

# Symplectic Geometry

2<sup>nd</sup> Semester 2001/2002

## Problems #11

( due on June 6 )

1. Consider the standard  $(S^1)^3$ -action on  $\mathbb{P}^3$ :

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3] .$$

- (a) What is the moment polytope for this action ?
- (b) Exhibit explicitly the subsets of  $\mathbb{P}^3$  for which the stabilizer under this action is  $\{1\}$ ,  $S^1$ ,  $(S^1)^2$  and  $(S^1)^3$ . Show that the images of these subsets under the moment map are the interior, the facets, the edges and the vertices, respectively.

2. What is the moment polytope for the  $\mathbb{T}^2$ -action on  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  as

$$(e^{i\theta}, e^{i\eta}) \cdot ([z_0 : z_1], [w_0 : w_1]) = ([z_0 : e^{i\theta} z_1], [w_0 : e^{i\eta} w_1]) ?$$

3. Show that an effective hamiltonian action of a torus  $\mathbb{T}^n$  on a  $2n$ -dimensional symplectic manifold gives rise to an integrable system.

**Hint.** Show that the coordinates of the moment map give  $n$  linearly independent commuting integrals of motion.

- 4.a) Classify all 2-dimensional Delzant polytopes with 3 vertices, i.e., triangles, up to translation, change of scale and the action of  $\mathrm{SL}(2; \mathbb{Z})$ .

**Hint.** By a linear transformation in  $\mathrm{SL}(2; \mathbb{Z})$ , we can make one of the angles in the polytope into a square angle. How are the lengths of the two edges forming that angle related?

- 4.b) Classify all 2-dimensional Delzant polytopes with 4 vertices, up to translation and the action of  $\mathrm{SL}(2; \mathbb{Z})$ .

**Hint.** By a linear transformation in  $\mathrm{SL}(2; \mathbb{Z})$ , we can make one of the angles in the polytope into a square angle. Check that automatically another angle also becomes  $90^\circ$ .

- 4.c) What are all the 4-dimensional symplectic toric manifolds that have four fixed points?