$, then μ_i is induced by the inclusions of intersections (i.e. as in $M(sk_n G, n)$);
(vii) otherwise μ_i is trivial.
Finally define $h : M_A \times M_B \to M_{A \cup B}$ by
(viii) if $A = \emptyset$ and $B = \langle m \rangle$ with $m > n$ then as $M_\emptyset = G_{n-1}$ and $M_B = C(G)_m$, if $a \in M_\emptyset$ and $b \in M_B$,

$$h(a, b) = [s_0^{m-n+1}(a), b] \in M_B;$$

similarly if $A = \langle m \rangle$ and $B = \emptyset$;
(ix) if $A, B \subseteq \langle n \rangle$, h is defined as in $M(sk_n G, n)$;
(x) otherwise h is trivial.
This data defines a crossed $\mathbb{N}$-cube which is, in fact, a crossed n-cube complex.

Proof: Much of this can be safely 'left to the reader'. It uses results from earlier parts of the notes. Note, however, that (viii) and (x) effectively say that it is only the $s_0^{n-1}G_0$ part of G_{n-1} that acts on any $M_{\langle m \rangle}$ and even then the image of $d_0 : NG_1 \to G_0$ acts trivially. To see this note that any $a \in G_{n-1}$ that is in some $Ker\, d_i$ is in the image of some μ_i, hence $a = \mu_i x$ say, but then

$$\begin{aligned} h(a, b) &= h(\mu_i x, b) \\ &= h(x, \mu_i b) \\ &= 1, \end{aligned}$$

by necessity if the structure is to be crossed $\mathbb{N}$-cube. Thus to check that the h-maps, and, in particular, those involved with part (viii) of the definition, satisfy the axioms, it suffices to use the methods mentioned earlier for checking that $\mathsf{C}(G)$ was a crossed complex, see [106]. ■

We might denote this crossed n-cube complex by $\mathsf{C}(G, n)$, as it combines both the technology of the $M(G, n)$ and the $\mathsf{C}(G)$. These models have yet to be explored in any depth, but see [106] and below for some preliminary results.

4.5.3 From n to $n-1$: collecting up ideas and evidence

We noted earlier that given $M(G,n)$, the quotient crossed $(n-1)$-cube was $M(G,n-1)$. Is a similar result true here? Is there an epimorphism from $\mathsf{C}(G,n)$ to $\mathsf{C}(G,n-1)$? In fact this is linked with another problem. We have a nested sequence of full categories of $Crs^{\mathbb{N}}$,

$$Crs^1.Comp \subset Crs^2.Comp \subset \ldots \subset Crs^n.Comp \subset \ldots \subset Crs^{\mathbb{N}}.$$

Does the inclusion of $Crs^{n-1}.Comp$ into $Crs^n.Comp$ have a left adjoint, in other words, is $Crs^{n-1}.Comp$ a reflexive subcategory of $Crs^n.Comp$? We investigate this question here only for $n = 2$ as this is at the same time easiest to see and also one of the most useful cases.

In this case, the crossed square complexes can be neatly represented as

$$\mathsf{C} := \qquad \begin{array}{ccccccc} \cdots \longrightarrow & C_3 & \xrightarrow{\mu_3} & C_{\langle 2\rangle} & \xrightarrow{\mu_2} & C_{\langle 1\rangle} & , \\ & & & {\scriptstyle\mu_1}\big\downarrow & & \big\downarrow{\scriptstyle\mu_1} & \\ & & & C_{\{2\}} & \xrightarrow[\mu_2]{} & C_{\emptyset} & \end{array}$$

whilst those corresponding to crossed complexes look like

$$\mathsf{D} := \qquad \begin{array}{ccccccc} \cdots \longrightarrow & D_3 & \xrightarrow{\mu_3} & D_{\langle 2\rangle} & \xrightarrow{\mu_2} & D_{\langle 1\rangle} & . \\ & & & {\scriptstyle\mu_1}\big\downarrow & & \big\downarrow{\scriptstyle\mu_1} & \\ & & & 1 & \xrightarrow[\mu_2]{} & D_{\emptyset} & \end{array}$$

A map φ in $Crs^2.Comp$ from C to D, clearly, must kill off $C_{\{2\}}$ and hence must also kill off $\mu_2(C_{\{2\}})$, which is normal in $C_\emptyset$. That is not all. If $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2\rangle}$, then

$$\varphi(h(a,b)) = h(\varphi a, \varphi b) = 1,$$

and $\varphi a = 1$, thus φ must kill off the action of $C_{\{2\}}$ on $C_{\langle 2\rangle}$, and all elements of this form, $h(a,b)$ with $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2\rangle}$.

Example: To illustrate what is happening let us examine the case of an inclusion crossed square. Suppose G is a group and M, N normal subgroups, then

$$\mathsf{C} = \begin{pmatrix} M \cap N & \longrightarrow & M \\ \big\downarrow & & \big\downarrow \\ N & \longrightarrow & G \end{pmatrix}$$

is a crossed square. Any 2-truncated crossed complex also gives a crossed square

$$\mathsf{D} = \begin{pmatrix} D_2 & \longrightarrow & D_1 \\ \big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & D_0 \end{pmatrix},$$

and any map from C to D factors through

$$\begin{tikzcd} \frac{M\cap N}{[M,N]} \arrow[r] \arrow[d] & \frac{M}{[M,N]} \arrow[d] \\ 1 \arrow[r] & G/N \end{tikzcd}$$

Proposition 31 *The inclusion of Crs^1.Comp into Crs^2.Comp has a left adjoint, denoted L. This left adjoint is a reflection, fixing the objects of the subcategory.* ∎

The proof should be fairly obvious so we will leave it as an exercise.

From $\mathsf{C}(G,2)$ to $\mathsf{C}(G,1)$: What happens if we apply this L to $\mathsf{C}(G,2)$? The answer is not that much of a surprise!

Proposition 32 *If G is a simplicial group, then there is a natural isomorphism*

$$\mathsf{L}(\mathsf{C}(G,2)) \cong \mathsf{C}(G,1).$$

∎

(Of course, the 'crossed 1-cube complex', $\mathsf{C}(G,1)$, is just the crossed complex $\mathsf{C}(G)$ under another name.)

This does generalise to higher dimensions. We thus have a series of crossed approximations to homotopy types, each one reflecting nicely down to the previous one, but what do these crossed gadgets tell us about the spaces being modelled? To explore that we must go back to crossed modules and their classifying spaces. There is a two way process here, algebraic gadgets tell us information about spaces, but conversely spaces can inform us about algebra.

Chapter 5

Classifying spaces, and extensions

We will first look in detail at the construction of classifying spaces and their applications for the non-Abelian cohomology of *groups*. This will use things we have already met. Later on we will need to transfer some of this to a sheaf theoretic context to handle 'gerbes' and to look at other forms of non-Abelian cohomology.

5.1 Non-Abelian extensions revisited

We again start with an extension of groups:

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1.$$

From a section, s, we constructed a factor set, f, but this *is* a bit messy. What do we mean by that? We are working in the category of groups, but neither s nor f are group morphisms. For s, there is an obvious thing to do. The function s induces a homomorphism, k_1, from $C_1(G)$, the free group on the set, G, to E and

$$\begin{tikzcd} C_1(G) \arrow[r] \arrow[d, "k_1"] & G \arrow[d, "="] \\ E \arrow[r, "p"] & G \end{tikzcd}$$

commutes. One might be tempted to do the same for f, but f is partially controlled by s, so we try something else. When we were discussing identities among relations (page 39), we looked at the example of taking $X = \{\langle g\rangle \mid g \neq 1, g \in G\}$ and a relation $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle gg'\rangle^{-1}$ for each pair (g, g') of elements of G. (Here we will write $\langle g_1, g_2\rangle$ for r_{g_1,g_2}.)

We can use this presentation $\mathcal{P}$ to build a free crossed module

$$C(\mathcal{P}) := C_2(G) \to C_1(G).$$

We noted earlier that the identities were going to correspond to tetrahedra, and that, in fact, we could continue the construction by taking $C_n(G)$ = the free G-module on $\langle g_1, \ldots, g_n\rangle$, $g_i \neq 1$, i.e. the normalised bar resolution. This is very nearly the usual bar resolution coming from the nerve of G, but we have a crossed module at the base, not just some more modules.

We met this structure earlier when we were looking at syzygies, and later on with crossed n-fold extensions, but is it of any use to us here?

We know $pf(g_1, g_2) = 1$, so $f(g_1, g_2) \in K$, and $C_2(G)$ is a free crossed module ... . Also, $K \to E$ is a normal inclusion, so is a crossed module ... . Thinking along these lines, we try

$$k_2 : C_2(G) \to K$$

defined on generators by f, i.e., $i(k_2(\langle g_1, g_2\rangle) = f(g_1, g_2)$. It is fairly easy to check this works, that

$$\partial k_2(\langle g_1, g_2\rangle) = k_1\partial(\langle g_1, g_2\rangle),$$

and that the actions are compatible, i.e., $\mathbf{k} : C(\mathcal{P}) \to \mathcal{E}$, where will write $\mathcal{E}$ also for the crossed module (K, E, i).

In other words, it seems that the section and the resulting factor set give us a morphism of crossed modules, $\mathbf{k}$. We note however that f satisfies a cocycle condition, so what does that look like here? To answer this we make the boundary, $\partial_3 : C_3(G) \to C_2(G)$, precise.

$$\partial_3\langle g_1, g_2, g_3\rangle = {}^{\langle g_1\rangle}\!\langle g_2, g_3\rangle\langle g_1, g_2g_3\rangle\langle g_1g_2, g_3\rangle^{-1}\langle g_1, g_2\rangle^{-1}$$

and, of course, the cocycle condition just says that $k_2\partial_3$ is trivial.

We can use the idea of a crossed complex as being a crossed module with a tail which is a chain complex, to point out that $\mathbf{k}$ gives a morphism of crossed complexes:

$$\begin{tikzcd}
\mathsf{C}(G) : ... \arrow[r] \arrow[d] & C_3(G) \arrow[r] \arrow[d] & C_2(G) \arrow[r] \arrow[d, "k_2"] & C_1(G) \arrow[r] \arrow[d, "k_1"] & G \arrow[d] \\
\mathcal{E} : ... \arrow[r] & 1 \arrow[r] & K \arrow[r] & E \arrow[r] & G
\end{tikzcd}$$

where the crossed module $\mathcal{E}$ is thought of as a crossed complex with trivial tail.

Back to our general extension,

$$\mathcal{E} : \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

we note that the choice of a section, s, does not allow the use of an action of G on K. Of course, there is an action of E on K by conjugation and hence s does give us an action of $C_1(G)$ on K. If we translate 'action of G on a group, K', to being a functor from the *groupoid*, $G[1]$, to *Grps* sending the single object of $G[1]$ to the object K, then we can consider the 2-category structure of *Grps* with 2-cells given by conjugation, (so that if K and L are groups, and $f_1, f_2 : K \to L$ homomorphisms, a 2-cell $\alpha : f_1 \Longrightarrow f_2$ will be given by an element $\ell \in L$ such that

$$f_2(x) = \ell f_1(x)\ell^{-1}$$

for all $x \in K$). With this categorical perspective, s *does* give a lax functor from $G[1]$ to *Grps*. We essentially replace the action $G \to Aut(K)$, when s is a splitting, by a lax action (see Blanco, Bullejos and Faro, [15]);

$$\begin{tikzcd}
\arrow[r] & C_2(G) \arrow[r] \arrow[d, "k_2"] & C_1(G) \arrow[d, "k_1"] \\
& K \arrow[r] \arrow[d, "="] & E \arrow[d] \\
& K \arrow[r] & Aut(K).
\end{tikzcd}$$

Using this lax action and $\mathbf{k}$, we can reinterpret the classical reconstruction method of Schreier as forming the semidirect product $K \rtimes C_1(G)$, then dividing out by all pairs,

$$(k_2(\langle g_1, g_2\rangle), \partial_2(\langle g_1, g_2\rangle)^{-1}).$$

(We give Brown and Porter's article, [33], as a reference for a discussion of this construction.)

By itself this reinterpretation does not give us much. It just gives a slightly different viewpoint, however two points need making. This formulation is nearer the sort of approach that we will need to handle the classification of gerbes and the use of $K \to Aut(K)$ to handle the lax action of G reveals a problem and also a power in this formulation.

Dedecker, [48], noted that any theory of non-Abelian cohomology of groups must take account of the variation with K. Suppose we have two groups K and L and lax actions of G on them. What should it mean to say that some homomorphism $\alpha : K \to L$ is compatible with the lax actions?

A lax action of G on K can be given by a morphism of crossed modules / complexes $Act_{G,K} : \mathsf{C}(G) \to \mathsf{Aut}(K)$, but $\mathsf{Aut}(K)$ is not functorial in K, so we do not automatically get a morphism of crossed modules, $\mathsf{Aut}(\alpha) : \mathsf{Aut}(K) \to \mathsf{Aut}(L)$. Perhaps the problem is slightly wrongly stated. One might say α is compatible with the lax G-actions if such a morphism of crossed modules existed and such that $Act_{G,L} = \mathsf{Aut}(\alpha)Act_{G,K}$. It is then just one final step to try to classify extensions with a finer notion of equivalence.

Definition: Suppose we have a crossed module, $\mathsf{Q} = (K, Q, q)$. An extension of K by G of the type of Q is a diagram:

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\omega"] & G \arrow[r] & 1 \\
 & K \arrow[r, "q"'] & Q & &
\end{tikzcd}$$

where ω gives a morphism of crossed modules.

There is an obvious notion of equivalence of two such extensions, where the isomorphism on the middle terms must commute with the structural maps ω and ω'. The special case when $Q = Aut(K)$ gives one the standard notion. In general, one gets a set of equivalence classes of such extensions $Ext_{K\to Q}(G, K)$ and this can be related to the cohomology set $H^2(G, K \to Q)$. This can also be stated in terms of a category $\mathcal{E}xt_{\mathsf{Q}}(G)$ of extensions of type Q, then the cohomology set is the set of components of this category.

This latter object can be defined using any free crossed resolution of G as there is a notion of homotopy for morphisms of crossed complexes such that this set is $[\mathsf{C}(G), \mathsf{Q}]$. Any other free crossed resolution of G has the same homotopy as $\mathsf{C}(G)$ and so will do just as well. Finding a complete set of syzygies for a presentation of G will do.

Example:

$$G = (x, y \mid x^2 = y^3)$$

This is the trefoil group. It is a one relator presentation and has no identities so $C(\mathcal{P})$ is already a crossed resolution. A morphism of crossed modules $\mathbf{k} : C(\mathcal{P}) \to \mathsf{Q}$ is specified by elements $q_x, q_y \in Q$, and $a_r \in K$ such that $\mathbf{k}(a_r) = (q_x)^2(q_y)^{-3}$. Using this one can give a presentation of the E that results.

Remark: Extensions correspond to 'bitorsors' as we will see. These in higher dimensions then yields gerbes with action of a gr-stack and a corresponding cohomology. In the case of gerbes, as against extensions, a related notion was introduced by Debremaeker, [44–47]. This has recently been revisited by Milne [91] and Aldrovandi, [2], who consider the special case where both K and Q are Abelian and the action of Q is trivial. This links with various important structures on gerbes and also with Abelian motives and hypercohomology. In all these cases, Q is being viewed as the coefficients of the cohomology and the gerbes / extensions have interpretations accordingly. Another very closely related approach is given in Breen, [17, 19]. We explore these ideas later in these notes.

We can think of the canonical case $K \to Aut(K)$ as being a 'natural' choice for extensions by K of a group, G. It is the structural crossed module of the 'fibre'. The crossed modules case says we can restrict or, alternatively, lift this structural crossed module to Q. This may, perhaps, be thought of as analogous to the situation that we will examine shortly where geometric structure corresponds to the restriction or the lifting of the natural structural group of a bundle. Both restricting to a subgroup and lifting to a covering group are useful and perhaps the same is true here.

5.2 Classifying spaces

The classifying spaces of crossed modules are never far from the surface in this approach to cohomology and related areas. They will play a very important role in the discussion of gerbes, as, for instance, in Larry Breen's work, [17–19] and later on here.

Classifying spaces of (discrete) groups are well known. One method of construction is to form the nerve, $Ner(G)$, of the group, G, (considered as a small groupoid, $\mathcal{G}$ or $G[1]$, as usual). The classifying space is obtained by taking the geometric realisation, $BG = |Ner(G)|$.

To explore this notion and how it relates to crossed modules, we need to take a short excursion into some simplicially based notions.

A classifying space of a group classifies principal G-bundles (G-torsors) over a space, X, in terms of homotopy classes of maps from X to BG, using a universal principal G-bundle $EG \to BG$.

This is very topological! If possible, it is useful to avoid the use of geometric realisations, since (i) this restricts one to groups and groupoids and makes handling more general 'algebras' difficult and (ii) for algebraic geometry, the topology involved is not the right kind as a sheaf-theoretic, topos based construction would be more appropriate. Thus the classifying space is often replaced by the nerve, as in Breen, [19].

How about classifying spaces for crossed modules? Given a crossed module, $\mathsf{M} = (C, G, \theta)$, say, we can form the associated 2-group, $\mathcal{X}(\mathsf{M})$. This gives a simplicial group by taking the nerve of the groupoid structure, then we can form $\overline{W}$ of that to get a simplicial set, $Ner(\mathsf{M})$. To reassure ourselves that this *is* a good generalisation of $Ner(G)$, we observe that if C is the trivial group, then $Ner(\mathsf{M}) = Ner(G)$. But this raises the question:

What does this 'classifying space' classify?

To answer that we must digress to provide more details on the functors G and $\overline{W}$, we mentioned earlier.

5.2.1 Simplicially enriched groupoids

We denote the category of simplicial sets by $\mathcal{S}$ and that of simplicially enriched groupoids by $\mathcal{S}-Grpds$. This latter category includes that of simplicial groups, but it must be remembered that a simplicial object in the category of groupoids will, in general, have a non-trivial simplicial set as its 'object of objects', whilst in $\mathcal{S}-Grpds$, the corresponding simplicial object of objects will be constant. This corresponds to a groupoid in which each collection of 'arrows' between objects is a simplicial set, not just a set, and composition is a simplicial morphism, hence the term 'simplicially enriched'. We will often abbreviate the term 'simplicially enriched groupoid' to '$\mathcal{S}$-groupoid', but the reader should note that in some of the sources on this material the looser term 'simplicial groupoid' is used to describe these objects, usually with a note to the effect that this is not a completely accurate term to use.

Remark: Later, in section **??**, we will need to work with $\mathcal{S}$-categories, i.e. simplicially enriched categories. Some brief introduction can be found in [76], in the notes, [105] and the references cited there. We *will* give a fairly detailed discussion of the main parts of the elementary theory of $\mathcal{S}$-categories later.

The loop groupoid functor of Dwyer and Kan, [51], is a functor

$$G : \mathcal{S} \longrightarrow \mathcal{S}-Grpds,$$

which takes the simplicial set K to the simplicially enriched groupoid GK, where $(GK)_n$ is the free groupoid on the directed graph

$$K_{n+1} \underset{t}{\overset{s}{\rightrightarrows}} K_0 \ ,$$

where the two functions, s, source, and t, target, are $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$ with relations $s_0x = id$ for $x \in K_n$. The face and degeneracy maps are given on generators by

$$\begin{aligned} s_i^{GK}(x) &= s_{i+1}^K(x), \\ d_i^{GK}(x) &= d_{i+1}^K(x), \text{for } \ x \in K_{n+1}, 1 < i \leq n \end{aligned}$$

and

$$d_0^{GK}(x) = (d_0^K(x))^{-1}(d_1^K(x)).$$

This loop groupoid functor has a right adjoint, $\overline{W}$, called the *classifying space* functor. The details as to its construction will be given shortly. It is important to note that if K is reduced, i.e. has just one vertex, then GK will be a simplicial group, so is a well known type of object. This helps when studying these gadgets as we can often use simplicial group constructions, suitable adapted, in the $\mathcal{S}$-groupoid context. The first we will see is the Moore complex.

Definition: Given any $\mathcal{S}$-groupoid, G, its Moore complex, NG, is given by

$$NG_n = \bigcap_{i=1}^{n} Ker(d_i : G_n \longrightarrow G_{n-1})$$

with differential $\partial : NG_n \longrightarrow NG_{n-1}$ being the restriction of d_0. If $n \geq 1$, this is just a disjoint union of groups, one for each object in the object set, O, of G. If we write $G\{x\}$ for the simplicial group of elements that start and end at $x \in O$, then at object x, one has

$$NG\{x\}_n = (NG_n)\{x\}.$$

In dimension 0, one has $NG_0 = G_0$, so the $NG_n\{x\}$, for different objects x, are linked by the actions of the 0-simplices, acting by conjugation via repeated degeneracies.

For simplicity in the description below, we will often assume that the $\mathcal{S}$-groupoid is *reduced*, that is, its set O, of objects is just a singleton set $\{*\}$, so G is just a simplicial group.

Suppose that NG_m is trivial for $m > n$.

If $n = 0$, then NG_0 is just the group G_0 and the simplicial group (or groupoid) represents an Eilenberg-MacLane space, $K(G_0, 1)$.

If $n = 1$, then $\partial : NG_1 \longrightarrow NG_0$ has a natural crossed module structure.

Returning to the discussion of the Moore complex, if $n = 2$, then

$$NG_2 \xrightarrow{\partial} NG_1 \xrightarrow{\partial} NG_0$$

has a 2-crossed module structure in the sense of Conduché, [39] and above section 4.2. (These statements are for groups and hence for connected homotopy types. The non-connected case, handled by working with simplicially enriched groupoids, is an easy extension.)

In all cases, the simplicial group will have homotopy groups only in the range covered by the non-trivial part of the Moore complex.

Now relaxing the restriction on G, for each $n > 1$, let D_n denote the subgroupoid of G_n generated by the degenerate elements. Instead of asking that NG_n be trivial, we can ask that $NG_n \cap D_n$ be. The importance of this is that the structural information on the homotopy type represented by G includes structure such as the Whitehead products and these all lie in the subgroupoids $NG_n \cap D_n$. If these are all trivial then the algebraic structure of the Moore complex is simpler, being that of a crossed complex, and $\overline{W}G$ is a simplicial set whose realisation is the *classifying space of that crossed complex*, cf. [28]. The simplicial set, $\overline{W}G$, is isomorphic to the *nerve* of the crossed complex.

Notational warning. As was mentioned before, the indexing of levels in constructions with crossed complexes may cause some confusion. The Dwyer-Kan construction is essentially a 'loop' construction, whilst $\overline{W}$ is a 'suspension'. They are like 'shift' operators for chain complexes. For example G decreases dimension, as an old 1-simplex x yields a generator in dimension 0, and so on. Our usual notation for crossed complexes has C_0 as the set of objects, C_1 corresponding to a relative fundamental groupoid, and C_n abstracting its properties from $\pi_n(X_n, X_{n-1}, p)$, hence the natural topological indexing *has* been used. For the $\mathcal{S}$-groupoid $G(K)$, the set of objects is separated out and $G(K)_0$ is a groupoid on the 1-simplices of K, a dimension shift. Because of this, in the notation being used here, the crossed complex $\mathsf{C}(G)$ associated to an $\mathcal{S}$-groupoid, G, will have a dimension shift as well: explicitly

$$C(G)_n = \frac{NG_{n-1}}{(NG_{n-1} \cap D_{n-1})d_0(NG_n \cap D_n)} \qquad \text{for } n \geq 2,$$

$C(G)_1 = NG_0$, and, of course, C_0 is the common set of objects of G. In some papers where only the algebraic constructions are being treated, this convention is not used and C is given without this dimension shift relative to the Moore complex. Because of this, care is sometimes needed when comparing formulae from different sources.

5.2.2 Conduché's decomposition and the Dold-Kan Theorem

The category of crossed complexes (of groupoids) is equivalent to a reflexive subcategory of the category $\mathcal{S}-Grpds$ and the reflection is defined by the obvious functor : take the Moore complex of the $\mathcal{S}$-groupoid and divide out by the $NG_n \cap D_n$, see [52, 53]. We will denote by $C : \mathcal{S}-Grpds \longrightarrow Crs$ the resulting composite functor, Moore complex followed by reflection. Of course, we have the formula, more or less as before,

$$C(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)\ d_0(NG_{n+1} \cap D_{n+1})}.$$

The Moore complex functor itself is part of an adjoint (Dold-Kan) equivalence between the category $\mathcal{S} - Grpds$ and the category of hypercrossed complexes, [37], and this restricts to the Ashley-Conduché version of the Dold-Kan theorem of [7].

In order to justify the description of the nerve, and thus the related classifying space, of a crossed complex C, we will specify the functors involved, namely the Dold-Kan inverse construction and the $\overline{W}$. This will also give us extra tools for later use. We will first need the Conduché decomposition lemma.

Proposition 33 *If G is a simplicial group(oid), then G_n decomposes as a multiple semidirect product:*

$$G_n \cong NG_n \rtimes s_0NG_{n-1} \rtimes s_1NG_{n-1} \rtimes s_1s_0NG_{n-2} \rtimes s_2NG_{n-1} \rtimes \ldots s_{n-1}s_{n-2}\ldots s_0NG_0$$

∎

The order of the terms corresponds to a lexicographic ordering of the indices $\emptyset$; 0; 1; 1,0; 2; 2,0; 2,1; 2,1,0; 3; 3,0; ... and so on, the term corresponding to $i_1 > \ldots > i_p$ being $s_{i_1}\ldots s_{i_p}NG_{n-p}$.

The proof of this result is based on a simple lemma, which is easy to prove.

Lemma 14 *If G is a simplicial group(oid), then G_n decomposes as a semidirect product:*

$$G_n \cong Ker\ d^n_n \rtimes s^{n-1}_{n-1}(G_{n-1}).$$

∎

We next note that in the classical (Abelian) Dold-Kan theorem, (cf. [43]), the equivalence of categories is constructed using the Moore complex and a functor K constructed via the original direct sum /Abelian version of Conduché's decomposition, cf. for instance, [43].

For each non-negatively graded chain complex $\mathsf{D} = (D_n, \partial)$ in Ab, $K\mathsf{D}$ is the simplicial Abelian group with

$$(K\mathsf{D})_n = \oplus_a(D_{n-\sharp(a)}, s_a),$$

the sum being indexed by all descending sequences, $a = \{n > i_p \geq \ldots \geq\ i_1 \geq 0\}$, where $s_a = s_{i_p}\ldots s_{i_1}$, and where $\sharp(a) = p$, the summand D_n corresponding to the empty sequence.

The face and degeneracy operators in $K\mathsf{D}$ are given by the rules:
(1) if $d_is_a = s_b$, then d_i will map (D_{n-p}, s_a) to $(D_{(n-1)-(p-1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero;
(2) if $d_is_a = s_bd_0$, then d_i will map (D_{n-p}, s_a) to (D_{n-p-1}, s_b) as the homomorphism $\partial_{n-p} : D_{n-p} \rightarrow$

D_{n-p-1}; its components into other direct summands will be zero;
(3) if $d_i s_a = s_b d_j$, $j > 0$, then $d_i(D_{n-p}, s_a) = 0$;
(4) if $s_i s_a = s_b$, then s_i maps (D_{n-p}, s_a) to $(D_{(n+1)-(p+1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero.

This suggests that we form a functor

$$K : Crs \to \mathcal{S} - Grpds$$

using a semidirect product, but we have to take care as there will be a dimension shift, our lowest dimension being C_1:
if C is in Crs, set

$$K(\mathsf{C})_n = C_{n+1} \rtimes s_0 C_n \rtimes s_1 C_n \rtimes s_1 s_0 C_{n-1} \rtimes \cdots \rtimes s_{n-1} s_{n-2} \ldots s_0 C_1.$$

The order of terms is to be that of the proposition given above. The formation of the semidirect product is as in the proof we hinted at of that proposition, that is the bracketing is inductively given by

$$(C_{n+1} \ldots \rtimes s_{n-2} \ldots s_0 C_2) \rtimes (s_{n-1} C_n \rtimes \ldots \rtimes s_{n-1} \ldots s_0 C_1);$$

each $s_\alpha(C_{n+1-\sharp(\alpha)})$ is an indexed copy of $C_{n+1-\sharp(\alpha)}$; the action of

$$s_{n-1} C_{n-1} \rtimes \ldots \rtimes s_{n-1} \ldots s_0 C_0 \quad (\cong s_{n-1} K(\mathsf{C})_{n-1})$$

on $C_{n+1} \rtimes \ldots s_{n-2} \ldots s_0 C_1$, is given componentwise by the actions of each C_i and as C is a crossed complex, these are all via C_0. This implies, of course, that the majority of the components of these actions are trivial.

To see how this looks in low dimensions, it is simple to give the first few terms of the simplicial group(oid). As we are taking a reduced crossed complex as illustration, the result is a simplicial group, $K(\mathsf{C})$, having

- $K(\mathsf{C})_0 = C_1$
- $K(\mathsf{C})_1 = C_2 \rtimes s_0(C_1)$
- $K(\mathsf{C})_2 = (C_3 \rtimes s_0 C_2) \rtimes (s_1 C_2 \rtimes s_1 s_0 C_1)$
- $K(\mathsf{C})_3 = (C_4 \rtimes s_0 C_3 \rtimes s_1 C_3 \rtimes s_1 s_0 C_2) \rtimes (s_2 C_3 \rtimes s_2 s_0 C_2 \rtimes s_2 s_1 C_2 \rtimes s_2 s_1 s_0 C_1)$.

and so on.

The face and degeneracy maps are determined by the obvious rules adapting those in the Abelian case, so that if $c \in C_k$, the corresponding copy of c in $s_\alpha C_k$ will be denoted $s_\alpha c$ and a face or degeneracy operator will usually act just on the index. The exception to this is if, when renormalised to the form $s_\beta d_\gamma$ using the simplicial identities, γ is non-empty. If $d_\gamma = d_0$ then $d_\gamma c$ becomes $\delta_k c \in C_{k-1}$, otherwise $d_\gamma c$ will be trivial.

Lemma 15 *The above defines a functor*

$$K : Crs \to \mathcal{S} - Grpds$$

such that $\mathsf{C}K \cong Id$. ■

This extends the functor $K : CMod \to Simp.Grps$, given earlier, to crossed complexes as there $C_k = 1$ for $k > 2$.

5.2.3 $\overline{W}$ and the nerve of a crossed complex

We next need to make explicit the $\overline{W}$ construction. The simplicial / algebraic description of the nerve of a crossed complex, C is then as $\overline{W}(K(\mathsf{C}))$. We first give this description for a general simplicially enriched groupoid.

Let H be an $\mathcal{S}$-groupoid, then $\overline{W}H$ is the simplicial set described by

- $(\overline{W}H)_0 = ob(H_0)$, the set of objects of the groupoid of 0-simplices (and hence of the groupoid at each level);

- $(\overline{W}H)_1 = arr(H_0)$, the set of arrows of the groupoid H_0:

and for $n \geq 2$,

- $(\overline{W}H)_n = \{(h_{n-1}, \ldots, h_0) \mid h_i \in arr(H_i) \text{ and } s(h_{i-1}) = t(h_i), 0 < i < n\}$.

Here s and t are generic symbols for the domain and codomain mappings of all the groupoids involved. The face and degeneracy mappings between $\overline{W}(H)_1$ and $\overline{W}(H)_0$ are the source and target maps and the identity maps of H_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(h_{n-1}, \ldots, h_0) = (h_{n-2}, \ldots, h_0)$;

- for $0 < i < n$, $d_i(h_{n-1}, \ldots, h_0) = (d_{i-1}h_{n-1}, d_{i-2}h_{n-2}, \ldots, d_0h_{n-i}h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$;

and

- $d_n(h_{n-1}, \ldots, h_0) = (d_{n-1}h_{n-1}, d_{n-2}h_{n-2}, \ldots, d_1h_1)$;

whilst

- $s_0(h_{n-1}, \ldots, h_0) = (id_{dom(h_{n-1})}, h_{n-1}, \ldots, h_0)$;

and,

- for $0 < i \leq n$, $s_i(h_{n-1}, \ldots, h_0) = (s_{i-1}h_{n-1}, \ldots, s_0h_{n-i}, id_{cod(h_{n-i})}, h_{n-i-1}, \ldots, h_0)$.

Remark: We note that if H is a constant simplicial groupoid, $\overline{W}(H)$ is the same as the nerve of that groupoid for the algebraic composition order. Later on, when re-examining the classifying space construction we may need to rework the above definition in a form using the functional composition order.

To help understand the structure of the nerve of a (reduced) crossed complex, C, we will calculate $Ner(\mathsf{C}) = \overline{W}(K(\mathsf{C}))$ in low dimensions. This will enable comparison with formulae given earlier. The calculations are just the result of careful application of the formulae for $\overline{W}$ to $H = K(\mathsf{C})$:

- $Ner(\mathsf{C})_0 = *$, as we are considering a *reduced* crossed complex - in the general case, this is C_0;

- $Ner(\mathsf{C})_1 = C_1$, as a *set* of 'directed edges' or arrows - we will avoid using a special notation for 'underlying set of a group(oid)';

- $Ner(\mathsf{C})_2 = \{(h_0, h_1) \mid h_1 = (c_2, s_0(c_1)), h_0 = c_1', \text{ with } c_2 \in C_2, c_1, c_1' \in C_1\}$, and such a 2-simplex has faces given as in the diagram

$$\begin{tikzcd}
 & {} \arrow[dr, "c_1'"] \arrow[d, phantom, "(h_1,h_0)"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta c_2.c_1.c_1'"'] & {} & {}
\end{tikzcd}$$

  Note that $h_1 : c_1 \longrightarrow \delta c_2.c_1$ in the internal category corresponding to the crossed module, (C_2, C_1, δ), so the formation of this 2-simplex corresponds to a right whiskering of that 2-cell (in the corresponding 2-groupoid) by the arrow c_1';

- $Ner(\mathsf{C})_3 = \{(h_2, h_1, h_0) \mid h_1 = (c_3, s_0c_2^0, s_1c_2^1, s_1s_0c_1), h_1 = (c_2', s_0(c_1')), h_0 = c_1''\}$ in the evident notation. Here the faces of the 3-simplex (h_2, h_1, h_0) are as in the diagrams, (in each of which the label for the 2-simplex itself has been abbreviated):

$$d_3 : \begin{tikzcd}
 & {} \arrow[dr, "c_1'"] \arrow[d, phantom, "c_2^1"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta c_2^1.c_1.c_1'"'] & {} & {}
\end{tikzcd} ; \quad d_2 : \begin{tikzcd}
 & {} \arrow[dr, "\delta c_2'.c_1'.c_1''"] \arrow[d, phantom, "c_2^0c_2^1"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta(c_2^0c_2^1).c_1.\delta c_2'.c_1'.c_1''"'] & {} & {}
\end{tikzcd}$$

$$d_1 : \begin{tikzcd}
 & {} \arrow[dr, "c_1''"] \arrow[d, phantom, "\delta c_3.c_2^0.{}^{\delta c_2^1.c_1}c_2'"] & \\
{} \arrow[ur, "\delta c_2^1.c_1.c_1'"] \arrow[rr, "\delta(c_2^0c_2^1).c_1.\delta c_2'.c_1'.c_1''"'] & {} & {}
\end{tikzcd} ; \quad d_0 : \begin{tikzcd}
 & {} \arrow[dr, "c_1''"] \arrow[d, phantom, "c_2'"] & \\
{} \arrow[ur, "c_1'"] \arrow[rr, "\delta c_2'.c_1'.c_1''"'] & {} & {}
\end{tikzcd}$$

The only face where any real thought has to be used is d_1. In this the d_1 face has to be checked to be consistent with the others. The calculation goes like this:

$$\begin{aligned}
\delta(\delta c_3.c_2^0.{}^{\delta c_2^1.c_1}c_2').(\delta c_2^1.c_1.c_1').c_1'' &= \delta c_2^0.(\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}).\delta c_2^1.c_1.c_1'.c_1'' \\
&= \delta(c_2^0c_2^1).c_1.\delta c_2'.c_1'.c_1''
\end{aligned}$$

This uses (i) $\delta\delta c_3$ is trivial, being a boundary of a boundary, and (ii) the second crossed module rule for expanding $\delta({}^{\delta c_2^1.c_1}c_2')$ as $\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}$.

This diagrammatic representation, although useful, is limited. A recursive approach can be used as well as the simplicial / algebraic one given above. In this, $Ner(\mathsf{C})$ is built up via its skeleta, specifying a simplex in $Ner(\mathsf{C})_n$ as an element of C_n, together with the empty simplex that it 'fills', i.e. the set of compatible $(n-1)$-simplices. This description is used by Ashley, ([7], p.37). More on nerves of crossed complexes can be found in Nan Tie, [101, 102]. There is also a very neat 'singular complex' description, $Ner(\mathsf{C})_n = Crs(\pi(n), \mathsf{C})$, where $\pi(n)$ is the free crossed complex on the n-simplex, $\Delta[n]$. We will have occasion to see this later.

5.3 Simplicial Automorphisms and Regular Representations

The usual enrichment of the category of simplicial sets is given by :
for each $n \geq 0$, the set of n-simplices is

$$\underline{\mathcal{S}}(K,L)_n = \mathcal{S}(K \times \Delta[n], L),$$

together with obvious face and degeneracy maps.
Composition : for $f \in \underline{\mathcal{S}}(K,L)_n$, $g \in \underline{\mathcal{S}}(L,M)_n$, so $f : \Delta[n] \times K \to L$, $g : \Delta[n] \times L \to M$,

$$g \circ f := (\Delta[n] \times K \overset{diag \times K}{\longrightarrow} \Delta[n] \times \Delta[n] \times K \overset{\Delta[n] \times f}{\longrightarrow} \Delta[n] \times L \overset{g}{\to} M);$$

Identity : $id_K : \Delta[0] \times K \overset{\cong}{\to} K$.

Definition: The simplicial set $\underline{\mathcal{S}}(K,L)$ defined above is called the *simplicial mapping space of maps from K to L.*

For fixed K, $\underline{\mathcal{S}}(K,K)$ is a simplicial monoid, called the *simplicial endomorphism monoid of K* and $\mathsf{aut}(K)$ will be the corresponding simplicial group of invertible elements, that is the *simplicial automorphism group of K.*

If $f : K \times \Delta[n] \longrightarrow L$ is an n-simplex, then we can form a diagram

$$\begin{tikzcd} K \times \Delta[n] \arrow[rr, "(f,p)"] \arrow[dr] & & L \times \Delta[n] \arrow[dl] \\ & \Delta[n] & \end{tikzcd}$$

in which the two slanting arrow are the obvious projections, (so $(f,p)(k,\sigma) = (f(k,\sigma),\sigma)$). Taking $K = L$, $f \in \mathsf{aut}(K)$ if and only if (f,p) is an isomorphism of simplicial sets.

Given a simplicial set K, and an n-simplex x in K, there is a representing map

$$\mathbf{x} : \Delta[n] \longrightarrow K,$$

that send the top dimensional generating simplex of $\Delta[n]$ to x. The enrichment above is part of an adjunction

$$\mathcal{S}(K \times L, M) \cong \mathcal{S}(L, \underline{\mathcal{S}}(K,M))$$

in which, given $\theta : K \times L \longrightarrow M$ and $y \in L_n$, the corresponding simplicial map

$$\bar{\theta} : L \longrightarrow \underline{\mathcal{S}}(K,M)$$

sends y to the composite

$$K \times \Delta[n] \xrightarrow{K \times \mathbf{y}} K \times L \xrightarrow{\theta} M\ .$$

In a simplicial group G, the multiplication is a simplicial map, $\#_0 : G \times G \longrightarrow G$, and so by the adjunction, we get a simplicial map

$$G \longrightarrow \underline{\mathcal{S}}(G,G)$$

and this is a simplicial monoid morphism. This gives the right regular representation of G,

$$\rho = \rho_G : G \longrightarrow \mathsf{aut}(G).$$

This representation needs careful interpretation. In dimension n, an element $g \in G_n$ acts by multiplication on the right on G, but even in dimension 0, this action is not as simple as one might think. (NB. Here $\mathsf{aut}(G)$ is the simplicial group of 'simplicial automorphisms of the underlying simplicial set of G' as, of course, multiplication by an element does not give a mapping that respects the group structure.) Simple examples are called for:

Suppose $g \in G_1$, then $\rho(g) \in \mathsf{aut}(G)_1 \subset \underline{\mathcal{S}}(G,G)_1 = \mathcal{S}(G \times \Delta[1], G)$. In other words, $\rho(g)$ is a homotopy between $\rho(d_1 g)$ and $\rho(d_0 g)$. Of course, it is an invertible element of $\underline{\mathcal{S}}(G,G)_1$ and this will have implications for its properties as a homotopy, and to use a geometric term, we will loosely refer to it as an *isotopy*.

In general, 0-simplices give simplicial maps corresponding to multiplication by that element, so that for $g \in G_0$, and $x \in G_n$,

$$\rho(g)(x) = x\#_0 s_0^{(n)}(g).$$

In dimension 1, we have that elements give isotopies, and in higher dimensions, we have 'isotopies of isotopies', and so on.

5.4 $\overline{W}$, W and twisted Cartesian products

Suppose we have simplicial sets Y, a potential 'fibre' and B, a potential 'base' which will be assumed to be pointed by a vertex, $*$. Inspired by the sort of construction that works for the construction of group extensions, we are going to try to construct a fibration sequence

$$Y \longrightarrow E \longrightarrow B.$$

Clearly the product $E = B \times Y$ will give such a sequence, but can we somehow *twist* this Cartesian product to get a more general construction? We will try setting $E_n = B_n \times Y_n$ and will change as little as possible in the data specifying faces and degeneracies. In fact we will take all the degeneracy maps to be exactly those of the Cartesian product, and all but d_0 of the face maps likewise. This leaves just the zeroth face map.

In, say, a covering space considered as a fibration with discrete fibre, the fundamental group(oid) of the base acts by automorphisms / permutations on the fibre, and the fundamental group(oid) is generated by the edges, hence by elements of dimension one greater than that of the fibre, so we try a formula for d_0 of form

$$d_0(b, y) = (d_0 b, t(b)(d_0 y)),$$

where $t(b)$ is an automorphism of Y, determined by b in some way, hence giving a function $t : B_n \longrightarrow \mathsf{aut}(Y)_{n-1}$. Note here Y is an arbitrary simplicial set, not the underlying simplicial set of a simplicial group as was previously the case when we considered aut, but this makes no difference to the definition.

Of course, with these tentative definitions, we must still have that the simplicial identities hold, but it is easy to check that these will hold exactly if t satisfies the following equations

$$\begin{array}{rcll} d_i t(b) & = & t(d_{i-1}b) \quad \text{for} & i > 0, \\ d_0 t(b) & = & t(d_1 b) \#_0 t(d_0 b)^{-1}, & \\ s_i t(b) & = & t(s_{i+1} b) \quad \text{for} & i \geq 0, \\ t(s_0 b) & = & *. & \end{array}$$

A function t satisfying these equations will be called a *twisting function*, and the simplicial set E, thus constructed will be called a *regular twisted Cartesian product*. We write $E = B \times_t Y$.

Of course, a twisting function is not a simplicial map, but the formulae it satisfies look closely linked to those of the Dwyer-Kan loop group(oid) construction, given earlier, page 121. In fact:

Proposition 34 *A twisting function $t : B \longrightarrow \mathsf{aut}(Y)$ determines a unique homomorphism of simplicial groupoids $t : GB \to \mathsf{aut}(Y)$, and conversely.* ■

Of course, since G is left adjoint to $\overline{W}$, we could equally well note that t gave a simplicial morphism $t : B \longrightarrow \overline{W}(\mathsf{aut}(Y))$, and conversely.

Of course, we could restrict attention to a particular class of simplicially enriched groupoids such as those coming from groups (constant simplicial groups), or nerves of crossed modules, or of crossed complexes, etc. We will see some aspects of this in the following chapter, but we will be generalising it as well.

This adjointness gives us a 'universal' twisting function for any simplicial group, H. We have the general natural isomorphism

$$\mathcal{S}(B, \overline{W}H) \cong Simp.Grpds(G(B), H),$$

so, as usual in these situations, it is very tempting to look at the special case where $B = \overline{W}H$ itself and hence to get the counit of the adjunction from $G\overline{W}(H)$ to H corresponding to the identity simplicial map from $\overline{W}H$ to itself. By the general properties of adjointness, this map 'generates' the natural isomorphism in the general case.

