Geometry and Gauge Theory

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Abstract

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Chapter 1

Introduction

1.1 Some general BlaBla

- Usually have **determinism** in physics: Given some fields (tensor fields, sections of fiber bundles, position of point-particle, etc), initial conditions and equations of motion uniquely fix the time evolution (the solution in spacetime).
- Gauge Invariance of a Physical system:
  - time evolution contains time-dependent arbitrary functions (i.e. not fixed):
    e.g. $q_1(t) = v_1 t + \varepsilon(t)$, $q_2(t) = v_2 t - \varepsilon(t)$.
    Can be seen as coming from local symmetries of the action:
    Symmetries map solutions of eom's to solutions.
    $\Rightarrow$ local symmetries (i.e. with time-dependent transformation-parameter) thus allow arbitrary time dependent parameters in the solution, i.e. solution is not fixed.
  - determinism can be reinstalled by identifying solutions that differ by this shift (**mod out gauge orbits**)
  - or by '**fixing the gauge**', i.e. choosing a representative of the orbit.
  - In most cases one can think of these symmetries as a **change of coordinates or change of reference frame**
  - Note that one can also mod out discrete symmetries (e.g. **dualities**$\cong \mathbb{Z}^2$: mirror symmetry (T-duality), S-duality: electric magnetic duality, T-folds, U-folds)
- Examples:
  - **YM:** Yang Mills theories (including Maxwell U(1), QCD SU(3), electroweak SU(2)xU(1)):
    standard class of examples. Relation to **fiber bundles** manifest. Gauge trafo=transition functions (choice of basis in the fiber)
  - classical mechanics:
    Mostly related to implementation of reparametrization invariance, e.g. relativistic point particle (geodesic equations).
    But one can easily construct plenty of artificial examples.
  - **GR:** General relativity :
    Again reparametrization invariance. Metric (instead of connection) plays the role of the gauge field.
    $\Rightarrow$In flat space, metric can be gauged away completely (by fixing an orthonormal coordinate system). One then says that the metric is "pure gauge".
    In curved space this is possible only for a given point of the manifold (where one can even bring the Christoffel symbols to zero).
  - Pure math example: geodesic equations (minimal volumes) from variational principle. (Nambu-Goto action with Euclidean signature):
    Worldvolume-diffeomorphism-invariance
  - Similar, but with Minkowskian signature:
    - Relativistic point particle.
    - Or even superparticle (particle propagating through superspace)$\Rightarrow$super-diffeomorphism-invariance

- * **ST:** (bosonic) string theory:
    Worldsheet diffeo invariance. (with gauge field(metric): Polyakov-string, without: Nambu-Goto)
    Local Weyl-invariance (without gauge field)
    Gauge-constraints: Virasoro-Constraints -> Virasoro-algebra
  * Dp-branes (world-volume diffeomorphism invariant Dirac Born Infeld action + other gauge symmetries)
- **gauged WZW:** Wess Zumino (Novikov) Witten theory (2dim sigma-models with (coset) group manifold target space):
  Fields live in the group G itself. Can act on them with other group elements by either left or right multiplication.
  One of these is realized for a subgroup $H \subset G$ as local symmetry (gauge symmetry), the other just a global symmetry. Modding out the local symmetry leads effectively to a sigma-model on the coset G/H.
- higher gauge fields (B-field, TQFT):
  $\delta B^{(2)} = \mathbf{d}\Lambda^{(1)}$ (compare $\delta \boldsymbol{A}^{(1)} = \mathbf{d}\Lambda^{(0)}$)
  Fiber bundles not enough. Need **gerbes**.
  (Čech-2-Cocycles: $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$, $\delta g = g_{\beta\gamma}\delta g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta}\delta g^{-1}_{\alpha\beta\gamma} = 1$ (0?),
  $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}, \quad B_\beta - B_\alpha = dA_{\alpha\beta}$, $dB_\beta = dB_\alpha = H|_{U_\alpha}$. Hitchin-talk)
- topological models (Poisson sigma-model: no propagating degrees of freedom, →Kontsevich star product)
- can gauge any global symmetry (**Noether-method**)
  ⇒was used to derive **SUGRA** from SUSY