From our point of view, the two natural isomorphic sets are much better viewed as being $\mathsf{Tw}(B, H)$, the set of twisting functions $\tau : B \to H$, so the key case will be a 'universal' twisting function, $\tau_H : \overline{W}H \to H$ and hence a universal twisted Cartesian product $\overline{W}H \times_{\tau_H} H$. (Notational point: the context tells us that the fibre H is the underlying simplicial set of the simplicial group, H, but no special notation will be used for this here.) This universal twisted Cartesian product is called the *classifying bundle for H* and is denoted WH. We can unpack its definition from its construction, but will not give the detailed derivation (which is suggested as a **useful exercise**). Clearly

$$(WH)_n = H_n \times \overline{W}(H)_n,$$

so from our earlier description of $\overline{W}(H)$, we have

$$WH_n = H_n \times H_{n-1} \times \ldots \times H_0.$$

The face maps are given by

$$d_i(h_n, \ldots, h_0) = (d_i h_n, \ldots, d_0 h_{n-i}.h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$$

for all i, $0 \leq i \leq n$, whilst

$$s_i(h_n, \ldots, h_0) = (s_i h_n, \ldots s_0 h_{n-i}, 1, h_{n-i-1}, \ldots, h_0).$$

(It is noteworthy that $d_0(h_n, \ldots, h_0) = (d_0 h_n.h_{n-1}, h_{n-2}, \ldots, h_0)$ so the universal twist, τ_H, must somehow be built in to this. In fact τ_H is an 'obvious' map as one would hope. We have $\overline{W}(H)_n = H_{n-1} \times \ldots \times H_0$ and we need $(\tau_H)_n : \overline{W}(H)_n \to H_{n-1}$, since it is to be a twisting map and so has degree -1. The obvious formula to try is that τ_H is the projection map - and it works. The details are left to you. A glance back at the formula for the general d_0 in a twisted Cartesian product will help.)

An introduction to simplicial bundle theory can be found in Curtis' classical survey article, [43] section 6, but will need some related results. For the moment, we limit ourselves to a number of observations, based on the classical treatment:

1). The simplicial set, $W(H)$, is a Kan complex.

2). $W(H)$ is contractible, i.e., is homotopy equivalent to $\Delta[0]$.

3). The simplicial map,
$$W(H) \to \overline{W}(H),$$
is a Kan fibration with fibre the underlying simplicial set of H, (so the long exact sequence of homotopy groups together with point 2) shows that $\pi_n(\overline{W}H) \cong \pi_{n-1}(H)$).

4). If $p : E \to B$ is a principle H-bundle, that is, E is $H \times_t B$ for some twisting function, $t : B \to H$, then we have a simplicial map
$$f_t : B \to \overline{W}(H)$$
given by $f_t(b) = (t(b), t(d_0 b), \ldots, t(d_0^{n-1} b))$, and we can pull back $(W(H) \to \overline{W}(H))$ along f_t to get a principal H-bundle over B
$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & W(H) \arrow[d] \\ B \arrow[r, "f_t"'] & \overline{W}(H). \end{tikzcd}$$

We can, of course, calculate E' and p' precisely:
$$\begin{aligned} E' &\cong \{((h_n, h_{n-1}, \ldots, h_0), b) \mid h_{n-1} = t(b), \ldots h_0 = t(d_0^{n-1} b)\} \\ &\cong \{(h_n, b) \mid h_n \in H_n, b \in B_n\} \\ &= H_n \times B_n. \end{aligned}$$

It should come as no surprise to find that $E' \cong H \times_t B$, so is E itself up to isomorphism, and that p' is p in disguise.

The assignment of f_t to t gives a one-one correspondence between H-equivalence classes of principal H-bundles with base and the set of homotopy classes of simplicial maps from B to $\overline{W}(H)$.

Chapter 6

Non-Abelian Cohomology: Torsors, and Bitorsors

One of the problems to be faced when presenting the applications of crossed modules, etc., is that such is the breadth of these applications that they may safely be assumed to be potentially of interest to mathematicians of very differing backgrounds, algebraists of many different hues, geometers both algebraic and differential, theoretical physicists and, of course, algebraic topologists. To make these notes as useful as possible, some part of the more basic 'intuitions' from the background material from some of these areas has been included at various points. This cannot be 'all inclusive' nor 'universal' as different groups of potential readers have different needs. The real problems are those of transfer of 'technology' between the areas and of explanation of the differing terminology used for the same concept in different contexts. Often, essentially the same idea or result will appear in several places. This repetition is not just laziness on the authors behalf. The introduction of a concept bit-by-bit from various angles almost necessitates such a treatment.

For the background on bundle-like constructions (sheaves, torsors, stacks, gerbes, 2-stacks, etc.), the geometric intuition of 'things over X' or X-parametrised 'things' of various forms, does permeate much of the theory, so we will start with some fairly basic ideas, and so will, no doubt, for some of the time, be 'preaching to the converted', however that 'bundle' intuition is so important for this and later sections that something more than a superficial treatment is required.

(In the original lectures at Buenos Aires, I did assume that that intuition was understood, but in any case concentrated on the 'group extension' case rather than on 'gerbes' and their kin. By this means I avoided the need to rely too heavily on material that could not be treated to the required depth in the time available. However I cannot escape the need to cover some of that material here!)

Initially crossed modules, etc., will not be that much in evidence, *but* it is important to see how they do enter in 'geometrically' or their later introduction can seem rather artificial.

We start by looking at descent, i.e., the problem of putting 'local' bits of structure into a global whole.

6.1 Descent: Bundles, and Covering Spaces

(Remember, if you have met 'descent' or 'bundles', then you should 'skim' this section only / anyway.)

We will look at these structures via some 'case studies' to start with.

6.1.1 Case study 1: Topological Interpretations of Descent.

Suppose A and B are topological spaces and $\alpha : A \to B$ a continuous map (sometimes called a 'space over B' or loosely speaking a 'bundle over B', although that can also have a more specialised meaning later). An obvious and important example is a product, $A = B \times F$, with α being the projection.

If $U \subset B$ is an open set, then we get a restriction $\alpha_U : \alpha^{-1}(U) \to U$. If $V \subset B$ is another open set, we, of course, have $\alpha_V : \alpha^{-1}(V) \to V$ and over $U \cap V$ the two restrictions 'coincide', i.e. if we form the pullbacks

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \alpha^{-1}(U) \arrow[d] \\ U \cap V \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} ? \arrow[r] \arrow[d] & \alpha^{-1}(V) \arrow[d] \\ U \cap V \arrow[r] & V \end{tikzcd}$$

the resulting spaces over $U \cap V$ are 'the same'. (We have to be a bit careful since we formed them by pullbacks so they are determined only 'up to isomorphism' and we should take care to interpret 'the same' as meaning 'being isomorphic' as spaces over $U \cap V$. This care will be important later.) Now assume that for each $b \in B$, we choose an open neighbourhood $U_b \subset B$ of b. We then have a family

$$\alpha_b : A_b \to U_b \qquad\qquad b \in B,$$

where we have written A_b for $\alpha^{-1}(U_b)$, and we know information about the behaviour over intersections.

Can we reverse this process? More precisely, can we start with a family $\{\alpha_b : A_b \to U_b : b \in B\}$ of maps (with A_b now standing for an arbitrary space) and add in, say, information on the 'compatibility' over the intersections of the cover $\{U_b : b \in B\}$ so as to rebuild a space over B, $\alpha : A \to B$, which will restrict to the given family.

We will need to be more precise about that 'compatibility', but will leave it aside until a bit later. Clearly, indexing the cover by the elements of B is a bit impractical as usually we just need, or are given, some (open) cover, $\mathcal{U}$, of B, and then can choose, for each $b \in B$, some set of the cover containing b. This way we do not repeat sets unless we expressly need to. Thinking like this we have a cover $\mathcal{U}$ and for each U in $\mathcal{U}$, a space over U, $\alpha_U : A_U \to U$. To encode the condition on compatibility on intersections, we need some (temporary) notation: If $U, U' \in \mathcal{U}$, write $(A_U)_{U'}$ for the restriction of A_U over the intersection $U \cap U'$, similarly $(\alpha_U)_{U'}$ for the restriction of α_U to $U \cap U'$. This is given by the further pullback of α_U along the inclusion of $U \cap U'$ into U, so we also get a map

$$(\alpha_U)_{U'} : (A_U)_{U'} \to U \cap U'.$$

We noted that if the family $\{\alpha_U \mid U \in \mathcal{U}\}$ did come from a single $\alpha : A \to B$, then the α_Us agreed up to isomorphism on the intersections, i.e., we needed homeomorphisms

$$\xi_{U,U'} : (A_U)_{U'} \xrightarrow{\cong} (A_{U'})_U$$

over $U \cap U'$ if we were going to give an adequate description. (These are sometimes called the *transition functions* or gluing *cocyles.*) This, of course, means that

$$(\alpha_{U'})_U \circ \xi_{U,U'} = (\alpha_U)_{U'}.$$

Clearly we should require
1. $\xi_{U,U}$ = identity,
but also if U'' is another set in the cover, we would need
2. $\xi_{U',U''} \circ \xi_{U,U'} = \xi_{U,U''}$
over the triple intersection $U \cap U' \cap U''$.

(This condition 2. is a *cocycle condition*, similar in many ways to ones we have met earlier in apparently very different contexts.)

These two conditions are inspired by observation on decomposing an original bundle. They give us 'descent data', but are our 'descent data' enough to construct and, in general, to classify such spaces over B? The obvious way to attempt construction of an α from the data $\{\alpha_U; \xi_{U,U'}\}$ is to 'glue' the spaces A_U together using the $\xi_{U,U'}$. 'Gluing' is almost always a colimiting process, but as that can be realised using coproducts (disjoint union) and coequalisers (quotients by an equivalence relation), we will follow a two step construction

Step 1: Let $C = \sqcup_{U \in \mathcal{U}} A_U$ and $\gamma : C \to \sqcup_{U \in \mathcal{U}} U$, the induced map. Thus if we consider a specific U in $\mathcal{U}$, we will have inclusions of A_U into C and U into $\sqcup U$ and a diagram

$$\begin{tikzcd} A_U \arrow[r, hook] \arrow[d, "\alpha_U"'] & C = \sqcup A_U \arrow[d, "\gamma"] \\ U \arrow[r, hook] & \sqcup U \end{tikzcd} .$$

Remember that a useful notation for elements in a disjoint union is a pair, (element, index), where the index is the index of the set in which the element is. We write (a, U) for an element of C, then $\gamma(a, U) = (\alpha_U(a), U)$, since $a \in A_U$.

Step 2: We relate elements of C to each other by the rule:

$$(a, U) \sim (a', U')$$

if and only if
(i) $\alpha_U(a) = \alpha_{U'}(a')$,
and
(ii) we want to glue corresponding elements in fibres over the same point of B so need something like $\xi_{U,U'}(a) = a'$. Although intuitively correct, as it says that if a and a' are over the same point of $U \cap U'$ then they are to be 'related' or 'linked' by the homeomorphism, $\xi_{U,U'}$, a close look at the formula shows it does not quite make sense. Before we can apply $\xi_{U,U'}$ to a, we have to restrict a to be in $(A_U)_{U'}$ and the result will be in $(A_{U'})_U$. Perhaps the neatest way to present this is to look at another disjoint union, this time $\sqcup_{U,U'}(A_U)_{U'}$, and to map this to $C = \sqcup_{U \in \mathcal{U}} A_U$ in two ways. The first of these, $\mathbf{a}$, say, takes the component $(A_U)_{U'}$ and injects it into C via the injection of A_U. The second map, $\mathbf{b}$, first sends $(A_U)_{U'}$ to $(A_{U'})_U)$ using $\xi_{U,U'}$ then sends that second component to $(A_{U'})$ and thus into C. We thus get the correct version of the formula for (ii) to be:

there is an $x \in \sqcup_{U,U'}(A_U)_{U'}$ *such that* $\mathbf{a}(x) = a$ *and* $\mathbf{b}(x) = a'$.

The two conditions on the homeomorphisms ξ readily imply that this is an equivalence relation and that the α_U together define a map

$$\alpha : A = C/_{\sim} \to B$$

given by

$$\alpha[(a, U)] = \alpha_U(a),$$

on the equivalence class, $[(a, U)]$ of (a, U). For this to be the case, we only needed $\alpha_U(a) = \alpha_{U'}(a')$ to hold. Why did we impose the second condition, i.e., the cocycle condition? Simply, if we had not, we would risked having an equivalence relation that crushed C down to B. Each fibre $\alpha^{-1}(b)$ might have been a single point since each $\alpha_U^{-1}(a)$ could have been in a single equivalence class.

We now have a space over B, $\alpha : A \to B$ (with A having the quotient topology, which ensures that α will be continuous).

If we had started with such a space, decomposed over $\mathcal{U}$, then had constructed a 'new space' from that data, would we have got back where we started? Yes, up to isomorphism (i.e., homeomorphism over B). To discuss this, it helps to introduce the category Top/B of spaces over B. This has continuous maps $\alpha : A \to B$ (often written (A, α)) as its objects, whilst a map from (A, α) to $\alpha' : A' \to B$ will be a continuous map $f : A \to A'$ making the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & A' \arrow[dl, "\alpha'"] \\ & B & \end{tikzcd}$$

commutative. This, however, raises another question.

If we have such an f and an (open) cover $\mathcal{U}$ of B, we restrict f to $\alpha^{-1}(U)$ to get

$$f_U : A_U \to A'_U$$

which, of course, is in Top/U. If we have data,

$$\{\alpha_U : A_U \to U, \{\xi_{U,U'}\}\}$$

for (A, α) and similarly for (A', α'), and morphisms

$$\{f_U : A_U \to A'_U\},$$

when can we 'rebuild' $f : A \to A'$? We would expect that we would need a compatibility between the various f_U and the $\xi_{U,U'}$ and $\xi'_{U,U'}$. The obvious condition would be that whenever we had U, U' in $\mathcal{U}$, the diagram

$$\begin{tikzcd} (A_U)_{U'} \arrow[r, "(f_U)_{U'}"] \arrow[d, "\xi_{U,U'}"'] & (A'_U)_{U'} \arrow[d, "\xi'_{U,U'}"'] \\ (A_{U'})_U \arrow[r, "(f_{U'})_U"] & (A'_{U'})_U \end{tikzcd}$$

should commute, where we have extended our notation to use $(f_U)_{U'}$ for the restriction of f_U to $\alpha^{-1}(U \cap U')$. To codify this neatly we can form each category Top/U for $U \in \mathcal{U}$, then form the category, D, consisting of families of objects, $\{\alpha_U : U \in \mathcal{U}\}$, of $\prod Top/U$ together with the extra structure of the $\xi_{U,U'}$. Morphisms in D are families $\{f_U\}$ as above, compatible with the structural isomorphisms $\xi_{U,U'}$.

Remark: This category is called the *category of descent data relative to the cover $\mathcal{U}$*. We will look at both it and its higher dimensional relatives in quite a lot of detail later on. The reason for the use of the word 'descent' is that in many geometric situations, structure is easily encoded on some basic 'patches'. This structure, that is locally defined, 'descends' to the space giving it a similar structure. In many cases the A_U have the fairly trivial form $U \times F$ for some fibre F. This fibre often has extra structure and the $\xi_{U,U'}$ have then to be structure preserving automorphisms of the space, F. The term 'bundle' is often used in general, but some authors restrict its use to this *locally trivial* case. The classic case of a locally trivial bundle is a Möbius band as a bundle over the circle. Locally, on the circle, the band is of form $U \times [-1, 1]$, but globally one has a twist.

6.1.2 Case Study 2: Covering Spaces

This is a classic case of a class of 'spaces over' another space. It is also of central importance for the development of possible generalisations to higher 'dimensions', (cf. Grothendieck's *Pursuit of Stacks*, [62].) We have a continuous map

$$\alpha : A \to B$$

and for any point $b \in B$, there is an open neighbourhood U of b such that $\alpha^{-1}(U)$ is the disjoint union of open subsets of A, each of which is mapped homeomorphically onto U by α. The map α is then called a *covering projection*. On such a U, $\alpha^{-1}(U)$ is $\sqcup U_i$ over some index set which can be taken to be $\alpha^{-1}(b) = F_b$, the fibre over b. Then we may identify $\alpha^{-1}(U)$ with $U \times F_b$ for any $b \in U$. This F_b is 'the same' up to isomorphism for all $b \in U$. If B is connected then for any b, $b' \in B$, we can link them by a chain of pairwise intersecting open sets of the above form and hence show that $F_b \cong F_{b'}$. We can thus take each $\alpha^{-1}(U) \cong U \times F$ and F will be a discrete space provided B is nice enough. The descent data in this situation will be the local covering projections

$$\alpha_U : U \times F \to U$$

together with the homeomorphisms

$$\xi_{U,U'} : (U \cap U') \times F \to (U \cap U') \times F$$

over $(U \cap U')$. Provided that $(U \cap U')$ is connected, this $\xi_{U,U'}$ will be determined by a permutation of F.

We often, however, want to allow for non-connected $(U \cap U')$. For instance take B to be the unit circle S^1, $F = \{-1, 1\}$,

$$U_1 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x > -0.1\}$$

$$U_2 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x < 0.1\}.$$

The intersection, $U_1 \cap U_2$, is not connected, so we specify ξ_{U_1,U_2} separately on the two connected components of $U_1 \cap U_2$. We have

$$U_1 \cap U_2 = \{(x, y) \in S^1 \mid |x| < 0.1, y > 0\} \cup \{(x, y) \mid |x| < 0.1, y < 0\}.$$

Let $\xi_{U_1,U_2}((x, y), t) = \begin{cases} ((x, y), t) & \text{if } y > 0 \\ ((x, y), -t) & \text{if } y < 0, \end{cases}$
so on the part with negative y, ξ exchanges the two leaves. The resulting glued space is either viewed as the edge of the Möbius band or as the map,

$$S^1 \to S^1$$

$$e^{i\theta} \mapsto e^{i2\theta}.$$

Remark: The well known link between covering spaces and actions of the fundamental group $\pi_1(B)$ on Sets is at the heart of this example.

A neat way to picture the n-fold covering spaces of S^1 for $n \geq 2$ is to consider a knot on the surface of a torus, $S^1 \times S^1$, for instance the trefoil. The projection to the first factor of $S^1 \times S^1$ gives a covering, as does the second projection. It is **also instructive** to consider the covering space $\mathbb{R}^2 \to S^1 \times S^1$, working out what the various transitions are for a cover. We note the way that quotients of $\mathbb{R}^n$ by certain geometrically defined group actions, yields neat examples of coverings (although some may be 'ramified', an area we will not stray into here.)

In general, when we have a local product structure, so $\alpha^{-1}(U) \cong U \times F$, the homeomorphisms $\xi_{U,U'}$ have a nicer description than the general one, since being 'over' the intersection, they have to have the form that interprets at the product levels as being $\xi_{U,U'}(x, y) = (x, \xi'_{U,U'}(x)(y))$ where $\xi'_{U,U'} : U \cap U' \to Aut(F)$. In the case of covering spaces F is discrete, so $\xi'_{U,U'}(x)$ will give a permutation of F.

6.1.3 Case Study 3: Fibre bundles

The examples here are to introduce / recall how torsors / principal fibre bundles are defined topologically and also to give some explicit instances of how fibre bundles arise in geometry.

(Often in this context, the terminology 'total space' is used for the source of the bundle projection.)

First some naturally occurring examples.

(i) Let S^n denote the usual n-sphere represented as a subspace of $\mathbb{R}^{n+1}$,

$$S^n = \{\underline{x} \in \mathbb{R}^{n+1} \big| \ \|\underline{x}\| = 1\},$$

where $\|\underline{x}\| = \sqrt{\langle \underline{x} \mid \underline{x} \rangle}$ for $\langle \underline{x} \mid \underline{y} \rangle$, the usual Euclidean inner product on $\mathbb{R}^{n+1}$. The *tangent bundle* of S^n, τS^n is the 'bundle' with total space,

$$TS^n = \{(\underline{b}, \underline{x}) \ \big| \ \langle \underline{b} \mid \underline{x} \rangle = 0\} \subset S^n \times \mathbb{R}^{n+1}.$$

We thus have a projection

$$p : TS^n \to S^n$$

given by $p(\underline{b}, \underline{x}) = \underline{b}$, as a space over S^n.

Similarly the *normal bundle*, νS^n, of S^n is given with total space,

$$NS^n = \{(\underline{b}, \underline{x}) \ \big| \ \underline{x} = k\underline{b} \text{ for some } k \in \mathbb{R}\} \subset S^n \times \mathbb{R}^{n+1}.$$

The projection map $q : NS^n \to S^n$ gives, as before, a space over S^n, $\nu S^n = (NS^n, q, S^n)$.

Another example extends this to a geometric context of great richness.

(ii) The *Stiefel variety* of k-frames in $\mathbb{R}^n$, denoted $V_k(\mathbb{R}^n)$, is the subspace of $(S^{n-1})^k$ such that $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ if and only if each $\langle v_i \mid v_j \rangle = \delta_{i,j}$, so that it is 1 if $i = j$ and is zero otherwise. Note $V_1(\mathbb{R}^n) = S^{n-1}$.

The *Grassman variety* of k-dimensional subspaces of $\mathbb{R}^n$, denoted $G_k(\mathbb{R}^n)$, is the set of k-dimensional subspaces of $\mathbb{R}^n$. There is an obvious function,

$$\alpha : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

mapping $(v_1, \ldots, v_k)$ to $span_{\mathbb{R}}\langle v_1, \ldots, v_k\rangle \subseteq \mathbb{R}^n$, that is, the subspace with $(v_1, \ldots, v_k)$ as basis. We give $G_k(\mathbb{R}^n)$ the quotient topology defined by α. (For $k = 1$, we have $G_1(\mathbb{R}^n)$ is the real projective space of dimension $n - 1$.)

This setting also produces other examples of 'bundles'. Consider the subspace of $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ given by those (V, x) with $x \in V$. Using the projection $p(V, x) = V$ gives the bundle

$$\gamma_k^n = (\gamma_k^n, p, G_k(\mathbb{R}^n)).$$

Similarly the orthogonal complement bundle $^*\gamma_k^n$ has total space consisting of those (V, x) with $\langle V \mid x\rangle = 0$, i.e. x is orthogonal to V. All of these 'bundles' have vector space structures on their fibres. They are all locally trivial (so in each case $\alpha^{-1}(U) \cong U \times F$ for suitable open subsets U of the base), and the resulting $\xi_{U,U'}$ have form

$$\xi_{U,U'}(x, t) = (x, \xi'_{U,U'}(x))(t)$$

where $\xi'_{U,U'} : U \cap U' \to Gl_M(\mathbb{R})$ for suitable M. (As usual, $Gl_M(\mathbb{R})$ is the (topological) group of non-singular $M \times M$ matrices over $\mathbb{R}$.) Such *vector bundles* are prime examples of the situation in which the fibres have extra structure.

Even more structure can be encoded, for instance, by giving each fibre an inner product structure with the requirement that the $\xi'_{U,U'}$ take values in $O_M(\mathbb{R})$, the orthogonal group, hence that they preserve that extra structure. Abstracting from this we have a group G which acts by automorphisms on the space, F, and have our descent data isomorphisms $\xi_{U,U'}$ of the form $\xi_{U,U'}(x, t) = (x, \xi'_{U,U'}(x))(t)$ for some continuous $\xi'_{U,U'} : U \cap U' \to G$.

As usual, if G is a (topological) group, by a G-space we mean a space X with an action (left action):

$$G \times X \to X,$$

$$(g, x) \to g.x.$$

The action is *free* if $g.x = x$ implies $g = 1$. The action is *transitive* if given any x and y in X there is a $g \in G$ with $g.x = y$. Let X^* be the subspace

$$X^* = \{(x, g.x) : x \in X, g \in G\} \subseteq X \times X,$$

(cf. our earlier discussion of action groupoids on page 11).

There is a function (called the *translation function*)

$$\tau : X^* \to G$$

such that $\tau(x, x')x = x'$ for all $(x, x') \in X^*$. We note

(i) $\tau(x, x) = 1$,

(ii) $\tau(x', x'')\tau(x, x') = \tau(x, x'')$,

(iii) $\tau(x', x) = \tau(x, x')^{-1}$

for all $x, x', x'' \in X$.

A G-space X is called *principal* provided X is a free, transitive G-space with *continuous* translation function $\tau : X^* \to G$.

Proposition 35 *Suppose X is a principal G-space, then the mapping*

$$G \times X \to X \times X$$

$$(g, x) \to (x, g.x)$$

is a homeomorphism.

Proof: The mapping is continuous by its construction. Its inverse is (τ, pr_1), which is also continuous. ∎

This is often taken as the definition of a principal G-space, so you could try to prove the converse. We, in fact, need a fibrewise version of this.

Given any G-space, X, we can form a quotient X/G with a continuous map $\alpha : X \to X/G$. A bundle $\mathsf{X} = (X, \alpha, B)$ is called a *G-bundle* if X has a G-action, so that B is homeomorphic to X/G compatibly with the projections from X. The bundle is a *principal G-bundle* if X is a principal G-space *over* B. What does this mean? In a G-bundle, as above, the fibres of α are orbits of the G-action, so the action is 'fibrewise'. We can replace G by $\underline{G} = G \times B$ and, thinking of it as a space over B, perhaps rather oddly, write the action within the category Top/B. We replace the product in Top by that in Top/B which is just the pullback along projections in Top. The action is thus

$$\underline{G} \times_B X \to X$$

over B, or just $\underline{G} \times \mathsf{X} \to \mathsf{X}$ in the notation valid in Top/B. Now 'principalness' will say that the action if free and transitive, and that the translation function is a continuous map *over* B. A neater way to handle this is to use the above proposition and to define X to be a principal G-bundle if the corresponding morphism over B,

$$\underline{G} \times \mathsf{X} \to \mathsf{X} \times \mathsf{X}$$

is an isomorphism in Top/B. We will not explore this more here as that *is*, more or less, the way we will define G-torsors later on, except that we will be using a bundle or sheaf of groups rather than simply $\underline{G}$.

We note that if $\xi = (X, p, B)$ is a principal G-bundle then the fibre $p^{-1}(b)$ is homeomorphic to G for any point $b \in B$. It is usual in topological situations to require that the bundle be locally trivial. For the moment, we can summarise the idea of principal G-bundle as follows:

A principal G-bundle is a fibre bundle $p : X \to B$ together with a continuous left action $G \times X \to X$ by a topological group G such that G preserves the fibers of p and acts freely and transitively on them.

Later we will see other more categorical views of principal G-bundles. As we have mentioned, they will reappear as 'G-torsors' in various settings. For the moment we need them to provide the link to the general notion of fibre bundle.

For F, a (right) G-space with action $G \times F \to F$, we can form a quotient, X_F, of $F \times X$ by identifying (f, gx) with (fg, x). The composite

$$F \times X \overset{pr_2}{\to} X \to X/G$$

factors via X_F to give $\beta : X_F \to X/G$, where $\beta(f, x)$ is the orbit of x, i.e. the image of x in X/G. The earlier examples of 'bundles' were all examples of this construction. The resulting (X_F, β, B) is called a *fibre bundle* over B $(= X/G)$.

Note: The theory of fibre bundles was developed by Cartan and later by Ehresmann and others from the 1930s onwards. Their study arose out of questions on the topology and geometry of manifolds. In 1950, Steenrod's book, [109], gave what was to become the first reasonably full treatment of the theory. Atiyah, Hirzebruch and then, in book form, Husemoller, [69] in 1966 linked this theory up with K-theory, which had come from algebraic geometry. The books contain much of the basic theory including the local coordinate description of fibre bundles which is most relevant for the understanding of the descent theory aspects of this area (cf. Chapter 5 of Husemoller, [69]). The restriction of looking at the local properties relative to an open cover makes this treatment slightly too restrictive for our purposes. It *is* sufficient, it seems, for many of the applications in algebraic topology, differential geometry and topology and related areas of mathematical physics, however as Grothendieck points out (SGA1, [63], p.146), in algebraic geometry *localisation of properties*, although still linked to certain types of "base change" (as here with base change along the map

$$\sqcup\, \mathcal{U} \to B$$

for $\mathcal{U}$ an open cover of B), needs to consider other families of base change. These are linked with some problems of commutative algebra that are interesting in their own right and reveal other aspects of the descent problem, see [16]. For these geometric applications, we need to replace a purely topological viewpoint by one in which *sheaves* take a front seat role.

(The Wikipedia entries for principal G-space, principal bundle and 'fiber' bundle are good places to start seeing how these concepts get applied to problems in geometry. For a picture of how to build a fibre bundle out of wood, see http://www.popmath.org.uk/sculpmath/pagesm/fibundle.html.)

6.1.4 Change of Base

This is a theme that we will revisit several times. Suppose that we have a good knowledge of 'bundles' over some space B', but want bundles over another space B. We have a continuous map $f : B \to B'$ and hope to glean information on bundles on B by comparing them with those on B', using f in some way. (We could be looking to transfer the information the other way as well but this way will suffice for the moment!)

What we have used when restricting to open subsets of a base space was pullback and that works here as well. Suppose $p' : A' \to B'$ is a principal G-bundle over B' then we form the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Categorically the pullback, as it is characterised by a universal property, is only determined up to isomorphism, but we can pick a definite model for A in the form

$$A' \times_{B'} B = \{(a, b) \mid p'(a) = f(b)\},$$

with $a \in A'$ and $b \in B$. The projection of A onto B is given by sending (a, b) to b and the map from A to A' by the obvious other projection. As we have an action of G on the left of A' it is tempting to see if there is one on A and the obvious thing to attempt is $g.(a,b) = (g.a, b)$. Does this make sense? Yes, because $p'(g.a) = p'(a)$, since B' is the space of orbits of the action of G on A'. Is $A \to B$ then a principal G-bundle? Again the answer is yes. To gain some idea why look at the fibres. We know the fibres of a principal G bundle are copies of the *space* G, and fibres of the pullback are the same as fibres of the original. The action is concentrated in the fibres as the orbit space of the action *is* the base.

The one question is whether the map

$$\underline{G} \times_B A \to A \times_B A$$

is an isomorphism. You can see that it is in two ways. The elements of A are pairs (a,b), as above. The map is $((g,b),(a,b)) \mapsto ((a,b),(g.a,b)$ and this is clearly in the fibres as the second component in each pair is the same. It has an inverse surely, (since an element in $A \times BA$, has the form $((a_1,b),(a_2,b))$ and since A' is a principal bundle we can continuously find g such that $a_2 = g.a_1$). The alternative approach is to note that the map fits into a diagram with lots of pull back squares and to note that is is induced from the corresponding map for (A', B', p').

We thus have, it would seem, that $f : B \to B'$ induces a 'functor' from the category of principal G-bundles over B' to the corresponding one over B. (The word 'functor' is given between inverted commas since we have not discussed morphisms between bundles of this form. That is left to you both to formulate the notion and to check that the inverted commas can be removed. In any case we will be considering this in the more general setting of G-torsors slightly later in this chapter.)

We thus have induced bundles $f^*(\mathsf{A}')$, but different maps f can lead to isomorphic bundles. More precisely suppose f and g are two maps from B to B', then if f and g are homotopic (under mild compactness conditions on the spaces) it is fairly easy to prove that for any (principal) bundle A' on B', the two bundles $f^*(\mathsf{A}')$, and $G^*(\mathsf{A}')$, are isomorphic. We will not give the details here as they are in most text books on the area, but the idea is that if $H : B \times I \to B'$ is a homotopy between f and g, we get a bundle $H^*(\mathsf{A})$ with base $B \times I$. You now use local triviality of the bundle to cover $B \times I$ by open sets over which this bundle trivialises. Using compactness of B we get a sequence of points t_i in I and an open cover of $B \times I$ made up of open sets of the form $U \times (t_i, t_{i+2})$. Now we work our way up the cylinder showing that the bundle over each slice $B \times \{t_i\}$ is isomorphic to that on the previous slice. (There are lots of details left vague here and you should look them up if you have not seen the result before.)

This result shows that categories of principal bundles over homotopically equivalent spaces will be equivalent, and in particular that over any contractible space, all principal bundles are isomorphic to each other and hence are all isomorphic to the product principal bundle. It also shows that if we can cover B with an open cover made up of contractible open sets that all bundles trivialise over that cover.

6.2 Descent: Sheaves

(As for the previous section, this should be 'skimmed' if you have met sheaves before. A good accessible account and brief introduction to this is Ieke Moerdijk's Lisbon notes, [94]. These also are useful for alternative developments of later material and are thoroughly to be recommended.)

6.2.1 Introduction and definition

Sheaves provide a useful alternative to bundles when handling 'local-to-global' constructions. The intuition is, in many ways, the same as that of bundles. We have a space B and for each $b \in B$, a 'fibre' over b, i.e. a set F_b, and we want to have F_b varying in some continuous way as we vary b continuously. In other words, naively a sheaf is a continuously varying family of 'sets'.

That is much too informal to use as a definition as it has employed several terms that have not been defined. Before seeing how that intuition might be encoded more exactly, we will return to the 'spaces over B'. Let $\alpha : A \to B$ be a space over B as before, and, once again, let $U \subset B$ be an open set. This time we will not consider $\alpha^{-1}(U)$, but will look at *local sections of* α *over* U. A *(local) section* of α, over U is a continuous map $s : U \to A$ such that, for all $x \in U$, $\alpha s(x) = x$, that is, $s(x)$ is always in the fibre over x. We write $\Gamma_A(U)$ for the set of such local sections.

If $V \subset U$ is another open set of B and $s : U \to A$ is a local section of α over U, then the restriction, $s|_V$, of s to V is a local section of α over V. We thus get, from $V \subset U$, an induced 'restriction' map

$$\mathsf{res}^U_V : \Gamma_A(U) \to \Gamma_A(V).$$

Of course, if $W \subset V$ is another such

$$\mathsf{res}^U_V \circ \mathsf{res}^V_W = \mathsf{res}^U_W.$$

There is a little teasing problem here. Suppose V is empty. Of course, the empty set is a subset of all the other open sets, so what should $\Gamma_A(\emptyset)$ be? The empty space is the initial object in the category of spaces so there is a unique map from it to A and, of course, this is a local section! (You can either check the condition at all points of the domain or argue that composition of this empty local section with the projection p yields the unique map from $\emptyset$ into B, as required.)

Back to the generalities, there is, again of course, a neat, and well known, categorical description of this setting.

Let $Open(B)$ denote the partially ordered set of open sets of B with the usual order coming from inclusion, and consider it as a category in the usual way. The above construction just gave a functor

$$\Gamma_A : Open(B)^{op} \to Sets,$$

a *presheaf* on B. Any functor $F : Open(B)^{op} \to Sets$ is called a *presheaf*, but not all presheaves come from 'spaces over B' by the local sections construction, as it is fairly clear that Γ_A has some special properties, for instance, we saw that such a presheaf must send $\emptyset$ to the singleton set, but we also have the gluing property:

Suppose $s_1 \in \Gamma_A(U_1)$ and $s_2 \in \Gamma_A(U_2)$ are two local sections and

$$\mathsf{res}^{U_1}_{U_1\cap U_2}(s_1) = \mathsf{res}^{U_2}_{U_1\cap U_2}(s_2),$$

so these local sections agree on the intersection of their domains, then define

$$s : U_1 \cup U_2 \to A$$

by

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in U_1 \\ s_2(x) & \text{if } x \in U_2. \end{cases}$$

It is easy to prove that s is continuous and so gives a local section over $U_1 \cup U_2$. We need not stop with just two local sections. If we have any family of local sections, over a family of open sets, that coincide on pairwise intersections, then they can be glued together, just as above, to give a unique local section on the union of those open sets, restricting to the given ones with which we started on their original domains. This gluing property is the defining property of the sheaves amongst the presheaves on B:

Definition: A presheaf $F : Open(B)^{op} \to Sets$ is a *sheaf* if given any family $\mathcal{U}$ of open sets of B, say $\mathcal{U} = \{U_i\}_{i\in I}$, and elements $s_i \in F(U_i)$ for $i \in I$, such that for $i, j \in I$ $\mathsf{res}^{U_i}_{U_i\cap U_j}(s_i) = \mathsf{res}^{U_j}_{U_i\cap U_j}(s_j)$, there is a *unique* $s \in F(U)$, for $U = \bigcup U_j$, such that $\mathsf{res}^U_{U_i}(s) = s_i$ for all i.

Query: Does this gluing property imply the normalisation condition that $F(\emptyset)$ is a singleton? **For you to investigate!**

For later purposes and comparisons, we will note that a *compatible family s_i of local elements*, as above, gives an element $\underline{s}$ in the product set $\prod\{F(U_i) : i \in I\}$. Not just any family of elements however. We also have a product of the parts over the intersections. We write $U_{ij} = U_i \cap U_j$ and get a product $\prod\{F(U_{i,j}) : i, j \in I\}$. There are two functions, which we will call a and b for convenience only, defined from $\prod\{F(U_i) : i \in I\}$ to $\prod\{F(U_{ij}) : i, j \in I\}$. To specify these we see how they project onto the factors $F(U_{ij})$. (Technically, we have maps $\prod F(U_{ij}) \stackrel{p_{ij}}{\to} F(U_{ij})$, being the $\{ij\}^{th}$ projection of the product.) The specifications are

$$p_{ij}a(\underline{s}) = res^{U_i}_{U_{ij}}(s_i),$$

whilst

$$p_{ij}b(\underline{s}) = res^{U_j}_{U_{ij}}(s_i).$$

We can now give the compatibility condition as $\underline{s}$ is a compatible family of local elements exactly if $a(\underline{s}) = b(\underline{s})$:

$$Eq(a, b) \longrightarrow \prod F(U_j) \overset{a}{\underset{b}{\rightrightarrows}} \prod F(U_{ij}) \ ,$$

i.e., $\underline{s}$ is in the *equaliser* $Eq(a, b)$ of a and b. This equaliser is sometimes called the *set of descent data for the presheaf relative to the cover.*

From this perspective, we note that the restriction maps give a map

$$c : F(U) \to \prod F(U_i),$$

with $p_i c(s) = res^U_{U_i}(s)$ and we know $ac = bc$. We thus get a function from $F(U)$ to $Eq(a, b)$ assigning $c(s)$ to s. We have F is a sheaf exactly when this is a bijection; it is a *separated presheaf* when this map is one-one, see below.

This scenario is quite useful for sheaves, but it really comes into its own when we look at higher dimensional analogues such as stacks.

We will note quite a lot of facts about sheaves and presheaves, but will not give a detailed development, since here is not a suitable place to give a lengthy treatment of sheaf theory.

6.2.2 Presheaves and sheaves

The category, $Sh(B)$, of sheaves on a space, B, is a reflective subcategory of the category, $Presh(B) = [Open(B)^{op}, Sets]$, of presheaves on B.

We first note a half-way house between general presheaves and sheaves.

The presheaf F is *separated* if there is at most one $s \in F(U)$ such that $res^U_{U_i}(s) = s_i$ for all i. ('Sheafness' would also require this, but, in addition, asks for the existence of such an s, not just uniqueness if it exists.) In fact:

The functors

$$Sh(B) \to Sep.Presh(B) \to Presh(B)$$

have left adjoints.

If F is a presheaf, we will write $s(F)$ for the corresponding separated presheaf and $a(F)$ for the associated sheaf. We can give explicit constructions of $s(F)$ and $a(F)$.

- Define an equivalence relation $\sim_U$ on $F(U)$, where, if $a, b \in F(U)$, then $a \sim b$ if and only if $res^U_{U_i}(a) = res^U_{U_i}(b)$ for all i, then $s(F)$ given by $s(F)(U) = F(U)/\sim_U$ is a separated presheaf. (For you to check the presheaf structure.)

- Suppose F is separated (if not replace it by $s(F)$ and rename!) Form $F_{\mathcal{U}}$, the set of compatible families (relative to $\mathcal{U}$) of elements in the $F(U_i)$. If $\mathcal{V} < \mathcal{U}$ is a finer cover of U, (so for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ with $V \subseteq U$), then there is a function $res^{\mathcal{U}}_{\mathcal{V}} : F_{\mathcal{U}} \to F_{\mathcal{V}}$ where $res^{\mathcal{U}}_{\mathcal{V}}(\underline{s})_j = res^{U_i}_{V_j}(s_i)$ if $V_j \subseteq U_i$. (**Check** it is well defined.)

  Varying $\mathcal{U}$, we get a diagram of sets and form

  $$a(F)(U) = colim_{\mathcal{U}} F_{\mathcal{U}}.$$

  Explicitly we generate an equivalence relation on the union of the $F_{\mathcal{U}}$s by

  $$\underline{s}_{\mathcal{U}} \sim \underline{s}_{\mathcal{V}}$$

  if $\mathcal{V} < \mathcal{U}$ and $res^{\mathcal{U}}_{\mathcal{V}}(\underline{s}_{\mathcal{U}}) = \underline{s}_{\mathcal{V}}$, and then form the quotient.

(The details are well known and, if you have not met them before should be checked or looked up, e.g. in a related context, [16], p.268. The sort of constructions used will be useful throughout this chapter. It is a good idea to try to rewrite this in terms of the equaliser description given earlier, to see what is happening there.)

6.2.3 Sheaves and étale spaces

The category $Sh(B)$ is equivalent to the category of étale spaces over B.

A continuous map $f : X \to Y$ between topological spaces is *étale* if, for every $x \in X$, there is an open neighbourhood U of x in X and an open neighbourhood, V, of $f(x)$ in Y such that f restricts to a homeomorphism $f : U \to V$. We also say that X is an *étale space over* Y.

Given a presheaf, F on B and $b \in B$, let

$$F_b = colim_{b \in U} F(U).$$

and $\mathsf{germ}_b : F(U) \to F_b$, be the natural map. The colimit is constructed using a disjoint union followed by using an equivalence relation. This germ map just send an element to its equivalence calss. More precisely: the set, F_b is the 'stalk' of F at b. It is made up of equivalence classes of 'germs' of *locally defined elements*, i.e., (U, b, x), where b is the point at which we are looking, U is an open set with $b \in U$ and $x \in F(U)$. If (U, b, x_U) and (V, b, x_V) are two such germs, they are equivalent if there is a $W \subset U \cap V$, again open in B, such that

$$res^U_W(x_U) = res^V_W(x_V),$$

i.e.,x_U and x_V agree 'near to b'. Now let $E(F) = \bigsqcup_{b \in B} F_b$ be the disjoint union with $\pi : E(F) \to B$, the obvious projection.

The topology on $E(F)$ is given by basic open sets: if $x \in F(U)$, $B(x) = \{\mathsf{germ}_b(x) \mid b \in U\}$ is to be open. (The idea is that we make x into a continuous local section of $E(F)$ over U by this means.) This makes $(E(F), \pi)$ an étale space over B.

We could construct $a(F)$ in (i) as $\Gamma_{E(F)}$, i.e., the sheaf of local sections of $E(F)$.

6.2.4 Covering spaces and locally constant sheaves

A covering space is an étale space which is locally trivial and it then corresponds to a locally constant sheaf on B.

For any set S, there is a constant sheaf, defined by the presheaf $F(U) = S$ for all $U \in Open(B)$. The corresponding étale space is $B \times S$ with its projection onto B and where S is given the discrete topology. A sheaf is *locally constant* if for each $b \in B$, there is an open set U_b containing b such that the restriction of F to U_b is a constant sheaf or, more strictly speaking, is isomorphic to a constant sheaf.

We can rephrase this in a neat way that introduces viewpoints that will be useful later on. The open sets U_b give us an open cover of B, so we could pick a subcover with the same trivialising property. We thus assume that we have a cover $\mathcal{U}$ and form a space $\bigsqcup \mathcal{U}$ by taking the disjoint union of the open sets in $\mathcal{U}$. (Recall that a convenient way of working with $\bigsqcup \mathcal{U}$ is to denote its elements by pairs (b, U) with $b \in U$ and $U \in \mathcal{U}$. We then have a copy of each b for each open set from the cover of which it is an element.) There is an obvious projection map

$$p : \bigsqcup \mathcal{U} \to B,$$

which is $p(b, U) = b$, and this is, fairly obviously, an étale map. We pull back F along p to get a sheaf on $\bigsqcup \mathcal{U}$ and, of course, this pulled back sheaf is constant.

This trick of turning a (topological) open cover into a map is very important. It forms the basis of the theory of Grothendieck topologies. In that theory, one replaces $Open(B)$ by a category $\mathcal{C}$, so a presheaf on $\mathcal{C}$ is just a functor $F : \mathcal{C}^{op} \to Sets$. The sheaf condition is adapted to this setting by specifying what (families of) morphisms in $\mathcal{C}$ are to be considered 'coverings' with an axiomatisation of their desired properties. For instance, for an open covering, $\mathcal{U}$ of B, if for each $U \in \mathcal{U}$, we pick an open covering of it and then combine these open coverings together we get an open covering of B. That is mirrored by a condition on the covering families in the Grothendieck topology.

We will not treat Grothendieck topologies in great detail here as, once again, that might take us too far away from the 'crossed menagerie' and the related issues of cohomology. We will give a definition shortly. It *will* be necessary, however, to have such a definition of a Grothendieck topos, i.e., the category of sheaves for such a Grothendieck topology and we will attempt to show how it relates to some of the topics we are considering. For greater detail from a very approachable viewpoint, the approach from Borceux and Janelidze's book, [16], is suggested, but we warn the reader that they also avoid very lengthy discussions of the topic, as their aim is not topos theory *per se*, but generalised Galois theory.

6.2.5 A siting of Grothendieck toposes

Definition: A *Grothendieck topos* is a category, $\mathcal{E}$, which is equivalent to a full reflective subcategory

$$\mathcal{E} \underset{}{\overset{a}{\leftrightarrows}} [\mathcal{C}^{op}, Sets]$$

of a presheaf category, $Presh(\mathcal{C}) = [\mathcal{C}^{op}, Sets]$, where the left adjoint, a, preserves finite limits.

The reflective nature of this category means that when considering morphisms *from* a (pre)sheaf to a sheaf, it is enough to give them at the presheaf level, since they will automatically be sheafified.

We had early on in our discussion of sheaves, the statement: *The category, $Sh(B)$, of sheaves on a space, B, is a reflective subcategory of the category, $Presh(B) = [Open(B)^{op}, Sets]$, of presheaves on B.* We can now rephrase this as a proposition:

Proposition 36 *The category, $Sh(B)$, of sheaves on a space, B, is a Grothendieck topos.* ∎

In addition to the category of sheaves on a space, B, we also have several other important examples of the notion.

Example: (i) For any $\mathcal{C}$, the presheaf category, $Presh(\mathcal{C})$, is itself a full reflective subcategory of itself! It thus is a Grothendieck topos.

In particular, the category, $\mathcal{S}$, of simplicial sets is a Grothendieck topos (by taking $\mathcal{C} = \mathbf{\Delta}$). Later we will consider sheaves and bundles of groups, i.e., group objects in the topos of sheaves on a (base) space B. Equally well, we could look at group objects in presheaf toposes such as $[\mathcal{C}^{op}, Sets]$, and these are the group valued presheaves, and thus, in particular, *Simp.Grps* is just the category of presheaves of groups on $\mathbf{\Delta}$.

We can take this 'analogy' further. If we have an étale space, $\alpha : A \to B$, over B, then a local section is a map $s : U \to A$ for $U \in Open(B)$, such that $\alpha s(x) = x$ for all $x \in U$. A presheaf $F : Open(B)^{op} \to Sets$ is thought of as having $F(U)$ as being the local sections over U of 'something' over B. That does not quite give an idea which is wholly expressed within the category of (pre)sheaves itself, as we needed to talk about U itself as well, but, from U, we can get a presheaf, much as above, namely the representable presheaf

$$\hat{U} = Open(B)(-, U).$$

This presheaf takes value a singleton on V if $V \subseteq U$ and is empty otherwise. The inclusion of U into B is the étale map that corresponds to this, so our local section $s : U \to A$ is the analogue of, (in fact, corresponds exactly to), a map of presheaves

$$s : \hat{U} \to \Gamma_A$$

and if $F : Open(B)^{op} \to Sets$ is arbitrary, $F(U) = Presh(B)(\hat{U}, F)$ by the Yoneda lemma, with each presheaf morphism φ from $\hat{U}$ to F yielding an element $\varphi_U(id_U) \in F(U)$. (Remember presheaf morphisms are merely natural transformations between the corresponding functors.)

Example: (ii) Another very important example of a presheaf topos, as above, comes from any group, G. We can, as we have done several times already, consider G as a one object groupoid, $G[1]$. It is then a suitable instance of a small category, which can be fed into the machine of the previous example. The category, $Presh(G[1])$, will be a Grothendieck topos, but what is the interpretation of these objects? From a straightforward perspective, they are set valued functors on $G[1]^{op}$. Suppose that $X : G[1]^{op} \to Sets$ is one such, then, abusing notation like mad, write $X = X(*)$ for the image of the single object $*$ of $G[1]^{op}$, and if $g \in G$, and $x \in X$, write $X(g)(x) = x.g$, then (and this is *left to you*) we can easily check that X is a *right G-set*. Conversely any right G-set, gives a presheaf on $G[1]$ and this sets up an equivalence of categories. (You should also check on morphisms.) If you prefer left G-sets, replace G by the opposite group, G^{op}.

This example is important as it provides the bridge between the cohomology of groups and the cohomology of spaces via a cohomology of toposes. We will see the above argument several times in what follows. (Following the idea that the reader should be able to 'dip' into these notes, we may repeat the point again and again!)

Example: (iii) Any category with a Grothendieck topology on it leads to a Grothendieck topos. We need a definition.

Definition: A *Grothendieck topology* on a category $\mathcal{C}$ is an assignment of families of 'coverings', $\{U_\alpha \to U\}_\alpha$ for each object U in $\mathcal{C}$ such that

- If $\{U_\alpha \to U\}_\alpha$ and $\{U_{\alpha\beta} \to U_\alpha\}_\beta$ are coverings, so is $\{U_{\alpha\beta} \to U\}_{\alpha\beta}$, i.e., 'coverings of coverings are coverings';
- If $\{U_\alpha \to U\}_\alpha$ is a covering family and $V \to U$ is a morphism in $\mathcal{C}$, then the pullback family $\{U_\alpha \times_U \to V\}_\alpha$ is a covering family for V, i.e., 'coverings are pullback stable';
- If $\{V \xrightarrow{\cong} U\}$ is an isomorphism, then this singleton family is a covering family.

A category together with a Grothendieck topology is called a *site*.

Given a site based on $\mathcal{C}$, a presheaf $F : \mathcal{C}^{op} \to Sets$ is called a *sheaf* on the site if for any object U and covering family $\{U_\alpha \to U\}_\alpha$, the sequence

$$F(U) \longrightarrow \prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \times_U U_\beta) \ ,$$

is an equaliser. (If the left hand morphism is merely injective then F will be a 'separated presheaf' in this context'.) The category of sheaves for a given site gives a Grothendieck topos.

Returning to the general case of $[\mathcal{C}^{op}, Sets]$, the Yoneda lemma shows the importance of the representable presheaves. In our key example with $\mathcal{C} = \mathbf{\Delta}$, these representable presheaves are just the simplices $\Delta[n] = \mathbf{\Delta}(-, [n])$. Our observations above point out that if K is a simplicial set, $K_n = K[n] \cong \mathcal{S}(\Delta[n], K)$ and this is the analogue of $F(U)$, i.e., the analogue of the set of local sections of F. Of course, there is no notion of topological continuity in the classical sense here, and as, in the 'presheaf topos' $\mathcal{S}$, all presheaves are sheaves, we have that in some sense 'all sections are as if they were continuous'. (The topological language is being pushed to breaking point here, so the corresponding intuitions would need refining if we were to follow them up properly. One *can* do this with the language of Grothendieck topologies, but we will not explore that further here. To some extent this is done in [16] with a different end point in mind. Here our purpose is to explain loosely why $\mathcal{S}$ is a topos, and why that may be useful and, reciprocally, what do the simplicial ideas, seen from that presheaf/sheaf viewpoint, suggest about general toposes.)

One further fact worth noting is that if $\mathcal{E}$ is a topos and B is an object in $\mathcal{E}$, then the 'slice category', $\mathcal{E}/B$, is also a topos. It thus is Cartesian closed, i.e., not only does it have finite limits, but the functor $- \times A : \mathcal{E} \to \mathcal{E}$, which sends an object X to $X \times A$ for some fixed object A, has a right adjoint $(-)^A$ thought of as being the object of maps from A to whatever. General results can be found in the various books on topos theory, which give very general constructions of these mapping space objects in settings such as the slice toposes. We will need some elementary ideas about Cartesian closed categories later.

6.2.6 Hypercoverings and coverings

It is sometimes necessary to mention 'hypercoverings', instead of 'coverings' when looking at generalisations of sheaves.

In any topos $\mathcal{E}$, there is a precise sense in which $\mathcal{E}$ behaves like a generalisation of the category of sets, but with a logic that replaces the two truth values $\{0, 1\}$ of ordinary Boolean logic by a more general object of truth values. In the topos $Sh(B)$ of sheaves on a space B, this truth value object is the lattice of open sets, $Open(B)$. This may seem a bit weird, but in fact works beautifully. (The logic is non-Boolean in general, so occasionally you need to take care with classical arguments.) This allows one to do things like simplicial homotopy theory *within* $\mathcal{E}$. This replaces the category, $\mathcal{S}$, of simplicial sets by $Simp(\mathcal{E})$ and if $\mathcal{E} = Sh(B)$, then the objects are just simplicial sheaves on B, i.e., sheaves of simplicial sets on B.

Any open cover $\mathcal{U}$ of a space B yields $\bigsqcup \mathcal{U}$, as before, and one can take repeated pullbacks to construct a simplicial sheaf on B from that cover. It is fun to view this in another way as it illustrates some of the ideas working within the topos $\mathcal{E}$ and, in particular, within $Sh(B)$.

Firstly, in $Sets$, there is a terminal object, 1, 'the one point set'. In a topos $\mathcal{E}$, there is a terminal object, $1_{\mathcal{E}}$, and, for $\mathcal{E} = Sh(B)$, this is the constant sheaf with value the one point set. Viewed as an étale space, it is just the identity map, $B \xrightarrow{id} B$. (This multitude of viewpoints may initially seem to lead to confusion, but it does give a beautifully rich context in which to work, with different intuitions and analogies interacting and combining.)

Within $\mathcal{E}$, we have a product, so if A_1, $A_2 \in \mathcal{E}$, we can form $A_1 \times A_2$. What does this looks like for $\mathcal{E} = Sh(B)$? The A_i gives étale spaces $\alpha_i : A_i \to B$, $i = 1, 2$ and $A_1 \times A_2$ corresponds to the pullback

$$A_1 \times_B A_2 \to B.$$

In particular, if $\mathcal{U}$ is an open covering of B, write $U \to 1$ for $\mathcal{U}$ viewed as a sheaf / étale space, $\bigsqcup \mathcal{U} \to B$, within $Sh(B)$, then the product

$$\begin{tikzcd} U \times U \arrow[r, shift left=2] \arrow[r] & U \arrow[l, shift left=2] \end{tikzcd}$$

makes U into a groupoid / equivalence relation within $\mathcal{E} = Sh(B)$. The simplicial object defined by multiple pullbacks is just the nerve of this groupoid, which will be denoted $N(U)$, or more often

$N(\mathcal{U})$. In low dimensions, this looks like

$$N(U): \quad \dots \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; U \times \cdots \times U \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; \cdots \underset{d_2}{\overset{d_0}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}}} U \times U \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \xrightarrow{p} 1.$$

(In the case when B is a manifold and $\mathcal{U}$ is an open covering by contractible open sets such that all the finite intersections of sets from $\mathcal{U}$ are also contractible (sometimes called a 'Leray cover', cf. [83]), the groupoid above is called a 'Leray groupoid', see the same cited paper.)

(In terms of étale spaces over B, you just replace $\times$ by $\times_B$ and 1 by B.) In cases where B is not a 'locally nice space', the simplicial sheaf given by $\mathcal{U}$ is too far away from being an internal Kan complex and so we have to replace the nerve of a cover by a 'hypercovering', which is a 'Kan' simplicial sheaf, K, with an 'augmentation map' $K \to 1$, which is a 'weak homotopy equivalence'. (Look up papers on hypercoverings for a much more accurate treatment of them than we have given here.) Of course, this is very like the situation in group cohomology, where one starts with a 'resolution' of G. This is a resolution of B or better of 1 by a simplicial object.

It will be useful later on to give a 'down-to-earth' description of the various levels of $N(\mathcal{U})$. The zeroth level $N(\mathcal{U})_0$ is just the sheaf $\mathcal{U} = \sqcup\{U : U \in \mathcal{U}\}$, or rather the local sections of this over B. A point in this étale space can be represented by a pair (b, U) where $b \in U$, i.e., the point b of B indexed by U. The projection to B, of course, sends (b, U) to b. This notation is one way of labelling points in a disjoint union, namely the point and an index labelling in which of the sets of the collection is it being considered to be for that part of the disjoint union. Now a point of the pullback over B will be a pair of such points with the same b, so is easily represented as (b, U_0, U_1) where (b, U_0) and (b, U_1) are both points in the above sense. This however implies that $b \in U_0 \cap U_1$, and here, and in higher levels, this idea works: a point in the multiple pullback occurring at level n is of the form $(b, U_0, \dots, U_n)$, where $b \in \bigcap_{i=0}^{n} U_i$.

6.2.7 Base change at the sheaf level

Changing the base induces a pair of adjoint functors.

It is often necessary to examine what happens when we 'change the base space' for our sheaves. Suppose X is a space and $Sh(X)$ the corresponding category of sheaves on X. We might have a subspace A of X, and ask for the relationship between $Sh(X)$ and $Sh(A)$, for instance: Is there an induced functor? In which direction? If so, when does it have nice properties? and so on. More generally, if $f : X \to Y$ is a continuous map, then we expect to have some 'induced functors' between $Sh(X)$ and $Sh(Y)$.

First take a look at presheaves, and so naturally we need to look at the behaviour of f on open sets. The partially ordered sets $Open(X)$ and $Open(Y)$ can be thought of as categories as we already have done, and since continuity of f is just : if V is open in Y, then $f^{-1}(V)$ is open in X, f corresponds to a functor

$$f^{-1} : Open(Y) \to Open(X).$$

(You should **check functoriality**. It is routine.)

As a presheaf F on X is just a functor $F : Open(X)^{op} \to Sets$, we can precompose with $(f^{-1})^{op}$ to get a presheaf on Y, i.e., we have a presheaf, $f_*(F)$. This is then given by $f_*(F)(V) = F(f^{-1}(V))$. If $\mathcal{V} = \{V_i\}$ is an open cover of V, then $f^{-1}(\mathcal{V}) = \{f^{-1}(V_i)\}$ is an open cover of $f^{-1}(V)$, so it is easy to check that, if F is a sheaf on X, $f_*(F)$ is a sheaf on Y. (An interesting exercise is to consider the inclusion, f, of a subspace, A, into Y and a sheaf F on A. What is the value of $f_*(F)(V)$ if

$A \cap V = \emptyset$ and why?) The sheaf $f_*(F)$ is often called the *direct image* of F under f, but this is not always a good name as it is not really an 'image'.

The construction gives a functor

$$f_* : Sh(X) \to Sh(Y),$$

and, clearly, if $g : Y \to Z$ as well, then $(gf)_* = g_* f_*$, whilst $(Id_X)_* = Id_{Sh(X)}$. (Note we are saying that f_* is a functor, but also that writing $Sh(f)$ for f_* would give us a 'sheaf category functor'. That is more or less true, but things are, in fact, richer and more complex than just this.) The richness of the situation is that f also induces a functor going in the other direction, that is from $Sh(Y)$ to $Sh(X)$. This is easier to see if we change our view of sheaves back from special presheaves to étale spaces over the base.

Suppose we have a space over Y, $p : A \to Y$, then we can form the pullback $X \times_Y A$. This is, in fact, 'only specified 'up to isomorphism' as it is defined by a universal property. (You should check up on this point if you are unsure, although we will discuss it in some more detail as we go along.) There is a 'usual construction' of it namely as a subspace of the product $X \times A$:

$$X \times_Y A = \{(x, a) \mid f(x) = p(a)\},$$

but this is not 'the' pullback, just a choice of representing object within the class of isomorphic objects satisfying the specifying universal pullback property - and we also need the structural maps $p_X : X \times_Y A \to X$ and $X \times_Y A \to A$ in order to complete the picture. Of course, for instance, $p_X(x, a) = x$. There is no canonical choice of pullback possible and the resulting coherence situation is the source of much of the higher dimensional structure that we will be meeting later.

We will find it useful to use the universal property more or less explicitly, so it may be good to recall it here:

We have a square

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p_X"'] & A \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

such that (i) it commutes: $pf' = fp_X$, and (ii) given any object B and maps $q : B \to A$ such that $pg = qf$, then there is a *unique* morphism $\alpha : B \to P$ such that $p_X\alpha = q$ and $f'\alpha = g$.

We repeat that this property determines P, p_X and f' up to isomorphism only. Our construction of P as $X \times_Y A$ for the situation in the category of spaces shows that such a P exists, but does not impose any odour of 'canonisation' on the object constructed.

We next look at local sections of (P, p_X). We have $s : U \to P$ such that $p_X s(x) = x$ for all $x \in U$. This means that s determines, and is determined by, a map from U to A, namely $f's$, such that $f(x) = pf's(x)$ for all $x \in U$. This looks a bit like a local section of $A \xrightarrow{p} Y$ over $f(U)$, but we do not know if $f(U)$ is open in Y. To make things work, we can take $f^*(F)(U) = colim\{F(V) : V \text{ open in } Y, f(U) \subseteq V\}$, so we have the elements of $f^*(F)(U)$ are germs of local sections of F, whose domain contains $f(U)$. (You should check this works in giving us a sheaf on X, and that it is functorial, giving us a functor

$$f^* : Sh(Y) \to Sh(X).$$

See why it works yourself, but looks up the details in a sheaf theory textbook.) Of course, warned by previous comments, you will want to check that if $g : Y \to Z$, $(gf)^*$ and f^*g^* will be naturally isomorphic, (but usually not 'equal'). This will be very important later on.

If $F \in Sh(X)$, the sheaf we have just constructed is variously called the *pullback of F along f*, the *inverse image sheaf* or if f is the inclusion of a subspace into Y, the *restriction of F to X*. This construction is also said to lead to *induced sheaves* or sometimes *co-induced sheaves* depending on the style of terminology being used.

Now suppose $f : X \to Y$ and so we have

$$f_* : Sh(X) \to Sh(Y),$$

and

$$f^* : Sh(Y) \to Sh(X).$$

These functors must be related somehow! In fact if $F \in Sh(Y)$ and $G \in Sh(X)$, then

$$Sh(X)(f^*(F), G) \cong Sh(Y)(F, f_*(G)).$$

We sketch a bit of this, leaving the details to be looked for. Suppose $\varphi : F \to f_*(G)$ in $Sh(Y)$, then for an open set V in Y, we have

$$\varphi_V : F(V) \to G(f^{-1}(V)).$$

Now suppose U is open in X and $V \supseteq f(U)$, then $f^{-1}(V) \supseteq U$, so we have

$$F(V) \xrightarrow{\varphi} G(f^{-1}(V)) \to G(U),$$

and passing to the colimit we get a map from $f^*(F)(U)$ to $G(U)$. The other way around is similar, so is left for you to worry out for yourselves.

Of course, the above natural isomorphism says f^* is left adjoint to f_*, and this implies a lot of nice properties that are often used.

This makes for quite a lot of 'facts' about sheaves and their uses, but we need one more observation before passing to other things. Often geometric information is encoded by a sheaf, sometimes 'of rings', sometimes 'of modules' or 'of chain complexes'. For instance, on a differential manifold, one has a sheaf of differential functions and also the de Rham complex which is a sheaf of differential graded algebras. In algebraic geometry, the usual basic object is a scheme, which is a space together with a sheaf of commutative rings on it that is 'locally' like the prime spectrum of a commutative ring. There are many other examples. We will also be looking at sheaves of groups and sheaves of crossed modules.

It would have been nice to show how a sheaf theoretic viewpoint provides the link between covering space theory and Galois theory, but again this would take us too far afield so we refer to Borceux and Janelidze, [16], and the references therein.

6.3 Descent: Torsors

(Some of the best sources for the material in this section are in the various notes and papers of Breen, [17, 18] and, in particular, his Astérisque monograph, [19] and his Minneapolis notes, [20].)

The demands of algebraic geometry mean that principal G-bundles for G a (topological) group are not sufficient to handle all that one would like to do with such things. One generalisation is to vary G over a base. This may be to replace G by a sheaf of groups or by a group object in Top/B, i.e., a group bundle. (This is the topological analogue of a group scheme.) The situation that we considered earlier then corresponds to a constant sheaf of groups or the group bundle $G_B := (B \times G \to B)$ given by projection from the product. It also includes the vector bundles that we briefly saw earlier. The more general case, however, does not change things much. We have a parametrised family of groups G_b, $b \in B$, acting on a parametrised family of spaces, X_b, $b \in B$. The sheaf of groups viewpoint corresponds to an étale space on B and thus to a group bundle on B with each G_b discrete as a topological group. We will let, in the following, G be a bundle of groups on a space B. (We will on occasion abuse notation and write G instead of G_B for the 'constant G' example.)

Technically we will need to be working in a setting where we can talk of a bundle of locally defined maps from one bundle to another. This is fine in the sheaf theoretic setting, and will be assumed to be the case in the general case of a suitable category of bundles within the ambient category, Top/B. It corresponds to the functor $- \times A$ always having a right adjoint $(-)^A$, the function bundle of locally defined maps from A to whatever. Technically we are assuming that our category of bundles on B, Bun/B is a Cartesian closed category.

6.3.1 Torsors: definition and elementary properties

Definition: A left G-*torsor* on B is a space $P \xrightarrow{\pi} B$ over B together with a left group action

$$G \times_B P \to P$$
$$(g, p) \longmapsto g.p$$

such that the induced morphism

$$\phi : G \times_B P \to P \times_B P$$
$$(g, p) \longmapsto (g.p, p)$$

is an isomorphism. In addition we require that there exists a family of local sections, $s_i : U_i \to P$, for some open cover $\mathcal{U} = (U_i)_{i \in I}$ of B.

A right G-torsor is defined similarly with a right G-action. If P is a left G-torsor, there is an associated right G-torsor, P^o, with action $p.g = g^{-1}.p$.

When we refer to a G-torsor, without mentioning 'left' or 'right', we will mean a left G-torsor.

The connection with our earlier definition of principal G-bundle can be made more evident if we note that, on writing $\theta = \phi^{-1} : P \times_B P \to G \times_B P$, then the analogue of the *translation function* of page 137, is the *translation morphism*, $\tau : P \times_B P \to G$, given by $pr_1 \circ \theta$. The morphism θ then equals (τ, pr_2).

The effect of the requirement that local sections exist is to ensure that the bundle $P \xrightarrow{\pi} B$ is locally trivial, i.e., locally like $G \to B$. This is a consequence of the following lemma.

Lemma 16 *Suppose $P \xrightarrow{\pi} B$ is a G-torsor for which there is a global section*

$$s : B \to P$$

of π, then there is an isomorphism

$$G \xrightarrow{f} P$$

of spaces over B.

Proof: Define a function $f : G \to P$ by $f(g) = (g.s(b))$, where $g \in G_b$. As the projection of the group bundle G is continuous, f is continuous. To get an inverse for f, consider the map

$$P \xrightarrow{\pi} B \xrightarrow{s} P.$$

For any $p \in P$, $s\pi(p)$ is in the same fibre as p itself, so we get a continuous map

$$P \xrightarrow{(id,s\pi)} P \times_B P \xrightarrow{\cong} G \times_B P$$

on composing with the inverse of the torsor's structural isomorphism. Finally projecting on to G gives a map $h : P \to G$. This is continuous and checking what it does on fibres shows it to be the required inverse for f. ■

This does not, of course, transfer a group structure to P, but says that P is like G with 'an identity crisis'. It no longer knows what its identity is!

The group bundle, $G \to B$, considered as a space over B is naturally a G-torsor with multiplication on the left giving the G-action. Check the conditions. It has a global section, since we required it to be a group object in Top/B, so there is a continuous map, e, over B from the terminal object of Top/B to G, which plays the role of the identity. As that terminal object is (isomorphic to) the identity on B, $B \to B$, this splits $G \to B$,

$$\begin{tikzcd} B \arrow[rr, "e"] \arrow[dr, "="'] & & G \arrow[dl] \\ & B & \end{tikzcd}$$

This trivial G-torsor will be denoted T_G.

Applying this to a general G-torsor, the local section $s_i : U_i \to P$ makes $P_{U_i} = \pi^{-1}(U_i)$, the restricted torsor over the open set U_i, into the trivial G_{U_i}-torsor over U_i, so P is *locally trivial.* It is important to note again that this means that P looks locally like G, (but if G is not a product bundle, P will not be locally a product). The way that P differs globally from G is measured by cohomology. (An important visual example is, once again, the boundary circle of the Möbius band, i.e., the double cover of the circle, S^1, that twists as you go around that base circle. It is locally a product $U \times \{-1, 1\}$, but not globally so.)

The next observation is very important for us as it shows how the language of G-torsors starts to interact with that of groupoids. First an obvious definition.

Definition: If P and Q are two left G-torsors, then a *morphism* $f : P \to Q$ *of* G*-torsors* (over B) is a continuous map over B such that $f(g.p) = g.f(p)$ for all $g \in G$, $p \in P$.

Here and elsewhere, it is to be understood that we only write $g.p$ if $g \in G_b$ and $p \in P_b$ for the same b. This avoids our constantly repeating mention of the base space and its points. If working with sheaves on a site, i.e., a category $\mathcal{C}$, with a Grothendieck topology, the g and p correspond to locally defined 'elements' in some $G(C)$ and $P(C)$ respectively, so the same (abusive) notation suffices.

Lemma 17 *Any morphism* $f : P \to Q$ *is an isomorphism.*

Proof: We have trivialising covers, $\mathcal{U}$ for P, and $\mathcal{V}$ for Q, on which local sections are known to exist. By taking intersections, or any other way, we can get a mutual refinement on which both P and Q trivialise so we can assume $\mathcal{U} = \mathcal{V}$. We thus are looking at a morphism f and local sections $s : U \to P$, $t : U \to Q$, which (locally) determine isomorphisms to T_G over U. We thus have reduced the problem, at least initially, to showing that $f : T_G \to T_G$ is always an isomorphism, but

$$f(1_G) = g.1_G$$

for some $g \in G_B$, i.e., for some global element of G. Moreover g is uniquely determined by f. Now it is clear that the morphism sending 1_G to $g^{-1}.1_G$ is inverse to f. (Although it is probably an obvious comment, we should point out that saying where a single global element goes determines the morphism, and, within T_G, any (locally defined) element is given by multiplication of the global section 1_G by that element, but now regarded as an element of G itself.)

Back to our original $f : P \to Q$, on each U, we have $f_U : P_U \to Q_U$, its restriction to the parts of P and Q over U, is an isomorphism, so we construct the inverse locally and then glue it into a single f^{-1}.

Remark on descent of morphisms: Although we have not yet completed the proof, it is instructive to go into this in a bit more detail, since it introduces methods and intuitions that here should be more or less clear, but later, in more 'lax' or 'categorified' settings will need both good intuition and the ability to argue in detail with (generalisations of) local sections.

If we use s and t, then with respect to these local sections over U, every local element of P_U has the form $g_U.s_U$ for some unique locally defined $g_U : U \to G$ (or in sheaf theoretic notation $g_U \in G(U)$). Similarly in Q_U, local elements looks like $g_U.t_U$, but then

$$f(g_U.s_U) = g_U.f(s_U),$$

so we only need to look at $f(s_U)$. As $f(s_U) \in Q_U$, it determines some unique local element $h_U \in G(U)$ with

$$f(s_U) = h_U.t_U,$$

and checking for behaviour when composing morphisms, it is then clear that

$$f_U^{-1}(t_U) = h_U^{-1}.s_U$$

with continuity of f^{-1} handled by the continuity of inversion, that of t and of multiplication.

As the construction of f_U^{-1} is done using maps defined locally over U, f_U^{-1} is in Top/U (or alternatively, is a map of sheaves on U). We now have to check that this locally defined morphism 'descends' from $\bigsqcup \mathcal{U}$ to B.

Of course, it is 'clear' that it must do so! Each h_U is uniquely defined so ... . That *is* true, but when we go to higher dimensional situations we will often not have uniqueness, merely uniqueness up to isomorphism, or equivalence, so we will spell things out in all the 'gory detail'.

We need to check what happens on intersection $U_1 \cap U_2$ of local patches in our trivialising cover, $\mathcal{U}$. Write $f_i = f_{U_i}$, $i = 1, 2$, etc. for simplicity. The local sections s_1 and s_2 (resp. t_1 and t_2) will not, in general, agree on $U_1 \cap U_2$, so we have

$$f_1(s_1) = h_1.t_1,$$

$$f_2(s_2) = h_2.t_2,$$

but the key local elements $h_1|_{U_1\cap U_2}$ and $h_2|_{U_1\cap U_2}$ need not agree. A bit more notation will probably help. Let us denote by s_{12} the restriction of $s_1 : U_1 \to P$ to the intersection $U_1 \cap U_2$ and similarly $s_{21} = s_2|_{U_1\cap U_2}$, extending this convention to other maps when needed.

We then have some $g_{12} \in G_{U_1\cap U_2}$ for which

$$s_{21} = g_{12}.s_{12}, \text{ (and } s_{12} = g_{21}.s_{21}, \text{ so } g_{12} = g_{21}^{-1}),$$

but then, over $U_1 \cap U_2$,

$$f(s_{21}) = g_{12}.f(s_{12}).$$

We thus have

$$t_{21} = h_{21}^{-1} g_{12} h_{12} t_{12}.$$

Now turning to f^{-1}, defined locally by $f_i^{-1} : Q_{U_i} \to P_{U_i}$, $i = 1, 2$ with

$$f_i^{-1}(t_i) = h_i^{-1}.s_i,$$

then over $U_1 \cap U_2$, $f_{ij}^{-1}(t_{ij}) = h_{ij}^{-1} s_{ij}$, but we also have $f_j^{-1}(t_{ji}) = h_{ji}^{-1} s_{ji}$ and we have to check that on $Q_{U_i\cap U_j}$, $f_{ij}^{-1} = f_{ji}^{-1}$. To do this, it is sufficient to calculate $f_{ji}^{-1}(t_{ij})$ and to compare it with $f_{ij}^{-1}(t_{ij})$ as both are defined on the same generating local section and so extend via their G-equivariant nature. We have

$$\begin{aligned} f_{ji}^{-1}(t_{ij}) &= f_{ji}^{-1}(h_{ij}^{-1} g_{ji} h_{ji} t_{ji}) \\ &= h_{ij}^{-1} g_{ji} h_{ji} f_{ji}^{-1}(t_{ji}) \\ &= h_{ij}^{-1} g_{ji} h_{ji} h_{ji}^{-1}.s_{ji} \\ &= h_{ij}^{-1} g_{ji} g_{ij} s_{ij} \\ &= h_{ij}^{-1} s_{ij} \\ &= f_{ij}^{-1}(t_{ij}), \end{aligned}$$

so the two restrictions *do* agree over the intersection and hence *do* give a morphisms from Q to P inverse to f. (This last point is easy to check.) ■

If we denote the category of left G-torsors on B by $Tors(B, G)$ (or $Tors(G)$ if B is understood), then we have

Proposition 37 *$Tors(B, G)$ is a groupoid.* ■

6.3.2 Torsors and Cohomology

In the above discussion, we saw how a choice of local sections $s_i : U_i \to P$ gave rise to a map $g_{ij} : U_{ij} \to G$. (Here we will again abbreviate: $U_i \cap U_j = U_{ij}$. This notation will be extended to give $U_{ijk} = U_i \cap U_j \cap U_k$, etc.)

The maps g_{ij} are to satisfy

$$s_i = g_{ij} s_j$$

on U_{ij} and for all indices i, j. The map g_{ij} gives the translation from the description using s_i to that using s_j. Of course, as g_{ij} is invertible, it can also translate back again. These elements are uniquely determined by the sections, so over a triple intersection, U_{ijk}, we have the 1-cocycle equation,

$$g_{ij}g_{jk} = g_{ik}.$$

If we use different local sections, say s'_i, assumed to be on the same open cover, there will be local elements, $g_i : U_i \to G$, such that $s'_i = g_i.s_i$ for all $i \in I$. The corresponding cocycles g_{ij} and g'_{ij} will be related by a coboundary relation

$$g'_{ij} = g_i g_{ij} g_j^{-1}.$$

These equations will determine an equivalence relation on the set, $Z^1(\mathcal{U}, G)$, of 1-cocyles for $\mathcal{U}$, as before, the (fixed) open cover. The set of equivalence classes will be denoted $H^1(\mathcal{U}, G)$. To remove the dependence on the open cover, one passes to the limit on finer covers to get the Čech non-Abelian cohomology set, $\check{H}^1(B, G) = colim_{\mathcal{U}} H^1(\mathcal{U}, G)$ which, by its construction classifies isomorphism classes of G-torsors on B. The trivial left G-torsor, T_G, gives a natural distinguished element to $\check{H}^1(B, G)$.

This looks quite good. We have started with a torsor and seem to have classified it, up to isomorphism, by cocycles. The one deficiency is that we need to know that cocycles give torsors, i.e., a (re)construction process of P from the cocycle (g_{ij}), but without prior knowledge of P itself.

The method we will use will take the basic ingredients of the group bundle, G, and will twist them using the g_{ij}. First if we have $\gamma \in \check{H}^1(B, G)$, by the basic construction of colimits, we can pick an open cover $\mathcal{U}$ and a $g_{\mathcal{U}} = (g_{ij})$, whose cohomology class represents γ in the colimit. Next taking this $\mathcal{U} = \{U_i\}$, and g_{ij}, let

$$P = \bigsqcup_i G(U_i)/\sim .$$

As we are once again using a disjoint union, we will give our points an index, (g, i), and, of course,

$$(g, i) \sim (gg_{ij}, j).$$

We have a projection $P \to B$ induced from the bundle projections $G(U) \to B$. (For you to check that it works.) This is continuous if P is given the quotient topology. Moreover the multiplications

$$G(U) \times G(U) \to G(U)$$

give a left action

$$G \times P \to P$$

making P into a left G-torsor as hoped for.

To sum up:

Theorem 9 *The set, $\check{H}^1(B, G)$, is in one-one correspondence with the set of isomorphism classes of G-torsors on B, that is, with the set $\pi_0 Tors(B; G)$ of connected components of the groupoid, $Tors(B; G)$.* ■

The relationship for isomorphisms is **left for you to check**.

6.3.3 Change of base

This link with cohomology suggests that we should see what might happen if we changed the base space B in the above. As cohomology is about maps *out of* the space, we should expect that if $f : B \to B'$ is a continuous map then we would get an induced map going from $\check{H}^1(B', G)$ to $\check{H}^1(B, f^*(G))$, but what would this look like through the G-torsors perspective? Suppose we have a G-torsor, Q, over B', then Q is a sheaf on B', so we have an induced sheaf $f^*(Q)$ on B given by pullback, as above, page 150. Strictly speaking as G is a sheaf or bundle of groups on B', $f^*(Q)$ cannot be a G-torsor, but might be a $f^*(G)$-torsor.

We have checked some of what has to be examined before, in the simpler case of principal G-bundles. We will repeat some of the results, but with slightly more categorical proofs as the very element based approach we used is fine for that topological setting, but is here beginning to be less optimal with a sheaf of groups as coefficients. (We will not, however, go to a elegant, fully categorical proof as we have not treated geometric morphisms of toposes.)

First we need an action of $f^*(G)$ on $f^*(Q)$. We have the action of G on Q. There is a quick derivation of this which we will sketch. The functor f^* is a left adjoint and so preserves colimits ..., which is useless to us in this situation! It is also a right adjoint of another functor which we have not discussed. It therefore preserves products and thus actions. A way to see that $f^*(G \times_{B'} Q) \cong f^*(G) \times_B f^*(Q)$, without producing the left adjoint of f^* is via the étale space description of sheaves. In that description, $f^*(G)$, etc., are all given by pullbacks. We draw a diagram:

$$\begin{tikzcd}
 & f^*(G \times_{B'} Q) \arrow[rr] \arrow[dr] \arrow[ddl] & & G \times_{B'} Q \arrow[dr] \arrow[dd] & \\
 & & f^*(Q) \arrow[rr] \arrow[ddl] & & Q \arrow[dd] \\
f^*(G) \arrow[rrr] \arrow[dr] & & & G \arrow[dr] & \\
 & B \arrow[rrr] & & & B'
\end{tikzcd}$$

Each face of the resulting cube is a pullback, as is the vertical square given by the diagonals of the two ends plus the top and bottom maps, but the same would be true of the equivalent diagram with $f^*(G \times_{B'} Q)$ replaced by $f^*(G) \times_B f^*(Q)$, so these two objects are isomorphic.

If we now look at what happens to the action then the original action of G on Q induces one of $f^*(G)$ on $f^*(Q)$ as hoped for. (The detailed verification is left to you as usual.) As the first condition of the definition of torsor again involves pullbacks, it is now fairly routine to check it for $f^*(Q)$. The other condition is the existence of local sections and we have to use a slightly different approach for this. We know that there is an open cover $\mathcal{U}$ of B' over which local sections exist, say, $s_i : U_i \to Q$, $U_i \in \mathcal{U}$. The obvious open cover for B is $f^{-1}(\mathcal{U})$, so we look for sections $f^{-1}(U_i) \to f^*(Q)$. As $f^*(Q)$ is given by a pullback, we will get such a map if we specify maps $f^{-1}(U_i) \to Q$ and $f^{-1}(U_i) \to B$ making the obvious square commute. The map $f^{-1}(U_i) \to B$ 'must' be the inclusion ... what else could it be, so we will try that. Composing that with f gives a map $f^{-1}(U_i) \to B'$, which can also be written as the composite of f restricted to $f^{-1}(U_i)$ followed by the inclusion of U_i into B', so we can compose that restriction of f with s_i to get a map to Q. Since s_i is a section over U_i of the map $Q \to B'$, it is now easy to check that the 'obvious square' commutes. (**Left to you**.) We have built a local section over $f^{-1}(U_i)$. We thus have

Proposition 38 *If Q is a G-torsor over B', then $f^*(Q)$ is a $f^*(G)$-torsor over B.* ■

The new torsor $f^*(Q)$ would here loosely be called the *induced torsor of Q along f.*

We have a cocycle description of torsors. If we have one for Q, what will be the one for $f^*(Q)$? In a sense, we know what the answer is without doing any calculation. The cocycle description of Q gives a class in $H^2(B', G)$ and the induced map from that to $H^2(B, f^*(G))$ must surely be given by composition with f. The fact that the coefficients change as well as the space should come out 'in the wash'. We would, from this perspective, also expect the maps induced from homotopic maps to be the same. We know what to expect but what about the details!

Suppose we pick local sections s_i for Q over the various U_i in a cover $\mathcal{U}$ of B', and we get the $g_{ij} \in G(U_{ij})$ as above. These satisfy

$$s_i = g_{ij}s_j.$$

We have just seen that suitable local sections over the $f^{-1}(U_i)$ are given by the pairs of maps $(s_if, inc) : f^{-1}U_i \to Q \times_{B'} B$, but these are determined just be the first component. Likewise the sections g_{ij} over pairwise intersections of G, correspond by composition to the corresponding elements $g_{ij}f$ over the pairwise intersections of $f^{-1}(\mathcal{U})$, and, of course, these are the transition cocycles for the s_if. That they are cocycles follows since the g_{ij} satisfy the cocycle condition.

To summarise: the cocycle data for $f^*(Q)$ can be derived from that for Q merely by precomposing by the relevant restrictions of f to the sets of the cover $f^{-1}(\mathcal{U})$ and their intersections. Just as we expected.

Having seen that homotopic maps induced isomorphic principal bundles in an earlier section, it is natural to expect the same thing to happen here. It does, but rather than explore that here we will put it aside for a little while until we have a simplicial description of torsors in sections 6.3.5 and 6.4.5. That will make life a lot easier.

We have changed the base, what about changing the 'coefficients'?

6.3.4 Contracted Product and 'Change of Groups'

In Abelian cohomology, one would expect the cohomology 'set' (there a group) to vary nicely with the coefficient sheaf of groups, G. Something like that occurs here as well and determines some essential structure on the torsors. Suppose $\varphi : G \to H$ is a homomorphism of sheaves of groups, then one expects there to be induced functors between $Tors(G)$ and $Tors(H)$ in one direction or the other. Thinking of the better known case of a ring homomorphism, $\varphi : R \to S$, and modules over R or S, then we could, for an S-module, M, form an R-module by restriction along φ. The analogue works for an H-set X as one gets a G-set by defining $g.x = \varphi(g).x$, but there is no reason to expect the resulting G-set to be principal, so this does not look so feasible for torsors. There is, however, another module construction. Suppose that N is a left R-module, and make S into a *right* R-module, S_R by $s.r = s\varphi(r)$, then we can form $S_R \otimes_R N$, and the left S-action by multiplication is nicely behaved. The point is that S is behaving here as a two sided module over itself, and also as a (S, R)-bimodule. The corresponding idea in torsor theory is that of a bitorsor, explored in depth by Breen in [17], which we will examine later in this chapter.

Before looking at this in a bit more detail, we will look at the contracted product, which replaces the tensor product here. Suppose we have a category, $\mathcal{C}$, and an internal group, G, in $\mathcal{C}$. Here we have various examples in mind. If $\mathcal{C} = Sh(B)$, G will be a sheaf of groups; if $\mathcal{C}$ is the category of groupoids, G will be an internal group in that category, i.e., a *(strict) gr-groupoid*, and will correspond to a crossed module, and, if we combine the two ideas, $\mathcal{C}$ is a category of sheaves of groupoids, so G is a sheaf of gr-groupoids, corresponding to a sheaf of crossed modules, and so on in various variants.

A left G-object in $\mathcal{C}$ is an object X together with a morphism, (left action),

$$\lambda : G \times X \to X,$$

satisfying obvious rules. Similarly a right G-object Y comes with a morphism, (right action),

$$\rho : Y \times G \to Y.$$

The *contracted product* of Y and X is, intuitively, formed from $Y \times X$ by dividing by an equivalence relation

$$(y.g, g^{-1}.x) \equiv (y, x).$$

The usual notation is $Y \wedge^G X$, *but* this is often inadequate as it assumes X, (resp. Y), stands for the object *and* the G-object, unambiguously, whilst, of course, X really stands for (X, λ) and Y for (Y, ρ). It is sometimes useful, therefore, to add the action into the notation, but only when confusion would occur otherwise, so $Y_\rho \wedge^G {}_\lambda X$ is the full notation, but variants such as $Y_\rho \wedge^G X$ would be used if it was clear what λ was, etc.

We gave an element based description of $Y \wedge^G X$, but how can we adapt this to work within our general $\mathcal{C}$? There are obvious maps

$$Y \times G \times X \underset{(Y,\lambda)}{\overset{(\rho,X)}{\rightrightarrows}} Y \times X \ ,$$

and we can form their coequaliser. (As usual, we assume that the category $\mathcal{C}$ has all limits and colimits that we need to make constructions, and to enable definitions to make sense, but we do not constantly remind the reader of these hidden conditions!) Of course, we met this construction earlier when considering a left principal G-bundle and a right G-space (fibre), F, forming the fibre bundle $X_F = F \wedge^G X$; it was also at the heart of the regular twisted Cartesian product construction from our discussion of simplicial twisting maps.

Example: Suppose $\varphi : G \to H$ is a morphism of group bundles on B, then we can give H a right G-action by

$$H \times_B G \xrightarrow{H \times \varphi} H \times_B H \to H$$

where the second map is multiplication. If P is a G-object such as a G-torsor, we have a contracted product $H_\varphi \wedge^G P$.

Lemma 18 *If P is a G-torsor, then $H_\varphi \wedge^G P$ is an H-torsor.*

Proof: Writing $Q = H_\varphi \wedge^G P$, we check the usual map,

$$H \times_B Q \to Q \times_B Q,$$

is an isomorphism. This is merely checking that the 'obvious' fibrewise formula is well defined. This sends a pair $([h,p],[h_1,p])$ to $(hh_1^{-1},[h_1,p])$. That verification is left to the reader.

Local sections of P immediately yield local sections of Q, so Q *is* an H-torsor. ■

A group homomorphism

$$\varphi : G \to H$$

thereby gives us a functor

$$\varphi_* : Tors(G) \to Tors(H) \qquad\qquad \varphi_*(P) = H_\varphi \wedge^G P.$$

Of course, there are still some details (**for you**) to check, namely relating to behaviour on morphisms of G-torsors. (These are probably 'clear', but **do need checking**.)