1.2 Plan

- Definition of gauge transformations
- Examples (including a first glance on Yang Mills)
- Back to general discussion: Noether theorem and Noether identities
- **Hamiltonian** description:
  - Phase space T^*M with natural Poisson bracket
  - $H: \quad T^*M \to \mathbb{R}$
  - Constraints in the phase space (1st and 2nd class).
    (presymplectic Hamiltonian systems)
  - Koszul Tate differential δ reduces to the corresponding submanifold.
  - BRST differential (longitudinal exterior derivative) mods out gauge transformations (generated by the BRST charge via the Poisson bracket).
- For both $\updownarrow$ need: **homological perturbation theory**: $\underbrace{\mathbf{s}}_{\text{BRST}} = \underbrace{\delta}_{\text{Koszul Tate}} + \underbrace{\mathbf{d}_l}_{\text{longitudinal}} + \dots$ exterior derivative
- **Lagrangian** description (**Legendre**-transformation $T^*M \to TM$)
  - Tangent space TM a priori without natural Poisson bracket
  - $L: \quad TM \to \mathbb{R}$
  - Space of paths $\mathcal{P}M$ in M (determined by initial conditions on TM and eom's).
  - Eom's are constraints in $\mathcal{P}M$
  - extend M to antifields. Functions on extended space$\cong \Gamma\left(\bigwedge^\bullet TM\right)$ (multivector-fields)
    Graded Lie algebra with natural "odd Poisson bracket": Schouten-bracket or antibracket
  - Koszul tate differential δ reduces to the surface given by the solution to the equations of motion ("on-shell")
  - BRST differential mods out gauge transformations (generated by the action S via the antibracket).

- Fiber bundle geometry and relation to YM
  - characteristic classes (fiber bundle obstructions)
  - Index theorem (anomalies)
- Maybe more on:
  - Anomalies
  - Fermions
  - Solitons (magnetic monopole)
  - ...

1.3 Definition of a gauge transformation

- [Henneaux,p.67]: Consider a manifold (?) M and the space of all paths $\mathcal{P}M$. I.e. an arbitrary path $q \in \mathcal{P}M$ is of the form

$$\begin{aligned} q: \quad \mathbb{R} \supset [a,b] &\rightarrow M \\ t &\mapsto q(t) \quad \text{(with coords } q^i(t), \quad i \in \{1,\ldots,\dim M\}) \end{aligned} \tag{1.1}$$

  where we assume the paths to be smooth. Consider further a functional on this space

$$\begin{aligned} S: \quad \overbrace{\mathcal{P}M}^{\text{all paths on } M} &\rightarrow \mathbb{R} \\ q &\mapsto S[q] \equiv \int_a^b dt \quad L(q(t),\dot{q}(t),\ddot{q}(t),\ldots,t) \end{aligned} \tag{1.2}$$

  which will be called an **action functional** . (It is called **local** if the integrand contains only a finite number of derivatives)

- The equations

$$0 \overset{!}{=} \delta S[q] = \int dt\, \delta q^i(t) \frac{\delta}{\delta q^i(t)} S[q] \quad \forall \underline{\text{variations }} \delta q \qquad (\tilde{q}(t) = q(t) + \delta q(t)) \tag{1.3}$$

  which extremize S are called the **equations of motion (eom)**

- If there exists a particular (infinitesimal) variation $\delta_\varepsilon q$ (parametrized by some **transformation parameters** ε^a) with

$$\delta_\varepsilon S[q] = 0 \quad \underline{\forall q} \tag{1.4}$$

  it is called a **symmetry transformation** of the action.

- Or finite (more general, as there might not be an infinitesimal version, e.g. for discrete symmetries): A map

$$f: \quad \mathcal{P}M \rightarrow \mathcal{P}M \tag{1.5}$$

$$q \mapsto \tilde{q} = f(q) \quad (= q + \delta_\varepsilon q) \tag{1.6}$$

  with

$$S[f(q)] = S[q] \quad \forall q \tag{1.7}$$

`finite-sym`

  is called a symmetry trafo.

- Symmetry transformations of an action form a group:
  - identity is always a symmetry
  - Composition of two symmetries is always a symmetry

$$S[f(q)] = S[q], \quad S[g(q)] = S[q] \quad \forall f,g \qquad \Rightarrow S[f(g(q))] = S[g(q)] = S[q] \tag{1.8}$$

– Inverse is always a symmetry (plug $f^{-1}(q)$ into (1.7)

$$S[q] = S[f^{-1}(q)] \tag{1.9}$$

– counterexample to last: $S[q_1, q_2] = \int dt \quad L(q_1, \dot{q}_1)$ (action does not depend on q_2). The function $f: q_2 \mapsto 0$ is a non-invertible symmetry transformation of this action.