Another point from this calculation is that we could work with 'elements' as if in a G-set. This can be thought of either as working, carefully, in each fibre of the torsor or using local sections or as a heuristic to obtain a formula that is then encoded purely in terms of the structural maps. All of these viewpoints are valid and all are useful.

Now suppose $\mu, \nu : G \to H$ are two group homomorphisms, thus giving us two functors,

$$\mu_*, \nu_* : Tors(G) \to Tors(H).$$

When is there a natural transformation $\eta : \mu_* \to \nu_*$? The answer is neat and very useful.

Lemma 19 *(cf. Breen, [19], Lemma 1.5)*
A natural transformation $\eta : \mu_* \to \nu_*$ is determined by a choice of a section h of H such that

$$\nu = h^{-1}\mu h.$$

Proof: Suppose that P is a G-torsor, then $\mu_*(P) = H_\mu \wedge^G P$, similarly for $\nu_*(P)$ and $\eta_P : H_\mu \wedge^G P \to H_\nu \wedge^G P$.

If we look locally

$$\eta_P([\mu(g), p]) = h.[\nu(g), p]$$

for some h, since $\eta_P(\mu(g), p)$ is of form $[h_1, p]$ for some h_1 and as $\nu_*(P)$ is an H-torsors, etc.

(Unfortunately we need to know h does not depend on g, and is defined globally, so this suggests looking at the special case where global sections *do* exist, i.e., $P = T_G$, the trivial G-torsor. There we can assume $g = 1_G$, so

$$\eta_{T_G}([1_G, p]) = h.[1_H, p],$$

giving us a possible h. We know that η_P is H-equivariant and natural as well as being 'well-defined'. We use these properties as follows:

If $g \in G$,

$$\begin{aligned} \eta_{T_G}[\mu(g), p] &= \eta_{T_G}[1_H, g.p] \\ &= h[1_H, g.p] \\ &= h[\nu(g), p] \\ &= h.\nu(g)[1_H, p], \end{aligned}$$

whilst also

$$\begin{aligned} \eta_{T_G}[\mu(g), p] &= \eta_{T_G}(\mu(g).[1_H, p]) \\ &= \mu(g)\eta_{T_G}[1_H, p] \\ &= \mu(g)h[1_H, p], \end{aligned}$$

using that η_{T_G} is H-equivariant. We thus have a globally defined h with

$$\mu(g)h = h\nu(g)$$

for all $g \in G$,

$$\text{or} \quad \mu = i_h \circ \nu \quad \text{or} \quad \nu = i'_h \circ \mu,$$

where i_h is inner automorphism by h and i'_h, that by h^{-1}.

Conversely given such an h, we can *define* η by our earlier formula, extending it by H-equivariance and naturality. Checking well definition is quite easy, but instructive, and so is left to you. ■

Aside: For any groupoids G, H, the functor category H^G has groupoid morphisms as its objects and the natural transformations can be seen to be 'conjugations'. (This is a **useful calculation to do, if you have not seen it before**.) In particular if $G = H$ is a group, the full subcategory $aut(G)$ of G^G given by the automorphisms of G is an internal group object in the category of groupoids, so corresponds to a crossed module. What crossed module? What else, $\mathsf{Aut}(G)$, that is,

$$i : G \to Aut(G).$$

Two automorphisms μ, ν are related by a natural transformation if and only if there is a g such the $\mu = i_g \circ \nu$, where i_g is inner automorphism by g. The similarity with our current setting is *not* coincidental and can be exploited!

Another fairly obvious result is that, if P is a G-torsor, then

$$G \wedge^G P \cong P,$$

since locally we have each representative (g, p) is equivalent to $(1_G, g.p)$. The details are **left as an almost trivial exercise**.

This notation is 'dangerous' however, as we pointed out earlier. We are using the right multiplication of G on itself to give us the contracted product, but we could also make G act on itself by conjugation *on the right*: for $g \in G$, $x \in G$, with G being considered as a bundle,

$$x.g = g^{-1}xg.$$

We will write this action as i', for 'inner', so have $G_{i'} \wedge^G P$ as well. This is, in fact, a very useful object. It is related to automorphisms of P in the following way:

Suppose that $\alpha : P \to P$ is a locally defined automorphism of G-torsors, i.e., a local section of $Aut_G(P)$. Continuing to work locally, pick a section (local element) p. As α is 'fibrewise',

$$\alpha(p) = g_p.p$$

for some local elements g_p of G, and as α is G-equivariant,

$$\alpha(g.p) = g\alpha(p) = gg_p.p.$$

Assigning, to each pair (g, p) in $G \times P$. the automorphism given by

$$\alpha(g_1, p) = g_1g.p$$

gives a map

$$\lambda : G \times P \to Aut_G(P), \quad \lambda(g,p)(p) = g.p,$$

and this is an epimorphism by our previous argument. 'Obviously'

$$\lambda(g,p) = \lambda(gg', (g')^{-1}p),$$

so the map λ passes to the quotient $G \wedge^G P$ **-or does it**? We have not actually examined the definition of $\lambda(g,p)$ that closely.

Look at this from another direction. Examine $\lambda(g, g'p)$ as an automorphism of P. To work out $\lambda(g, g'p)(p)$, we have first to convert p:

$$\lambda(g, g'p)(p) = \lambda(g, g'p)((g')^{-1}g'.p),$$

as $\lambda(g, g'p)$ is specified by what it does to its basic P-part. Now

$$\lambda(g, g'p)((g')^{-1}g'.p) = (g')^{-1}\lambda(g, g'p)(g'.p)$$

by G-equivariance, and so equals

$$(g')^{-1}gg'.p,$$

which is $\lambda((g')^{-1}gg', p)(p)$.

Thus our initial impulse was hasty. We do have $Aut_G(P)$ as a contracted product, $G \wedge^G P$, but not with right multiplication as the action of G on itself, rather it uses right conjugation. We have proved

Lemma 20 *For any G-torsor P, there is an isomorphism*

$$\lambda : G_{i'} \wedge^G P \xrightarrow{\cong} Aut_G(P),$$

where $i' : G \to Aut(G)^o$, $i'(g)(g') = g^{-1}g'g$, yielding the right conjugation action of G on itself. ■

Perhaps something more needs to be said about $Aut_G(P)$ here. We are working with sheaves or bundles and so have an essentially Cartesian closed situation, in other words function objects exist. For each pair of sheaves, X, Y on B, $Hom(X,Y)$ is a sheaf. In particular $End(X)$ is a sheaf and $Aut(X)$ a subsheaf of it. It thus makes basic sense to have that $Aut_G(P)$ is a G-torsor. Of course, it is also a group object, since automorphisms (gauge transformations) of P are invertible. This group is sometimes written P^{ad}. It is the group (bundle) of G-equivariant fibre preserving automorphisms of P; it is also called the *gauge group* of P. (The precise origin in the thoughts of Hermann Weyl of the use of 'Gauge' are fun to look up, but they make me think that the term is very much over used in mathematical physics, as Weyl's use seems to have been beautifully simple and down to earth, whilst the mystique of the modern use by comparison may be tending to obscure the simple idea from a simple minded mathematician's viewpoint.)

In the isomorphic $G_{i'} \wedge^G P$ version, it is instructive to explore the group structure, but this is left for you to do. This group operates on the *right* of P, by the rule

$$p.\alpha = \alpha^{-1}(p),$$

and makes P into a right P^{ad}-torsor. (Exploration of these statements is well worth while and is **left as an exercise**. It, of course, presupposes that P^{ad} is seen as a bundle /sheaf of groups, which itself needs 'deconstructing' before you start. The overall intuition should be fairly clear *but* the technicalities, detailed verifications, etc., **do need mastering**.)

A cohomological perspective on change of groups. We have that $\check{H}^1(B,G)$ is the set of isomorphism classes of G-torsors on B, i.e., $\pi_0 Tors(G)$, the set of connected components of the groupoid $Tors(G)$. We have now seen that if $\varphi : G \to H$ is a homomorphism of group bundles and P is a G-torsor, then $H_\varphi \wedge^G P = \varphi_*(P)$ is an H-torsor and that this gives a functor $\varphi_* : G \to H$. This will, of course, induce a function on sets of connected components and hence, as one might expect, an induced function

$$\varphi : \check{H}^1(B,G) \to \check{H}^1(B,H).$$

There is another obvious way of inducing such a function, as the elements of $\check{H}^1(B,G)$ are classes of cocycles (g_{ij}) and so composing with φ sends $[(g_{ij})]$ to $[\varphi(g_{ij})]$. It is standard to check that this does induce a function from $H^1(\mathcal{U},G)$ to $H^1(\mathcal{U},H)$ and, by its independence from $\mathcal{U}$, it is then routine to check that it induces a corresponding map on Čech non-Abelian cohomology.

It is easy to see that these two induced maps are the same. (It would be surprising if they were not!) Pick a set of local sections, $\{s_i\}$ for P over a trivialising cover $\mathcal{U}$ and we get $\{[1,s_i]\}$ is a set of local sections for $H_\varphi \wedge^G P$. Changing patches, $s_i = g_{ij}s_j$, and so

$$[1,s_i] = [1,g_{ij}sj] = [\varphi(g_{ij}).1,s_j] = \varphi(g_{ij})[1,s_j],$$

and the transition functions for $\varphi_*(P)$ are exactly as expected. (The rest of the details are left as an exercise.) The important thing for later use is the identification of the cocycles for $\varphi_*(P)$. This will be especially important when discussing G-bitorsors in the next section.

6.3.5 Simplicial Description of Torsors

As usual we look at a sheaf or bundle of groups, G, on a space, B, and suppose P is a G-torsor. We then know there is an open cover, $\mathcal{U}$, of B and trivialising local sections, $s_i : U_i \to P$, over the various different open sets U_i of $\mathcal{U}$. We have seen that over the intersections U_{ij}, the restrictions of the two local sections s_i and s_j must be related and this gives us transition cocycles $g_{ij} : U_{ij} \to G$ such that

$$s_i = g_{ij}s_j,$$

where, over triple intersections, the 1-cocycle condition

$$g_{ij}g_{jk} = g_{ik}$$

must be satisfied.

The information on intersections in $\mathcal{U}$ is neatly organised in the simplicial sheaf, $N(\mathcal{U})$, (cf. page 148 in section 6.2.6). We also know that from a sheaf of groups we can construct various simplicial sheaves. Is there a way of viewing the cocycles g_{ij} from this simplicial perspective?

From a group, G, (no sheaves for the moment), we earlier saw the uses of models for the classifying space, BG, of G. We could use the nerve of G as a group or rather its nerve as a single object groupoid, $G[1]$. We could alternatively take the constant simplicial group, $K(G,0)$ (so $K(G,0)_n = G$ for all $n \geq 0$, with all face and degeneracies, being the identity isomorphism of G). If we then formed $\overline{W}(K(G,0))$, we get $Ner(G[1])$ back.

These different approaches all yield a simplicial set (and if you really want a space, you just take its geometric realisation). This simplicial set will be denoted BG, even though that notation is often restricted to that corresponding space. We have to be a bit careful about the order of composition in the groupoid, $G[1]$, if it is to be consistent with the construction K, which was the nerve of an internal groupoid in the category of groups. We also have to be careful about our use of *left* actions and the assumption that that makes about the order of composition being 'functional' rather than algebraic (which latter order works best with right actions). That being said we have

- $BG_0 =$ a singleton set, $\{*\}$;
- $BG_1 = G$, as a set, and in general,
- $BG_n = \underbrace{G \times \ldots \times G}_{n}$

Writing $\mathbf{g} = (g_n, \ldots, g_1)$ for an n-simplex of BG, we have

$$\begin{aligned} d_0\mathbf{g} &= (g_n, \ldots, g_2), \\ d_i\mathbf{g} &= (g_n, \ldots, g_{i+1}g_i, \ldots, g_0), \quad 0 < i < n, \\ d_n\mathbf{g} &= (g_{n-1}, \ldots, g_1), \end{aligned}$$

with the degeneracy maps, s_j given by insertion of 1_G in the j^{th} place, shifting later entries one place to the right. (Warning: multiple use of the label s_j here *may* cause some confusion, but each use is the natural one in that context!)

We have already seen this several times (but repetition *is* useful). The key diagram is usually that indicating a 2-simplex, $\mathbf{g} = (g_2, g_1)$, namely

$$\begin{tikzcd} & * \arrow[dr, "g_2"] & \\ * \arrow[ur, "g_1"] \arrow[rr, "g_2g_1"'] & & * \end{tikzcd}$$

Back to G being a sheaf of groups, and we get BG will be a sheaf of simplicial sets. We now have two simplicial sheaves, $N(\mathcal{U})$ and BG. Curiosity alone should suggest that we compare these via a simplicial morphism and for our purposes, it should be a simplicial sheaf map $f : N(\mathcal{U}) \to BG$.

Looking back at $N(\mathcal{U})$ and its construction (page 148), the zero simplices are formed by the open sets and as BG_0 is trivial, f_0 is not much of interest!

At the next level $f_1 : N(\mathcal{U})_1 \to BG_1$ so consists - yes, of course, - of local sections over the intersections U_{ij}, hence g_{ij} in $G(U_{ij})$ or G_{ij}. Over triple intersections U_{ijk}, f_2 will give a 2-simplex, as above, so $g_{ij}g_{jk} = g_{ik}$, given by $f_2 : U_{ijk} \to G \times G$, $f_2 = (g_{jk}, g_{ij})$.

We thus have our 1-cocyle condition is automatic from the simplicial structure.

What about change of the choice of local sections of P, i.e., $s_i : U_i \to P$. If we change these, we get elements $g_i \in G_i$ such that $s' = g_i s_i$ and the new g'_{ij} are related to the old by a sort of conjugacy rule:

$$g'_{ij} = g_i g_{ij} g_j^{-1},$$

which can be visualised as a square

$$\begin{tikzcd} {} \arrow[r, "g_{ij}"] \arrow[d, "g_j"'] & {} \arrow[d, "g_i"] \\ {} \arrow[r, "g'_{ij}"'] & {} \end{tikzcd}$$

This is reminiscent of a homotopy, and, in fact, defines one from our f (relative to the $\{s_i\}$) to f' (relative to the $\{s'_i\}$). In other words, we are identifying isomorphism classes of G-torsors that trivialise over $\mathcal{U}$ with homotopy classes, i.e., elements of $[N(\mathcal{U}), BG]$. We will return to this later when we discuss passing to refinements of $\mathcal{U}$ to get a homotopy description of all G-torsors, so we will not give the details here.

Several questions should come to mind at this stage. Given our recent description of 'change of groups', an obvious thing to do is to view that from a simplicial perspective. Suppose $\varphi : G \to H$ is a homomorphism of sheaves of groups. It is easy to see that φ induces a map of simplicial sheaves, $B\varphi : BG \to BH$, so we get, for given $\mathcal{U}$, an induced map

$$[N(\mathcal{U}), B\varphi] : [N(\mathcal{U}), BG] \to [N(\mathcal{U}), BH].$$

If we start off with a G-torsor, P, and use our change of groups methods above, what is the link between $\varphi_*(P)$ and the image of the isomorphisms class of P as represented by some map from $N(\mathcal{U})$ to BG. Of course, we have just seen that if $\{g_{ij}\}$ represents P then $\{\varphi(g_{ij})\}$ represents $\varphi_*(P)$ - but this is exactly the image under $[N(\mathcal{U}), B\varphi]$. There is thus yet another good way of interpreting the change of groups functor from $Tors(G)$ to $Tors(H)$, namely as a simplicial induced map from BG to BH. (Later we will see that $Tors(G)$ is the stack completion of BG or equivalently of $G[1]$ and this yields a variant of this simplicial viewpoint.)

Picking up an earlier problem, what about change of base. If we have the above simplicial description of isomorphism classes of those G-torsors on a base B that trivialise over some open cover $\mathcal{U}$, in terms of homotopy classes of maps from $N(\mathcal{U})$ to BG, and then we change the base along a continuous map, how does this look from a simplicial viewpoint?

To start with we rename some objects to get things into line with our earlier discussion. We will consider two spaces B and B' and a continuous map $f : B \to B'$. We have a sheaf or bundle of groups G on B' and hence an induced pullback sheaf $f^*(G)$ on B. We assume given some open cover $\mathcal{U}$ of B', and hence an open cover $f^{-1}(\mathcal{U})$ of B, and will be interested in those $f^*(G)$-torsors that trivialise over $f^{-1}(\mathcal{U})$ and which are induced from G-torsors that trivialise over $\mathcal{U}$.

6.3.6 Torsors and exact sequences

One classical method of analysing the cohomology and in so doing of providing interpretations of cohomology classes, is to vary the coefficients within an exact sequence. For instance, if

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to 1$$

is an exact sequence of sheaves of groups, then one might try to relate torsors over L, M and N. The usual techniques would then be to see what is the likelihood of having something like a long exact sequence of the cohomology 'sets' or groups. Where should it start?

We will, to start with, look at the Abelian case, but will try not to use commutativity so as to get as general a result as possible. Sheaf cohomology with coefficients in sheaves of Abelian groups, etc., is considered as measuring the non-exactness of the global sections functor. Given a sheaf L of Abelian groups on B, $\Gamma_B(L)$ is one of several notations used for the Abelian group of global sections of L. Another is $L(B)$, of course. If the exact sequence above had been of Abelian sheaves, we would have had a long exact sequence

$$0 \to L(B) \to M(B) \to N(B) \to \check{H}^1(B, L) \to \check{H}^1(B, M) \to \check{H}^1(B, N) \to \check{H}^2(B, L) \to \dots,$$

and so on. It is to be noted that the induced map, $v_* : M(B) \to N(B)$, need not be onto, so $\check{H}^1(B, L)$ picks up the obstruction to 'lifting' a global section of N to one of M. This is particularly interesting to us here since we have linked $\check{H}^1(B, L)$ with L-torsors in the general situation - and, of course, that interpretation is also valid in the Abelian case.

To see how $\check{H}^1(B, L)$ arises naturally in this situation, suppose given a global section h of N. As our exact sequence above was of sheaves, we have to examine what that means. This can be viewed from several angles. An exact sequence of sheaves may not be exact as a sequence of presheaves. The functor that forgets that sheaves are sheaves has a left adjoint namely 'sheafification', so will itself be 'left exact', e.g., will preserve monomorphisms. (If you do not know of this type of result, try to prove it yourself.) It need not preserve epimorphisms. Sheafification itself will preserve epimorphisms, but not all epimorphisms need be the sheafification of an epimorphism at the presheaf level. An epimorphism of sheaves will give an epimorphism on stalks. (We are thinking here of sheaves on a space, B rather than more general topos centred results.) This means epimorphisms are locally defined. Suppose we have a point $b \in B$, then if x is in the stalk of N above b, it means that x is representable as a pair (x_U, U), where $b \in U$, U is an open set and $x_U \in N(U)$, the group of local sections of N over U. (Recall, from page 144, section 6.2.3, that the stalk of a sheaf N at a point b is a colimit of the $N(U)$ for $b \in U$.) The morphism v being an epimorphism, there is an element y in the stalk of M at b, say $y = [(y_V, V)]$, such that over some open set $W \subseteq U \cap V$, $v(y_W) = x_W$.

Now start, not with an element in a stalk, but rather with a global section x of N. This does give an element in each stalk and we can find an open cover $\mathcal{U}$ such that over each U_i in $\mathcal{U}$, we can find a local section, y_i, mapping down to the restriction, x_i, of x to U_i, (but remember that different global sections will most likely need different covers, etc.). There is no reason these y_i should be compatible on intersections U_{ij}, so there will be (unique) elements $\ell_{ij} \in L_{ij} = L(U_{ij})$ such that

$$y_i = u(\ell_{ij})y_j,$$

since both y_i and y_j map to x_{ij} over U_{ij}. As u is a monomorphism, these ℓ_{ij} will satisfy the cocycle condition

$$\ell_{ij}\ell_{jk} = \ell_{ik}$$

and, as you no doubt now expect, if we change the local sections y_i within the L_i-coset of possible choices, then $y'_i = u(\ell_i)y_i$ and the ℓ_i define a coboundary.

In other words, there is an L-torsor, $P(x)$, which is constructed from the global section x of N, and which is trivial exactly when the y_i can be chosen compatibly, i.e., when there is a global section y mapping down to x. We can thus think of $P(x)$ as being the obstruction to lifting x to a global section of M. (Of course, the choices made have to be checked not to matter, up to isomorphism of $P(x)$ - but that can be safely **left to the reader**.)

There is thus an extension of the earlier sequence to

$$0 \to L(B) \to M(B) \to N(B) \to \pi_0(Tors(L)),$$

where the last term corresponds to $\check{H}^1(B, L)$. (The notation π_0 is, you may recall, to designate the set of connected components of a groupoid, simplicial set or space and $Tors(L)$ is a groupoid as we have seen.)

The next two terms in the long exact sequence, $\check{H}^1(B, M)$ and $\check{H}^1(B, N)$, are easy to handle geometrically. They give $\pi_0(Tors(M))$ and $\pi_0(Tors(N))$ respectively, and, of course, the induced maps are those given by the 'change of groups' along u and v. Exactness of the result is then routine to check, but

$$v_* : \pi_0(Tors(M)) \to \pi_0(Tors(N))$$

will not, in general, be onto. (You would not expect it to be as the standard homological machinery gives a $\check{H}^2(B, L)$ term.) Of course, none of the above depended on the sheaves involved being Abelian, but if they are not, $\check{H}^1(B, L)$ is not an Abelian group, it is just a pointed set. It is still given by $\pi_0(Tors(L))$, and $Tors(L)$ *is* always a groupoid, so there is a second layer that is hidden by the homological approach namely the automorphisms of the different objects in this groupoid.

6.4 Bitorsors

The fact that the left G-torsor is also a right P^{ad}-torsor suggests the notion of a bitorsor, the analogue of a left R-, right S-module for our non-Abelian setting. (Our basic reference for this will be Breen's Grothendieck Festschrift paper, [17] and his beautiful 'Notes on 1- and 2-gerbes', [20], based on his Minneapolis lectures.)

6.4.1 Bitorsors: definition and elementary properties

Definition: Let G, H be two bundles of groups on B or more generally two group objects in a topos, $\mathcal{E}$. A (G, H)-*bitorsor* on B is a space P over B together with fibre preserving left and right actions of G and H, respectively, on P, which commute with each other,

$$(g.p).h = g.(p.h),$$

and which define both a left G-torsor and a right H-torsor structure on P. If $G = H$, we say G-*bitorsor* rather than (G, G)-bitorsor.

There is an obvious extension of the notion to that of a (G, H)-bitorsor in a topos. We leave the exact formulation to you.

A family of local sections s_i of a (G, H)-bitorsor defines a local identification of P as *the* trivial left G-torsor and *the* trivial right H-torsor. It therefore determines a family of local isomorphisms $u_i : H_{U_i} \to G_{U_i}$, given by the rule $s_i h = u_i(h)s_i$, for $h \in H_{U_i}$. It is important to note that this does not mean that G and H are *globally* isomorphic.

Examples: a) The trivial (left) G-torsor T_G is also a right G-torsor (using right multiplication) and has a G-bitorsor structure.

b) Any left G-torsor, P, is a (G, P^{ad})-bitorsor, as above. Any G-torsor, P, is a (G, H)-bitorsor if and only if $H \cong P^{ad}$.

c) Let

$$1 \to G \xrightarrow{i} H \xrightarrow{j} K \to 1$$

be an exact sequence of bundles of groups on B. Form $G_K = G \times_B K$, which is again a bundle of groups, then H is a G_K-bitorsor over K. This needs a bit of working through. For a start, K is a bundle of groups so has a (hidden) structural projection, $K \to B$. Thinking of this as a cover as we have done previously, then G_K is the induced bundle of groups on K (as a space), so we have transferred attention from Top/B to Top/K or from $Sh(B)$ to $Sh(K)$. There are actions of G_K on H,

$$h \star (g,k) = hi(g).$$

(but note that requires us to use $H \xrightarrow{j} K$, as the structural projection of H over K, again, going to bundles on K,

$$(g,k).h = i(g).h,$$

but is only defined if $j(h) = k$, as we are 'over K,' in this equation).

This is somewhat simplified if we have $B = 1$, when it is simply an exact sequence of groups, G_K is $G \times K$ as a group over K, via projection, and so on.

There is an obvious notion of morphism of bitorsors and thus various categories, $Bitors(G,H)$, $Bitors(G) := Bitors(G,G)$, ... It should come as no surprise that if P is a (G,H)-bitorsor and Q is a (H,K)-bitorsor, both on B, then $P \wedge^H Q$ is a (G,K)-bitorsor. Moreover, P gives a (H,G)-bitorsor, P^o, (o for 'opposite') by reversing the two actions. (**For you to check out.**) We thus have that a (G,H)-bitorsor will induce a functor

$$Tors(H) \to Tors(G)$$

and that, for a given bundle of groups G, the category of G-bitorsors has a monoidal structure given by $P \wedge^G Q$ and with T_G as unit object. The opposite construction acts like an inverse,

$$P \wedge^G P^o \cong T_G \cong P^o \wedge^G P,$$

but note that these are isomorphisms *not* equality.

Lemma 21 *The category $Bitors(G)$ with contracted product is a group-like monoidal category, with the bitorsor T_G as unit and P^o, an inverse for P.*

Proof: This is **left as an exercise**, but here is a suggestion for the above isomorphisms: use local sections to send any $[p,p']$ in $P^o \wedge^G P$ to an element of G, now show independence of that element on the choice of local section. It is also necessary to check through the group-like monoidal category axioms, which are left for you to find in detail. ∎

A group-like monoidal category is often called a *gr-category*. We have already (essentially introduced on page 44) seen that strict gr-categories are 'the same as' crossed modules, so once again that crossed structure is lurking around just beneath the surface. It is interesting and useful (i.e., an **exercise left to the reader!**) to examine the above structure when G is a sheaf of *Abelian* groups, for instance to show that the monoidal structure is symmetric.

A very useful result, akin to Lemma 20 above, gives a similar interpretation of $Isom_G(P,Q)$, where P is a (G,H)-bitorsor and Q a left G-torsor. As P is thus also a left G-torsor and $Tors(G)$ is a groupoid, $Isom_G(P,Q)$ is just the sheaf of G-equivariant torsor maps from P to Q, all of which are invertible. The following lemma identifies this as a contracted product.

Lemma 22 *Let P be a (G,H)-bitorsor and Q a left G-torsor, then there is an isomorphism*

$$Isom_G(P,Q) \xrightarrow{\cong} P^o \wedge^G Q.$$

Proof: We start by noting a morphism in the other direction. Suppose we take a local element in $P^o \wedge^G Q$ given by $(p,q) \in P^o \times Q$, defined over an open set U. We have

$$(p,q) \equiv (p.g^{-1}, g.q),$$

but as $p \in P^o$, $p.g^{-1} = q.p$ with the original left G-action on P. We assign to (p,q) the isomorphism, $\alpha_{(p,q)}$, from P to Q defined over U, which sends p to q. Of course, $\alpha_{(p,q)}$ is to be extended to a G-equivariant map, $\alpha_{(p,q)}(g.p) = g.q$, but we effectively knew that fact already since

$$\alpha_{(p,q)} = \alpha_{(p.g^{-1},g.q)},$$

so it sends $p.g^{-1}$ to $g.q$. Of course, if $\beta : P_U \to Q_U$ is a local morphism defined over some U, then we can assume P_U has a local section p and that $\beta(p) = q$ for some local section q of Q. (If not, refine U by an open cover on which P trivialises and work on the open sets of that finer open cover.) However then we can assign $[p,q]$ in $P^o \wedge^G Q$ to the morphism β. The rest of the details should now be easy to check. ∎

6.4.2 Bitorsor form of Morita theory (First version):

Within the theory of modules and more generally of Abelian categories, there is a very important set of results known as Morita theory, describing equivalences between categories of modules. The idea is that if R and S are rings, then we can use a homomorphism as above to induce a right R, left S module structure on S itself and this is what induces, via tensor product, a functor from $Mod(S)$ to $Mod(R)$. We have seen the corresponding idea with torsors above. Not all functors between $Mod(R)$ and $Mod(S)$ are induced by morphisms at the ring level in this way however, but provided we look at equivalences between categories, this bimodule idea allows us to describe the equivalences precisely - and this does go across to the torsor context.

The first essential is to recall the definition of an equivalence of categories.

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories is an *equivalence* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms, $\eta : GF \Rightarrow Id_{\mathcal{C}}$ and $\eta' : FG \Rightarrow Id_{\mathcal{D}}$. We say G is *(quasi-)inverse* to F.

Proposition 39 *A (G,H)-bitorsor Q on B induces an equivalence*

$$Tors(H) \stackrel{\Phi_Q}{\to} Tors(G)$$
$$M \longmapsto Q \wedge^H M$$

between the corresponding categories of left torsors on B. In addition if P is a (H,K)-bitorsor on B, then there is a natural isomorphism of functors

$$\Phi_{Q\wedge^H P} \cong \Phi_Q \circ \Phi_P,$$

and, in particular, the equivalence Φ_{Q^o} is quasi-inverse to Φ_Q.

Proof: The last part follows from the statement on composites, which should be clear by construction and, of course, $T_H \wedge^H Q \cong Q$, as we saw earlier. This proof is thus just a compilation of earlier ideas - and so will be **left to the reader**! ∎

In fact it is now easy to give a weak version of the torsor Morita theorem.

Proposition 40 *If*

$$\Phi : Tors(H) \to Tors(G)$$

is an equivalence of categories, then there is a (G, H)-bitorsor, Q, which itself induces such an equivalence.

Proof: We will limit ourselves to pointing out that we can take $Q = \Phi(T_H)$. This inherits its right H-action from the right action of H on T_H. (You should **check** that it is a right H-torsor for this action.) ■

It is, in fact, the case that Φ is equivalent to the equivalence induced by Q, but this is more relevant in a later context, so will be revisited then.

6.4.3 Twisted objects:

Continuing our study of torsors and bitorsors, as such, we should mention the analogue of fibre bundles in this context.

Let P be a left G-torsor on B and E a space over B on which G acts on the right. We can again use the contracted product construction to form $E^P := E \wedge^G P$ over B. In this context we call E^P the *P-twisted form* of E.

Choice of a local section s of P over an open set U determines an isomorphism $\varphi_P : E^P_{|U} \cong E_U$, so E^P is locally isomorphic to E. (Beware, especially if you are used to the case where E is a product space over B, so $E = F \times B$, say. In that case E^P is locally trivial in a very strong sense, but this need not be so in general).

Suppose E_1 is now a space over B and there is an open cover $\mathcal{U}$ of B over which E_1 is locally isomorphic to E, then the sheaf or bundle $Isom_B(E_1, E)$ is a left torsor on B for the action of the bundle of groups, $G := Aut_B(E)$. This gives us a G-torsor and a space, E, on which G acts on the right.

These two constructions are inverse to each other.

In particular, if we are given G and have a second bundle of groups, H, on B, which is locally isomorphic to G, then $P := Isom_B(H, G)$ is a $Aut_B(G)$-torsor. It is worth pausing to think out the components of this fact. The object $Isom_B(H, G)$ exists, as before, because of the Cartesian closed assumption about our categories of bundles over B, (e.g. if we are interpreting bundles as sheaves, $Isom_B(H, G)$ is a subsheaf of the function sheaf, $Sh(B)(H, G)$, but although it would always have an action of $Aut_B(G)$, we need the 'H is locally isomorphic to G' condition to ensure the existence of local sections and hence to ensure it is a $Aut_B(G)$-torsor).

Look now at $G \wedge^{Aut(G)} P$ and the map

$$G \wedge^{Aut(G)} P \to H$$

$$(g, u) \mapsto u^{-1}(g).$$

(We make $Aut_B(G)$ act on the right of G, via the obvious left action.) This map is an isomorphism and so H is the P-twisted form of G for this right $Aut_B(G)$-action.

On the other hand, if G is a bundle of groups on B and P is a left G-torsor, $H := G \wedge^{Aut(G)} P$ is a bundle of groups on B locally isomorphic to G and this identifies P with the left $Aut_B(G)$-torsor, $Isom_B(H, G)$.

This provides a torsor's-eye-view of our examples on fibre bundles given in section 6.1.3, (Case study, page 136). We will sketch in a few more details:

A vector bundle, V, of rank n on B is locally isomorphic to $\mathbb{R}^n_B := \mathbb{R}^n \times B$. The group of automorphisms of this is the trivial bundle of groups, $G\ell(n,\mathbb{R})_B := G\ell(n,\mathbb{R}) \times B$. The left $G\ell(n,\mathbb{R})_B$-torsor on B associated to V is $Isom(V, \mathbb{R}^n_B)$ and this is just the *frame bundle*, P_V, of V. The vector bundle V is a bundle of groups, so the above discussion applies, showing it to be the P_V-twist of $\mathbb{R}^n_B$. Conversely for any $G\ell(n,\mathbb{R})_B$-torsor P on B, the twisted object $V = \mathbb{R}^n_B \wedge^{G\ell(n,\mathbb{R})_B} P$ is the rank n vector bundle associated to P and its frame bundle P_V is canonically isomorphic to P. (If you have not explored vector bundles and differential manifolds, a brief excursion into that area may be well worthwhile, as it reinforces the geometric origins and intuitions behind this area of cohomology.)

6.4.4 Cohomology and Bitorsors

Earlier, (page 155), we saw how local sections, s, of a torsor, P, over an open cover, $\mathcal{U}$, led to 'transition maps', or 'cocycles', $g_{ij} : U_{ij} \to G$, on the intersections. Changing local sections to $s'_i : U_i \to P$, $s'_i = g_i s_i$, we have that the corresponding cocycles g'_{ij} are related via the coboundary relation

$$g'_{ij} = g_i g_{ij} g_j^{-1},$$

to the earlier ones. This led to the set of equivalence classes, $H^1(\mathcal{U}, G)$, and eventually to the cohomology set $\check{H}^1(B, G)$, which classified isomorphism classes of G-torsors on B.

What would be the additional structure available if P was a (G,H)-bitorsor? The family of local sections $s_i : U_i \to P$ then would also determine a family of local isomorphisms $u_i : H_{U_i} \to G_{U_i}$, where

$$u_i(h)s_i = s_i.h.$$

Remark: This formula needs a bit of thought. That u_i *is* a bijection is clear, as it follows from the fact that P is a G-torsor, but that it *is* a homomorphism needs a bit more care. The defining equation is specifically using the local section s_i so, for instance, on a more general element $g.s_i$ we have to extend the formula using G-equivariance, (remember the two actions are independent), so $(g.s_i).h = g.u_i(h).s_i$. In particular, if h_1 and h_2 are two local section of H over U_i, then $s_i.(h_1h_2) = u_i(h_1).s_i.h_2 = u_i(h_1)u_i(h_2).s_i$, so $u_i(h_1h_2)$ does equal $u_i(h_1)u_i(h_2)$.

Over an intersection U_{ij} of the cover, $s_i = g_{ij}s_j$, so

$$u_i = i_{g_{ij}} u_j$$

with as usual, i the inner automorphism homomorphism from G to $Aut_B(G)$, sending g to i_g. The (u_i, g_{ij}) therefore satisfy the *cocycle conditions*

$$g_{ik} = g_{ij} g_{jk}$$

and

$$u_i = i_{g_{ij}} u_j.$$

Changing the local sections to $s'_i = g_i s_i$ in the usual way determines coboundary relations

$$g'_{ij} = g_i g_{ij} g_j^{-1}$$

and

$$u'_i = i_{g_i} u_i.$$

Isomorphism classes of (G, H)-bitorsors on B with given local trivialisation over $\mathcal{U}$, thus are classified by the set of equivalence classes of such cocycle pairs (g_{ij}, u_i) modulo coboundaries. In the most important case of G-bitorsors, the u_i are locally defined automorphisms of the G_{U_i} and so are local sections of $Aut(G)$.

We thus have from a G-bitorsor, P, a fairly simple way to get a piece of descent data, $\{(g_{ij}, u_i)\}$, with the right sort of credentials to hope for a 'reconstruction' process. We needed P to trivialise over the open cover $\mathcal{U} = \{U_i\}$ and then to chose local sections, $s_i : U_i \to P$. This gave $\{g_{ij} : U_{ij} \to G\}$ and $\{u_i : U_i \to Aut(G)\}$, so let us start off with these and see how much of P's structure we can retrieve.

Putting aside the u_is for the moment, we have a G-valued cocycle, $\{g_{ij}\}$, and we already have seen how to build a G-torsor from that information. Recall we take

$$P = \bigsqcup_i G(U_i)/\sim,$$

where $(g, i) \sim (gg_{ij}, j)$. (The basic relation is really that $(1_{U_i}, i) \sim (g_{ij}, j)$ with the left translation $G(U_{ij})$-action giving the more general form.) We thus have a lot of the structure already available. We are left to obtain a right G-action, which has to be 'independent' of the left action, i.e., to commute with it as in the first definition of this section. (To avoid confusion between the two actions, we will pass to the (G, H)-bitorsor case so $u_i : U_i \to Isom(H, G)$, and will denote local elements that act on the right by h_i, whilst any acting on the left by g_i.)

In our 'reconstructed' P, there is clearly a natural choice for a local section over U_i, namely the equivalence class of the identity element $1_{U_i} \in G(U_i)$, or, more exactly of $(1_{U_i}, i)$, then we could define

$$[g, i].h := [g.u_i(h), i].$$

It is clear that this is a right action, since u_i is a homomorphism, and that it does not interfere with the left $G(U_i)$-action, which is $g'[g, i] = [g'g, i]$. Of course, we have to check compatibility with the equivalence relation, and that is exactly what is needed for checking that it works on adjacent patches / open sets of the cover. The key case is to work with a local section h of G over an open set, U, and examine what h does on patches U_i, U_j and their intersection. (Of course, this presupposes that we are intersecting U_i, etc., with U, i.e., that we are effectively working with an open cover of U itself.)

We know how the U_i are related over the different patches, namely

$$u_i = i_{g_{ij}} u_j,$$

which on our local element, h, gives

$$u_i(h) = g_{ij} u_j(h) g_{ij}^{-1}.$$

As h is defined on U, the restrictions to the various U_i form a compatible family, (i.e., we do not need to worry about transitions for h in formulae), so

$$[g, i].h = [gu_i(h), i] = [g.u_i(h) g_{ij}, j],$$

on the one hand, and also

$$[g.g_{ij}, j].h = [gg_{ij}u_j(h), j].$$

The earlier identity shows that

$$u_i(h)g_{ij} = g_{ij}u_j(h),$$

so these are the same local element of P over U_{ij}.

The u_i were introduced as the way to link local right and left actions,

$$u_i(h).s_i = s_i.h.$$

They also have an interpretation if we seek to study when a given left G-torsor, P, has an additional G-bitorsor, or more generally, a (G, H)-bitorsor structure. The cocycle rules linking the u_i with the g_{ij} involve the group homomorphism $i : G \to Aut(G)$. The g_{ij} part of the cocycle family only uses the left G-torsor structure on P. It is perhaps only because of 'natural curiosity', but it does seem natural to look at the $Aut(G)$-torsor, $i_*(P)$. Our earlier calculations show that suitable cocycles for this are given by $\{i(g_{ij})\} = \{i_{g_{ij}}\}$, but the u_i now look very like a coboundary! In fact that key equation, $u_i = i_{g_{ij}}u_j$, can obviously be rewritten as

$$i_{g_{ij}} = u_i u_j^{-1},$$

or

$$i_{g_{ij}} = u_i.1.u_j^{-1},$$

so the class of $\{i_{g_{ij}}\}$ is 'cohomologically null', i.e., equivalent to 1 modulo coboundaries. In other words, $i_*(P) \cong T_{Aut(G)}$.

Conversely, if we have P and hence its cocycle representation, *and* a 0-cocycle trivialising $i_*(P)$, so $\{i_{g_{ij}}\}$ is a coboundary,

$$\{i_{g_{ij}}\} = \alpha_i\alpha_j^{-1},$$

then taking $u_i = \alpha_i$, we have a cocycle pair, (g_{ij}, u_i), giving P a G-bitorsor structure.

We clearly should look at this from the viewpoint of contracted products as they have a clearer *geometric* interpretation. The $Aut(G)$-torsor, $i_*(P)$, has a description as $Aut(G)_i \wedge^G P$, thus, by quotienting $Aut(G) \times P$ by the equivalence relation

$$(\alpha.g.p) \sim (\alpha \circ i(g), p).$$

The fact that $i_*(P)$ is locally trivial was given by the local sections induced by those $s_i : U_i \to P$ for P, namely

$$[(1, s_i)] : U_i \to Aut(G)_i \wedge^G P.$$

(Note this formulation is slightly different from that in Breen, [17], as he uses the opposite group $Aut^o(G)$ and i', but we can avoid that extra complication for our purposes here, since we really only need $\alpha = 1$ in the above.)

We can compare these local sections on overlaps U_{ij},

$$(1, s_i) \sim (1, g_{ij}s_j) \sim (i_{g_{ij}}, s_j) \sim (u_iu_j^{-1}),$$

but now our local sections $[(1, s_i)]$ are equivalent to others $t_i = [(u_i^{-1}, s_i)]$, which *agree* on overlaps

$$t_i = [(u_i^{-1}, s_i)] = [(u_i^{-1}u_iu_j^{-1}, s_i)] = t_j$$

over U_{ij}. These t_i thus form a global section for $i_*(P)$, which is hence the trivial torsor, up to isomorphism.

Reversing the argument, a global section of $i_*(P)$, together with the structural cocycle $\{g_{ij}\}$ for P gives a G-bitorsor structure on P. (We will return to this in more generality a bit later.)

We thus have that a G-bitorsor is a *relative* $\mathsf{Aut}(G)$*-torsor*, where $\mathsf{Aut}(G) = (G, Aut(G), \iota)$. It corresponds to a G-torsor, P, together with a trivialisation of $\iota_*(P)$. Using the fact that morphisms from the induced torsor $\iota_*(P)$ to $T_{Aut(G)}$ corresponds to morphisms over ι from P to $T_{Aut(G)}$, we get a second description, which is very useful for further generalisation.

6.4.5 Bitorsors, a simplicial view.

Pausing in our development, let us return to the simplicial viewpoint that we adopted earlier. The cover $\mathcal{U}$ gives a sheaf / bundle,

$$p : E = \sqcup\mathcal{U} \to B$$

and by repeated pullbacks, we get a simplicial sheaf / bundle,

$$N(\mathcal{U}) \to B.$$

The cocycle $\{(u_i, g_{ij})\}$ consists of a family $\{u_i\}$ giving a morphism,

$$\mathbf{g}_0 : N(\mathcal{U})_0 = \sqcup\mathcal{U} \to Aut(G),$$

together with a second family

$$\mathbf{g}_1 : N(\mathcal{U})_1 \to G \rtimes Aut(G).$$

This second piece of data is not quite as obvious as it might seem. The earlier model of the crossed view of group extensions used the crossed module, $\mathsf{Aut}(G) = (G, Aut(G), \iota)$ directly. Here we are using the cat^1-group / gr-groupoid / 2-group analogue, which can also be thought of simplicially as in our discussion of algebraic 2-types, page 77. Recall the face maps

$$d_i : G \rtimes Aut(G) \to Aut(G), \quad i = 0, 1,$$

are given by

$$\begin{aligned} d_1(g, \alpha) &= \alpha, \\ d_0(g, \alpha) &= i_g \circ \alpha \end{aligned}$$

and the degeneracy is

$$s_0(\alpha) = (1_G, \alpha).$$

The maps $\mathbf{g}_0$, $\mathbf{g}_1$ are to be hoped to be a part of a simplicial map from the simplicial sheaf $N(\mathcal{U})$ to the sheaf of simplicial groups, $K(\mathsf{Aut}(G))$, and to check that this is indeed the case, we need to recall that 'bundle-wise' the elements of $\sqcup\mathcal{U} = N(\mathcal{U})_0$ can usefully be thought of as pairs (x, U), where $U \in \mathcal{U}$ and $x \in U$. Of course, the projection maps p sends (x, U) to x itself. The 1-simplices of $N(\mathcal{U})$ therefore are given by triples (x, U_0, U_1) with $x \in U_0 \cap U_1$, so the corresponding face and degeneracy maps are

$$\begin{aligned} d_1(x, U_0, U_1) &= (x, U_0), \\ d_0(x, U_0, U_1) &= (x, U_1), \\ s_0(x, U) &= (x, U, U). \end{aligned}$$

We can thus see what this $\mathbf{g}$ must satisfy. We write $\mathbf{g}_1 = (g, \alpha)$ as before, and will try to identify what g and α must be. We have, then,

- $d_1\mathbf{g}_1 = \mathbf{g}_0 d_1$ means $\alpha = u_{|U_0} =: u_0$;
- $d_0\mathbf{g}_1 = \mathbf{g}_0 d_0$ means $i_g u_0 = u_{|U_1} =: u_1$;
- $s_o\mathbf{g}_0 = \mathbf{g}_1 s_0$ is a normalisation condition, which will make more sense when the first two conditions have been explored in more detail.

The obvious way to build $\mathbf{g}_1$, i.e., g itself, is thus to take

$$\mathbf{g}(x, U_0, U_1) = (g_{10}(x), u_0(x)),$$

and to require that g_{ii} is 1_G restricted to $U_{ii} = U_i \cap U_i$ for the normalisation.

To continue our simplicial description, we should look at triple intersections, i.e., $N(\mathcal{U})_2$, and the corresponding $K(\mathsf{Aut}(G))_2$. The points of $N(\mathcal{U})_2$ are, of course, represented by symbols such as (x, U_0, U_1, U_2), whilst those of $K(\mathsf{Aut}(G))_2$ above the point x, are of form $(g_2, g_1, \alpha)(x)$. The face maps of $N(\mathcal{U})$ are the obvious ones, $d_2(x, U_0, U_1, U_2) = (x, U_0, U_1)$, and so on, whilst

$$\begin{aligned} d_2(g_2, g_1, \alpha) &= (g_1, \alpha), \\ d_1(g_2, g_1, \alpha) &= (g_2 g_1, \alpha), \\ d_0(g_2, g_1, \alpha) &= (g_2, i_{g_1}\alpha), \end{aligned}$$

with the s_i inserting an identity in the appropriate place. (Of course, all these g_i, etc., are 'local elements', so are really local sections, and our formulae would have, over a given x, the values $g_2(x)$, etc., as above.)

We want $\mathbf{g}$ to be a simplicial morphism, so on 2-simplices we expect, for (x, U_0, U_1, U_2),

$$d_2\mathbf{g}_2 = \mathbf{g}_1 d_2,$$

etc., i.e., if $\mathbf{g}_2(x, U_0, U_1, U_2) = (g_2, g_1, \alpha)(x)$, the d_2-face $(g_1, \alpha)(x) = (g_{10}(x), u_0(x))$, so $g_1, = g_{10}$, $\alpha = u_0$, and then the d_0 face gives $g_2 = g_{21}$. Finally the d_1-face gives

$$g_2 g_1 = g_{20},$$

so this gives us the cocycle condition

$$g_{21} g_{10} = g_{20}$$

over U_{012}.

The other simplicial morphism rules give compatibility with degeneracies, but using simplicial identities, these then give that $g_{01} = g_{10}^{-1}$, i.e., again a normalisation condition.

We thus have

(i) the bundle of crossed modules $\mathsf{Aut}(G)$ given by $(G, Aut(G), \iota)$;

(ii) the corresponding bundle of simplicial groups, $K(\mathsf{Aut}(G))$;

(iii) the bundle / sheaf of simplicial sets, $N(\mathcal{U})$;
and

(iv) our local cocycle description of our bitorsor, P,

giving, it would seem, a simplicial map

$$\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{Aut}(G)).$$

Conversely such a simplicial map gives a cocycle (for **you to check**).

(Here we are abusing notation slightly, since the domain of $\mathbf{g}$ is a bundle of simplicial sets, whilst the right hand side is the underlying simplicial set bundle of the simplicial group bundle, not that simplicial group bundle itself, however we have not shown that in the notation. It is, however, an important point to note.)

Continuing with this quite detailed look at the 'cocycles for bitorsors' context, we clearly have next to look at the 'change of local sections' from this simplicial viewpoint.

Suppose we change to local sections, $s_i' = g_i s_i$, so, as before, get

$$g_{ij}' = g_i g_{ij} g_j^{-1}$$

and

$$u_i' = i_{g_i} u_i.$$

If we are describing cocycles as simplicial maps, then fairly naturally, we might hope that the equivalence relation coming from coboundaries, as here, was something like homotopy of simplicial maps. We can see immediately that this looks to be not that stupid an idea, by looking at the base of the corresponding simplicial objects.

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^{(2)} \rtimes Aut(G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \rtimes Aut(G) \arrow[r, shift left=1] \arrow[r, shift right=1] & Aut(G) \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & N(\mathcal{U})_2 \arrow[u, shift left=1, "g_2"] \arrow[u, shift right=1, "g_2'"'] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & N(\mathcal{U})_1 \arrow[u, shift left=1, "g_1"] \arrow[u, shift right=1, "g_1'"'] \arrow[r, shift left=1] \arrow[r, shift right=1] & N(\mathcal{U})_0 \arrow[u, shift left=1, "g_0"] \arrow[u, shift right=1, "g_0'"']
\end{tikzcd}$$

then we would expect that a homotopy between $\mathbf{g}$ and $\mathbf{g}'$ would pick out, for each (x, U_0) in $N(\mathcal{U})_0$, an element $(g, \alpha) \in G \rtimes Aut(G)$ with $g = d_1(g, \alpha) = g_0$, $d_0(g, \alpha) = g_0'$, i.e., $\alpha = u_0$ and $g_0' = u_0' = i_{g_0} \circ u_0$, exactly as needed. To see if this works in higher dimensions, we need to glance at simplicial homotopies. We will take a fairly naïve view of them to start with. We have already met them in passing in our discussion of simplicial mapping spaces in Chapter 5.3, page 127.

Given $f, g : K \to L$, two morphisms of simplicial sets, a *simplicial homotopy* from f to g is, of course, a map

$$h : K \times \Delta[1] \to L$$

such that if $e_0 : \Delta[0] \to \Delta[1]$ is the 0-end of $\Delta[1]$, (so is actually represented by the d_1 face - beware of confusion) and $e_1 : \Delta[0] \to \Delta[1]$, gives the 1-end, then

$$f = h \circ (K \times e_0),$$

$$g = h \circ (K \times e_1).$$

(More on such cylinder based homotopies in abstract settings can be found in Kamps and Porter, [76]. In a general context, simplicial homotopy does *not* give an equivalence relation on the set of simplicial maps as although it gives a reflexive relation symmetry and transitivity depend on the existence of fillers in the simplicial set of morphisms.)

This is the neat geometric way of picturing simplicial homotopies. There is an alternative 'combinatorial' way that is also very useful (see [76], p.184-186, for a discussion - but not for the formulae which were left as an exercise!) This gives h being specified by a family of maps,

$$h_i^n : K_n \to L_{n+1},$$

indexed by $n = 0, 1, \ldots$, and i with $0 \leq i \leq n$, and satisfying some face and degeneracy relations that we will give later on. For the moment we will only need to use these in low dimensions, so imagine the lowest dimension $h_0^0 : K_0 \to L_1$. For each vertex, k_0, we get an edge / 1-simplex in L_1 joining $f_0(k_0)$ and $g_0(k_0)$. Now if $k_1 \in K_1$, we expect a square in $K \times \Delta[1]$ looking like

$$\begin{tikzcd}
{} \arrow[r, "{(k_1,s_0 1)}"] & {} \\
{} \arrow[u, "{(s_0 d_1 k_1,\iota)}"] \arrow[r, "{(k_1,s_0 0)}"'] \arrow[ur, "\tau_0", "\tau_1"'] & {} \arrow[u, "{(s_0 d_0 k_1,\iota)}"']
\end{tikzcd}$$

with $\iota \in \Delta[1]_1$, the unique non-degenerate 1-simplex, corresponding to $id : [1] \to [1]$. (Remember the product of simplicial sets, K and L, has $(K \times L)_q = K_q \times L_q$.) The homotopy h has to thus give two 2-simplices of L. These will be $h_0^1(k_1) := h(\tau_0)$ and $h_1^1(k_1) := h(\tau_1)$ respectively. We first note that $d_1\tau_0 = d_1\tau_1$, so

$$d_1 h_0^1 = d_1 h_1^1.$$

Likewise the geometric picture tells us that $d_2 h_1^1 = f_1$ and $d_0 h_0^1 = g_1$ and finally that $d_0 h_0^1 = h_0^0 d_0$, whilst $d_2 h_1^1 = h_0^0 d_1$.

In our special case of that general square, $k_1 = (x, U_0, U_1)$ with $d_0 k_1 = (x, U_1)$, $d_1 k_1 = (x, U_0)$, thus our earlier choices should mean the horizontal edges are mapped to

$$\begin{aligned}
h((x, U_0, U_1), 0) &= (g_{10}(x), u_0(x)), \\
h((x, U_0, U_1), 1) &= (g'_{10}(x), u'_0(x)),
\end{aligned}$$

and the vertical ones,

$$\begin{aligned}
h((x, U_1), \iota) &= (g_1(x), u_1(x)), \\
h((x, U_0), \iota) &= (g_0(x), u_0(x)).
\end{aligned}$$

They match up as required.

We need to work out h_0^1 and h_1^1. These will map (x, U_0, U_1) to 2-simplices of $K(\mathsf{Aut}(G))$, i.e., to triples $(\gamma_2, \gamma_1, \alpha)$, with $\gamma_i \in G$ and $\alpha \in Aut(G)$. First we look at $h_0^1(x, U_0, U_1)$ and the faces we know of it.

Let $h_0^1(x, U_0, U_1) = (\gamma_2, \gamma_1, \alpha)$, then the two descriptions of $d_2 h_0^1$ give

$$(g_{10}(x), u_0(x)) = (\gamma_1, \alpha),$$

whilst for $d_0 h_0^1$, we have

$$(g_1(x), u_1(x)) = (\gamma_2, i_{\gamma_1} \circ \alpha).$$

We thus have $\gamma_1 = g_{10}(x)$, $\alpha = u_0(x)$ and $\gamma_2 = g_1(x)$ and we can check back that $i_{g_{10}}u_0 = u_1$ from earlier calculations. We have h_1^1 completely specified as

$$h_1^1(x, U_0, U_1) = (g_1(x), g_{10}(x), u_0(x)).$$

This gives $d^1h_1^1(x, U_0, U_1) = (g_1(x)g_{10}(x), u_0(x))$, which we will need shortly.

We next turn to $h_0^1(x, U_0, U_1)$ and reset the meaning of γ_i and α, so this is $(\gamma_2, \gamma_1, \alpha)$. We do a similar calculation and this gives

$$h_0^1(x, U_0, U_1) = (g'_{10}(x), g_0(x), u_0(x)).$$

This 'feels' right, but we have to check it matches h_0^1 on the diagonal:

$$d_1h_0^1(x, U_0, U_1) = (g'_{10}(x)g_0(x), u_0(x)),$$

but $g'_{10}(x) = g_1(x)g_{10}(x)g_0(x)^{-1}$, so this equals $(g_1(x)g_{10}(x), u_0(x))$, as hoped.

We have laboriously checked through the calculations of (h_0^1, h_1^1) to show how well behaved things really are. It is reasonably easy to extend the calculation to all dimensions. What needs to be retained is that h was completely specified by the coboundary and cocycle data and, conversely, if we were given any homotopy h between $\mathbf{g}$ and $\mathbf{g}'$, then $\mathbf{g}$ and $\mathbf{g}'$ will be equivalent. This suggests that the simplicial mapping sheaf or bundle $\underline{\mathcal{S}Sh_B}(N(\mathcal{U}), K(\mathsf{Aut}(G)))$, is what is really encoding the data in a neat way. (If you are hazy about simplicial mapping spaces, recall that if K and L are simplicial sets, $\underline{\mathcal{S}}(K, L)$ is the simplicial set of simplicial maps and (higher) homotopies, so

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

Using the constant simplicial sheaves, $\Delta[n]_B$, to replace the use of the $\Delta[n]$ gives a similar simplicial enrichment for the category of simplicial sheaves / bundles on B, but this can be localised to make $\underline{\mathcal{S}Sh_B}(K, L)$, a simplicial sheaf as well.)

Earlier we omitted the detailed description of homotopies as families of maps. To complete our picture here, that description will now be useful. We first give it for simplicial sets, so in the very classical setting.

Let K and L be simplicial sets, and $f, g : K \to L$ two simplicial maps, then a homotopy

$$h : K \times I \to L$$

between f and g can be specified by a family of functions

$$h_, = h_i^n : K_n \to L_{n+1},$$

satisfying various relations. To understand how these arise, we use some simple notation extending that which we used above.

The non-degenerate $(n+1)$-simplices of $\Delta[n] \times \Delta[1]$ are of form $(s_j\iota_n, s_{\hat{j}}\iota_1)$, where $\iota_n \in \Delta[n]_n$ is the unique non-degenerate n-dimensional simplex corresponding to $id_{[n]} : [n] \to [n]$ in the description of $\Delta[n]$ as $\mathbf{\Delta}(-, [n])$, ι_1 being similarly specified for $n = 1$, and where $s_{\hat{j}}$ is the multiple degeneracy corresponding to $\hat{j} = (0, \ldots, \hat{j}, \ldots, n)$, i.e., $s_n \ldots s_0$, but without s_j. (Any $(n+1)$ simplex of $\Delta[1]$ is given by an increasing map $[n+1] \to [1]$, so can be represented as a string $(0, \ldots, 0, 1, \ldots, 1)$, say with j zeroes. This will be $s_{\hat{j}}\iota_1$, since the first j degeneracies 'add in' 0s, whilst those after the $(j+1)^{st}$, that is, after the break, will add in 1s. The simplicial identities give $s_i s_j = s_j s_{i-1}$ if $i > j$, so $s_{\hat{j}}$ has a second useful description as $(s_{last})^{n-j}(s_0)^j$.)

For an n-simplex $k \in K$, we denote $(s_j k, s_{\hat{j}}\iota_1)$ by τ_j, or, more exactly, $\tau_j(k)$ if confusion might arise. We then encode our $h : K \times I \to L$ by $h^n_j(k) = h(\tau_j(k))$. The homotopy h is, of course, a simplicial map so, for any $0 \leq i \leq n+1$, we have $d_i h = h d_i$. These relations translate to give the following rules:

$$d_0 h_0 = g, \qquad\qquad d_{n+1}h_n = f,$$

$$\begin{cases} d_i h_j &= h_{j-1} d_i & \text{for } i < j, \\ d_{j+1}h_{j+1} &= d_{j+1}h_j, & \\ d_i h_j &= h_j d_{i-1} & \text{for } i > j+1, \end{cases}$$

and the corresponding degeneracy rules are

$$s_i h_j = h_{j+1} s_i, \qquad i \leq j,$$

$$s_i h_j = h_j s_{i-1}, \qquad i > j.$$

Of course, these h_js etc. are further indexed by a dimension h^n_j, so, for instance, $d_i h^n_j = h^{n-1}_{j-1} d_i$ is the full form of the second line of these.

Aside on Tensors and Cotensors: It is often the case, when considering simplicial objects in a category, $\mathcal{A}$, that one can form a 'tensor', $X \otimes I$, using a coproduct in each dimension, then one defines a homotopy to be a morphism

$$h : X \otimes I \to Y.$$

The construction of this 'tensor' is : given any simplicial set K, and a simplicial object X in $\mathcal{A}$, (where $\mathcal{A}$ has the coproducts that we will be using below),

$$(X \otimes K)_n = \bigsqcup_{k \in K_n} X_n(k) \text{ with each } X_n(k) = X_n,$$

i.e., a K_n-indexed copower of X_n. Using an element based notation, the usual way of denoting the copy of $x \in X_n$, in the k-indexed copy of X_n would be $x \otimes k$ and then face and degeneracy maps are given, in $X \otimes K$, by $d_i(x \otimes k) = d_i x \otimes d_i k$, etc., i.e. 'component-wise'. In this setting again $h : X \otimes \Delta[1] \to Y$ can be decomposed to give a family $\{h^n_j : X_n \to Y_{n+1}\}$. The same description works if instead of a tensor, we have a cotensor.

The setting is that of $\mathcal{S}$-enriched categories having enough (finite) limits. Suppose now $\mathcal{C}$ is $\mathcal{S}$-enriched, so for objects $X, Y \in \mathcal{C}$, we can form a simplicial set $\underline{\mathcal{C}}(X,Y)$ of 'morphisms' from X to Y. A homotopy between $f, g \in \underline{\mathcal{C}}(X,Y)_0$ will, of course, be a 1-simplex $h \in \underline{\mathcal{C}}(X,Y)$ with $d_1 h = f$, $d_0 h = g$. If $\mathcal{C}$ is *cotensored* then, for any simplicial set K, there is a *cotensor*, $\overline{\mathcal{C}}(K,Y)$, for each Y in $\mathcal{C}$, such that

$$\mathcal{S}(K, \underline{\mathcal{C}}(X,Y)) \cong \mathcal{C}(X, \overline{\mathcal{C}}(K,Y)).$$

Of particular use is the case $K = \Delta[1]$, as a 1-simplex $h \in \underline{\mathcal{C}}(X,Y)$ can be represented by an element in $\mathcal{S}(\Delta[1], \underline{\mathcal{C}}(X,Y))$ and thus by an element of $\mathcal{C}(X, \overline{\mathcal{C}}(\Delta[1],Y))$. In other words, a homotopy is a morphism

$$h : X \to \overline{\mathcal{C}}(\Delta[1], Y),$$

so $\overline{\mathcal{C}}(\Delta[1], Y)$ behaves like a path-space object or cocylinder on Y. The construction of $\overline{\mathcal{C}}(K, Y))$ uses limits and can be 'deconstructed' to give a family based description of homotopies, just as before. The nice thing about that description is, however, that it makes sense whatever category $\mathcal{A}$ is as it is merely governed by some small list of identities between composite maps. (For any $\mathcal{A}$, $Simp.\mathcal{A}$ is $\mathcal{S}$-enriched, so can be taken to be the $\mathcal{C}$ above; see Kamps and Porter, [76] for a discussion of some of these ideas, in particular on cylinders and cocylinders as a basis for 'doing' homotopy theory in some seemingly unlikely places! We will examine simplicially enriched categories more fully later on, starting on page **??**.) A word of caution, however, is in order. As we mentioned earlier, homotopies are not always composable, nor reversible. If we have a homotopy, in this abstract setting, between morphisms f_0 and f_1 and another between f_1 and f_2, then there may not be one directly from f_0 to f_2. This is annoying! It depends on Kan filling conditions in the simplicial hom-sets. Luckily in many of the cases that we need, the composition of homotopies does work, however once or twice we will have to be careful in the wording. Of course, we could generate the equivalence relation defined by 'direct' homotopy, but, whilst this is very useful, it does often require a chain or 'zig-zag' of explicit 'direct' homotopies if it is to be of maximal use. Conditions on $\mathcal{A}$ can be found that imply that homotopy in $Simp(\mathcal{A})$ is an equivalence relation, (but I do not know if optimal such conditions are known).

Remark: We are heading for a fairly simplicial description of cohomology. A very useful reference at this point is Jack Duskin's memoir, [49], although that emphasises the Abelian theory only, and also his outline of a higher dimensional descent theory, [50]. From this simplicially based theory, it is then a short journey to give a 'crossed' description of the bitorsor based, (and then gerbe based), non-Abelian cohomology.

Pause: At this point, it is a good idea to take stock of what we have shown. We have used local sections $\{s_i\}$ to get cocycles $\{(g_{ij}, u_i)\}$ and have constructed the beginnings of a simplicial morphism $\mathbf{g}$ from $N(\mathcal{U})$ to $K(\mathsf{Aut}(G))$. So far we have explicitly given $\mathbf{g}_n$ for $n \leq 2$ only, and so should check higher dimensions as well. (Intuitively it would be strange if something came adrift in higher dimensions, since $\mathsf{Aut}(G)$ '*is* a 2-type', but we should make certain!) We also have to check our interpretation of homotopies in higher dimensions.

Let us see what $\mathbf{g}_n : N(\mathcal{U}) \to K(\mathsf{Aut}(G))$ would have to satisfy. Let

$$\mathbf{g}_n(x, U_0, \ldots, U_n) = (g_n, \ldots, g_1, \alpha),$$

then

$$\begin{aligned} d_n\mathbf{g}_n(x, U_0, \ldots, U_n) &= (g_{n-1}, \ldots, g_1, \alpha), \\ d_0\mathbf{g}_n(x, U_0, \ldots, U_n) &= (g_n, \ldots, g_2, i_{g_1} \circ \alpha), \\ d_i\mathbf{g}_n(x, U_0, \ldots, U_n) &= (g_n, \ldots, g_{i+1}g_i, \ldots, g_1, \alpha), \end{aligned}$$

for $0 < i < n$, so we *can* thus read off $\mathbf{g}_n$ from a knowledge of its faces! In other words, our intuition was right and $\mathbf{g}_0$, $\mathbf{g}_1$ and $\mathbf{g}_2$ determined $\mathbf{g}_n$ in all dimensions.

A very similar calculation shows that $\mathbf{h} : N(\mathcal{U}) \times I \to K(\mathsf{Aut}(G))$ corresponds to the 1-cocycle $\{g_i\}$ and nothing more.

We thus have established a one-one correspondence between the set of isomorphism classes of G-bitorsors that trivialise over $\mathcal{U}$ and the set $[N(\mathcal{U}), K(\mathsf{Aut}(G))]$ of homotopy classes of simplicial sheaf maps from $N(\mathcal{U})$ to the underlying simplicial sheaf of the simplicial group, $K(\mathsf{Aut}(G))$.

We should continue our pause here and make some comments about the overall situation. This set can be interpreted as a type of zeroth non-Abelian hyper-cohomology of B relative to the cover $\mathcal{U}$. It is $H^0(N(\mathcal{U}), \mathsf{Aut}(G))$. But what is hyper-cohomology? We will have a look at its classical Abelian form below, but note that the coefficients, here, are in a sheaf of crossed modules, so will also need to look at that in more detail. We saw earlier a related situation (in section 5.1) where we replaces the crossed module $\mathsf{Aut}(G)$ by a general one $\mathsf{Q} = (K, Q, q)$, when discussing non-Abelian extensions of G by K 'of the type of Q'. We there obtained a cohomology set, there called $H^2(G, \mathsf{Q})$, identifiable as $[\mathsf{C}(G), \mathsf{Q}]$, and the correspondence was obtained by identifying the cocycles as maps of crossed complexes and, as $\mathsf{C}(G)$ is 'free', it sufficed to give them on the generating elements, in other words on the analogue of $N(\mathcal{U})$.

The reason given for introducing the notion of extension of type Q was to obtain functoriality in the coefficients. (Recall that if $\varphi : G \to H$ is a homomorphism of groups then it is not clear when there is a morphism of crossed modules from $\mathsf{Aut}(G)$ to $\mathsf{Aut}(H)$ which is φ on the 'top group'.) This also gave a good possibility of a finer classification of *all* extensions of G by H: some will be of the type of a particular Q, others not.

In our bitorsor situation, the functoriality is once again important, but the second aspect gains an additional geometric significance. A very important part of classical fibre bundle theory relates to the possibility of 'reducing the group'. For instance, suppose we have a n-dimensional real manifold, X, then its tangent bundle is a fibre bundle with each fibre a vector space of dimension n and with the transition functions taking their values in $Gl(n, \mathbb{R})$, i.e., a n-dimensional vector bundle. (Its associated $Gl(n, \mathbb{R})_X$-torsor is, as we saw, the frame bundle.) If X is at all 'nice', we can put a Riemannian metric on it, (i.e., additional structure of considerable geometric importance), and this corresponds to showing that our transition functions can be replaced by ones taking values in $O(n, \mathbb{R})$, the corresponding group of orthogonal matrices, as these are the ones that preserve the metric/inner product. Note that the tangent bundle naturally has an action by $Aut(F)$, that is the corresponding automorphism group of the fibre, F. (With our bitorsors, the corresponding acting object is a strict automorphism gr-groupoid, and we have used the corresponding crossed module, $\mathsf{Aut}(G)$.)

Other examples would correspond to other subgroups of general linear groups. Foliated structures, systems of partial differential equations, etc., correspond to sub-bundles of bundles of jets on X. These structures may be on X itself or on some given fibre bundle $E \to X$ over X. In each case, giving a G-structure on E, for a group, G, which is a subgroup of the natural group of automorphisms, corresponds to 'reducing' the $Aut(F)$-torsor to a G-torsor. Another type of structure corresponds to 'lifting' the transition functions from some given H to a G, where $\varphi : G \to H$ is a nice epimorphism. For instance, the special orthogonal group $SO(n, \mathbb{R})$ for $n \geq 2$, has a universal covering group, $Spin(n) \to SO(n, \mathbb{R})$, and extra structure of use for some applications, corresponds to *lifting* the $u_{ij} : U_{ij} \to SO(n, \mathbb{R})$ to take values in $Spin(n)$. Of course, this is not always possible. Obstructions may exist to doing it, depending in part on the topological structure of X.

All these examples were of Lie groups, i.e., groups in the category of differential manifolds, but a similar intuition was central to discussions in the 1960s and 1970s of the relationship between smooth and piecewise linear structures on topological manifolds, in which various *simplicial* groups of automorphisms were related and the obstructions to lifting transition functions of certain natural simplicial bundles were the key to the problem. Again analogous situations exist in algebraic geometry involving group schemes and their 'subgroups'. Here, as a group scheme over a fixed base $Spec(K)$ is in many ways a bundle of groups, the more general theory of group bundles and change of group bundles, rather than merely change of groups, as such, is what is important here.

It would almost be fair to say that, from a historical perspective, this is one modern interpretation of Klein's original intuition of what geometry is, i.e., the study of the automorphisms that preserve some 'structure'. What seems now to be emerging is the relationship between higher level 'automorphism gadgets' such as $\mathsf{Aut}(G)$ and classical invariants such as cohomology and consequently, some appreciation of higher level 'structure'. Many of the ingredients of the theory are still missing or are merely 'embryonic' in the crossed module / 2-group case as yet, but the plan of action is becoming clearer.

Returning to the detail, we therefore consider a sheaf or bundle of crossed modules, $\mathsf{M} = (C, P, \partial)$, and look at data of the form

$$\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M}),$$

so $g_0(x, U_i) = p_i(x)$ with $p_i : U_i \to P$, a local section of P over U_i and $g_1(x, U_i, U_j) = (c_{ji}(x), p_i(x))$, where $c_{ji} : U_{ji} \to C$ is a local section of C over the intersection U_{ji}. These local sections satisfy

$$\partial(c_{ji})p_i = p_j \text{ and } c_{kj}c_{ji} = c_{ki}$$

over the intersections. Corresponding to a change in local sections will be a coboundary rule of the form:

$$c'_{ij} = c_i c_{ij} c_j^{-1},$$

and

$$p'_i = \partial(c_i)p_i,$$

i.e., a homotopy between $\mathbf{g}$ and $\mathbf{g}'$. The equivalence classes will be in $H^0(N(\mathcal{U}), \mathsf{M})$ and, both in this general case and in the particular case of $\mathsf{M} = \mathsf{Aut}(G)$, it is natural to pass to the limit over coverings (or if working in a more general Grothendieck topos, over hypercoverings) to get the zeroth Čech hyper-cohomology set with values in M, denoted $\check{H}^0(B, \mathsf{M})$.

We have $H^0(N(\mathcal{U}), \mathsf{M}) = [N(\mathcal{U}), K(\mathsf{M})]$, and it is reasonably safe to think of $\check{H}^0(B, \mathsf{M})$ in these terms, but, in fact, one really needs to introduce the category $D(\mathcal{E}) = Ho(Simp(\mathcal{E}))$, obtained by taking the category of simplicial objects in the topos, $\mathcal{E}$, in our simplest case that of simplicial sheaves on B, and inverting the 'quasi-isomorphisms', i.e., those simplicial maps that induce isomorphisms on all homotopy groups. There are several detailed treatments of this type of construction in the literature - not all completely equivalent - so we will not give another one here!

We could, and later on will, go further. We could replace the crossed module M by a crossed complex, or, in general, could use a simplicial group, H, instead of $K(\mathsf{M})$. We will definitely keep this in mind, but just because it could be done, does not mean it needs doing *now*. The problem is that we, as yet, have only an embryonic understanding of the algebraic and geometric properties of the situation with M a crossed module or bundle / sheaf of such things. Past experience shows that the generalisation and abstraction *will be* worth doing, but we may not yet have the auxiliary concepts and intuitions to interpret what that theory will tell us, nor what are the *significant* new questions to ask and problems to solve. As yet, there are few signposts in that new land!

6.4.6 Cleaning up 'Change of Base'

Although we have considered change of base several times, we have not had available enough machinery to handle it really adequately. In particular, we have left the question of homotopic maps inducing 'isomorphic torsors' up in the air. Now we can give a reasonable treatment of that results and at the same time treat change of base for bitorsors, (and in such a way as to handle change of base for relative M-torsors as well, and we have not formally defined *them* yet).

One conceptual difficulty left over from earlier was that if f and f' were homotopic maps from B to B', and P was a G-torsor on B', we want to be able to say that somehow $f^*(P)$ and $(f')^*(P)$ are isomorphic, yet they are 'over' different groups bundles. The first is a $f^*(G)$-torsor, the second a $(f')^*(G)$-torsor. This problem did not arise with principal G-bundles as there the 'coefficient group' was just that, a group, corresponding to a constant sheaf of groups, so the two coefficient 'groups', $f^*(G)$ and $(f')^*(G)$ were the same. Both were trivial. Our first task is thus to look at a simplicial treatment of change of base, and once that is done, a lot of things will simplify!

Suppose that $f : B \to B'$ is a continuous map and $\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M})$ represents either a G-torsor, or a G-bitorsor or, looking forward to the next section, a relative M-torsor, for M a sheaf or bundle of crossed modules on B' and we assume that that object trivialises over the open cover $\mathcal{U}$. The continuous function f pulls back that cover to $f^{-1}(\mathcal{U})$. This can either be viewed as the result of pulling back each open set to get a cover, or, equivalently but perhaps better, by forming the sheaf / étale space, $\bigsqcup\mathcal{U}$ over B' and then pulling back that sheaf to $f^*(\bigsqcup\mathcal{U})$. The result is the same. In fact we saw earlier that f^* preserved pullbacks and so $N(f^*(\bigsqcup\mathcal{U}))$ is isomorphic to $f^*(N(\mathcal{U}))$. This isomorphism is given by examining local sections of the two simplicial sheaves, so local sections of $f^*(\bigsqcup\mathcal{U})$ are induced by composition of f with a local section of $\bigsqcup\mathcal{U}$. (A detailed treatment is not quite that simple. The map can better be examined at the level of germs of local sections as we did in our discussion of f^*, page 149.)

Remark: In situations where hypercoverings are needed to give an adequate cohomology theory, the functor f^* still works more or less as above. Of course, the detailed geometric nature of the construction is a bit different as ideas of germs of local sections, etc., have to be interpreted slightly differently, say, in a topos, however the intuition is much the same.

Viewed as a pullback construction, there is a canonical map from $f^*(N(\mathcal{U}))$ to $N(\mathcal{U})$, namely the projection, and this is 'over' f itself. At the level of elements, this sends $(x, f^{-1}U_0, \dots f^{-1}U_n)$ to $(x, U_0, \dots, U_n)$. Abusing notation we will call this f as well. The induced cocycle is then just the composite, $\mathbf{g}f : N(f^{-1}\mathcal{U}) \to K(\mathsf{M})$, and this gives the induced torsor, but that is a $f^*(G)$-torsor. Thus at the level of the simplicial description of the induced torsor, the work is done for us without too much pain! We just have composition with f, and that, of course, is what we expected.

The next thing to look at is the connection between the induced functors for homotopic maps. We will restrict to compact spaces to simplify the discussion. If $h : f \simeq f' : B \to B'$, and we are looking at a torsor on B' that trivialises over the open cover $\mathcal{U}$, then we can get an open cover $h^{-1}(\mathcal{U})$ on $B \times I$ and a torsor on that space just by thinking of h as a continuous map. Because of our simplifying assumption of compactness, it is possible to refine $h^{-1}(\mathcal{U})$ to a cover of the form, $\{U \times V \mid U \in \mathcal{U}', V \in \mathcal{V}\}$ for $\mathcal{U}'$ an open cover of B and $\mathcal{V}$ an open cover of the unit interval I. We will denote this cover by $\mathcal{U} \times \mathcal{V}$. We can assume that the nerve of $\mathcal{I}$ is a simplicial sheaf that is essentially a subdivision $\underline{Sd}(\Delta[1])$ of the constant simplicial sheaf on I with value $\Delta[1]$. (The cover $\mathcal{V}$ may need further refinement to get it to be of this form, and you should look at this point, but we also are using that I is contractible to get that we have a trivial sheaf.) The nerve of a product cover is isomorphic to the product of the nerves as can be seen by inspection. We thus have that $N(\mathcal{U}\times\mathcal{V})$ can be replaced by $N(\mathcal{U})\times\underline{Sd}(\Delta[1])$. The subdivided $\Delta[1]$ is a concatenation of a number of copies of $\Delta[1]$, end to end, so the map induced at the simplicial level from $N(\mathcal{U}\times\mathcal{V})$ to $K(\mathsf{M})$ gives us not only the two maps induced by f and f', but also a sequence of simplicial homotopies between intermediate maps. These can be composed to get a simplicial homotopy between the original induced maps. Notice none of this uses any information about the actual torsor involved except the initial assumption that it trivialises over $\mathcal{U}$. This does it! We have a description of isomorphism classes of torsors in terms of homotopic maps, we have homotopic maps so .... .

From this lots of good things flow. Homotopically equivalent spaces, say B and B', give equivalent categories of torsors over 'linked' sheaves of groups, and, in particular, if G is a constant sheaf of groups, or M a constant sheaf of crossed modules, then over the two spaces the induced sheaves are also constant, hence we can talk of G-torsors over B or over B' without fussing too much about the fact that we really mean $\underline{G}_B$- and $\underline{G}_{B'}$-torsors.

The situation for contractible spaces is then simple. All torsors over $\underline{G}_B$ are trivial, and as a consequence, if B is a space which has an open covering by contractible open sets, and such that all finite intersections of the open sets are also contractible, (i.e., a Leray cover), then we automatically have lots of local sections over that cover. As manifolds are examples of spaces with this property, this comes in to be very useful in applications of the torsors to geometry.

6.5 Relative M-torsors

(The basic references are Breen's paper [18], (but our conventions are different and so some of the results also look different), and also the papers of Jurčo, in particular, [75].)

6.5.1 Relative M-torsors: what are they?

What are the objects corresponding to a $\mathbf{g}: N(\mathcal{U})\to K(\mathsf{M})$? We saw that this consisted of some local sections

$$p_i : U_i \to P$$

and others

$$c_{ij} : U_{ij} \to C$$

satisfying some evident relations, one of which was the cocycle condition

$$c_{kj}c_{ji} = c_{ki}.$$

These c_{ji} will give us a left C-torsor, E, say. We can examine the induced P-torsor, $\partial_*(E)$, and - surprise, surprise - the p_i part of the cocycle pair, $\{(c_{ij}, p_i)\}$, provides a trivialising coboundary, since

$$p_i = \partial(c_{ij})p_j$$

yields

$$\partial(c_{ij}) = p_i p_j^{-1} = p_i.1.p_j^{-1}.$$

Conversely suppose we have a C-torsor, E, and we know that $\partial_*(E)$ is trivial, then we can find p_is satisfying the above equations and making E into an M-torsor. If we look back to our motivating case with $\mathsf{M} = \mathsf{Aut}(G)$, then we can adapt the argument given there (page 172) to get an explicit global section of $\partial_*(E) = P_\partial \wedge^C E$, namely, for local sections e_i of E, define $\mathbf{t} = \{t_i\} = \{[p_i^{-1}, e_i]\}$ to get a compatible family and hence a global section, t, of $\partial_*(E)$. This process can be reversed, so from t and a choice of e_i, one can obtain p_i. We will see a neat way of doing this shortly.

What happens if we choose different local sections e_i' of E? These e_i' will give some c_is such that $e_i' = c_ie_i$, and also $p_i' = \partial(c_i)p_i$, but then

$$[(p_i')^{-1}, e_i'] = [p_i^{-1}\partial(c_i)^{-1}, c_ie_i] = [p_i^{-1}, e_i],$$

so the global section does not change.

We saw earlier that contracted product gave the category of G-bitorsors the structure of a group-like monoidal category with inverses, a gr-groupoid. (If P and Q are in $Bitors(G)$, then $P \wedge^G Q$ gave the 'product', whilst P^o was 'inverse' to P. Of course, the trivial bitorsor, T_G, was the identity object.) There is an obvious category of M-torsors, which we will denote by $\mathsf{M}-Tors$, (so $\mathsf{Aut}(G)-Tors = Bitors(G)$), does this in general have any similar structure?

Before we attempt to answer that, we should give formal definitions of M-torsors, etc, as a base reference:

Definition: Let $\mathsf{M} = (C, P, \partial)$ be a bundle or sheaf of crossed modules over a space B, (or more generally a crossed module in a topos $\mathcal{E}$). By a *(relative)* M*-torsor*, or M*-relative torsor* we mean a left C-torsor together with a global section t of $\partial_*(E)$.

A morphism of M-torsors, $f : (E, t) \to (E', t')$, is a C-torsor morphism, $f : E \to E'$, such that

$$\begin{tikzcd} \partial_*(E) \arrow[rr, "\partial_*(f)"] & & \partial_*(E') \\ & B \arrow[ul, "t"] \arrow[ur, "t'"'] & \end{tikzcd}$$

commutes.

We will denote the category of M-torsors by $\mathsf{M}-Tors$.

Remark on terminology: The idea of a relative M-torsor lies between that of torsors and global sections and in the long exact sequences, the $\pi_0(\mathsf{M}-Tors)$-term is the transition from global section terms, $P(B)$, etc. to true torsor terms, $\pi_0(Tors(C))$. It is a Janus, looking back and forward. Various names have been applied to this. Breen in [17], following Deligne, used something of the form (C, P)-torsor, but that does not use the boundary map, ∂ and clearly various different crossed modules having the same C and P, but perhaps different actions or boundary maps might give differently behaved (C, P)-torsors. Aldrovandi, in conversation, favours a terminology that said, what we might write as $\pi_0(\mathsf{M}-Tors)$ or $\check{H}^0(B, K(\mathsf{M}))$, was a $\check{H}^0$-term so was the group of global sections of M. That is very good terminological reasoning, but it neglects the fact that the objects *are* C-torsors plus extra structure. It looks back in the sequence and neglects the future! Using the terminology of M-torsor, which I originally favoured, fails to look back and also hitls the problem that the corresponding gr-groupoid
$mathcalM = \mathsf{M} - \mathsf{tors}$ is used later on to build $\mathcal{M}$-torsors, which are stacks with a nice action of $\mathcal{M}$, and these live at the next 'janus step' of the exact sequence. There seems no really good choice here. We have used 'relative M-torsor' or 'M-relative torsor' in the definition, but will continue to use 'M-torsor' later on as 'relative M-torsor' is quite tedious to type!

At this point, we need to revisit an old intuition that we have used several times before, but without which 'life' will seem unduly complicated! That intuition is that a principal G-set is a copy of G with an 'identity crisis'. In more detail, in situations such as that of universal covering spaces, E over a space B, the fibre is a copy of $\pi_1(B)$, but without a definite element being chosen as the identity. The natural path lifting property of covering spaces gives that any loop γ at a chosen base-point b_0 in B will lift uniquely to a path in the covering space, *once a start point e_0 above b_0 has been chosen.* If you choose a different start point e_0', you, of course, get a different lifted path. The end point of the lifted path will give the image of e_0 under the action of the path class $[\gamma] \in \pi_1(B)$. Thus once e_0 is chosen $p^{-1}(b_0) = E_{b_0}$ can be mapped bijectively to $\pi_1(B)$. (Remember we did say E was a *universal* covering space.) Under this bijection, the identity element of $\pi_1(B)$ corresponds to e_0, but our alternative choice, e_0', will give a bijection with e_0' itself corresponding to $1_{\pi_1(B)}$. There is no canonical choice of start point in E_{b_0}, so no definitive identification of E_{b_0} with $\pi_1(B)$.

For a G-bitorsor, with a local section $e_i : U_i \to E$, we have essentially the same situation. The left and right G-actions are globally independent and yet are locally linked by the $u_i : G_{U_i} \to G_{U_i}$. To use these it *is* necessary to use the e_i to temporarily pick a 'start point' in each fibre of E. Thus the equation,

$$u_i(g).e_i = e_i.g,$$

interprets as both the definition of u_i given the right action *and* conversely, given the u_i, as a defining equation of a right action. This does need to be spelt out again: given any local element x of E over U_i, it has the form $x = g'e_i$ for some local element g' of G. Suppose we now operate with g on the right of x, then we get

$$x.g = g'e_i.g = g'u_i(g)e_i.$$

(This is very analogous to defining a linear transformation between vector spaces by transforming the elements of a chosen basis and then 'extending linearly'. Here we extend G-equivariantly for the *left* action, having transformed the 'basic' element e_i to $e_i.g$.)

The key transition equation for the u_is was

$$u_i' = i_{g_i} \circ u_i,$$

which emphasises this viewpoint. We changed e_i to e_i' using g_i, so $e_i' = g_ie_i$, but then, for right action by g,

$$e_i'g = u_i'(g)e_i' = u_i'(g)g_ie_i,$$

whilst also

$$e_i'g = g_ie_ig = g_iu_i(g).e_i,$$

giving the transition equation in the form $g_iu_i(g) = u_i'(g)g_i$.

We now need to translate this into a tool that can be used for M-torsors. The plan of action is to show that any M-torsor, E, has a natural C-bitorsor structure and for this we have to use $t : B \to \partial_*(E)$ to obtain a right C-action on E. In Lemma 16, (page 151), we saw how to go from a global section of a torsor to an identification of it as an 'identity-less' copy of the group bundle. We thus have that t allows us to identify $\partial_*(E)$ with T_P, i.e., with P itself (as left P-torsor). We can unpack the recipe in Lemma 16, (but beware the change of notation, P is here the basic group of our crossed module M, but was the torsor in that earlier discussion). Any local element of $\partial_*(E)$ over some U_i is of form $[p, e]$, with p a local section of P over U_i and e a local section of E, again over

U_i. We can get from t an expression $[p, e] = p'.t$ for some p' defined over U_i. Using the structural map of $\partial_*(E)$ as a P-torsor, we get

$$\partial_*(E) \stackrel{(t\pi, id)}{\rightarrow} \partial_*(E) \times \partial_*(E) \stackrel{\cong}{\rightarrow} P \times_B \partial_*(E) \stackrel{proj}{\rightarrow} P,$$

which, from $[p, e]$ gives the p'. (Recalling that, given e_i, the unadjusted choice of local sections is $[1, e_i]$, then this process picks out the corresponding p_i, so that $t = [p_i^{-1}, e_i]$.) Thus from t, we get a map from $\partial_*(E)$ to P.

In this 'game', it pays to go back-and-fore between the different descriptions and to revisit the special case, $\mathsf{M} = \mathsf{Aut}(G)$, for guidance, and, hopefully, inspiration. Our key equation defining the u_i was $u_i(g)e_i = e_i.g$. In our general case of $\mathsf{M} = (C, P, \partial)$, the rôle of the u_i is taken by the local elements p_i, which act on C (since, recall, that action is part of the crossed module structure) and the corresponding equation would be

$$^{p_i}c.e_i = e_i.c,$$

but $e_i.c$ is not defined, a least not yet! We will take this as its definition (and remember our earlier discussion of right actions, and what here would be the C-equivariant extension), then see if it works!

First let us underline what the equation actually says. An arbitrary local element of E_{U_i} has form $e = c_i.e_i$ and the expression for $e.c$ will be $c_i.{}^{p_i}c.e_i$ as the right action has to be left C-equivariant, now if $c_1, c_2 \in C_{U_i}$, then

$$(e_i.c_1).c_2 = {}^{p_i}c_1.e_i.c_2 = {}^{p_i}c_1.{}^{p_i}c_2.e_i = {}^{p_i}(c_1c_2).e_i = e_i.(c_1.c_2),$$

so it does define an action, at least locally. Next we have to check on intersections. Supposing that p_i on U_i and p_j on U_j satisfy $p_j = \partial(c_{ji})p_i$, where $e_j = c_{ji}e_i$, then over U_{ij},

$$e_j.c. = c_{ji}e_i.c = c_{ji}\,{}^{p_i}c.e_i = c_{ji}\,{}^{p_i}c_{ji}^{-1}.e_j$$

and also

$$e_j.c = {}^{p_j}c.e_j = {}^{\partial(c_{ji})p_i}c.e_j,$$

and the Peiffer rule for crossed modules gives

$$^{\partial c}c' = cc'c^{-1},$$

so the two local actions patch together neatly. We thus have an action of C on the right of E. Is it giving us a right C-torsor structure on C? This amounts to asking if locally the equation $x = yc$ can be solved uniquely for c in (some) terms of x and y over U_i, but $x = c'.y$ for a unique c', since E is a left C-torsor. The obvious element to try out as our required solution, c, is ${}^{p_i^{-1}}c'$ - **try it**! It works. We have proved:

Lemma 23 *If (E, t) is a M-torsor, then E is a C-bitensor.* ■

From another perspective, this is quite clearly due to the natural map from M to $\mathsf{Aut}(C)$, given by the identity on C and the action map

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d] & C \arrow[d] \\ P \arrow[r, "\alpha"'] & Aut(C) \end{tikzcd}$$

We would expect an M-torsor to be mapped to a $\mathsf{Aut}(C)$-torsor, that is, a C-torsor, via this morphism of crossed modules, so from this viewpoint the lemma may not seem surprising.