– In order to avoid this:

  * Restrict to trafos generated by infinitesimal ones (which are always invertible)
  * Or restrict the trafos to be elements of the diffeomorphisms on $\mathcal{P}M$. Then the symmetry group is a subgroup

- A symmetry of the action is also a symmetry of the equations of motion, in the sense that it <u>maps solutions of the equations to other solutions</u>:

$$\begin{aligned} \frac{\delta}{\delta q^i(t)} S[q_c + \delta_\varepsilon q] &= \frac{\delta}{\delta q^i(t)} S[q_c] + \int dt' \; \delta_\varepsilon q^j(t') \frac{\delta^2}{\delta q^j(t') \delta q^i(t)} S[q] = && (1.10) \\ &= \underbrace{\frac{\delta}{\delta q^i(t)} S[q_c]}_{=0} + \underbrace{\frac{\delta}{\delta q^i(t)} \; \delta_\varepsilon S[q]}_{=0} = 0 && (1.11) \end{aligned}$$

- **On shell** = either on the "shell" <u>defined by the equations of motion</u> (that's how I will call it, while probably writing "**eom**"), or on the constraint surface (I will say "on the constraint surface"), in particular on the mass-shell $p^2 = -m^2$ (which I will call – if ever – "on mass-shell").

- **Off shell** = Not on the "shell".

Definition 1.1. Assume we can expand the symmetry transformation $\delta_\varepsilon q^i(t)$ in some (infinitesimal) and maybe time-dependent parameter ε and its time derivatives:

$$\begin{aligned} \delta_\varepsilon q^i(t) &= \varepsilon^a(t) \underbrace{\delta_a^{(0)} q^i}_{R^i_{(0)a}}(t) + \dot{\varepsilon}^a(t) \delta_a^{(1)} q^i(t) + \ldots = && (1.12) \\ &= \int dt' \quad \varepsilon^a(t') \underbrace{\delta_a q^i(t', t)}_{R^i_a(t', t)} && (1.13) \end{aligned}$$

If $\delta_\varepsilon q^i(t)$ is a symmetry transformation for any (infinitesimal) function $\varepsilon^a(t)$ of t, i.e.

$$\delta_\varepsilon S[q] = 0 \quad \forall \varepsilon(t) \tag{1.14}$$

then δ_ε is called a **local symmetry of S** or (in particular if those q's related by a local transformation δ_ε are identified) a **gauge transformation**.
(some ambiguity for the latter: as soon as one uses a symmetry to identify solutions – could also be a discrete symmetry like dualities – one can call these symmetries gauge transformations, even in the discrete case. However, as a local symmetry enforces the identification, one tends to identify gauge symmetry with local symmetry)

Remarks:

- Allowing $\varepsilon(t)$ to be an arbitrary function of t is quite a strong condition on the symmetry. It is quite common that one has some restrictions on $\varepsilon(t)$. For example, $\delta_\varepsilon q^i(t)$ might be a symmetry transformation only if $\varepsilon(t) = \varepsilon = const$, in which case all the derivative terms drop. Such a symmetry is called a **global symmetry**.

- The definition of a local symmetry requires arbitrary ε, but not arbitrary form of $\delta_\varepsilon q^i(t)$. In particular there can be also local symmetries that don't have derivative terms, i.e. that are of the form $\delta_\varepsilon q^i(t) = \varepsilon^a(t) \delta_a^{(0)} q^i(t)$. Still they differ from global symmetries by the fact that $\varepsilon(t)$ has an arbitrary t-dependence.

- **Remark on the notation:** Note that M can not only be finite dimensional (for example for a point particle moving in space-time), but also infinite dimensional, like in field theorey where one should replace the discrete index i by a continuous (perhaps combined with a discrete). Take for example a co-vector field A in spacetime (section of the cotangent bundle of $\mathbb{R}^{3,1}$, where "3,1" means that the signature of the metric is $(-1,1,1,1)$). Its components are $A_\mu(x^0,x^1,\ldots,x^3) \equiv A_\mu(t,\vec{x})$. One can think of $\vec{x}$ as a continous "index", i.e. $q^i(t) \equiv A_{\mu,\vec{x}}(t)$. Many of the statements derived for finite dimensions carry over for infinite dimensions. Sometimes, however, it is nice to avoid the split of the spacetime coordinates into time t and space $\vec{x}$. So some of our discussion will be made explicitly for the field-theory case. When we don't want to specify, whether we are talking about vector fields or other tensor fields or whatever, we will collectively denote all the fields of some theory by $\phi^I(x)$ (where x is now space and time). One then obtains the particle case by splitting spacetime into time and space and taking the space-dimension to be zero. So $q^i(t)$ in these notes can be both, either the point particle case (space-dimension=0) of the general field theory or a **condensed notation** of the general field theory:

$$q^i(t) \equiv \phi^{\mathcal{I},\vec{x}}_{\text{all}}(t) \equiv \phi^{\mathcal{I}}_{\text{all}}(x) \quad (\text{i is then a cont } \infty\text{-dim index} : \sum_i(\ldots) = \sum_{\mathcal{I}}\int d^{dim}\vec{x}(\ldots)) \tag{1.15}$$

$$\text{or } q^i(t) \equiv q^i(t) = \phi^{\mathcal{I}}_{\text{all}}(t) \quad (i \text{ is discrete and finite dimensional}) \tag{1.16}$$

Rarely we might use even more condensed notation, where also time is included in the index:

$$q^i \hat{=} q^{i,t}, \quad \sum_i \hat{=} \sum_i \int dt \tag{1.17}$$

1.4 Examples

A few examples which we might use from time to time to illustrate some general concepts / techniques.

1.4.1 Stupid example

- $$S[q_1,q_2] = \int_a^b dt \quad \tfrac{1}{2}(\dot{q}_1 - \dot{q}_2)^2 - V(q_1 - q_2) \tag{1.18}$$
- Local symmetry
$$\delta_\varepsilon q_1(t) = \varepsilon(t), \quad \delta_\varepsilon q_2(t) = \varepsilon(t) \tag{1.19}$$
- Eom's ($\delta q|_a = 0 = \delta q|_b$ general variations, but with fixed end-points)
$$0 = \int dt \quad (\delta\dot{q}_1 - \delta\dot{q}_2)(\dot{q}_1 - \dot{q}_2) - (\delta q_1 - \delta q_2)V'(q_1 - q_2) = \tag{1.20}$$
$$= \int dt \quad (\delta q_2 - \delta q_1)\underbrace{((\ddot{q}_1 - \ddot{q}_2) + V'(q_1 - q_2))}_{\frac{\delta S}{\delta q_2(t)} = -\frac{\delta S}{\delta q_1(t)}} \tag{1.21}$$
- equations of motion are dependent (Noether identities):
$$\frac{\delta S}{\delta q_1(t)} + \frac{\delta S}{\delta q_2(t)} = 0 \tag{1.22}$$
- Solutions to eom's are not unique. Only the difference is unique after fixing initial conditions
$$q_1(t) = \tfrac{1}{2}q_0 + \tfrac{1}{2}vt + f(t) \quad (\text{arbitrary } f(t)) \tag{1.23}$$
$$q_2(t) = -\tfrac{1}{2}q_0 - \tfrac{1}{2}vt + f(t) \tag{1.24}$$
$$q_1(t) - q_2(t) = q_0 + vt \tag{1.25}$$
- Physically identify solutions that differ by a gauge transformation (equivalence relation)
$$(q_1,q_2) \sim (q_1 + \delta_\varepsilon q_1, q_2 + \delta_\varepsilon q_2) \tag{1.26}$$
- Can **gauge fix** (= choose a representative of the equivalence class of paths) e.g. via
$$f(t) \stackrel{!}{=} \tfrac{1}{2}q_0 + \tfrac{1}{2}vt \quad (q_2(t) = 0, \quad q_1 \text{ as effective variable}) \tag{1.27}$$
- This is an example for a gauge symmetry without the explicit appearance of a gauge connection ($\delta \boldsymbol{A} = \mathbf{d}\varepsilon + \ldots$).