A few pages ago, we set out to extend the contracted product to M-torsors. Now that we have this lemma, we can, at least, work with a contracted product of the associated C-bitorsors. In other words, if (E_1, t_1), (E_2, t_2) are M-torsors, then we might tentatively explore a definition of $(E_1, t_1) \wedge^{\mathsf{M}} (E_2, t_2)$ as being $(E_1 \wedge^C E_2, t)$ with t still to be described. Here is a suitable, almost heuristic, approach that tells us we are going in the right direction.

We have $\partial_*(E) = P_\partial \wedge^C E_1$, where P_∂ is the trivial (left) P-torsor with, in addition, a right C-action given by : if $x \in P_\partial$, $x = p.t$, where t is a global section (fixed for the duration of the calculation), then, for $c \in C$, $x.c = p.\partial(c).t$. Now if $\partial_*(E)$ is assumed to have a global section, it is easy to show that it is, itself, isomorphic to P_∂. Next look at (E_1, t_1), and (E_2, t_2) and let us examine $\partial_*(E_1 \wedge^C E_2)$. This is $P_\partial \wedge^C E_1 \wedge^C E_2 = (P_\partial \wedge^C E_1) \wedge^C E_2 \cong P_\partial \wedge^C E_2$ by the above calculation, using t_1 to trivialise $(P_\partial \wedge^C E_1)$, and finally this is trivial using t_2.

This argument, although valid, merely shows that t exists. It could be taken apart further to get an explicit formula, but we will, instead, approach that through cocycles. We pick local sections of E_1 and E_2 over the same open cover $\mathcal{U}$. These we will denote by $e_i^1 : U_i \to E_1$, $e_i^2 : U_i \to E_2$. Given t_1 and t_1, we get local elements of P, p_i^1 and p_i^2, so that

$$t_1 = [(p_i^1)^{-1}, e_i^1],$$

and similarly for t_2. These p_i^1s are those for the local cocycle description of E_1 as (c_{ij}^1, p_i^1), so are the parts of the contracting homotopy on $\partial_*(E_1)$, etc.

Now look at $E_1 \wedge^C E_2$. The obvious local sections of this would be $e_i = [e_i^1, e_i^2]$, and using these we want to work out the corresponding cocycle pair. We need to work out the relationship of e_i with $e_j = [e_j^1, e_j^2]$. We have $e_i^1 = c_{ij}^1 e_j^1$, $e_i^2 = c_{ij}^2 e_j^2$, so

$$\begin{aligned} (e_i^1, e_i^2) &= (c_{ij}^1 e_j^1, c_{ij}^2 e_j^2) \equiv c_{ij}^1 (e_j^1, c_{ij}^2 e_j^2) \\ &= c_{ij}^1 ({}^{p_j^1} c_{ij}^2 . e_j^1, e_j^2) = c_{ij}^1 \, {}^{p_j^1} c_{ij}^2 (e_j^1, e_j^2), \end{aligned}$$

and we have $e_i = c_{ij}^1 \, {}^{p_j^1} c_{ij}^2 . e_j$. This C-coefficient may look familiar (or not), but before we identify it, we should look for the p_is. The obvious ones to try are $p_i = p_i^1 p_i^2$, i.e., the product within P of the two values. We have a $c_{ij} = c_{ij}^1 . {}^{p_j^1} c_{ij}^2$, so can see if this works for the equation $p_i = \partial(c_{ij}) p_j$:

$$\begin{aligned} p_i = p_i^1 p_i^2 &= \partial(c_{ij}^1) p_j^1 . \partial(c_{ij}^2) p_j^2 \\ &= \partial(c_{ij}^1) p_j^1 . \partial(c_{ij}^2) (p_j^1)^{-1} p_j^1 p_j^2 = \partial(c_{ij}) p_j . \end{aligned}$$

The simplicial interpretation of the cocycles gave a map from $N(\mathcal{U})$ to $K(\mathsf{M})$, and in dimension 1, $K(\mathsf{M})$ is $C \rtimes P$. The multiplication in this semidirect product is

$$(c_1, p_1).(c_2, p_2) = (c_1 \, {}^{p_1} c_2, p_1 p_2).$$

In other words, if (E_1, t_1) corresponds to a simplicial map $\mathbf{g}_1 : N(\mathcal{U}) \to K(\mathsf{M})$ and similarly $\mathbf{g}_2$ corresponding (E_2, t_2), then $(E_1, t_1) \wedge^{\mathsf{M}} (E_2, t_2)$ is associated to the product $\mathbf{g}_1.\mathbf{g}_2$,

$$N(\mathcal{U}) \to K(\mathsf{M}) \times K(\mathsf{M}) \to K(\mathsf{M}),$$

using the multiplication map of the simplicial group $K(\mathsf{M})$ corresponding to the crossed module, M. Does this give us a gr-groupoid structure on $\mathsf{M}-Tors$? The above description of the multiplication as corresponding to contracted product tells us that we can use the inverse of that multiplication to construct an inverse for the contracted product. The detailed formula for the inverse of an M-torsor, (E,t), is **left as an exercise**.

Note that we have not checked certain necessary facts about the (c_{ij}, p_j), namely that $c_{ij}c_{jk} = c_{ik}$ and they transform correctly under change of local sections. The details of these are **left to the reader**. They use the crossed module axioms several times. We have proved the following:

Proposition 41 *Under the identification of $\pi_0(\mathsf{M}-Tors)$ and $\check{H}^0(B,\mathsf{M})$, the group structure on the first given by the contracted product coincides with that given on the second under the group structure of $K(\mathsf{M})$, the associated simplicial group bundle of the bundle of crossed modules, M.* ■

6.5.2 An alternative look at Change of Groups and relative M-torsors

When we discussed change of groups, we saw a neat induced torsor construction. Recall we had

$$\varphi : G \to H,$$

a morphism of sheaves of groups and a torsor E over G, we obtained $\varphi_*(E)$ by first forming H_φ, i.e. the (H,G)-object with right G-action given via φ and then $\varphi_*(E) = H_\varphi \wedge^G E$.

This construction has various universal properties that we have not yet made explicit nor exploited, yet which are very useful. We will need to recall that if P and Q are two G-torsors, a morphism $f : P \to Q$ is a map over B such that $f(g.p) = g.f(p)$ for all $g \in G$ and $p \in P$. In other words, it is a sheaf map $f : P \to Q$, which is G-equivariant. We can represent this by a diagram:

$$\begin{tikzcd} G \times_B P \arrow[r, "G\times f"] \arrow[d] & G \times_B Q \arrow[d] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

in which the vertical maps give the actions, and which is required to commute.

There is a neat notion from the theory of group actions (on sets), which adapts well to the torsor context. Suppose that $\varphi : G \to H$ is a homomorphism of ordinary groups, and (X, a_X) and (Y, a_Y) are a G-set and an H-set respectively, with $a_X : G \times X \to X$ and $a_Y : H \times Y \to Y$ being the actions. A map $f : X \to Y$ is said to be *over* φ if for all $x \in X$ and $g \in G$, we have $f(g.x) = \varphi(g).f(x)$. This is, of course easily represented by a similar commutative diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\varphi\times f"] \arrow[d, "a_X"'] & H \times Y \arrow[d, "a_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It thus follows that a G-map between G-sets is a slightly degenerate form of this notion.

Before we return to the situation of torsors, it will pay to note that φ makes H into a *right* G-set and that $\varphi_*(X)$ as being $H_\varphi \wedge^G X$, makes sense here as well. suppose $f : X \to Y$ is over

φ in the above sense, then we look at f and see if it induces an H-map from $\varphi_*(X)$ to T. The elements of $\varphi_*(X)$ will be equivalence classes of pairs (h,x), where $(h, g.x) \equiv (h\varphi(g), x)$. We write $[(h,x)]$ for the equivalence class and try to guess what form an map induced from f might take. The obvious form to try would seem to be to set $\tilde{f}[(h,x)] = h.f(x)$ and to see if this works. Even though this is easy, let us do it explicitly:

$$h.f(g.x) = h.\varphi(g)f(x),$$

since f is over φ, but $\tilde{f}[(h\varphi(g), x)] = h.\varphi(g)f(x)$ as well, so we are done. We note, however, that this is really the only sensible way to define such a $\tilde{f}$. This is thus well defined as an H-map from $\varphi(X)$ to Y. (The fact that it is an H-map **should be clear**.)

We now have $f : X \to Y$ and $\tilde{f} : \varphi_*(X) \to Y$, so is there a possible factorisation of f as a composite of some map $X \to \varphi_*(X)$ over φ followed by $\tilde{f}$? There is an obvious map from X to $\varphi_*(X)$ namely that which sends x to $[(1_H, x)]]$. This then sends $g.x$ to $[(1_H, g.x)]$, which is the same as $[(\varphi(g), x)]$, which is $\varphi(g)[(1_H, x)]$, by the definition of the left H-action on $H_\varphi \wedge^G X$. This is thus a map over φ as expected and does not depend on f itself.

Going back to $\tilde{f}$, we hinted that this might be unique in some sense. What sense? First let us give a name to the map that we have just examined, say $\varphi_\sharp : X \to \varphi_*(X)$. We noted that $f = \tilde{f}\varphi_\sharp$ - but did not **check it**. That done, suppose we had some 'other' H-map $f' : \varphi_*(X) \to Y$, so that $f = f'\varphi_\sharp$, then $f'[(1,x)] = f(x)$, but f' is assumed to be an H-map, so $f'[(h,x)] = f'(h.[(1,x)]) = h.f(x)$ and $f' = \tilde{f}$.

If we write $Maps_\varphi(X,Y)$ for the set of maps from X to Y over φ, we have shown it to be isomorphic to $H - Sets(\varphi_*(X), Y)$. As both are functorial in Y, and (**for you to check**), the isomorphism is natural, we have shown that $Maps_\varphi(X, -)$ is a representable functor with $\varphi_*(X)$ as a representing object. There are still more things to work through and question here. What happens if we change X, for instance? But these can be **left to the reader**.

We did the above in the easy case of $Sets$, now transport the idea across to $Sh(B)$, or better still, to an arbitrary topos, $\mathcal{E}$. We have our original situation of a morphism, $\varphi : G \to H$, of sheaves of groups. We suppose E is a G-torsor and E' an H-torsor.

Definition: A sheaf map $f : E \to E'$ is said to be a *morphism of torsors over* φ if the diagram:

$$\begin{tikzcd} G \times E \arrow[r, "\varphi\times f"] \arrow[d, "a_E"'] & H \times E' \arrow[d, "a_{E'}"] \\ E \arrow[r, "f"'] & E' \end{tikzcd}$$

commutes, the vertical arrows representing the actions.

We can equally well state this in terms of 'local elements'. (The choice of the approach used is largely a question of taste and is left to you.It is advisable to be able to follow and use any of the different methods when handling such discussions - although you may prefer one, say the diagrammatic one, to some other.)

We will write $Sh(B)_\varphi(E, E')$ for the sheaf of morphisms over φ from E to E'. (This is sloppy as E and E' really have to have the actions included in their labeling, but this is fairly anodyne sloppiness.) It should now be easy to prove:

Proposition 42 *(i) For any E, E' as above, there is a natural isomorphism of sheaves*

$$Sh(B)_\varphi(E,E') \cong Tors(H)(\varphi_*(E),E').$$

(ii) The functor $Sh(B)_\varphi(E,-)$ is representable. ■

Although easy, there are quite a lot of things to **check** here!

We thus have a neat universal property for φ_* as a functor from $Tors(G)$ to $Tors(H)$. We can now apply it to the case of relative M, where $\mathsf{M} = (C,P,\partial)$ is a sheaf of crossed modules. We had a description of a relative M-torsors as a C-torsor, E, together with a specified trivialisation $t : \partial_*(E) \stackrel{\cong}{\to} T_P$.

Proposition 43 *Suppose E is a C-torsor and $t : E \to T_P$, a morphism over ∂, then $(E,\tilde{t})$ is an M-torsor. Conversely f $E,f)$ is a relative M-torsor, then E is a C-torsor and $f\partial_\sharp : E \to T_P$ is a morphism of torsors over ∂.*

Proof: this is mostly just a corollary of our earlier result. The one point is that $\tilde{t} : \partial_*(E) \to T_P$ is a morphism of H-torsors, and hence is an isomorphism, hence, also, $\tilde{t}^{-1}(1_P)$ is a global section of $\partial_*(E)$. ■

We can use this to get a separate description of the category of M-torsors, which incidentally justifies the choice of name 'relative M-torsors' as they are somehow 'relative to T_P in a controlled way. In this description a morphism of M-torsors is a C-torsor morphism, f, making

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "t"'] & & E' \arrow[dl, "t'"] \\ & T_P & \end{tikzcd}$$

commute. (Here f is a C-torsor map, but t and t' are maps over φ. This diagram thus 'lives' in the category of sheaves on B.)

We will categorify this description later replacing M by a lax gr-groupoid, and, in fact, in a particular case by $\mathsf{M}-Tors$ itself, but all that requires stacks for a thorough handling, so must wait.

6.5.3 Examples and special cases

Right at the start of our discussion of crossed modules, in section 2.1, we gave various different examples. One was the $(G, Aut(G), \partial)$ case, where ∂ sending g to the inner automorphism determined by g. Others were normal subgroups and P-modules. We based the definition of (relative) M-torsor on that of G-bitorsor and thus on the first of these. What about the others?

(i) To take an almost silly example, let $\mathsf{M} = (1,P,inc)$, that is, the case $C = 1$. If $\mathcal{C}$ is our open cover, then the cocycle description of M-torsors gives us a family of local sections of P, say, $u_i : U_i \to P$, satisfying $p_i = p_j$ on intersections, $U_i \cap U_j$, but that means that the family glues to a global section of P. Conversely any global section of P gives a morphism from $N(\mathcal{U})$ to M. (We leave to the reader the examination of how this corresponds to a 1-torsor that yields a trivial

P-torsor on application of ∂_*.) Thus in this case, M-torsors are just global sections of P and $\check{H}^0(B, \mathsf{M}) \cong \check{H}^0(B, P)$. (There is no question of coboundaries or equivalent cocycles as there is nothing above dimension 0 in M.)

(ii) The other extreme case is when C is Abelian and P is trivial. (We will sometimes write this as $C[1] = (C \to 1)$. It is a 'suspended' or 'shifted' form of C.) Here we just have a C-torsor E, and, of course $\partial_*(E)$ is a 1-torsor! There is not much choice of trivialisation, so we just have that C-torsor. In this case, we have $\check{H}^0(B, \mathsf{M}) \cong \check{H}^1(B, C)$, that is, cohomology in the old sense of Abelian cohomology.

(iii) The next obvious case is 'inclusion crossed modules' or 'normal subgroup pairs'. In other words, suppose N is a normal subgroup of P and M is the corresponding crossed module. (We write ∂ for the inclusion of N into P.) We would expect that, writing G for P/N, an M-torsor would be more or less the same, up to equivalence perhaps, as a $(1 \to G)$-torsor, i.e., to a global section of G. The conditions on the local sections p_i over some cover $\mathcal{U}$, and the corresponding n_{ij} are now

$$p_i = n_{ij} p_j,$$

as well as $n_{kj} n_{ji} = n_{ki}$.

Remark: There is a morphism of crossed modules with kernel $(N, N, =)$ giving a short exact sequence,

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 1 \arrow[d] \\ N \arrow[r] & P \arrow[r, "\varphi"] & G \end{tikzcd}$$

we know that this will give a short exact sequence of simplicial groups and that M-torsors correspond to maps from $N(\mathcal{U})$ to $K(\mathsf{M})$ if they trivialise over the open cover $\mathcal{U}$. Our observation that M-torsors might lead to global sections of G relates to composition with the quotient map φ from M to $(1, G, inc)$. (This raises the question of maps of crossed modules inducing functors between the corresponding categories of torsors, in general. We will return to this shortly.)

Looking in more detail, suppose we have a M-torsor specified by a cocycle pair (p_i, n_{ij}) over some open cover $\mathcal{U}$, and we write g_i for $\varphi(p_i)$, then the g_is *do* form a global section of G, since they are compatible over the intersections. Conversely, given a global section g of G, we know that φ is an epimorphism of sheaves, so would like to lift g to something in P. This situation is one we have encountered before and will do so again later. An epimorphism of sheaves need not be an epimorphism of the underlying presheaves. In our spatial context, it will be an epimorphism on stalks, however. We thus do not know if there is a global section p of P satisfying $\varphi(p) = g$, but, thinking about the idea of stalk, for any $b \in B$, and any open set U containing b, there is a representative (g_U, U) of the element $g_b = g(b)$, which is in the stalk over b. As φ is an epimorphism on stalks, we can choose U such that there is a $p_u \in P(U)$ with $\varphi_U(p_U) = g_U$. This gives us an open cover $\mathcal{U}$ of B and a family of local section of P over $\mathcal{U}$. Next look at the intersections, $U \cap V$, of sets from $\mathcal{U}$. There the restrictions of p_U and p_V need not agree, but as their images are the same under φ, there is a $n_{U,V}$ in N over $U \cap V$, which satisfies $p_U = n_{U,V} p_V$, and the family of these ns satisfy the cocycle condition, so from our global section of G, we have constructed a cocycle pair for an M-torsor. Different liftings of g give local sections that agree up to a coboundary, n_u, (possibly on a joint refinement of the covers), so M-torsors do give global sections of G, and *vice versa*.

(iv) The last case is $\mathsf{M} = (M, G, 0)$, i.e. M is a sheaf of G-modules. Here we have that cocycle pairs, (g_i, m_{ij}), must satisfy

$$g_i = \partial(m_{ij})g_j,$$

but ∂ is trivial, so the g_is give a global section, whilst the m_{ij} give a M-torsor in the usual sense.

This example is good because it links M-torsors in this case with M-torsors *and* global sections, i.e., some sort of 'extension', $G(B) \to \mathsf{M}-Tors \to Tors(M)$, or perhaps in the other order? We have not analysed the effect of the action of G on M. Does this mean that we have some sort of 'G-equivariant' cohomology, or cohomology of the sheaf of groups G with coefficients in the G-module M, ... and what about the gr-category structure. The detailed examination of all the structures involved is interesting and useful to do, so is, once again, **left as an exercise**.

This class of examples is also very important as amongst the examples of this type are, of course, the G-bitorsors with G a sheaf of Abelian groups, since for such a G, we have that $\mathsf{Aut}(G)$ is of the form $(G, Aut(G), 0)$. The best known example is where G is $U(1)$ or equivalently $Gl(1, \mathbb{C})$, the group of unit modulus complex numbers. We will return to this later.

The above discussion suggests some interesting areas to explore. Reaction of these M-torsors to 'change of M', short exact sequences of sheaves of crossed modules and their 'reflection' in the behaviour of the M-torsors, etc. One particular short exact sequence is

$$\begin{tikzcd} K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N \arrow[d] \\ 1 \arrow[r] & P \arrow[r] & P, \end{tikzcd}$$

where $K = Ker\,\partial$ and $N = Im\,\partial$. It suggests that $\mathsf{M} - Tors$ is an extension of $G(B)$ by a category of K-torsors for an Abelian group sheaf, K, somehow twisted by the G-action. After examining one or two related subjects, we will be able to give a bit more insight and precision about this idea.

6.5.4 Change of crossed module bundle for 'bitorsors'.

We now have a very thorough knowledge of G-bitorsors and the more general (relative) M-torsors, via the link with simplicial maps from $N(\mathcal{U})$ to $K(\mathsf{M})$, but, of course, that link makes change of 'coefficients' more or less obvious.

First it should be noted, once again that the identification of $\check{H}^0(B, \mathsf{Aut}(G))$ as a second non-Abelian cohomology group of B with coefficients in G, runs foul of non-functoriality in G, but that this is not due to some subtle deep property of non-Abelian cohomology, rather it is due to the banal failure of $Aut(G)$ to be functorial in G, in other words, to a low level group theoretic fact, low level but in fact fundamental. It is here group theoretic, but generally automorphism groups do not vary functorially - and that opens the way to crossed modules.

If $\varphi : G \to H$ is a morphism of group bundles, then there may, or may not, be a morphism $\varphi' : Aut(G) \to Aut(H)$ such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "i"'] & H \arrow[d, "i"] \\ Aut(G) \arrow[r, "\varphi'"'] & Aut(H) \end{tikzcd}$$

is a morphism of crossed modules.

There is an induced morphism on $\check{H}^0(B, \mathsf{Aut}(G))$ if such a φ' does exist, and, of course, in more generality, if we have that $\varphi : \mathsf{M} \to \mathsf{N}$ is a morphism of crossed modules, then there is an induced homomorphism of groups

$$\varphi_* : \check{H}^0(B, \mathsf{M}) \to \check{H}^0(B, \mathsf{N}).$$

(It could happen that two crossed modules of the form $\mathsf{Aut}(G)$ could be linked by a zig-zag of other crossed modules so that the morphisms in the reverse direction were weak equivalences / quasi-isomorphisms in our earlier sense, and then there would be an induced map between the two $\check{H}^0(B, \mathsf{Aut}(G))$ groups. We will explore this more fully later on, using the beautiful theory of 'butterflies' as developed by Noohi, [**?** **?**].)

Exploring the above at a gr-groupoid level, i.e., on $\mathsf{M}-Tors$ with contracted product, rather than at connected component / cohomology level, we get an induced gr-functor between $\mathsf{M}-Tors$ and $\mathsf{N}-Tors$, since it uses the functor K from crossed modules to simplicial groups. Explicitly $\varphi : \mathsf{M} \to \mathsf{N}$ induces $K(\varphi) : K(\mathsf{M}) \to K(\mathsf{N})$, a morphism of simplicial groups, but then our identification of the contracted product structure on $\mathsf{M}-Tors$ as being induced from the simplicial group structure of $K(\mathsf{M})$ immediately implies that $K(\varphi)$ induces a functor from M–$Tors$ to N–$Tors$ compatibly with the gr-groupoid structures.

6.5.5 Representations of crossed modules.

In the classical group based case, the naturally occurring vector bundles such as the tangent and normal bundles had the general linear group of some dimension as the basic G over which one worked. Extra structure corresponded to restricting to a subgroup or lifting to some 'covering group'. We recalled earlier, e.g., page 137, that the fibres of the bundles were vector spaces with an action of the chosen group, i.e., a matrix representation of that group. What is, or should be, the representation theory 'of crossed modules'? There are several tentative answers.

A representation of a (discrete) group G and thus an action of G on some object, can be thought of in different ways. For instance, as a group homomorphism $G \to H$, where H is some group of permutations or matrices in which we can use methods from outside group theory, perhaps combinatorics, perhaps linear algebra, to analyse more deeply the properties of the elements of G. We could also consider this as a functor from $G[1]$, the corresponding groupoid with one object, to $Sets$ for the permutation representations, or to some category of vector spaces or modules in the linear case.

The generalisations are to 'categorify' this second description by taking $\mathcal{X}(\mathsf{M})$, the 2-groupoid with one object (i.e., the 2-group) of M, and looking for a nice category of '2-vector spaces' or '2-modules'. (The permutation version has not been that well explored yet, but we will see some ideas later on.) Some doubt exists as to what is the 'best' category of '2-vector spaces' to use, in fact the discussion is really about what that term should mean. We mention two possibilities here, but there may be others. The first is due independently to Forrester-Barker, [58], and to Baez and Crans, [8]. The second is based on an idea of Kapranov and Voevodsky, [79], using more monoidal category theory than we have been assuming so far.

Here we will adopt the simpler version, more as an illustration then as a claim that this is the 'correct' version. The motivation for the definition, used by Forrester-Barker and by Baez and Crans, is that, as crossed modules are category objects in the category of groups, for a linear representation theory of such things, it is natural to try category objects in the category of vector spaces, but such objects are equivalent to short complexes of vector spaces. The idea is also that some of the potential applications of the structures that we have been studying use chain complexes as coefficients. (We will see this more clearly in the later discussion of hyper-cohomology.) Keeping things simple, we look at chain complexes of vector spaces (or more generally of modules) of length 1. (Warning: for us here 'length 1' means *one morphism*, $C_1 \to C_0$, not 'one group' so our objects are linear transformation between vector spaces and our morphisms are commutative squares.) These are highly Abelian versions of crossed modules, so we will use similar notation such as C, D, etc., for them.)

We recall that chain complexes have a natural 'internal hom' construction, well known from classical homological algebra. (We will see this again in our discussion of hyper-cohomology so will treat it in more detail there.) The chain complex, $Ch(\mathsf{C}, \mathsf{D})$, has graded maps of degree n in dimension n, so, for instance, has chain homotopies in dimension 1. Putting $\mathsf{D} = \mathsf{C}$ and looking at the invertible maps gives an automorphism group, $\mathsf{Aut}(\mathsf{C})$, which is also a chain complex of groups, i.e. we get a crossed module. If we have a general (discrete) crossed module M, we can consider a morphism $\mathsf{M} \to \mathsf{Aut}(\mathsf{C})$ as a representation of M, and can talk of M acting on C by 'linear maps'. We will not explore this further here, but note that we are very near the idea of representing a simplicial group as a simplicial group of simplicial automorphisms, somewhat as in section 5.3. At present the available discussions of 2-group representations of this form include the thesis, [58], and papers, [8]. A more extensive use of monoidal category theory would allow us to consider a variant that considers 2-vector spaces to mean the 2-categorical version of the monoidal category of vector spaces. We will return to this later.

Chapter 7

Hypercohomology and exact sequences

7.1 Hyper-cohomology

7.1.1 Classical Hyper-cohomology.

We have several times mentioned this subject and so should provide some slight introduction to the basic ideas. We will go right back to basics, even though we have already used some of the ideas previously, usually without comment. Most of this first part may be well known to you.

The basic idea is that of a graded, or more precisely $\mathbb{Z}$-graded, group and variants such as graded vector spaces, or graded modules, or sheaves of such on some space, B or in some topos $\mathcal{E}$.

Definition (First form): A *$\mathbb{Z}$-graded vector space* (gvs) is vector space together with a direct sum decomposition, $\mathsf{V} = \bigoplus_{p\in\mathbb{Z}} V_p$. The elements of V_p are said to be *homogeneous of degree p*. If $x \in V_p$, write $|x| = p$.

A graded vector space could equally well be defined as a family $\{V_i\}_{i\in\mathbb{Z}}$ of vector spaces, since we could then form their direct sum and obtain the first version.

Definition (Second form): A *$\mathbb{Z}$-graded vector space* (gvs) is a $\mathbb{Z}$-indexed family, $\{V_i\}_{i\in\mathbb{Z}}$, of vector spaces.

(The definitions are, pedantically, not completely equivalent as one can have a constant family with all V_i equal, but that is really a smokescreen and causes no problem.)

Both versions are useful. For example, if K is a simplicial set, we can define a graded vector space using the second version by taking V_n to be the vector space with basis indexed by the elements of K_n if $n \geq 0$ and to be the trivial vector space if $n < 0$. From our treatment of simplicial sets, it would be somewhat artificial to define $\mathsf{V} = \bigoplus_{i\in\mathbb{Z}} V_i$. For another example, the other description fits better. The polynomial ring, $\mathbb{R}[x]$, is a graded vector space with V_n having basis $\{x^n\}$, i.e., V_n is the subspace of degree n monomials over $\mathbb{R}$. The whole space, $\mathbb{R}[x]$, is here by far the more natural object.

For graded groups, etc., just substitute 'group' etc. for 'vector space' and correspondingly, 'direct product' for 'direct sum'.

Definition: A morphism $f : \mathsf{V} \to \mathsf{W}$ of graded vector spaces is *homogeneous* if $f(V_p) \subseteq W_{p+q}$ for all p and some common q, called the *degree of* f. The set of such morphisms of given degree is $Hom(\mathsf{V}, \mathsf{W})_q = \prod_p Hom(V_p, W_{p+q})$.

An endomorphism, $d : \mathsf{V} \to \mathsf{V}$, of degree -1 is called a *differential* or *boundary* (which depending largely on the context) if $d \circ d = 0$.

A gvs with a differential is really just a chain complex, where $d_n : V_n \to V_{n-1}$ and $d_{n-1}d_n = 0$.

Definition: A graded vector space together with a differential is variously called a *differential graded vector space* (dgvs), or a *chain complex*. Some authors reserve that latter term for a positively graded differential vector space, or module, or .... . The elements of V_n are called *n-chains*, those of $Ker\, d_n$, *n-cycles*, and those of $Im\, d_{n+1}$, *n-boundaries*.

A graded vector space V is *positively graded* if $V_i = 0$ for all $i < 0$. It is, on the other hand, *negatively graded* if $V_i = 0$ for $i > 0$.

The classical convention is to write V^{-n} instead of V_n for all n in the negatively graded case. This, of course, has the effect that if (V, d) is a differential graded vector space which is negatively graded, then d has apparent degree + 1, $d^n : V^n \to V^{n+1}$. In the usual terminology that will give a *cochain complex*. For some purposes, it is usual to adapt the terminology somewhat, for instance to use chain complex as a synonym for dgvs without mention of positive or negative, but then also to use cochain complex for what is essentially the same type of object, but with 'upper index' notation, so $\mathsf{V} = (V^n, d^n)$ with $d^n : V^n \to V^{n+1}$. Terms such as 'bounded above', 'bounded below' or simply 'bounded' are also current where they correspond respectively to $V_n = 0$ for large positive n, or large negative n or both. We will make little use, if any, of these in the context of these notes, but it is a good thing to be aware of the existence of the various conventions and to check before assuming that a given source uses exactly the same one as that which you are used to!

For simplicity of exposition, we will initially concentrate our attention on general dgvs, which we will often call *chain complexes* and will attempt to be reasonably consistent - although that is virtually impossible! We will extend that terminology to dg-modules and dg-groups if and when needed.

- The elements of a chain complex are called *chains*. If $c \in C_n$, it is an *n-chain*. If $dc_n = 0$, it is called an *n-cycle* and, if $c \in Im\, d_{n+1}$, an *n-boundary*. If 'n' is not important, or is understood, it may be omitted.
- A *chain map* $f : \mathsf{V} \to \mathsf{W}$ of chain complexes is a graded map of degree 0, $\{f_n : V_n \to W_n\}$ compatible with the differentials, so, for all n,

$$d_n^W f_n = f_{n-1} d_n^V,$$

  and, of course, we will drop the V and W superfixes whenever possible. The category of differential vector spaces and chain maps will be variously denoted dgvs, or Ch_k with variants $\mathsf{dgk}-\mathsf{mod}$, $\mathsf{dgk}-\mathsf{mod}_{\geq 0}$, Ch_k^+ and so on, denoting the k-module version, a positively graded variant, and an alternative notation. (These, and other, notations are all used in the literature with the precise convention usually evident from the context. To some extent the choice, say of dgvs as against Ch is determined by the use intended, but this is not completely consistent.)

- A *chain homotopy* between two chain maps $f, g : \mathsf{V} \to \mathsf{W}$ is a graded map of degree 1, $s : \mathsf{V} \to \mathsf{W}$ such that
$$f_n - g_n = d_{n+1}s_n + s_{n-1}d_n.$$

- The *homology* of a chain complex, $\mathsf{V} = (V, d)$, is the graded object
$$H_n(\mathsf{V}) = \frac{Ker\, d_n}{Im\, d_{n+1}}.$$
If we are using upper indices, for whatever reason, the more usual term will be 'cohomology',
$$H^n(\mathsf{V}^*) = \frac{Ker(d^n : V^n \to V^{n+1})}{Im(d^{n-1} : V^{n-1} \to V^n)}.$$
This most often occurs in the situation where C is a chain complex and A is a vector space / module or similar, then we form $Hom(\mathsf{C}, A)$, by applying the functor $Hom(-, A)$ to C. Of course, $d_n : C_n \to C_{n-1}$ induces a differential
$$Hom(C_{n-1}, A) \to Hom(C_n, A)$$
and the elements of $Hom(C_n, A)$ are called *cochains*, with *cocycles*, and *coboundaries* as the corresponding elements of kernels and images. The notation $Hom(\mathsf{C}, A)^n$ is used for the object $Hom(C_{-n}, A)$, so this 'dual' has negative grading if C has positive grading, and is given upper indexing. The homology of $Hom(\mathsf{C}, A)$ is then called the *cohomology of* C *with coefficients in* A. (We will try to follow usual terminology as given in standard homological algebra texts, e.g. the classic [85].)

- More generally, if C and D are both chain complexes (of modules), then we can form the graded Abelian group, $Hom(\mathsf{C}, \mathsf{D})$, with $Hom(\mathsf{C}, \mathsf{D})_n$ being the Abelian group of graded maps of degree n from C to D. This means, of course,
$$Hom(\mathsf{C}, \mathsf{D})_n = \prod_{p=-\infty}^{\infty} Hom(C_p, D_{p+n}),$$
as before.

  We make this into a chain complex by specifying, for $f \in Hom(\mathsf{C}, \mathsf{D})_n$, its 'boundary' ∂f by, if $c \in C_p$,
$$(\partial f)_p c = \partial^{\mathsf{D}}(f_p c) + (-1)^{n+1} f_{p-1}(\partial^{\mathsf{C}} c).$$
(In the event that you have not seen this before, check that (i) $\partial\partial = 0$, (ii) if f is of degree 0, then it is a chain map if and only if $\partial f = 0$ and (iii) a chain homotopy, s, between two chain maps, $f, g \in Hom(\mathsf{C}, \mathsf{D})_0$, is precisely an $s \in Hom(\mathsf{C}, \mathsf{D})_1$ with $\partial s = f - g$.)

  The homology of $Hom(\mathsf{C}, \mathsf{D})$ is called the *hyper-cohomology of* C *with coefficients in* D. The case where $D_0 = A$ and $D_n = 0$ if $n \neq 0$ is the cohomology we saw earlier. In general, $H^0(Hom(\mathsf{C}, \mathsf{D}))$, i.e., chain maps modulo coboundaries, is just the group of chain homotopy classes of chain maps by (ii) and (iii) above. As is usual in homological (and homotopical) algebra, we usually need good conditions on C and D to get really good invariants from this construction - typically C needs to be 'projective' or D 'injective', or C needs to be 'fibrant' or D 'cofibrant'. Our use of this will be somewhat hidden by the situations we will be considering.

7.1.2 Čech hyper-cohomology

The main type of application for us will be the 'hyper'-version of Čech cohomology. In this, or at least in its simplest form, we have a space, X, and we form the colimit over the open covers, $\mathcal{U}$, of X of the hyper-cohomology groups $H^n(C(\mathcal{U}), \mathsf{D})$. In more detail:

The classical Čech cohomology of X with coefficients in a sheaf of R-modules, A, is defined via open covers $\mathcal{U}$ of X. If $\mathcal{U}$ is an open cover of X, then we form the chain complex, $C(\mathcal{U})$, by taking $N(\mathcal{U})$, the nerve of $\mathcal{U}$, and letting $C(\mathcal{U})_n$ be the sheaf of free R-modules generated by $N(\mathcal{U})_n$ with $\partial = \sum_{k=0}^n (-1)^k d_k$ being the differential. This can either be thought of as a complex of (sheaves of) R-modules or in the straight forward module version. We take coefficients in another sheaf of R-modules, A, and form $H^n(C(\mathcal{U}), A)$.

If $\mathcal{V}$ is a finer cover than $\mathcal{U}$, there is a chain map from $C(\mathcal{V})$ to $C(\mathcal{U})$. Recall if $\mathcal{V} < \mathcal{U}$, for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ with $V \subseteq U$, and $(x, V_0, \ldots, V_n) \in N(\mathcal{V})_n$, we can map it to a corresponding $(x, U_0, \ldots, U_n) \in N(\mathcal{U})_n$ with each $V_i \subseteq U_i$. This is not well defined as several Us might work for a particular V, so the construction of the chain map involves a choice, however it does induce, firstly, a chain map from $C(\mathcal{V})$ to $C(\mathcal{U})$, which is determined up to (coherent) homotopy and thus a *well defined* map on cohomology, $H^*(C(\mathcal{U}), A) \to H^*(C(\mathcal{V}), A)$.

The Čech cohomology, $\check{H}^*(X, A) = colim_{\mathcal{U}} H^*(C(\mathcal{U}), A)$, was the first, historically, of the sheaf type cohomologies. Others apply to a topos rather than merely a space. The obvious hyper-variant of this replaces A by a sheaf of chain complexes (of whatever variety you like, provided they are 'Abelian'), so $H^n(C(\mathcal{U}), \mathsf{D}) = H^n(Hom(C(\mathcal{U}), \mathsf{D}))$ and then $\check{H}^*(X, \mathsf{D}) = colim_{\mathcal{U}} H^*(C(\mathcal{U}), \mathsf{D})$.

We should 'deconstruct' this a bit to see why it is relevant to us.

To simplify our lives no end, we will assume D is a presheaf of chain complexes of R-modules which is positive, ($D_n = 0$ if $n < 0$). By the method of construction of colimits of modules, etc., we can find for any element of $\check{H}^*(X, \mathsf{D})$, an open cover $\mathcal{U}$ of X and a representing element in $H^*(C(\mathcal{U}), \mathsf{D})$. We can thus, further, find a representing n-cocycle from $C(\mathcal{U})$ to D, i.e., an element in $\prod_p Hom(C(\mathcal{U})_p, D_{n+p})$.

To simplify still further, we look at low values of n:

- for $n = 0$, we have some $\mathbf{f} = \{f_p : C(\mathcal{U})_p \to D_p\}$, which satisfies $\partial \mathbf{f} = 0$, so $\mathbf{f}$ forms a chain map. In some of our most interesting cases, D is usually very short, e.g. $D_n = 0$ if $n > 1$, so $\mathsf{D} = (D_1 \to D_0)$ with zeroes elsewhere in other dimensions. Then the only f_ps that contribute to $\mathbf{f}$ are f_0 and f_1. Over an open set, U_i, of the cover, f_0 will be a local section, $f_{0,i}$, of D_0, since 0-simplices of $N(\mathcal{U})$ have form (x, U_i) over $x \in U_i$. Similiarly 1-simplices are, of course, represented by (x, U_i, U_j) with $x \in U_{ij}$, so f_1 corresponds to local sections $f_{1,ij} : U_{ij} \to D_1$. The boundary in $C(\mathcal{U})$ of (x, U_i, U_j) is $(x, U_j) - (x, U_i)$, so

$$d^{\mathsf{D}} f_{1,ij} = f_{0,j}(x) - f_{0,i}(x),$$

or

$$f_{0,j}(x) = d^{\mathsf{D}} f_{1,ij} + f_{0,i}(x).$$

If we look at the non-Abelian analogue of this, it gives

$$f_{0,j}(x) = d^{\mathsf{D}} f_{1,ij}.f_{0,i}(x),$$

which 'is' the equation $p_j = \partial(c_{ij})p_i$. (You could explore the cases where D is slightly longer, or what about a non-Abelian version?)

- for $n = 1$, we expect to find a formula corresponding to the coboundaries that we met on 'changing the local sections' for M-torsors. If h, (yes, 'h' as in 'homotopy') is a degree 1 map in $Hom(C(\mathcal{U}), \mathsf{D})$ and D has length 1 as above, the only case that contributes is $h_0 : C(\mathcal{U})_0 \to D_1$ and hence $h_{0,i} : U_i \to D_1$. You are **left to check** that this does give (the Abelian version of) the coboundary / chain homotopy formula.

7.1.3 Non-Abelian Čech hyper-cohomology.

The idea should be fairly obvious in its general form. We replace our overall structural viewpoint of chain complexes or sheaves of such, by our favorite non-Abelian analogue. For instance, we could take D to be a sheaf of simplicial groups, or crossed complexes, or n-truncated simplicial groups or ... . These would really include sheaves of 2-crossed modules and clearly we might try sheaves of 2-crossed complexes, and so on. Some of these classes of coefficient are very likely to turn out to be useful in the future if recent developments in algebraic and differential geometry are any indication. We cannot consider all of them here. The first is the easiest to deal with and to some extent includes the others. It is not structurally the neatest, but ... .

If D is a sheaf of simplicial groups, then we might be tempted to replace $C(\mathcal{U})$ by the free simplicial group sheaf on $N(\mathcal{U})$. It is very important to note that this is NOT the same as $\mathcal{G}(N(\mathcal{U}))$ and we should pause to consider this point.

Let K be a simplicial set and G a simplicial group. The set of simplicial maps from K to the underlying simplicial set of G is isomorphic to $Simp.Grps(FK, G)$ by the standard adjunction between the free group functor, F, and the forgetful functor, U from $Grps$ to $Sets$. Complications might seem to arise if one tries to work with $\underline{\mathcal{S}}(K, UG)$ and $\underline{Simp.Grps}(FK, G)$, as initially it needs to be noted that $\underline{\mathcal{S}}(K, UG) = \mathcal{S}(K \times \Delta[n], UG)$ and one has to think of the relationship between $F(K \times \Delta[n])$ and $F(K) \otimes \Delta[n]$, the latter in the sense of our earlier discussion of tensoring in simplicially enriched categories, page 178. (This problem is, in fact, not really there, as although F does not preserve products, the product $K \times \Delta[n]$ is actually being thought of, and constructed, as a *colimit* and F, as a left adjoint, behaves nicely with respect to such.) We will not explore that further here and will, in fact, stick with $\underline{\mathcal{S}}(N(\mathcal{U}), \mathsf{D})$ rather than use F. (Note that by a useful result of Milnor, FK and $\mathcal{G}SK$ are isomorphic for a reduced simplicial set K, where S is the reduced suspension; see [43] and the paper, [93], which can be found in Adams, [1].) The relationship between $\underline{\mathcal{S}}(K, UG)$ and other related constructions such as $\underline{\mathcal{S}}(K, \overline{W}G) \cong \underline{\mathcal{S}-Grpds}(\mathcal{G}(K), G)$, is given by the induced fibration sequence

$$\underline{\mathcal{S}}(K, UG) \to \underline{\mathcal{S}}(K, WG) \to \underline{\mathcal{S}}(K, \overline{W}G)$$

coming from the fibration

$$UG \to WG \to \overline{W}G.$$

If we work within our favourite topos $\mathcal{E}$, or with bundles over B, this still holds true. It is also the case that WG is (naturally) contractible.

Back with hyper-cohomology, let D be a sheaf of simplicial groups and form $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$. We put forward the homotopy groups of this simplicial group as being one analogue of $H^*(C(\mathcal{U}), \mathsf{D})$ in this context. (If D is Abelian, it will be KD for some sheaf of chain complexes, D, and the Dold-Kan theorem, plus the freeness of $C(\mathcal{U})$, give a correspondence between the elements in the two cases. Since we have $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$ is a simplicial Abelian group in that case, its homotopy is its homology and the detailed correspondence passes down to homology without any pain. We thus do have a generalisation of the Abelian situation with our formula.)

We have $\pi_n(\mathcal{U}, \mathsf{D}) := \pi_n(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$ is thus a candidate for a 'non-Abelian' Čech cohomology relative to $\mathcal{U}$ with coefficients in D. (If $n > 1$, it is an Abelian group, which makes it suspiciously well behaved - in fact too well behaved! We really need not these π_n, but rather the various algebraic models for the various k-types of the homotopy type $\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D}))$, i.e., we could do with examining $M(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), U(\mathsf{D})), k)$, the crossed k-cube of that simplicial group. (For those of you who hanker for the simple life, it should be pointed out that when discussing extensions, we already had that there was a groupoid of extensions $\mathcal{E}xt(G, K)$, and although we could extract information from that groupoid to get cohomology groups, the natural invariant is really that groupoid, not the cohomology group as such. We can extract information from such an invariant, just as we can extract homotopy information from a homotopy type. To keep the information tractable we often truncate, or kill off, some of the structure to make the extraction process more amenable to calculation.)