1.4.2 Geodesic equation from the variational principle

- Embedding $\Sigma \overset{X}{\to} M$
- $\dim \Sigma = 1$: Geodesic:

$$S[X] = -\int_\Sigma d\sigma \quad \underbrace{\sqrt{|\, \dot{X}^m g_{mn}(X)\dot{X}^n \,|}}_{\sqrt{X^* g}} \tag{1.28}$$

Variation yields in proper-time parametrization (affine parametrization) the geodesic equation:

$$\delta S = 0 \quad \Longleftrightarrow \quad \ddot{X}^m - \dot{X}^k \dot{X}^l \Gamma_{kl}^m = 0 \tag{1.29}$$

(e.g. point-particle in GR-background)

- $\dim \Sigma$ arbitrary:

$$S[X] = -\int_\Sigma d^d\sigma \quad \underbrace{\sqrt{|\det(\partial_\mu X^m g_{mn} \partial_\nu X^n)\,|}}_{\sqrt{|\det(X^* g)|}} \tag{1.30}$$

(e.g. bosonic string, Dp-brane). Variation yields a generalized geodesic equation (again in a particular parametrization)

$$\partial_\mu \partial^\mu X^m - \partial^\mu X^k \partial_\mu X^l \Gamma_{kl}^m = 0 \tag{1.31}$$

- Gauge symmetry: local worldsheet-reparametrizations:

$$\delta_\varepsilon X^m = \mathcal{L}_\varepsilon X^m \quad = \quad \varepsilon^\kappa(\sigma)\partial_\kappa X^m(\sigma) \tag{1.32}$$

- E.g. for 1-dim case can gauge fix $\sigma^0 = X^0 \Rightarrow \dot{X}^0 = 1$, and assume that the remaining $\dot{X}^i \ll 1$ (where 1 can be thought of as the speed of light). Then we can expand the square root as follows:

$$\sqrt{|\, \dot{X}^m g_{mn}(X)\dot{X}^n \,|} = \sqrt{1 - \dot{X}^i g_{ij}(X)\dot{X}^j} = 1 - \tfrac{1}{2}\dot{X}^i g_{ij}(X)\dot{X}^j + \ldots \tag{1.33}$$

The second term is just the kinetic energy of a point particle in Newtonian mechanics which serves in this case as a Lagrangian.

- Above examples are without worldvolume metric. So one has gauge invariance without a "gauge field".
- With independent worldvolume-metric (Polyakov-action, works at least in 2dim)

$$S[X] = \int_{\Sigma^{(2)}} d^2\sigma \sqrt{h} \quad \tfrac{1}{2} h^{\mu\nu} \partial_\mu X^m g_{mn} \partial_\nu X^n \tag{1.34}$$

In 1d one has to add an extra term containing only h (or the vielbein e).
$0 \overset{!}{=} \frac{\delta S}{\delta h_{\underline{mn}}} \Rightarrow$ eom for $h_{\mu\nu}$, can be plugged back. One obtains back the old action.

Remark on condensed notation:

In case of the worldsheet (dimΣ=2)

$$\Sigma^{(2)} \cong \mathbb{R} \times \Sigma^{(1)} \tag{1.35}$$

$$\delta_\varepsilon X^m(\underbrace{\sigma^0}_{\tau}, \sigma^1) \quad = \quad \varepsilon^\kappa(\sigma^0, \sigma^1)\partial_\kappa X^m(\sigma^0, \sigma^1) = \tag{1.36}$$

$$\delta_\varepsilon X^{m,\sigma^1}(\tau) \quad = \quad \int d\tilde{\sigma}^1 \varepsilon^{\kappa,\tilde{\sigma}^1}(\tau) \underbrace{\delta(\sigma' - \sigma^1)\partial_\kappa X^m(\tau, \sigma^1)}_{\delta^{(0)}_{\kappa,\tilde{\sigma}^1} X^{m,\sigma^1}(\tau)} \tag{1.37}$$

Compare to

$$\delta_\varepsilon q^i(t) \quad = \quad \varepsilon^a(t) \underbrace{\delta_a^{(0)} q^i}_{R^i_{(0)a}}(t) + \dot{\varepsilon}^a(t)\delta_a^{(1)} q^i(t) + \ldots = \tag{1.38}$$

Local in σ^0 $\rightarrow$gauge symmetry. Local in σ^1 $\rightarrow$infinite global symmetries (conserved charges). Usually local in both.

1.4.3 General Relativity

Einstein-Hilbert action without matter

$$S_{EH}[g] \equiv \tfrac{1}{16\pi G}\int d^dx \quad \sqrt{|\det g|}\underbrace{R}_{\text{Ricci}-\text{scalar corresp to}\, g_{mn}(x)} \tag{1.39}$$

It is diffeomorphism invariant (gauge invariance)

$$\delta_\xi g_{\mu\nu}(x) = \mathcal{L}_\xi g_{\mu\nu}(x) = \xi^\kappa(x)\partial_\kappa g_{\mu\nu}(x) + 2\partial_{(\mu|}\xi^\kappa(x) g_{|\kappa)\nu}(x) \tag{1.40}$$

Equations of motion give Ricci-flatness

$$0 \stackrel{!}{=} \frac{\delta}{\delta g_{\mu\nu}(x)}S \propto \quad R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \tag{1.41}$$

$$\Longleftrightarrow 0 = R_{\mu\nu} \tag{1.42}$$

1.4.4 Yang Mills I - a first glance

The aim of this subsection is to quickly introduce the Yang Mills action in order to have it at hand as an example. This will be done without introducing in detail fiber bundle geometry (this will be done later). Some names will be dropped in order to specify what the objects are, but without yet carefully defining these.

- Conventions:

here	[Nak]	[Kugo]	[Hitchin]	[thesis]
$\boldsymbol{A}, \boldsymbol{D}, F$ tr	$\mathcal{A} = iA, \mathcal{D}, \mathcal{F} = iF$	$\frac{g}{i}A, D, \frac{g}{i}F$	A, D, F $\frac{1}{2}$tr	$-\boldsymbol{\Omega}, \nabla, -R$

- $\boldsymbol{A}$: Lie algebra valued one form, or to be more precise a local section of $T^*M \otimes \mathfrak{g}$.
  In fact it is the pullback of the so called connection 1-form (which is an element of $\Omega^1(P(M,G)) \otimes \mathfrak{g}$) of a principle fiber bundle along a local section.
  F: pullback of the corresponding curvature 2-form [Nak, 355]

- Action[1]

$$S[\boldsymbol{A}] \equiv \int_{\mathbb{R}^{D-1,1}} (-)^{p(D-p)}\epsilon_{(D,p)}\tfrac{1}{4}\mathrm{tr}\, F\wedge\star F = \tag{1.43}$$

$$= \int_{\mathbb{R}^{D-1,1}} d^Dx \quad \tfrac{1}{4}\mathrm{tr}\, F^{mn}F_{mn} \quad ({\scriptstyle m,n} \in \{0,1,\ldots,D-1\}) \tag{1.44}$$

with [Nak,p.353]

$$F = \mathbf{d}\boldsymbol{A} + \boldsymbol{A}\wedge\boldsymbol{A} \tag{1.45}$$

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] \tag{1.46}$$

[1] [thesis, p.169, (D.24)] Define the components of the ε-tensor (volume-form) for Minkowskian signature via

$$\varepsilon_{m_1\ldots m_D} \equiv \sqrt{|g|}\epsilon_{m_1\ldots m_D}, \quad \epsilon_{01\ldots(D-1)} \equiv 1 = i\epsilon_{D,1\ldots(D-1)} = \underbrace{(-)^{D-1}i\epsilon_{1\ldots D}}_{\equiv\epsilon^{(E)}_{1\ldots D}}$$

The last step defines the Euclidean version. The actual D-form thus reads

$$\varepsilon \equiv \tfrac{1}{D!}\varepsilon_{m_1\ldots m_D}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_D} = (-)^D i\varepsilon^{(E)}$$

Its components obey the following identity:

$$\tfrac{1}{p!}\epsilon_{a_1\ldots a_{D-p}c_1\ldots c_p}\epsilon^{b_1\ldots b_{D-p}c_1\ldots c_p} = -\check{\delta}^{b_1\ldots b_{D-p}}_{a_1\ldots a_{D-p}}, \quad \tfrac{1}{p!}\epsilon^{(E)}_{a_1\ldots a_{D-p}c_1\ldots c_p}\epsilon_{(E)}^{b_1\ldots b_{D-p}c_1\ldots c_p} = \check{\delta}^{b_1\ldots b_{D-p}}_{a_1\ldots a_{D-p}}$$