We are, however, running before we can walk here! The case we have met earlier is for $n = 0$, i.e., $[N(\mathcal{U}), \mathsf{D}]$, and we could pass to the colimit over covers to get $\check{H}^0(B, \mathsf{D})$. This is without restriction on the sheaf of simplicial groups, D. Our earlier example was with $D = K(\mathsf{M})$ for $\mathsf{M} = (C, P, \partial)$, a sheaf of crossed modules. (Breen in [17] calls this the *zeroth cohomology of the crossed module,* M, but as it varies covariantly in M perhaps his later terminology, [20], as the *zeroth Čech non-Abelian cohomology of B with coefficients in* M, is more appropriate.)

What about $\check{H}^1(B, \mathsf{M})$?

This will be $colim_{\mathcal{U}} H^1(N(\mathcal{U}), \mathsf{M})$, which is $colim_{\mathcal{U}} \pi_1(\underline{Simp.\mathcal{E}}(N(\mathcal{U}), K(\mathsf{M}))$. From the long exact fibration sequence, this will be isomorphic to $colim_{\mathcal{U}}[N(\mathcal{U}), \overline{W}K(\mathsf{M})]$ and so should classify some sort of simplicial $K(\mathsf{M})$-bundles on B. It does, but we need to wait until a later chapter for the details.

The set $[N(\mathcal{U}), \overline{W}K(\mathsf{M})]$ has elements which are homotopy classes of maps from $N(\mathcal{U})$ to $\overline{W}K(\mathsf{M})$ and by the properties of the loop groupoid construction, $\mathcal{G}$ of section 5.2.1, page 121, each such is adjoint to a morphism of sheaves of $\underline{\mathcal{S}}$-groupoids from $\mathcal{G}(N(\mathcal{U}))$ to $K(\mathsf{M})$. The category of crossed modules is equivalent, via K and $M(-, 2)$, to a full reflective subcategory / variety of $\underline{\mathcal{S}}-Grpds$, and this extends to sheaves, so the elements of $[N(\mathcal{U}), \overline{W}K(\mathsf{M})]$ correspond to homotopy classes of crossed module morphisms from $M(\mathcal{G}N(\mathcal{U}), 2)$ to M. In particular, for nice spaces, B, one would expect there to be 'nice' covers $\mathcal{U}$, such that $N(\mathcal{U})$ corresponded, via geometric realisation, to B itself, then taking $\mathsf{M} = M(\mathcal{G}N(\mathcal{U}), 2)$ itself, one would have a sort of universal element in $\check{H}^1(B, \mathsf{M})$, corresponding in this level, to a universal simplicial sheaf over B, extending in part the construction and properties of the universal covering space. This argument is one form of the 'evidence' for believing Grothendieck's intuition in 'En Poursuite des Champs / Pursuing Stacks', [62]. There seems no good reason why, for any nice class of simplicial groups that form a variety, $\mathcal{V}$, with perhaps having some stability with respect to homotopy types, there should not be a 'universal $\mathcal{V}$-stack' over B. The above corresponds to the case of crossed modules, but crossed complexes and many of the other types of crossed objects that we have met earlier would seem to be relevant here. The main hole in our understanding of this is not really how to do it, rather it is how to interpret the theory once it is there. This form of crossed homotopical algebra would extend Galois theory to higher 'levels', but what do the invariants tell us algebraically?

That provides some overview of this general case, but in our earlier situation, with extensions of groups, we used a crossed resolution of a group, G, not a simplicial one. We have also mentioned once or twice that the category, Crs, of crossed complexes is monoidal closed. This would suggest (i) that given a topos $\mathcal{E}$, and, in particular, given a space B and $\mathcal{E} = Sh(B)$, the category of crossed complexes *in* $\mathcal{E}$, denoted $Crs_{\mathcal{E}}$, would be monoidal closed, (ii) there would be a free crossed complex on a cover / hypercover in $\mathcal{E}$, i.e., if we have a simplicial object K in $\mathcal{E}$, we would get a crossed complex object, $\pi(K)$, and if $K \to 1$ is a 'weak equivalence' then there would be a local contracting homotopy on $\pi(K)$, i.e., $\pi(K) \to 1$ would be a 'weak equivalence' of crossed complex bundles (recall 1 is the terminal object of $\mathcal{E}$, so in the case of $\mathcal{E} = Sh(B)$ is the singleton sheaf), then (iii) if $\mathrm{C{\scriptsize RS}}_{\mathcal{E}}$ denotes the internal 'hom' of crossed complex bundles, we would be looking at the model $\mathrm{C{\scriptsize RS}}_{\mathcal{E}}(\pi(K), \mathsf{D})$ for a crossed complex, D, in $\mathcal{E}$ and would want the homotopy colimit of these over (hyper-)covers, K, so as to get a well-structured model. Of course, if $\mathcal{E} = Sh(B)$ and we have 'nice' (hyper-)covers K, then we would expect the homotopy type of this to stabilise, up to homotopy, so $\mathrm{C{\scriptsize RS}}_{\mathcal{E}}(\pi(K), \mathsf{D})$ would be the same, up to homotopy, as that homotopy colimit. This plan almost certainly works, but in detail has not been followed through as yet. The first part looks very feasible given the construction of $\mathrm{C{\scriptsize RS}}(\mathsf{C}, \mathsf{D})$ for (set based) crossed complexes, C and D. (A source for this is Brown and Higgins, [26] and it is discussed with some detail in Kamps and Porter, [76], p.222-227.) We will not give the details here. The other parts also look to work as the set based originals are given by explicit constructions, all of which generalise to $Sh(B)$. If that does all work then one has a Crs-based 'hyper-cohomology' crossed complex, $\check{\mathrm{C{\scriptsize RS}}}(B, \mathsf{D}) = hocolim_K \mathsf{Crs}(\pi(K), \mathsf{D})$, whose homotopy groups represent the analogue of hyper-cohomology.

If you are wary of not having a group or groupoid as an 'answer' for what is this 'hypercohomology', think of various analogous situations. For instance, for total derived functor theory, in homological and homotopical algebra, from a functor you get a complex, but it is the homotopy type of that complex which is used, not just its homotopy groups. In algebraic K-theory, it is quite usual to refer to the algebraic K-theory of a ring as being the (homotopy type of) a simplicial set or space. The algebraic K-groups are then the homotopy invariants of that simplicial set. In other words, in 'categorifying', one naturally ends up with an object whose homotopy type encapsulates the invariants that you are mostly used to, but that object is the thing to work with, not just the invariants themselves.

7.2 Mapping cocones and Puppe sequences

Exact sequences in cohomology can be constructed in various ways. One of these is related to the fibration and cofibration seqences of homotopy theory. If one has a fibration of spaces, then it leads to a long exact sequence of homotopy groups. Of course, not all maps are fibrations, but any map, $f : X \to Y$, can be replaced, up to homotopy, by a fibration, and its fibre Γ_f, then codes up homotopy information about f. This fibre is usually called the *homotopy fibre* of f and we have already met it in our list of common examples leading to crossed modules; see page 34. Later on we will need to use the construction to extend our simplicial interpretations of non-Abelian cohomology, but, by way of introduction, to start with both that construction (mapping cocylinders and mapping cocones/homotopy fibres) and the resulting homotopy exact sequences (Puppe sequences) will be looked at in a much simpler setting, namely that of chain complexes. Initially we will concentrate on the dual situation as that is slightly easier to understand geometrically.

(A very useful concise introduction to this theory can be found in May's book, [90], starting about page 55, and, for results on chain complexes, page 90.)

7.2.1 Mapping Cylinders, Mapping Cones, Homotopy Pushouts, Homotopy Cokernels, and their cousins!

We need various 'homotopy kernels', 'homotopy fibres' and more general 'homotopy limits' for our discussion. We have also already mentioned 'homotopy colimits' in passing several times, and so it seems a good idea to examine this general area from an elementary point of view.

We will work with a chain map $\mathsf{f} : \mathsf{C} \to \mathsf{D}$ of chain complexes of modules over some ring R. We will use a *cylinder* $\mathsf{C} \otimes \mathsf{I}$. This is given by tensoring C with the chain complex, I,

$$0 \longrightarrow R \xrightarrow{\partial} R \oplus R \longrightarrow 0,$$

$$\partial(e_1^1) = e_1^0 - e_0^0.$$

There is one generator, e_1^1, in dimension 1, and two in dimension zero, corresponding to the interval $I = [0, 1]$ or $\Delta[1]$ having one 1-cell and two 0-cells, e_1^0 and e_1^0, the superfix denoting the dimension of that generator . We should give a formal definition of a tensor product of chain complexes, even though you may have met this before.

Definition: If C and D are chain complexes, their tensor product $\mathsf{C} \otimes \mathsf{D}$ has

$$(\mathsf{C} \otimes \mathsf{D})_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and boundary / differential given on generators by

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^{|c|} c \otimes (\partial' d),$$

where $|c|$ is the degree of c, (that is, $c \in C_{|c|}$).

We note the connection between $\otimes$ and Hom, namely that, given chain complexes, C, D, and E, there are natural isomorphisms

$$Hom(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \cong Hom(\mathsf{C}, Hom(\mathsf{D}, \mathsf{E})),$$

so $- \otimes \mathsf{D}$ and $Hom(\mathsf{D}, -)$ are adjoint.

Example:

$$\begin{aligned} (\mathsf{C} \otimes \mathsf{I})_n &\cong C_n \otimes I_0 \oplus C_{n-1} \oplus I_1 \\ &\cong C_n \oplus C_n \oplus C_{n-1} \end{aligned}$$

(We will denote elements in this direct sum as column vectors, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, but will usually write $(x, y, z)^t$, or even (x, y, z) if we are being lazy!)

The isomorphism matches $c_n \otimes e_0^0$ with $(c_n, 0, 0)^t$, $c_n \otimes e_1^0$ with $(0, c, 0)^t$ and $c_{n-1} \otimes e_1^1$ with $(0, 0, c_{n-1})^t$. We can therefore calculate $\partial(x, y, z)^t$ explicitly for $(x, y, z)^t \in C_n \oplus C_n \oplus C_{n-1}$.

$$\begin{aligned} \partial(x, 0, 0)^t &= (\partial x, 0, 0)^t \\ \partial(0, y, 0)^t &= (0, \partial y, 0,)^t \end{aligned}$$

and, as $(0, 0, z)^t$ corresponds to a "$c_{n-1} \otimes e_1^1$", its boundary is

$$\begin{aligned} \partial(c_{n-1} \otimes e_1^1) &= \partial(c_{n-1}) \otimes e_1^1 + (-1)^{n-1} c_{n-1} \otimes \partial(e_1^1) \\ &= \partial(c_{n-1}) \otimes e_1^1 + (-1)^{n-1} c_{n-1} \otimes e_1^0 + (-1)^n c_{n-1} \otimes e_0^0 \end{aligned}$$

i.e. $\partial(0, 0, z)^t = ((-1)^n z, (-1)^{n+1} z, \partial z)^t$. This allows us to use, if we want to, a matrix representation of the boundary in $\mathsf{C} \otimes \mathsf{I}$ as

$$\begin{pmatrix} \partial & 0 & (-1)^{n-1} \\ 0 & \partial & (-1)^n \\ 0 & 0 & \partial \end{pmatrix}$$

and thus would allow us to use such a description to *define* a cylinder $\mathsf{C} \otimes \mathsf{I}$ for C, a chain complex in a more abstract setting such as that of an arbitrary Abelian category.

There are obvious chain maps,

$$\mathsf{e}_i : \mathsf{C} \to \mathsf{C} \otimes \mathsf{I},$$

$i = 0, 1$, corresponding to the ends of the cylinder, and a projection,

$$\sigma : \mathsf{C} \otimes \mathsf{I} \to \mathsf{C},$$

corresponding to 'squashing' the cylinder onto the base.

This, of course, leads to a notion of homotopy between chain maps.

Definition: A *(cylindrical) homotopy*, h, between two chain maps, $\mathsf{f}, \mathsf{g} : \mathsf{C} \to \mathsf{D}$, is a chain map,

$$\mathsf{h} : \mathsf{C} \otimes \mathsf{I} \to \mathsf{D},$$

with $\mathsf{he}_0 = \mathsf{f}$ and $\mathsf{he}_1 = \mathsf{g}$.

This notion of a 'cylindrical' homotopy, h, between two chain maps is easy to analyse. We have $h_n : C_n \oplus C_n \oplus C_{n-1} \to D_n$ and the conditions $\mathsf{he}_0 = \mathsf{f}$ and $\mathsf{he}_1 = \mathsf{g}$ become, in terms of coordinates, $h_n(x, 0, 0) = f_n(x)$, and $h_n(0, y, 0) = g_n(y)$, thus the 'free' or 'unbound' information for h is contained in $h_n(0, 0, z)$. This map, h , restricted to the C_{n-1}-summand gives a degree one map $\mathsf{h}' = \{h'_{n-1} : C_{n-1} \to D_n\}$. We have assumed that h is a chain map, so with our convention for the boundary on $\mathsf{C} \otimes \mathsf{I}$, we get:

$$\begin{aligned} \partial h'_{n-1}(z) = \partial h_n(0, 0, z) &= h\partial(0, 0, z) \\ &= h((-1)^{n-1} z, (-1)^n z, \partial z) \\ &= (-1)^{n-1}(f_{n-1}(z) - g_{n-1}(z)) + h'(\partial z). \end{aligned}$$

We thus have that, if we put $s_n = (-1)^n h'_n$, we will get a chain homotopy $\mathsf{s} : \mathsf{C} \to \mathsf{D}$, from f to g. Conversely any chain homotopy will yield a cylindrical homotopy.

Notational comment: The convention on signs that we have adopted is not the only on $\mathsf{C} \otimes \mathsf{I}$ and, as you can easily check, this will determine a different boundary on the chain complex, although the individual terms of the complex are still isomorphic to $C_n \oplus C_n \oplus C_{n-1}$.

Later we will consider the suspension $\mathsf{C}[1]$ of C and this has $C[1]_n = C_{n-1}$. Different sources on differential graded objects may adopt different conventions as to the form of the boundary for $\mathsf{C}[1]$. Quite often the convention chosen is $\partial_n^{\mathsf{C}[1]} = (-1)^n \partial_{n-1}^{\mathsf{C}}$, as this absorption of the $(-1)^n$ makes certain graded maps that naturally occur into chain maps and thus greatly simplifies the formulae and to some extent the theory.

These sign conventions are extremely useful in the study of differential graded algebras as in rational homotopy theory, cf. [57]. We are using chain complexes here mainly as an illustrative example, so will not need to adopt those conventions here. The reader is, however, advised that if working with differential graded (dg) structures, attention to the compatibility between the simplicial and 'dg' conventions is essential if your calculations are not going to look wrong! There is no essential difference in the geometric intuitions between the approaches, but confusion can easily arise if this is not recognised early on in work at this interface.

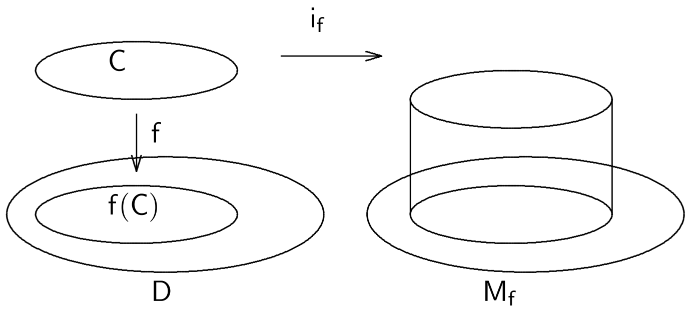
Given our chain map, $\mathsf{f} : \mathsf{C} \to \mathsf{D}$, we can form a *mapping cylinder* on f by the pushout

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathsf{f}"] \arrow[d, "\mathsf{e}_0"'] & \mathsf{D} \arrow[d, "\mathsf{j}_\mathsf{f}"] \\ \mathsf{C} \otimes \mathsf{I} \arrow[r, "\pi_\mathsf{f}"'] & \mathsf{M}_\mathsf{f} \end{tikzcd}$$

and we can set $\mathsf{i}_\mathsf{f} = \pi_\mathsf{f} \mathsf{e}_1$. The fact that the e_i are split by $\mathsf{s} : \mathsf{C} \otimes \mathsf{I} \to \mathsf{C}$ means that we can form a commutative square

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathsf{f}"] \arrow[d, "\mathsf{e}_0"'] & \mathsf{D} \arrow[d, "="] \\ \mathsf{C} \otimes \mathsf{I} \arrow[r, "\mathsf{fs}"'] & \mathsf{D} \end{tikzcd}$$

and obtain an induced map $\mathsf{p}_\mathsf{f} : \mathsf{M}_\mathsf{f} \to \mathsf{D}$ satisfying $\mathsf{p}_\mathsf{f} \mathsf{j}_\mathsf{f} = id_\mathsf{D}$ and $\mathsf{p}_\mathsf{f} \pi_\mathsf{f} = \mathsf{fs}$. The second equation then gives $\mathsf{p}_\mathsf{f} \mathsf{i}_\mathsf{f} = \mathsf{f}$, as an easy consequence.



In addition, $\mathsf{j}_\mathsf{f} \mathsf{p}_\mathsf{f} : \mathsf{M}_\mathsf{f} \to \mathsf{M}_\mathsf{f}$ is homotopic to the identity by a homotopy that is constant on composition with j_f, i.e., D is a strong deformation retract of M_f.

Note that we have not shown this last fact. That is **left for you to do**. We should also note that most of this does not use any specific properties of chain complexes nor of the cylinder that we have been using. The same arguments would work for any ‘reasonable’ cylinder functor on a category with pushouts. The construction of a homotopy from $\mathsf{j}_\mathsf{f}\mathsf{p}_\mathsf{f}$ to the identity *does* use a few more properties. (**Try to investigate what is needed.** A quite detailed discussion of this from one point of view can be found in Kamps and Porter, [76], in a form fairly compatible with that used here.) We will need to use this mapping cylinder construction several times more in different contexts, so abstraction is useful.

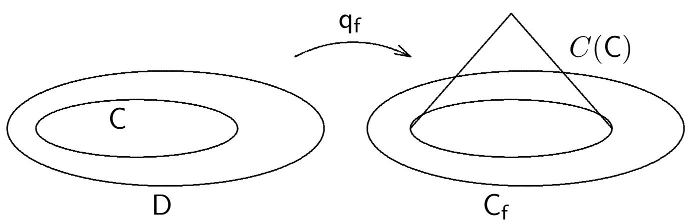
Aside: In [76], you will also find a proof that i_f satisfies a *homotopy extension property*, i.e., it is a *cofibration*. The description above shows that any f can be factored as a cofibration composed with a strong deformation retraction.

Before we leave mapping cylinder-type constructions as such, we also need to comment on the dual situation, as that is really what we need for our immediate task. In many situation we can form a cocylinder, D^I, either instead of, or as well as, a cylinder. For instance, in the setting of chain complexes, we can set $\mathsf{D}^\mathsf{I} = Hom(\mathsf{I}, \mathsf{D})$ and then, as is easily checked, ${\mathsf{D}^\mathsf{I}}_n \cong D_n \oplus D_n \oplus D_{n+1}$. The boundary is left to you to write down. The adjointness isomorphism gives the connection with the cylinder and also with chain homotopies. We can form a *mapping cocylinder* by a pullback:

$$\begin{tikzcd}
\mathsf{M}^\mathsf{f} \arrow[r, "\pi^\mathsf{f}"] \arrow[d, "\mathsf{j}^\mathsf{f}"'] & \mathsf{D}^\mathsf{I} \arrow[d, "\mathsf{e}^0"] \\
\mathsf{C} \arrow[r, "\mathsf{f}"'] & \mathsf{D}.
\end{tikzcd}$$

There is a morphism $\mathsf{p}^\mathsf{f} : \mathsf{C} \to \mathsf{M}^\mathsf{f}$ splitting j^f, so $\mathsf{j}^\mathsf{f}\mathsf{p}^\mathsf{f} = id$, and also $\mathsf{p}^\mathsf{f}\mathsf{j}^\mathsf{f} \simeq id$. Writing $\mathsf{i}^\mathsf{f} = \mathsf{e}_1\pi^\mathsf{f}$, we have $\mathsf{i}^\mathsf{f}\mathsf{p}^\mathsf{f} = \mathsf{f}$. This map i^f is a fibration, even in the abstract case under reasonable conditions on the context and the properties of the cocylinder functor, and we find, for instance in the topological setting, the method we used to replace an arbitrary map into a fibration, up to homotopy, (look back to page 34).

Returning now to mapping cylinders, we have $\mathsf{i}_\mathsf{f} : \mathsf{C} \to \mathsf{M}_\mathsf{f}$ inserting C as the ‘top’ of the cylinder part of M_f. The *mapping cone*, C_f, (or, sometimes, $C(\mathsf{f})$) of f is obtained by quotienting out by the image of i_f. (This is usually visualised by imagining C_f as a copy of D together with a cone, $C(\mathsf{C})$ on C glued to it using f.)



We note that the map $\mathsf{j}_\mathsf{f} : \mathsf{D} \to \mathsf{M}_\mathsf{f}$ composed with the quotient $q : \mathsf{M}_\mathsf{f} \to \mathsf{C}_\mathsf{f}$ gives a map, $\mathsf{q}_\mathsf{f} : \mathsf{D} \to \mathsf{C}_\mathsf{f}$ and that the cone structure provides a homotopy between the composite, $\mathsf{C} \to \mathsf{D} \to \mathsf{C}_\mathsf{f}$, and the trivial map, $\mathsf{C} \to \mathsf{C}_\mathsf{f}$. We should look at this more closely.

If we compose the cylindrical homotopy given by the identity on $\mathsf{C} \otimes \mathsf{I}$ with π_f, we get a homotopy between $\pi_\mathsf{f}\mathsf{e}_0$ and $\pi_\mathsf{f}\mathsf{e}_1$, but $\pi_\mathsf{f}\mathsf{e}_0 = j_\mathsf{f}\mathsf{f}$ and $\pi_\mathsf{f}\mathsf{e}_1 = \mathsf{i}_\mathsf{f}$. Finally composing everything with $\mathsf{q} : \mathsf{M}_\mathsf{f} \to \mathsf{C}_\mathsf{f}$, we have a homotopy between $\mathsf{q}j_\mathsf{f}\mathsf{f} = \mathsf{q}_\mathsf{f}\mathsf{f}$ and $q\mathsf{i}_\mathsf{f}$, which latter map is trivial.

Dually we can get a *homotopy cocone*: we take the homotopy cocylinder M^{f} and the map $\mathsf{i}^{\mathsf{f}} : \mathsf{M}^{\mathsf{f}} \to \mathsf{D}$ and form its fibre over the 'basepoint', that is the zero, of D. Of course that 'fibre' is just the kernel of i^{f} in our chain complex case study.

Aside on homotopy cokernels, etc.

In discussion on kernels and cokernels in Abelian and additive categories, it is quite often noted that the cokernel of a map, $\varphi : A \to B$, say in an Abelian category, gives a pushout

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & Coker\,\varphi \end{tikzcd}$$

and that the pushout square property is exactly the universal property defining cokernels. The construction of the mapping cone gives a similar square:

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathsf{f}"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \mathsf{D} \arrow[d, "\mathsf{q_f}"] \\ 0 \arrow[r] & C_{\mathsf{f}} \end{tikzcd}$$

but it is only homotopy commutative (or rather homotopy coherent as there is the natural *explicit* homotopy, $\mathsf{h_f} : \mathsf{q_f}\mathsf{f} \Rightarrow 0$). This homotopy coherent square has a universal property with respect to homotopy cohernet squares based on $0 \leftarrow \mathsf{C} \xrightarrow{\mathsf{f}} \mathsf{D}$. This makes it reasonable to call the result a *homotopy pushout* and then to say that C_{f} is the *homotopy cokernel* or sometimes the *homotopy cofibre* of f. It is, of course, an example of a homotopy colimit, but note that it is necessary to give not only C_{f} plus $\mathsf{q_f}$ to get the full universal property (as would be the case for an ordinary colimit), but also $\mathsf{h_f}$.

Exercise: The construction of the mapping cylinder is also a homotopy pushout. Try to formulate a good notion of homotopy pushout and identify that construction as an example of one such. The main idea is to start with two maps

$$B \xleftarrow{b} A \xrightarrow{c} C$$

with common domain and to form a homotopy coherent square

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "\Swarrow"] & C \arrow[d, "b'"] \\ B \arrow[r, "c'"'] & D, \end{tikzcd}$$

where h is a homotopy $A \times I \to D$ between $b'c$ and $c'b$. For instance, use a repeated pushout operation on the diagram

$$\begin{tikzcd}
 & A \arrow[r, "c"] \arrow[d, "e_0"] & C \arrow[dd, dashed] \\
A \arrow[r, "e_1"] \arrow[d, "b"'] & A \times I & \\
B \arrow[rr, dashed] & & D
\end{tikzcd}$$

to construct its colimit, which will be a *double mapping cylinder*. The homotopy h is then clear. Specialise down to the case of b being the identity to complete. Note that homotopy pushouts are determined 'up to homotopy', not 'up to isomorphism', so you may not quite get what you expect and different construction may give different, but homotopic, models for it!

This discussion of homotopy cokernels is almost 'general'. It works, more or less, in any setting where there is a null object, corresponding to 0, having a nice cylinder that preserves pushouts, and, of course, enough pushouts. In our well behaved case study of chain complexes, we can track the construction in the direct sum decomposition if we so wish.

Homotopy commutative v. homotopy coherent: It is quite important to note a sort of theme that occurs both here and earlier in our discussion of bitorsors and M-torsors. An M-torsor was a C-torsor, E together with a definite choice of global section for $\partial_*(E)$. We did not just say the $\partial_*(E)$ is trivialisable, we specified a trivialisation as part of the structure.

Here with homotopy pushouts, we do not just have a homotopy commutative square, but specify a definite choice of homotopy linking the two composite maps around the square, i.e., we give a 'homotopy coherent square'. This passage from 'there is a homotopy such that ...' to specifying one is of prime importance in interpreting non-Abelian cohomology.

We have concentrated, so far, on the case of chain complexes. We do need to caste a glance at the topological case. The above description in terms of homotopy cokernels goes through for pointed spaces.

Suppose $f : X \to Y$ is a map of pointed spaces, we can form M_f and factorise f as $p_f i_f = f$, where i_f is a cofibration and p_f is the retraction part of a strong deformation retraction, so in particular is a homotopy equivalence.

Using the cofibration $i_f : X \to M_f$, we divide out, identifying its image to a point, to get C_f as a quotient space, or dierctly as a homotopy pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & Y \arrow[d, "q_f"] \\
* \arrow[r] & C_f,
\end{tikzcd}$$

where $q_f = qj_f$ with $q : M_f \to C_f$ the quotient map.

7.2.2 Puppe exact sequences

The map q_f is a cofibration, under reasonable conditions on the spaces involved, and we can form the quotient of C_f by identifying the image of this map to a point: $SX \cong C_f/Y$, giving the (reduced) suspension, SX, on X. This can be defined directly as $(X \times I)/(X \times \{0,1\} \cup * \times I)$, where $*$ is the base point of X. It is also the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & * \arrow[d] \\ * \arrow[r] & SX, \end{tikzcd}$$

where the homotopy is the quotient map from $X \times I$ to SX.

This gives us a sequence of maps

$$X \xrightarrow{f} Y \to C_f \to SX \xrightarrow{Sf} SY \to SC_f \to S^2X \to \ldots,$$

where we have extended the bit that we have actually constructed by applying S to it and grafting it to the old part. This sequence is known, variously, as the *long cofibre sequence* of f, the *Puppe sequence* of f or the *cofibre Puppe sequence*. It is 'homotopy exact' - what does that mean?

Recall that in an exact sequence, say, of Abelian groups, the kernel of one map is the image of the previous one, so in particular, the composition of pairs of maps in the sequence is always trivial. In the above sequence of pointed spaces, there is an *explicit* null-homotopy from each composition of pairs of adjacent maps to the corresponding trivial map that send the domain to the base point of the codomain. This is clear for the first composable pair $X \xrightarrow{f} Y \to C_f$ as that is exactly what C_f was designed to do! (Some treatments of these sequences in fact construct them by repeating that basic construction of C_f from f for subsequent maps starting with $Y \to C_f$, and then showing that the resulting terms match, up to homotopy, with those of the above sequence. We do not adopt that approach here, although it has some very good points to it.)

The next pair $Y \to C_f \to SX$ is trivial anyway. The checking that $C_f \to SX \xrightarrow{Sf} SY$ is homotopy exact is omitted. It can be found in the literature or you can attempt it yourself. This is thus the analogue of the composites being trivial in an exact sequence. The arguments used for these also show that an analogue of the other part of 'exactness' also holds. For this it seems advisable to indicate a more precise statement. (The temptation to use the words 'exact statement' here must be resisted!) That statement is the usual one here, and goes as follows. (It will need a certain amount of commentary, which will be given shortly.)

For any pointed space, Z, applying the functor $[-, Z]$ to the above sequence yields a long exact sequence of groups and pointed sets,

$$\ldots \to [S^2X, Z] \to [SC_f, Z] \to [SY, Z] \to [SX, Z] \to [C_f, Z] \to [Y, Z] \to [X, Z].$$

We have already recalled the meaning of exactness for sequences of groups. The extension of that to pointed sets should be clear: we replace 'kernel' by 'preimage of the base point' whilst 'image' has the same meaning. If we examine the exactness at $[Y, Z]$, this says that if $g : Y \to Z$ is such that gf is null homotopic, (that is, there is some $h : gf \simeq *$), then there is some $\overline{g} : C_f \to Z$ such that $g = \overline{g}q_f$, and conversely. But that is just what the construction of C_f does, as the nullhomotopy extends the map on Y to the cone on the X part of C_f. In fact, of course, different nullhomotopies will extend to different maps on C_f and you are left to think about the way in which these different null homotopies are, or are not, 'observed' by the sequence. To start you thinking, if $h, h' : gf \simeq *$, then we have a self homotopy of $*$, intuitively, '$hh'^{(-1)}$'. The map

$hh'^{(-1)} : X \times I \to Z$ sends both ends of the cylinder to the basepoint and as it is constructed from pointed homotopies, it also sends $* \times I$ there. It thus induces a map from SX to Z, giving a possible link back to $[SX, Z]$. Again the theme of homotopy coherence v. homotopy commutativity is nearby as if we record the possible null homotopies then we get other information cropping up elsewhere in the sequence.

In this discussion of 'homotopy exact sequences', we have still to complete our discussion of the cofibre sequence of a chain map and also we will have need not so much of this cofibre form of the Puppe sequence, but rather the Puppe 'fibre' long exact sequence of a map. We start with the chain cofibre sequence.

So far we have

$$\mathsf{C} \to \mathsf{D} \to \mathsf{C_f}$$

and, in elementary terms,

$$(\mathsf{C_f})_n \cong D_n \oplus C_{n-1},$$

i.e., the pushout of D and a cone on C. (The differential / boundary is **left to you**.) There is an inclusion of D into $\mathsf{C_f}$, and, surprise surprise, the quotient is $\mathsf{C}[1]$, it has C_{n-1} in dimension n, so is the chain complex analogue of the suspension. (Here we must repeat the warning about sign conventions. The suspension is often considered to have boundary $(-1)^n\partial_n$, corresponding to the needs for the 'suspension map' to be a chain map. This is just due to a different convention on the boundary map of the cylinder. As we need this as a step to understanding the *simplicial* situation, our convention is slightly more appropriate.)

Of course, if E is another chain complex, then applying $[-, \mathsf{E}]$ should give us a long exact sequence. (All is not really as simple as that here as it is usually better to work in what is called the *derived category* of chain complexes rather than just dividing out by homotopy. Initially you should try this for chain complexes of free modules as you cannot always create the maps you want in more general contexts. This general situation *is* important and will be needed in certain aspects later on, but we will ignore the complication here. It is a very useful exercise to show the long exactness for chain complexes of free (or projective) modules, before trying to understand the complication if the freeness condition is removed.)

Now we turn to 'fibre Puppe sequences' in the topological case: we have our $f : X \to Y$ and form the *mapping cocylinder*, M^f, with $i^f : M^f \to Y$ being a fibration and $M^f \simeq X$ in a controlled way, (homotopy coherence again - and, yes, M^f *is* given by a homotopy pullback.) We form the fibre of i^f, and this is $C^f = F_h(f)$, the homotopy fibre of f that we have met before (cf. page 34). This is also a homotopy pullback:

$$\begin{tikzcd}
C^f \arrow[r] \arrow[d, "f^f"'] \arrow[dr, phantom, "\Swarrow"] & * \arrow[d] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

wher q^f is the composite $C^f \to M^f \to X$. We can realise this very neatly by first using the pullback

$$\begin{tikzcd}
\Gamma Y \arrow[r] \arrow[d] & Y^I \arrow[d, "e_0"] \\
* \arrow[r] & Y
\end{tikzcd}$$

giving the object of paths that start at $*$. This has a second map to Y induced by e_1, giving $\Gamma Y \to Y$, which is a fibration. This is the dual analogue of the cone on X in this dual context.

(The notation ΓY is 'traditional', but is also traditional for the set of global sections of a bundle! No confusion should arise!) This space ΓY is contractible in a geometrically pleasing way - the homotopy reduces the 'active' part of each path until it does nothing: if $\alpha : I \to Y$ with $\alpha(0) = *$, then $\alpha_t(s) = *$ if $s \leq t$ and is $\alpha(s - t)$ if $t \leq s \leq 1$. The α_t form a homotopy, essentially a path, from α to the constant path at $*$. We can realise C^f as the pullback:

$$\begin{tikzcd} C^f \arrow[r] \arrow[d] & \Gamma Y^I \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

(A useful observation here is that this pullback absorbs the homotopy of the homotopy pullback by replacing the $*$ by a contractible space. That *is* an example of a general process, a 'rectification' or 'rigidification' process, but this will not be explored until much later in these notes.)

Example 1: The neat example that illustrates the importance of this homotopy fibre construction is to take Y to be an arcwise connected space, X a proper subspace (so the inclusion f is very far from being a fibration). The fibre of f over a point $y \in Y$ is either a single point, if $y \in X$, or empty, if it is not. We think of y as being a map $y : * \to Y$, picking out that element, and change y along a path y_t, from being in X, say y_0, to not being in X, at y_1. That path is a homotopy between the maps y_0 and y_1, so although y_0 and y_1 are homotopic maps, the fibre over y_t changes homotopy type as t varies. On the other hand, the homotopy fibre has the same homotopy type along the whole of y_t. (We saw earlier (page 34) that the fundamental group of $F_h(f)$ was $\pi_2(Y, X)$ and does not change, up to specified isomorphisms, as one varies t between 0 and 1.)

Example 2: This first example was with f far from being a fibration. What if f *is* a fibration? (We, as usual, want to concentrate on the intuitions behind the facts here so will not explore this in depth, but it will be useful to have some picture of what is happening, leaving details either **to the reader** to provide or to find, as the results are fairly easy to find in the literature.)

First note the obvious

$$f^{-1}(*) = \{x \mid f(x) = *\},$$

whilst

$$C^f = \{(x, \lambda) \mid \lambda \in \Gamma Y, \lambda(0) = *, \lambda(1) = f(x)\},$$

so, in particular, there is a map from $f^{-1}(*)$ to C^f, mapping x to (x, c), where c is the constant path at $*$. We would like to see when this map is a homotopy equivalence. We have that underlying it, in some sense, is the map sending $*$ to $c \in \Gamma Y$, which is a homotopy equivalence, in fact a strong deformation retraction. If you try to see if this will induce in some way a retraction from C^f to $f^{-1}(*)$, then you hit the problem of what path an element (x, λ) should trace out in order to get to some $(x', c) \in f^{-1}(*)$. This would have to project down onto a path in X and in general there will not be one. If we assume that f is a fibration however, we can see more clearly what to do. (Recall that a fibration has a homotopy lifting property and it is that we will use.)

Examine the following diagram:

$$\begin{tikzcd} C^f \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ C^f \times I \arrow[r] & Y. \end{tikzcd}$$

The bottom horizontal map here is the composite $C^f \times I \to \Gamma Y \to Y$. The first of these is the inclusion, then the second is the homotopy retracting ΓY to a point, composed with the projection onto Y. The top horizontal map is q^f, so the diagram commutes. As f is assumed to be a fibration, there is a lift of the bottom map to a homotopy $C^f \times I \to X$, extending q^f on its 'zero' end. Its other end gives a map which has image in the fibre of f, so we have what we want - except for **checking details**!

This is very useful as it says: if f is a fibration, we do not need to turn it into one before taking its fibre! Why is that useful? Look at the fibre Puppe sequence so far

$$C^f \to X \to Y.$$

We said that ΓY is a fibration, so $q^f : C^f \to X$ is also a fibration. We can take its homotopy fibre, which will look messy to say the least, or its fibre, which is a lot easier to calculate!

$$\begin{aligned}(q^f)^{-1}(*_X) &= \{(\lambda, x) \mid \lambda(0) = *_Y, \lambda(1) = f(x), x = *_X\} \\ &= \{\lambda \mid \lambda(0) = \lambda(1) = *_Y\},\end{aligned}$$

so $(q^f)^{-1}(*_X) \cong \Omega Y$, the space of loops, at the base point,of Y. (This is neat, of course, as Ω is a functor, which is adjoint to S, the reduced suspension. Whether it is **right or left adjoint is left to you**! Thus we have a linkage between the right and left Puppe sequence constructions.) That fact gives us the tool to open up the whole of the sequence. It goes

$$\ldots \to \Omega^2 Y \to \Omega C^f \to \Omega X \xrightarrow{\Omega f} \Omega Y \to C^f \to X \xrightarrow{f} Y.$$

Given a pointed space Z, we can apply $[Z, -]$ to this sequence to get our long exact sequence

$$\ldots \to [Z, \Omega^2 Y] \to [Z, \Omega C^f] \to [Z, \Omega X] \xrightarrow{[Z,\Omega f]} [Z, \Omega Y] \to [Z, C^f] \to [Z, X] \xrightarrow{[Z,f]} [Z, Y],$$

(and once **you have sorted out** right or left adjunctions, you will find many terms you recognise from the other type of Puppe sequence).

Our treatment here has been deliberately informal. The importance of these sequences for cohomology cannot be over emphasised and we **suggest that you look** at some formal treatments, both for the algebraic case (via derived and triangulated categories, e.g. Neeman, [103]) and via the topological case consulting, say, May, [90] in the first instance before looking into the theory in other sources. There are abstract versions in homotopical algebra and a neat categorical treatment in Gabriel and Zisman, [60].

One final point before passing from descriptions of Puppe sequences to using them is the interpretation of exactness at the various points in the sequence. For instance, at $[Z, C^f]$, an element is represented by a map, g say, to C^f, and as C^f is given by a pullback, g decomposes via the two projections into a pair (g_X, g_Γ) with $g_X : Z \to X$ and $g_\Gamma : Z \to \Gamma Y$ such that $fg_X = e_1 g_\Gamma$. Going one step further, $\Gamma Y \subset Y^I$, so g_Γ gives a homotopy between $*$, the constant map to the basepoint, and fg_X. Now suppose $[Z, f] : [Z, X] \to [Z, Y]$ sends a homotopy class $[k]$ to the basepoint, then fk is homotopic to $*$ and we can build a $g : Z \to C^f$ from k and that homotopy. The more difficult part of the exactness at $[Z, X]$ follows. Back to $[Z, C^f]$, suppose our $g = (g_X, g_\Gamma)$ gets sent to the 'point' of $[Z, X]$, then $q^f g_X$ must be null homotopic. Pick such a null homootpy $h : Z \times I \to X$ and use the fact that q^f is a fibration to lift h to $\overline{h} : Z \times I \to C^f$. The 'other end ' of $\overline{h}$, i.e., $\overline{h}e_1$ is such that $q^f \overline{h} e_1$ is $*$, so $\overline{h} e_1$ is into the fibre of q^f, but that is ΩY. It remains to put the various pieces together. The details can be found in many sources, but what is important to retain is the way of constructing a corresponding element in the previous stage. A trivialisation of an element yields a class in another stage. This should remind you of M-torsors, of categorisation and of homotopy cohenrence.

7.3 Puppe sequences and classifying spaces

7.3.1 Fibrations and classifying spaces

In his discussion of bitorsors, etc., in [17], Breen makes use of Puppe sequences of maps between classifying spaces. Suppose $v : H \to G$ is a morphism of simplicial groups, then we get an induced map of classifying spaces $Bv : BH \to BG$. We can take BG to be $\overline{W}G$ as being the neatest construction from our simplicial viewpoint. (Detailed calculations with $\overline{W}G$, etc., are quite easy in the simple case that we will need, but do get complicated if G has lots of non-trivial terms in its Moore complex. Another point worth making is that the detailed formulae for $\overline{W}G$ given earlier, page 125, use the algebraic composition order and therefore sometimes seem to reflect 'right actions'. This can be got around in either of two ways. The formulae for both $\overline{W}$ and G, the Dwyer-Kan $\mathcal{S}$-groupoid functor, can easily be reversed to get equivalent ones using the other composition order. This may be needed later when considering cocycles, etc., however the second argument uses that $\overline{W}G$ determines a Kan complex that is determined up to homotopy type - so either method will lead to the same $[-, \overline{W}G]$ and thus *most* of the time we can ignore the composition order. To ignore it, or forget it, completely is not a good idea, but we can face the problem, if and when it is needed.)

We thus are looking at $Bv : BH \to BG$. If v is not surjective, then we can use the mapping cocylinder construction, suitably adapted, to replace it by a fibration and fibrations of simplicial groups are exactly the surjective morphisms. We can thus study, without loss of generality, the surjective case and, of course, that means using the exact sequence

$$K \xrightarrow{u} H \xrightarrow{v} G$$

of simplicial groups and studying the effect of the functor B on it.

We 'clearly' get a long Puppe sequence, ending with

$$\ldots \to \Omega BH \to \Omega BG \to C^{Bv} \to BH \to BG.$$

Such a Puppe sequence can be constructed from the 'obvious' cocylinder functor, $\mathcal{S}_*(\Delta[1], -)$, but only works really well if applied to Kan complexes. Luckily these simplicial sets *are* Kan, so we can proceed accordingly. We note that as v is a fibration of simplicial groups, Bv is a fibration of simplicial sets, so we can hope that C^{Bv} can be more easily calculated than would be the case in general.

To see why Bv is a fibration, imagine we have a $\underline{g} \in BG_n$ and thus $\underline{g}$ has the form $(g_{n-1}, \ldots, g_0)$ with $g_i \in G_i$. We can find $h'_i \in H_i$ such that $v(h'_i) = g_i$, $i = 0, \ldots, n-1$. If we are given a (n,k)-horn, $\overline{h}$, in BH that maps down to the (n,k)-horn, $(d_n\underline{g}, \ldots, \widehat{d_k\underline{g}}, \ldots, d_0\underline{g})$, of $\underline{g}$ (using the traditional $\widehat{\ }$ notation for an omitted element), then $\underline{h}^{-1}.\overline{h}'$ gives a horn over the trivial (n,k)-horn of BG, that is, we can *translate* the filling problem to the identity, where it is essentially that of proving that $\overline{W}G$ is a Kan complex, which is easier to handle and we will do so in a moment. Note this argument uses a transversal in each dimension, although we did not explicitly label it as being one, namely $g_i \mapsto h'_i$, which is suggestive of other uses of transversals in these notes.

An indirect, but neat, proof that $\overline{W}$ preserves fibrations and weak equivalences is to be found on p. 303 of the book, [61], by Goerss and Jardine. They note that this implies that $\mathcal{G}$ preserves cofibrations and weak equivalences, which is also very useful.

Postponing the proof that classifying spaces are Kan for the moment, the last thing to identify is the fibre of Bv, but this is easy, since we have an explicit description of Bv. It sends $\underline{h} = (h_{n-1}, \ldots, h_0)$ to $(v(h_{n-1}), \ldots, v(h_0))$, so its fibre is exactly the image by Bu of BK. We can thus use that, for fibrations, the fibre and homotopy fibre coincide up to equivalence, to conclude $C^{Bv} \simeq BK$ and our Puppe sequence now looks like

$$\ldots \to \Omega BH \to \Omega BG \to BK \to BH \to BG.$$

7.3.2 $\overline{W}G$ is a Kan complex

We have left this aside because we want to examine it in some detail, and those details were not needed at that point in our discussion. As an example of what might be done, suppose that G satisfies some extra condition such as the vanishing of its Moore complex in certain dimensions or that it satisfies the thin filler condition above some dimension, then the constructive description of $\overline{W}G$ suggests that it might be feasible to analyse $\overline{W}G$ to see if it satisfies some similar condition.