Using this tensor, we can define the Hodge dual as ([thesis,p.171,fn3], with redef $\omega^{(p)}_{mmm} \to \frac{1}{p!}\omega^{(p)}_{mmm}$, $\varepsilon \to d!\varepsilon$)

$$(\star\omega^{(p)})_{m_1\ldots m_{D-p}} \equiv \tfrac{\epsilon_{(D,p)}}{p!}\varepsilon_{m_1\ldots m_{D-p}}{}^{k_1\ldots k_p}\omega^{(p)}_{k_1\ldots k_p} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$

where $\epsilon_{(D,p)}$ is some p and D dependent sign-factor which can be chosen in any convenient way. Natural choices are $1, (-)^{p(D-p)}, (-)^{\frac{p(p-1)}{2}}$ or $(-)^{\frac{p(p-1)}{2}}(-)^{p(D-p)}$. The second is most common in literature ([Nak]), the last was used by myself in [thesis]. (The factor $(-)^{p(D-p)}$ corresponds to contracting $\omega^{(p)}$ with the first instead of the last p indices of the ε-tensor, the factor $(-)^{\frac{p(p-1)}{2}}$ reverses the order of $\omega^{(p)}$'s indices.)

or expressed in terms of the **covariant derivative**

$$F = [\boldsymbol{D}, \boldsymbol{D}] \tag{1.47}$$
$$\boldsymbol{D} = \mathbf{d} + [\boldsymbol{A}, .] \tag{1.48}$$

- Bianchi identity for general (nonabelian) group

$$\mathbf{d}F = [\mathbf{d}\boldsymbol{A}, \boldsymbol{A}]_\wedge \tag{1.49}$$
$$= [F - \boldsymbol{A} \wedge \boldsymbol{A}, \boldsymbol{A}]_\wedge \tag{1.50}$$
$$\boldsymbol{D}F \equiv \mathbf{d}F - [\boldsymbol{A}, F]_\wedge = 0 \quad (\iff [\boldsymbol{D}, [\boldsymbol{D}, \boldsymbol{D}]] = 0) \tag{1.51}$$

- Gauge transformation (bundle transition maps)

$$\tilde{\boldsymbol{A}} = g\boldsymbol{A}g^{-1} + \underbrace{g\mathbf{d}g^{-1}}_{-\mathbf{d}g\, g^{-1}} \quad , \quad \tilde{F} = gFg^{-1} \tag{1.52}$$

Or for an infinitesimal gauge transformation $g = 1 + \alpha$:

$$\delta\boldsymbol{A} = [\alpha, \boldsymbol{A}] - \mathbf{d}\alpha = -\boldsymbol{D}\alpha \quad , \quad \delta F = [\alpha, F] \qquad (g = 1 + \alpha) \tag{1.53}$$

- Equations of motion

$$\delta F = \mathbf{d}\delta\boldsymbol{A} + [\boldsymbol{A}, \delta\boldsymbol{A}] = \boldsymbol{D}\delta\boldsymbol{A} \tag{1.54}$$

$$\delta S = \int -\tfrac{1}{4}\mathrm{tr}\,\delta F \wedge \star F - \tfrac{1}{4}\underbrace{\mathrm{tr}\, F \wedge \star\delta F}_{\mathrm{tr}\,\delta F\wedge\star F} = \tag{1.55}$$
$$= \int -\tfrac{1}{2}\mathrm{tr}\,\boldsymbol{D}\delta\boldsymbol{A} \wedge \star F \tag{1.56}$$
$$= \int -\tfrac{1}{2}\mathrm{tr}\,\delta\boldsymbol{A} \wedge \boldsymbol{D} \star F + \int \mathbf{d}\big(\tfrac{1}{2}\mathrm{tr}\,\delta\boldsymbol{A} \wedge \star F\big) \tag{1.57}$$

The Hodge-dual of 1 is simply

$$(\star 1)_{m_1\ldots m_D} \equiv \epsilon_{(D,0)}\varepsilon_{m_1\ldots m_D} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$
$$\star 1 = \epsilon_{(D,0)}\varepsilon = \varepsilon \quad (\text{for all above versions of } \epsilon_{(D,p)})$$

The square of the Hodge-star operation is either +1 or -1, depending on dimension D and (convention-dependent) also on the form-degree p.

$$\star^2 = -(-)^{p(D-p)}\epsilon_{(D,p)}\epsilon_{(D,D-p)} \quad (\star^2_{(E)} = (-)^{p(D-p)}\epsilon_{(D,p)}\epsilon_{(D,D-p)})$$
$$\star^2 1 = -\underbrace{\epsilon_{(D,0)}\epsilon_{(D,D)}}_{\in\{1,\epsilon_{(D)}\}}, \quad \star\varepsilon = -\underbrace{\epsilon_{(D,D)}}_{\in\{1,\epsilon_{(D)}\}}$$