We will give the verification for a simplicial group, however, in many of the applications, we will need the construction for a simplicial group object in a topos, $\mathcal{E}$. This will allow us to talk of the classifying space of a sheaf of simplicial groups without worrying about the context. All the structure, however, is specified in a constructive way, and so goes across without any pain to a general topos.

For convenience, we repeat the formulae for $\overline{W}G$, from page 125, making small adjustments, since we will not be looking at the groupoid case here, so let G be a simplicial group.

The simplicial set, $\overline{W}G$, is described by

- $(\overline{W}G)_0$ is a single point, so $\overline{W}(G)$ is a reduced simplicial set;
- $(\overline{W}G)_n = G_{n-1} \times \ldots G_0$, as sets, for $n \geq 1$.

The face and degeneracy mappings between $\overline{W}(G)_1$ and $\overline{W}(G)_0$ are the source and target maps and the identity maps of G_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(g_{n-1}, \ldots, g_0) = (g_{n-2}, \ldots, g_0)$;
- for $0 < i < n$, $d_i(g_{n-1}, \ldots, g_0) = (d_{i-1}g_{n-1}, d_{i-2}g_{n-2}, \ldots, d_0g_{n-i}g_{n-i-1}, g_{n-i-2}, \ldots, g_0)$;

and

- $d_n(g_{n-1}, \ldots, g_0) = (d_{n-1}g_{n-1}, d_{n-2}g_{n-2}, \ldots, d_1g_1)$,

whilst

- $s_0(g_{n-1}, \ldots, g_0) = (1, g_{n-1}, \ldots, g_0)$;

and,

- for $0 < i \leq n$, $s_i(g_{n-1}, \ldots, g_0) = (s_{i-1}g_{n-1}, \ldots, s_0 g_{n-i}, 1, g_{n-i-1}, \ldots, g_0)$.

Let us start in a low dimension to see what problems there may be. For $n = 2$, suppose we had a $(2, 2)$ box in $\overline{W}G$, so we have a pair, (x_0, x_1), of elements of $\overline{W}G_1$, which fit together, so $d_0x_0 = d_0x_1$. (We think of this as $(x_0, x_1, -)$, a list of possible faces, with a gap in the d_2-position.) We want some $y \in \overline{W}G_2$ such that $d_0y = x_0$ and $d_1y = x_1$.

Expanding things (in fact this is purely formal here, but lays down notation for later), we thus have $x_0 = (x_{0,0})$, $x_1 = (x_{1,0})$. (The condition on the faces happens to be trivial here since $\overline{W}G_0$ is a single point.) These $x_{i,0}$ are in G_0, for $i = 0, 1$. Similarly y will be of form (y_1, y_0), and we can examine what the desired conditions imply

$$\begin{aligned} x_{0,0} &= d_0y = y_0 \\ x_{1,0} &= d_1y = d_0y_1.y_0. \end{aligned}$$

We thus already know y_0 and need to find a y_1 with $d_0y_1 = x_{1,0}x_{0,0}^{-1}$. Clearly, we can find one, for instance, $s_0(x_{1,0}x_{0,0}^{-1})$ will do and we can even find *all* such, since any other suitable y_1 will have form $ks_0(x_{1,0}x_{0,0}^{-1})$ for some $k \in Ker\, d_o$. In other words, we really do know a lot about the possible fillers for our horn, even being able to count them if G is a finite simplicial group!

Next in line, we suppose that we have $(x_0, -, x_2)$ and want y such that $d_0y = x_0$, $d_2y = x_2$. Expanding these, using the same notation as before, we have, once again, that $x_{0,0} = d_0y = y_0$, but now

$$x_{2,0} = d_2y = d_1y_1.$$

Again we have y_0 and can solve $d_1y_1 = x_{2,0}$, using $y_1 = s_0(x_{2,0})$, and, to get all fillers, $ks_0(x_{2,0})$ with $k \in Ker\, d_1$.

That was easy! What about $(2, 0)$-horns? These *are* slightly harder, as the other types did give us d_0y and thus handed us y_0 'on a plate', but it is only '*slightly*'.

We have $(-, x_1, x_2)$, $x_i = (x_{i,0})$ and want $y = (y_1, y_0)$. We thus know

$$\begin{aligned} x_{1,0} &= d_1y = d_0y_1.y_0 \\ x_{2,0} &= d_2y = d_1y_1. \end{aligned}$$

We do not know y_0, but do know d_1y_1 and can solve to get $y_1 = ks_0(x_{2,0})$ with $k \in Ker\, d_1$ as before. We then have $y_0 = (d_0(k)x_{2,0})^{-1}x_{1,0}$ for the general filler.

Although that is simple, it is also easy to see that it can be extended, with modifications, to higher dimensions.

If we have a (n, n)-horn in $\overline{W}G$, then we have $(x_0, \ldots, x_{n-1}, -)$ with $x_i = (x_{i,n-2}, \ldots, x_{i,0}) \in \overline{W}G_{n-1}$. for $i = 0, 1, \ldots, n-1$. The compatibility condition is non-trivial here, so we note that

$$d_ix_j = d_{j-1}x_i$$

if $i < j$.

We need to find all $y = (y_{n-1}, \ldots, y_0)$ with $d_i y = x_i$ for all $i < n$. We thus have

$$x_0 = d_0 y = (y_{n-2}, \ldots, y_0),$$

but this means that we know all but the top dimensional element of the string that is y. Next

$$x_1 = d_1 y = (d_0 y_{n-1}.y_{n-2}, \ldots, y_0),$$

so we glean the information that

$$d_0 y_{n-2} = x_{1,n-2}.x_{0,n-2}^{-1}.$$

Continuing, we get, for $k > 1$ and in the range $k < n$, that

$$x_k = d_k y = (d_{k-1} y_{n-1}, d_{k-2} y_{n-2}, \ldots, d_0 y_{n-k}.y_{n-k-1}, \ldots, y_0),$$

and here the only new information is that which we get on $d_{k-1} y_{n-1}$, which can be read off as being $x_{k,n-2}$.

We should note that the compatibility condition tells us that there will be no inconsistencies in the rest of this string. For instance, we seem to have

$$x_{k,n-k-1} = d_0 y_{n-k}.y_{n-k-1}.$$

As we know y_{n-k-1} and y_{n-k}, we can check that we do not have a conflict:

$$\begin{aligned} y_{n-k} &= x_{0,n-k} \\ y_{n-k-1} &= x_{0,n-k-1}, \end{aligned}$$

but then $x_{k,n-k-1}$ needs to be $d_0 x_{0,n-k}.x_{0,n-k-1}$, which is the $(n-k-1)$-component of $d_k x_0$. The compatibility condition tells us

$$d_0 x_k = d_{k-1} x_0,$$

and we leave the reader to check that the $(n-k-1)$-component of this equation is exactly

$$x_{k,n-k-1} = d_0 x_{0,n-k}.x_{0,n-k-1},$$

as hoped for.

Collecting things up, we know $d_\ell y_{m-1}$ for $\ell = 0, \ldots, n-2$, i.e., we have a $(n-1, n-1)$-horn in G itself. We know not only that G is a Kan complex, but how to fill horns algorithmically, so can find a suitable y_{n-1} and hence a filler, y for the original (n,n)-horn in $\overline{W}G$.

The intermediate cases of (n,i)-horns in $\overline{W}G$ for $0 < i < n$ are very similar and are **left to you**. In each case, as we have $d_0 y = x_0$, we just have to work on the first element, y_{n-1} in the string givng us y. The other parts give us a horn in G, which encodes the available information on the faces of y_{n-1}. We fill this horn to get y_{n-1}, and hence to fill the original horn in $\overline{W}G$. In each case, we can fill because we know that the underlying simplicial set of G is a Kan complex. We have the algorithm for fillers and so can analyse the set of fillers for a given horn, the algorithm giving a definite coset representative. For instance, in the (n,n)-horn, above, we found y exactly except in the first, highest dimensional position, y_{n-1}. We use the algorithm to find *one* filler / solution for y_{n-1}, then know any other will differ from it by an element of $\bigcap_{i=0}^{n-2} Ker\, d_i$. This latter group is essentially a 'translate' of NG_{n-1} using the argument that Carrasco used to simplify Ashley's group T-complex condition (see the comment in the discussion of group T-complexes, page 30).

We still have to handle the $(n, 0)$-horn case, so should not be too pleased with ourselves yet! That was the slightly awkward case for the $n = 2$ situation that we studied earlier, as we do not have y_{n-2} given us initially.

Suppose $(-, x_1, \ldots, x_n)$ is the horn and we have to find a $y \in \overline{W}G_n$ satisfying $d_i y = x_i$ for $i = 1, \ldots, n$. Using the same notation, we have

$$x_1 = d_1 y = (d_0 y_{n-1}.y_{n-2}.y_{n-3}, \ldots, y_0)$$

and we get all the y_i except y_{n-1} and y_{n-2}. We then have

$$x_i = d_i y = (d_{i-1} y_{n-1}, \ldots, y_0)$$

and so get all the faces of y_{n-1}, except that zeroth one. We can thus fill the resulting $(n-1, 0)$-box in G (using the algorithm) to find a suitable y_{n-1}. We still do not have y_{n-2}, but as we now have y_{n-1}, we can read off $d_0 y_{n-1}$ from our solution to get

$$y_{n-2} = (d_0 y_{n-1})^{-1}.x_{1,n-1}.$$

We thus do get a filler for our $(n, 0)$-horn *and* can analyse the set of fillers / solutions if we need to.

Theorem 10 *For any simplicial group, G, the classifying space, $\overline{W}G$, is a Kan complex* ■

Perhaps it occurs to you that it should be possible to adapt this constructive proof to give a proof that, if $f : G \to H$ is a surjection of simplicial groups, and thus a fibration, then $\overline{W}f$ will be a Kan fibration. We know already that $\overline{W}f$ *is* a fibration, as we saw this earlier, quoting some results in Goerss and Jardine, [61], but it should not be too difficult to construct a proof which took transversals in the necessary dimensions and *found* lifts for horns accordingly. This is left as a bit of a **challenge to the reader**. It is not just an exercise for amusement, however, as the analysis of fillers could give some interesting results in some cases.

We mentioned that most of this went across 'without pain' to the case of simplicial objects in a topos, $\mathcal{E}$, and hence to simplicial sheaves on a space. Perhaps a few words are needed, however, to show how this can be done. We start by thinking about how to talk about the Kan fibrations in $\mathcal{E}$, or more generally in any category with finite limits. For any object K in $Simp(\mathcal{E})$, we can form an object of $\mathcal{E}$ corresponding to the 'set of (n, k)-horns' in K. To see how to think about this, we look at $(2, 1)$-horns. These correspond, in the set based case, to pairs of 1-simplices, (x_0, x_2), with $d_0 x_2 = d_1 x_0$, so are elements of the pull back:

$$\begin{tikzcd}
\Lambda^0[2](K) \arrow[r] \arrow[d] & K_1 \arrow[d, "d_1"] \\
K_1 \arrow[r, "d_0"'] & K_0
\end{tikzcd}$$

More generally, for a simplicial set K, $\Lambda^k[n](K)$, the set of (n,k)-horns in K is given by an iterated pullback or limit of a diagram. (If you have not seen this before, or ever handled it yourself, do try to formulate the diagram in as neat a way as possible - 'neat' is a question of taste! It is technically quite easy, but gives good practice in converting concepts across to diagrams and hence to finite limit categories.)

We thus can mimic this to get an object, $\Lambda^k[n](K)$, and an induced map, $K_n \to \Lambda^k[n](K)$, which maps an n-simplex to the (n,k)-horn of its faces other than the k^{th} one.

Definition: If $\mathcal{E}$ is a finite limit category, a morphism, $p : E \to B$, in *Simp*$(\mathcal{E})$ is a *Kan fibration* if the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all epimorphisms in $\mathcal{E}$.

We can equally obtain the meaning of a *Kan object* in *Simp*$(\mathcal{E})$.

Beke, [**?**], uses the term *local Kan fibration* for what has been called a Kan fibration in $\mathcal{E}$ above. That 'local' terminology is especially good when talking about the topos case, but with, later on in these notes, a use of 'locally Kan' enriched category, it did seem a bit risky to over use 'local Kan'!

We now return to the case of simplicial groups in the usual sense.

Corollary 5 *Suppose that $NG_{n-1} = 1$, then, for any i, with $0 \leq i \leq n$, any (n,i)-horn in $\overline{W}G$ has a unique filler.*

Proof: We noted that different fillers for an (n,i)-horn differed by elements of NG_{n-1}, or its translates, thus if that group is trivial, ... ■

Of course, we expect $\overline{W}G$ to have the same homotopy groups as G, displaced by one dimension, since there is the fibration sequence

$$G \to WG \to \overline{W}G$$

with WG contractible, so this corollary comes as no surprise. What is interesting is the detail that it gives us. If $NG_k = 1$, then clearly $\pi_k(G) = 1$ and hence $\pi_{k+1}(\overline{W}G)$ is trivial as well, but that there are unique fullers in the structure is perhaps a bit surprising, at least until one sees why.

Suppose that, as usual, G is a simplicial group and $D = (D_n)_{n\geq 1}$ is the graded subgroup of products of degeneracies. Within $\overline{W}G_n$, let

$$T_n = D_{n-1} \times G_{n-2} \times \ldots \times G_0,$$

be the subset of those elements whose first component is a product of degenerate elements, yielding a graded subset of $\overline{W}G$.

Corollary 6 *If G is a group T-complex, then $(\overline{W}G, T)$ is a simplicial T-complex.*

Proof: There is not that much to check. We know, by the proof of the theorem, that every horn has a filler in T. Uniqueness follows from the fact that G is a group T-complex. The other conditions are as easy to check as well, so are **left to you**. ■

Corollary 7 *If G is thin in dimensions greater than n, then $\overline{W}G$ has a unique T-filler for all horns above dimension $n+1$.* ■

The property of being a T-complex involves all dimensions and here we are meeting some sort of weaker 'filtered' condition. This condition was studied extensively by Duskin, and used in various forms in [49**?**] and in later work. It was also used by his students Glenn, [**?**], and Nan Tie, [101, 102], who looked at some of the links with T-complexes. They are also used, more recently, by Beke, [**?**], and we will examine his approach in a bit more detail.

A good reason for briefly looking at this is that it introduces several useful concepts and the linked terminology. These in the main are due to Jack Duskin, who developed them for the study of simplicial objects in a topos. We will give the definitions and subsequent discussion within the classical setting of $Sets$, but this is really only because we have not given a thorough and detailed treatment of toposes earlier. The basic point is that if the arguments used in the development are 'constructive' then, usually with some minor changes, the theory will generalise from a category of sets, to one of sheaves, and eventually to any Grothendieck topos. To make that statement more precise would require quite a lot more discussion, and would take us away from our main themes, so investigation is **left to you**.

We start with a slight variant of the Kan fibration definition that we met near the end of the first chapter, (see page 26).

Definition: A simplicial map $p : E \to B$ is a *Kan fibration*, or *satisfies the Kan lifting condition, in dimension* n if, in every commutative square (of solid arrows) of form

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

a diagonal map (indicated by the dashed arrow) exists, i.e., there is an $f : \Delta[n] \to E$ such that $pf = f_0$, $f.inc = f_1$, so f lifts f_0 and extends f_1.

We thus have that p is a Kan fibration if it is one in *all dimensions*. To capture the sort of behaviour that we have just seen, we can refine the above (following Duskin, [**?**]).

Definition: A simplicial map $p : E \to B$ *satisfies the exact Kan lifting condition in dimension* n if, in every commutative square (as above), precisely one diagonal map f exists.

Starting with the Kan fibration condition, we singled out the Kan complexes as being those simplicial sets for which the unique map $K \to \Delta[0]$ was a Kan fibration. We clearly can do a similar thing here. Again the definition is intended to mirror attributes of results that we have seen in recent pages.

Definition: A simplicial set K is an *exact* n*-type*, or n*-hypergroupoid*, if $K \to \Delta[0]$ is a Kan fibration that is exact in dimensions greater than n.

The definition of n-hypergroupoid used by Glenn, [**?**], is slightly different from this as it only requires the (exact) Kan condition in dimensions greater than n, so not requiring K to 'be' a Kan complex in lower dimensions. The n-hypergroupoid terminology is due to Duskin, [**?**], whilst 'exact n-type' is Beke's, [**?**].

If we need a version of these ideas in $Simp(\mathcal{E})$, then we can easily adapt our earlier discussion of horns and Kan objects in that context. For instance:

Proposition 44 *If $\mathcal{E}$ is a finite limit category, a morphism, $p : E \to B$, in $Simp(\mathcal{E})$ is an exact Kan fibration in dimension n if, and only if, the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all isomorphisms in $\mathcal{E}$.* ■

Corollary 8 *In $Simp(\mathcal{E})$, an object K is an exact n-type (or n-hypergroupoid) if, and only if, the natural map, $K_k \to \Lambda^j[k](K)$, is an epimorphism for $k \leq n$ and an isomorphism for $k > n$.* ■

To begin to take 'exact n-types' apart, we will need to look at look at the coskeleta functors that we mentioned, but did not fully define, nor study in depth, in our discussion of 'truncation', (see section 4.2.1, page 92). The reasons for not handling those functors more fully there included the possible confusion between the various forms of truncation. There we considered truncating the Moore complex, but the natural *simplicial* truncation at level n would be to throw away all 'stuff' in dimensions greater than n. We next turn to this, so as to give enough detail on the coskeleta functors.

The category, $\mathbf{\Delta}$, consists of all finite ordinals and all order reserving maps between them. Given any natural number n, we can form a full subcategory, $\mathbf{\Delta}[0,n]$, with objects the ordinals $[0], \ldots, [n]$, and all order preserving maps between *them*. As the category of simplicial sets is $\mathcal{S} = Sets^{\mathbf{\Delta}^{op}}$, there is a restriction functor, call *n-truncation* or, more fully, *simplicial n-truncation*,

$$tr_n : \mathcal{S} \to Sets^{\mathbf{\Delta}[0,n]^{op}},$$

which, to a simplicial set, K, assigns the n-truncated simplicial set, $tr_n(K)$, with the same data in dimensions less than $n+1$, but forgets all information on higher dimensions.

This truncation functor has a right adjoint, which is denoted $cosk_n$. (Our previous use of this notation was, in fact, for the composite $cosk_n \circ tr_n$. This multiple use should not cause any problems as which one is being used should be clear from the context.) We thus have, for an n-truncated object, L, a natural isomorphism

$$Sets^{\mathbf{\Delta}[0,n]^{op}}(tr_n K, L) \overset{\cong}{\to} \mathcal{S}(K, cosk_n L).$$

From this, we can get the description of $cosk_n L$ that was essentially given earlier, via the isomorphism $(cosk_n L)_k \cong Sets^{\mathbf{\Delta}[0,n]^{op}}(tr_n(\Delta[k]), L)$.

It is very useful for our purposes to have a description of when a simplicial set, K, is isomorphic to its own n-coskeleton, or, more exactly, to $cosk_n tr_n(K)$. The following summary is actually adapted from Beke's paper, [**?**], but is quite well known and moderately easy to prove, so the proof will be **left as an exercise**.

Proposition 45 *For a simplicial set, K, the following are equivalent:*

1) K is isomorphic to an object in the image of $cosk_n$.

2) The natural morphism $K \to cosk_n tr_n(K)$ is an isomorphism.

3) Writing $\partial\Delta_k(K)$ for the set

$$\partial\Delta_k(K) = \{(x_0, \dots, x_k) \mid x_i \in K_{k-1} \text{ and, whenever } i < j,\ d_i x_j = d_{j-1} x_i\},$$

(so $\partial\Delta_k(K) \cong \mathcal{S}(\partial\Delta[k], K)$), the natural 'boundary' map $b_k(x) = (d_0 x, \dots, d_k x)$, from K_k to $\partial\Delta_k(K)$ is a bijection for all $k > n$.

4) The natural map $K_k \to Sets^{\mathbf{\Delta}[0,n]^{op}}(tr_n\Delta[k], tr_n(K))$, which sends a k-simplex x of K, considered as its 'name', $\ulcorner x \urcorner : \Delta[k] \to K$, to the n-truncation, of $\ulcorner x \urcorner$, is a bijection for all $k > n$.

5) For and $k > n$, and any pair of (solid) arrows

$$\begin{tikzcd} \partial\Delta[k] \arrow[r] \arrow[d] & K \\ \Delta[k] \arrow[ur, dashed] & \end{tikzcd}$$

there is precisely one *(dotted) arrow making the diagram commute.* ■

As we said, the proof is **left to you**, as it is just a question of translating between different viewpoints.

Definition: If K satisfies any, and hence all, of the above conditions, it is called *n-coskeletal.*

The first two conditions can be transferred verbatim for simplicial objects in any category with finite limits, and thus for simplicial objects in a topos. Condition 3 can also be handled in those contexts using iterated pullbacks to construct $\partial\Delta_k(K)$. Condition 4) can also be used if the category of simplicial objects has finite cotensors (see the discussion of tensors and cotensors in simplicially enriched categories in section **??**, page **??**). A similar comment may be made about 5), since using cotensors allows one to 'internalise' the condition - but it ends up then being 3) in an enriched form. The details will not be needed in our later discussion, so are **left to you if you need them**.

We use this notion of n-coskeletal object in the following way

Proposition 46 *(cf. Beke, [**?**], proposition 1.3) (i) If K satisfies the exact Kan condition above dimension n, then K must be $(n+1)$-coskeletal.*
(ii) If K is n-coskeletal, then it satisfies the exact Kan condition above dimension $n+1$.
(iii) If K is an n-coskeletal Kan complex, then it has vanishing homotopy groups in dimensions n and above.
(iv) An exact n-type has vanishing homotopy groups above dimension n.

Before we prove this, it needs noting that there is an internal version in $Simp(\mathcal{E})$ for $\mathcal{E}$ a topos, see [**?**]. We have refrained from giving it only to avoid the need to define the homotopy groups of such an object internally.

Proof: (i) Suppose we are given a map $b : \partial\Delta[k] \to K$ for $k > n+1$, then we can omit $d_0 b$ to get a $(k, 0)$-horn in K. By assumption, this horn has a filler, $f : \Delta[k] \to K$, so we consider both $d_0 f$ and $d_0 b$. As they have the same boundary and since K satisfies the exact Kan condition above dimension n, they must coincide. We have thus that f is a filler for b. By exactness, we have that it is unique.

(ii) If $m > n+1$, $tr_n(\Lambda^k[m]) \to tr_n(\Delta[m])$ is fairly obviously an isomorphism. Now $cosk_n(K)$ satisfies the exact Kan condition in dimension m if, and only if, for any horn, $\underline{x} : \Lambda^k[m] \to K$, there is a diagram

$$\begin{tikzcd} \Lambda^k[m] \arrow[r, "\underline{x}"] \arrow[d, "inc"'] & cosk_n K \arrow[d] \\ \Delta[k] \arrow[r] \arrow[ur, dashed, "\exists!"] & 1 \end{tikzcd}$$

with unique diagonal. Using the adjunction, this gives a diagram

$$\begin{tikzcd} tr_n\Lambda^k[m] \arrow[r, "\overline{x}"] \arrow[d, "inc"'] & K \arrow[d] \\ tr_n\Delta[k] \arrow[r] \arrow[ur, dashed, "?"] & 1 \end{tikzcd}$$

and we have noted that the left hand side is an isomorphism if $m > n+1$.

(iii) If K is Kan, the topological description of homotopy groups goes over to K, i.e., as the group of homotopy classes of maps from $\partial\Delta[n]$ to K mapping a vertex to chosen basepoint. Such a map will fill in dimensions $k \geq n$, so all the $\pi_k(K)$ will be trivial for any base point. (**You should fill in the details of this argument.**)

(iv) This just combines (i) and (iii). ■

We note that (iv) above says that exact n-types are n-types!

7.3.3 Loop spaces and loop groups

We now turn to ΩBG. Although not strictly necessary, it will help to shift our perspective slightly and talk a bit more on some generalities. Let S^0 be the pointed simplicial set with two vertices and only degenerate simplices in dimensions higher than 1. In other words, it is the 0-sphere. The reduced suspension SS^0 is S^1, the circle, which can also be realised as $\Delta[1]/\partial\Delta[1]$, the circle realised as the interval with the ends identified to a single point. The loop space, ΩK, on a pointed connected simplicial set, K, is then $\underline{\mathcal{S}}_*(S^1, K)$, or more briefly, K^{S^1}, the simplicial set of pointed maps from S^1 to K. (It will be a Kan complex if K is one.) As in the topological case, ΩK has the structure of an 'H-space'. This refers to a compositional structure *up to homotopy*, so we have

$$\mu : \Omega K \times \Omega K \to \Omega K,$$

given by composition of loops. Topologically this is just that: first do one loop, then the other, then rescale to get a map from $[0, 1]$ again. The rescaling means that this μ is not associative, but is associative up to a homotopy. There are also 'reverses', which are inverses up to homotopy, and it all fits together to make ΩK a 'group up to homotopy'. (Again the homotopies can be linked together to make a homotopy coherent version of a group.) The same can be done in the simplicial case provided that K is Kan. (This is a **good exercise to attempt**, to see once more the use of 'fillers' as a form of algebraic structure.)

If K is not reduced, we can replace it by a homotopy equivalent reduced simplicial set. (In fact we want $K = \overline{W}G$ and that *is* reduced.) For such a K, the simplicial group GK is often called the *loop group* of K. (Look back to page 121, if you need to review the construction of GK.) What is the connection between ΩK and GK?

It is clear there should be one as the free group construction involved in the definition of GK uses concatenation of strings of simplices and that is the algebraic analogue of composition of paths, however it is associative, has inverses, etc., as it gives a group. It looks like an abstract algebraic model of ΩK, which replaces the homotopy coherent multiplication by an algebraic one, but, as a result, gets a much bigger structure. Even in dimension 0, $\Omega K_0 \cong K_1$, whilst GK_0 is the free group on K_1. (This is again a **useful place** to see what the two structures look like, in low dimensions, and to see if there is a 'natural' map between them.) If we could replace Ω by G, our life would simplify as G is left adjoint to $\overline{W}$ and so, for any simplicial group, H, there is a natural map

$$G\overline{W}H \to H,$$

which is a weak equivalence, i.e., it induces isomorphisms on all homotopy groups, then we would be able to identify three more terms of the Puppe sequence. In fact for any reduced K, GK and ΩK *are* weakly equivalent. We will not give the proof, referring instead to the discussion in Goerss and Jardine, [61], in particular the proof on p. 285. (This is very neat for us as it uses both ΓK, there called PK, and induced fibrations in a very similar way to our earlier treatment of the Puppe sequence.) If G is more interesting and is not reduced, then GK is equivalent to a disjoint union, indexed by $\pi_0(G)$, of simplicial sets that 'look like' copies of ΩG, namely loops, not at the identity element, but at some representative of a connected component of G. This will shortly be linked up with the décalage construction.

Putting all this together, we get that if

$$K \xrightarrow{u} H \xrightarrow{v} G$$

is a short exact sequence of simplicial groups, then the Puppe sequence of Bv ends:

$$\Omega G \to K \xrightarrow{u} H \xrightarrow{v} G \to BK \xrightarrow{Bu} BH \xrightarrow{Bv} BG.$$

We need to add what might be considered a cautionary note. To emphasise the *ideas* behind this sequence, we have handled the case of simplicial groups. For many of the applications, we have to work with sheaves of simplicial groups or, more generally, simplicial group objects in some topos, $\mathcal{E}$. In those cases the meaning of such terms as 'fibration' or 'weak equivalence' needs refining, much as the notion of 'equivalence' between categories needs adjusting before it can be used to its full potential with the 'stacks' that we will meet in the next chapter. The category in which one 'does' one's homotopy is then naturally to be considered with a Quillen model category structure and $[-,-]$ is replaced by $Ho(Simp(\mathcal{E}))(-,-)$, the 'hom-set' in the category obtained from that of simplicial objects in $\mathcal{E}$ by inverting the weak equivalences. These technicalities *do* complicate things to quite a large amount and are very non-trivial to describe in detail, however the idea is the same and the technicalities are there just to bring that idea to its most rigorous form. We have left out these technicalities to concentrate on the intuition, but they cannot be completely ignored. (Some idea of the possible detailed approaches to this can be found in Illusie's thesis, [72, 73], Jardine's paper, [74] and various more recent works on simplicial sheaves.)

7.3.4 Applications: Extensions of groups

Suppose we have our old situation, namely an extension of groups, or rather of sheaves of groups,

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to 1$$

(as in section 6.3.6). We can replace each by a constant simplicial group, L by $K(L,0)$, etc. (To simplify notation we will, in fact, abbreviate $K(L,0)$ back to L, whenever this is feasible.) We now apply the classifying space construction and take the corresponding Puppe sequence. The result will be

$$1 \to L \xrightarrow{u} M \xrightarrow{v} N \to BL \to BM \to BN.$$

(Here we are abusing notation even more, as the first three terms are the underlying simplicial sheaves of the corresponding sheaves of simplicial groups, which are , ... and so on, but writing $U(K(L,0))$ seems silly and it would get worse, so ... .)

Note that in this sequence, we have that $\Omega^2 BN$ is equivalent to ΩN, which is contractible, which explains the 1 on the left hand end.. The classifying spaces are the nerves of the corresponding groupoids, $BL = Ner(L[1])$, etc.

All this is happening in $Sh(B)$ (or, more generally, in a topos, $\mathcal{E}$). Given an open cover $\mathcal{U}$ of B, with nerve $N(\mathcal{U})$, we get a long exact sequence of groups and pointed sets:

$$1 \to [N(\mathcal{U}), L] \to [N(\mathcal{U}), M] \to [N(\mathcal{U}), N] \to [N(\mathcal{U}), BL] \to [N(\mathcal{U}), BM] \to [N(\mathcal{U}), BN],$$

and passing to the colimit over coverings, this gives

$$1 \to L(B) \to M(B) \to N(B) \to \check{H}^1(B, L) \to \check{H}^1(B, M) \to \check{H}^1(B, N).$$

This is exactly the exact sequence that we discussed earlier, again in section 6.3.6. Note that we have not yet got our hands on any substitute for the $\check{H}^2(B, L)$, that exists in the Abelian case.

7.3.5 Applications: Crossed modules and bitorsors

Suppose $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules. It would be good to examine the simplicial view of relative M-torsors in a similar way. We have a sheaf of simplicial groups given by $K(\mathsf{M})$ and have identified $colim[N(\mathcal{U}), K(\mathsf{M})] = colim H^0(N(\mathcal{U}), \mathsf{M})$ with $\pi_0(\mathsf{M}{-}Tors)$, which is a group. We also showed that any M-torsor, (E, t), had that E is a C-torsor with t a trivialisation of $\partial_*(E)$. This suggests some sort of exact sequence:

$$\pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C)) \xrightarrow{\partial_*} \pi_0(Tors(P)),$$

i.e., anything in $Tors(C)$ that is sent to the base point (that is, the class of the trivial torsor) in $Tors(P)$, comes from an M-torsor. We can see this geometrically as we saw earlier. What is neat is that if (E, t) and (E', t') are M-torsors, with E and E' equivalent as C-torsors, then we can assume $E = E'$ and can use the trivialisations t and t' to obtain a global section, p, of P such that $t' = p.t$. The implication is that

$$P(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C))$$

is also exact. This can also be seen from the Puppe sequence.

First a very useful bit of the simplicial toolkit. We form the décalage of $K(\mathsf{M})$. (Recall $K(\mathsf{M})$ is the simplicial group associated to M, that is, it is formed as the internal nerve of the internal category corresponding to M, that it has P in dimension 0, $C \rtimes P$ in dimension 1, etc. It also has a Moore complex which is of length 1 and is exactly $C \xrightarrow{\partial} P$.)

What is the décalage?

Definition: The *décalage* of an arbitrary simplicial set, Y, is the simplicial set, $Dec\,Y$, defined by shifting every dimension down by one, 'forgetting' the last face and degeneracy of Y in each dimension. More precisely

- $(Dec\,Y)_n = Y_{n+1}$;
- $d_k^{n,Dec\,Y} = d_k^{n+1,Y}$;
- $s_k^{n,Dec\,Y} = s_k^{n+1,Y}$.

This comes with a natural projection, $d_{last} : Dec\,Y \to Y$, given by the 'left over' face map. (Check it is a simplicial map.) We will denote this by p, for 'projection'. Moreover this map gives a homotopy equivalence

$$Dec\,Y \simeq K(Y_0, 0),$$

between $Dec\,Y$ and the constant simplicial set on Y_0. The homotopy can be constructed from the 'left-over' degeneracy, s^Y_{last}. (A full discussion of the décalage can be found in Illusie's thesis, [72, 73] and Duskin's memoir, [49]. Be aware, however, some sources may use the alternative form of the construction that forgets the *zeroth* face rather than the *last* one. This works just as well. The translation between the two forms is quite easy, if sometimes a bit time consuming!)

Of course, this same construction works for simplicial objects in any category. We need it mainly for (sheaves of) simplicial groups and, in particular, as hinted at earlier, we need $Dec\,K(\mathsf{M})$. We list some properties of this simplicial group:

(i) $Dec\,K(\mathsf{M})_0 \cong C \rtimes P$, $Dec\,K(\mathsf{M})_1 \cong C \rtimes C \rtimes P$, and in general, $Dec\,K(\mathsf{M})_n \cong C^{(n+1)} \rtimes P$. The face maps are given by

$$\begin{array}{rcll} d_0(c_n, \ldots, c_0, p) & = & (c_n, \ldots, c_1, \partial c_0.p) & \\ d_i(c_n, \ldots, c_0, p) & = & (c_n, \ldots, c_i c_{i-1}, \ldots, c_0, p) & 0 < i < n \\ d_0(c_n, \ldots, c_0, p) & = & (c_n c_{n-1}, \ldots, c_0, p) & \end{array}$$

with degeneracies given by suitable insertions of identities.

(ii) $Dec\,K(\mathsf{M})$ has Moore complex isomorphic to one of the form

$$C \to C \rtimes P.$$

Here we clearly have $Ker\,d_1 = \{(c_1, c_0, p) \mid c_1 = c_0^{-1}, p = 1\} \cong C$. We also have a boundary, induced by d_0, so the boundary sends $(c^{-1}, c, 1)$ to $(c^{-1}, \partial c\}$. If this looks strange, **just check** that $(c^{-1}, c, 1)((c')^{-1}, c', 1) = ((cc')^{-1}, cc', 1)$. (Don't forget the Peiffer identity!)

(iii) The boundary is a monomorphism and its image is the kernel of the homomorphism from $C \rtimes P$ to P that sends (c, p) to $\partial c.p$. (That makes sense as that is the target / codomain map of the internal category or cat^1-group associated to M.)

(iv) $Dec\,K(\mathsf{M})$ is homotopy equivalent to the constant simplicial group on P. (This can be seen from the Moore complex, but also from the retraction of $Dec\,K(\mathsf{M})$ onto the subsimplicial group given by all $(1, \ldots, 1, p)$. That map is a deformation retraction with the 'extra degeneracy', $s_{\ell ast}$, of the décalage construction giving the homotopy, (**for you to check**). This is neat, because it is explicit and natural and thus can provide a more geometric picture than merely stating that there is a weak equivalence of simplicial groups between $Dec\,K(\mathsf{M})$ and $K(P, 0)$.)

(v) The morphism $\mathsf{p} : Dec\,K(\mathsf{M}) \to K(\mathsf{M})$ is an epimorphism, hence is a fibration. (It is, in fact, split at each level by the last degeneracy map of $K(\mathsf{M})$.) We can give p explicitly by $\mathsf{p}(c_n, \ldots, c_0, p) = (c_{n-1}, \ldots, c_0, p)$, hence:

(vi) The kernel of p is given by $Ker\,\mathsf{p} = \{(c, 1, \ldots, 1, 1) \mid c \in C\}$ with the face and degeneracy maps given by the restrictions of the above, so $Ker\,\mathsf{p}$ is isomorphic to $K(C, 0)$.

(vii) Within the context of our much earlier discussion of crossed modules as being given by fibrations (page 35), we had that if G is a simplicial group and $N \lhd G$ a normal simplicial subgroup, then applying π_0 to the inclusion of N into G gave us a crossed module. The proof that, up to isomorphism, all crossed modules arise in this way was left to the reader! Here it is:

If we take $G = Dec\,K(\mathsf{M})$, and $N = Ker\,\mathsf{p}$, then $\pi_0(N) \to \pi_0(G)$ is $\partial : C \to P$ and the actions agree, (all 'up to isomorphism', of course).

This is at the heart of the algebraic proof of Loday's theorem (see 4.4) that cat^n-groups / crossed n-cubes model all connected homotopy $(n+1)$-types. Its appearance here is not accidental.

We thus have an exact sequence of simplicial groups arising from M:

$$1 \to Ker\,\mathsf{p} \to Dec\,K(\mathsf{M}) \to K(\mathsf{M}) \to 1$$

corresponding to

$$K(C, 0) \to K(P, 0) \to K(\mathsf{M}),$$

(which is not exact!).

At a crossed module level, we get

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] & P \arrow[r] & P \end{tikzcd}$$

is homotopy exact, or, more exactly (pun intended!) that

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] & C \rtimes P \arrow[r] & P \end{tikzcd}$$

is exact.

If we pass to the Puppe sequence, it will end

$$\Omega K(\mathsf{M}) \to C \to P \to K(\mathsf{M}) \to BC \to BP \to BK(\mathsf{M}).$$

Going through the usual process of applying $[N(\mathcal{U}), -]$ for an open cover $\mathcal{U}$ of the base space B, followed by the colimit over such $\mathcal{U}$s, we get

Proposition 47 *For any crossed module, M, there is an exact sequence*

$$1 \to \check{H}^{-1}(B, \mathsf{M}) \to C(B) \to P(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(C)) \to \pi_0(Tors(P)) \to \check{H}^1(B, \mathsf{M}).$$

$\blacksquare$

There are two 'mysterious' terms here. The second is the 1st Čech hypercohomology of B with coefficients in M. We have, sort of, met this earlier. It is

$$\check{H}^1(B, \mathsf{M}) = colim_{\mathcal{U}}[N(\mathcal{U}), BK(\mathsf{M})].$$

The treatment we have given it here, and the language we have available, is however not yet rich enough to yield a good geometric interpretation. For that we will need stacks and gerbes, and we will start on them in the next chapter!

The other strange term is $\check{H}^{-1}(B, \mathsf{M})$, which comes from the various $[N(\mathcal{U}), \Omega K(\mathsf{M})]$. We can calculate $\Omega K(\mathsf{M})$ explicitly using its description as the simplicial group of maps from S^1_* to $K(\mathsf{M})$.

Lemma 24 *(i) There are isomorphisms* $\Omega K(\mathsf{M}) \cong K(\pi_1(\mathsf{M}), 0)$, *the constant simplicial group on the kernel* $\pi_1(\mathsf{M}) = Ker(\partial : C \to P) \cong \pi_1(K(\mathsf{M}))$.

(ii) There are isomorphisms $\check{H}^{-1}(B, \mathsf{M}) = \check{H}^0(B, \pi_1(\mathsf{M})) \cong \pi_1(\mathsf{M})(B)$, *the group of global sections of* $\pi_1(\mathsf{M})$.

Proof: This is just a question of calculation so is left to you the reader. ∎

7.3.6 Examples and special cases revisited

We can use the analyses of Puppe sequences and their applications to refine a bit more the information on relative M-torsors for the 'examples and special cases'. We first apply our exact sequence of the previous paragraph.

The first example is when $\mathsf{M} = (1, P, inc)$ and the exact sequence confirms the isomorphism between $P(B)$ and $\pi_0(\mathsf{M}{-}Tors)$. When M is $A[1] = (A \to 1)$ for Abelian A, the sequence gives, as expected, confirmation that $\pi_0(\mathsf{M}{-}Tors) \cong \pi_0(Tors(A))$ and that the latter has a group structure.

For an inclusion crossed module / normal subgroup pair, we can compare the exact sequence coming from $1 \to N \to P \to G \to 1$ with that from $\mathsf{M} = (N, P, \partial)$, with ∂ the inclusion. The induced maps give us a map of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & N(B) \arrow[r] \arrow[d, "="'] & P(B) \arrow[r] \arrow[d, "="'] & \pi_0(\mathsf{M}{-}Tors) \arrow[r] \arrow[d] & \pi_0(Tors(N)) \arrow[r] \arrow[d, "\cong"'] & \pi_0(Tors(P)) \arrow[r] \arrow[d, "\cong"'] & \check{H}^1(B, \mathsf{M}) \arrow[d, "?"'] \\
1 \arrow[r] & N(B) \arrow[r] & P(B) \arrow[r] & G(B) \arrow[r] & \check{H}^1(B, N) \arrow[r] & \check{H}^1(B, P) \arrow[r] & \check{H}^1(B, G)
\end{tikzcd}$$

which again gives $\pi_0(\mathsf{M}{-}Tors) \cong G(B)$, and suggests that the mysterious $\check{H}^1(B, \mathsf{M})$, in this special case, is our better known $\check{H}^1(B, G)$, i.e. $\pi_0(Tors(G))$.

The last case we looked at was $\mathsf{M} = (M, G, 0)$. The long exact sequence has the induced map, ∂_*, trivial, so gives us

$$1 \to G(B) \to \pi_0(\mathsf{M}{-}Tors) \to \pi_0(Tors(M)) \to 1.$$

To examine the other situation considered on page 192, we need to apply our analysis of exact sequences of simplicial groups to another case.

7.3.7 Devissage: analysing $\mathsf{M}-Tors$

We saw that for any (sheaf of) crossed module(s) M, we had a short exact sequence

$$\begin{tikzcd} K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N \arrow[d] \\ 1 \arrow[r] & P \arrow[r] & P, \end{tikzcd}$$

or

$$\pi_1(\mathsf{M})[1] \to \mathsf{M} \to \pi_0(\mathsf{M})$$

if you prefer, (as $\pi_0(\mathsf{M}) = \pi_0(K(\mathsf{M})) = P/N$). (We only saw this for a crossed module, but clearly the argument goes through with only trivial changes in any topos, given suitable definitions!) Applying the associated simplicial group functor, K, this gives that

$$K(\pi_1(\mathsf{M}), 1) \to K(\mathsf{M}) \to K(\pi_0(\mathsf{M}), 0)$$

is an exact sequence of simplicial groups.

Theorem 11 *For any crossed module,* M*, there is an exact sequence*

$$1 \to \pi_0(Tors(\pi_1(\mathsf{M}))) \to \pi_0(\mathsf{M}-Tors)) \to \pi_0(\mathsf{M})(B) \to$$
$$\check{H}^2(B, \pi_1(\mathsf{M})) \to \check{H}^1(B, \mathsf{M}) \to \pi_0(Tors(\pi_0(\mathsf{M}))).$$

Proof: The proof merely is to identify the various terms from the Puppe sequence. Firstly the general form of such sequences, seen above, gives

$$\to \check{H}^{-1}(B, \pi_0(\mathsf{M})) \to \check{H}^0(B, \pi_1(\mathsf{M})[1]) \to \check{H}^0(B, K(\mathsf{M})) \to \check{H}^0(B, \pi_0(\mathsf{M})) \to \check{H}^1(B, \pi_1(\mathsf{M})[1]) \to \ldots$$

The first of these terms is trivial since for a general crossed module, $\Omega K(\mathsf{N})$ is $K(Ker\partial, 0)$, up to equivalence, so in our case in which $\mathsf{N} = (1 \to \pi_0(\mathsf{M}))$, it will be trivial. (Remember $\check{H}^{-1}(B, \mathsf{N}) = colim_{\mathcal{U}}[N(\mathcal{U}), \Omega K(\mathsf{N})]$.)

The next term $\check{H}^0(B, \pi_1(\mathsf{M})[1]) \cong \check{H}^1(B, \pi_1(\mathsf{M})) \cong \pi_0(Tors(\pi_1(\mathsf{M})))$, by our earlier calculations (case (ii) above). The next two terms are routine to handle, whilst that $\check{H}^1(B, \pi_1(\mathsf{M})[1])$ is isomorphic to $\check{H}^2(B, \pi_1(\mathsf{M}))$ is a classical result that is easy to check anyhow. Finally the remaining terms are standard. ■

Note that this gives some new information on M–$Tors$, indicating the difference between this category for general M and for the particular special cases considered earlier.

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