The Hodge dual of a wedge product of two forms leads to the contraction of its indices. If we denote by $\tilde{\omega}$ the p-vector obtained by raising all p indices of the p-form $\omega^{(p)}$, then we can write

$$\star(\omega^{(p)} \wedge \eta^{(q)}) = (-)^{pq+p(D-p)}\epsilon_{(D,p+q)}\epsilon_{(D,q)}(-)^{p(p-1)/2}\imath_{\tilde{\omega}^{(p)}} \star \eta^{(q)} = (-)^{q(D-q)}\epsilon_{(D,p+q)}\epsilon_{(D,p)}(-)^{q(q-1)/2}\imath_{\tilde{\eta}^{(q)}}\omega^{(p)}$$

Replacing $\eta^{(q)}$ by $\star\eta^{(p)}$ (of degree $q = D - p$) and acting with another star, yields a **symmetric inner product**

$$(\omega^{(p)} \wedge \star\eta^{(p)}) = (-)^{p(D-p)}\epsilon_{(D,p)}\underbrace{(\imath_{(-)^{p(p-1)/2}\tilde{\omega}^{(p)}}\eta^{(p)})}_{\text{posdef for Euklid}}\varepsilon = (\eta^{(p)} \wedge \star\omega^{(p)})$$

Above we actually have a volume form, so that the proper inner product is its integral over the manifold.

For $\epsilon_{(D,p)} = (-)^{p(D-p)}(-)^{p(p-1)/2}$ we have

$$(\star\omega^{(p)})_{m_1\ldots m_{D-p}} \equiv \tfrac{1}{p!}\omega^{(p)}_{k_p\ldots k_1}\varepsilon^{k_1\ldots k_p}{}_{m_1\ldots m_{D-p}} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$
$$(\star 1)_{m_1\ldots m_D} \equiv \varepsilon_{m_1\ldots m_D} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$
$$\star^2 = -(-)^{D(D-1)/2}$$
$$(\omega^{(p)} \wedge \star\eta^{(p)}) = (\imath_{\tilde{\omega}^{(p)}}\eta^{(p)})\varepsilon = (\eta^{(p)} \wedge \star\omega^{(p)})$$

For $\epsilon_{(D,p)} = (-)^{p(D-p)}$ (e.g. [Nak])

$$(\star\omega^{(p)})_{m_1\ldots m_{D-p}} \equiv \tfrac{1}{p!}\omega^{(p)}_{k_1\ldots k_p}\varepsilon^{k_1\ldots k_p}{}_{m_1\ldots m_{D-p}} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$
$$(\star 1)_{m_1\ldots m_D} \equiv \varepsilon_{m_1\ldots m_D} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$
$$\star^2 = -(-)^{p(D-p)}, \quad \star^2_{(E)} = (-)^{p(D-p)}$$
$$(\omega^{(p)} \wedge \star\eta^{(p)}) = (\imath_{(-)^{p(p-1)/2}\tilde{\omega}^{(p)}}\eta^{(p)})\varepsilon = (\eta^{(p)} \wedge \star\omega^{(p)}) \qquad \diamond$$

$$\delta S[\boldsymbol{A}] = 0 \Rightarrow \quad \boldsymbol{D} \star F = 0 \quad (\boldsymbol{D}_m F^{mn} = 0) \quad \& \quad F_{mn} n^n|_{\partial M} = 0 \tag{1.58}$$

- Gauge group U(1) (commutative, linear eom's): Maxwell electromagnetic field $\boldsymbol{E}$:1-form on $\mathbb{R}^3$,

$$E_i = F_{i0} \tag{1.59}$$

$\boldsymbol{B}$:Hodge dual of a 2-form $B^{(2)}$ (in 3dim: 1-form $\cong$1-vector),

$$B^i = \frac{1}{2}\epsilon^{ijk} \underbrace{F_{jk}}_{\equiv B_{jk}} \tag{1.60}$$

So F written in terms of electric and magnetic field reads

$$F = B + c\boldsymbol{E} \wedge \mathbf{d}t \tag{1.61}$$

This is a closed 2-form in $\mathbb{R}^4$ upon some of the Maxwell-equations (the Bianchi-part of Maxwell eqs)

$$\mathbf{d}F = \mathbf{d}B + \mathbf{d}t \wedge \mathbf{d}\boldsymbol{E} = \tag{1.62}$$

$$= \mathbf{d}t \wedge \mathbf{d}x^i \wedge \mathbf{d}x^j \left(\tfrac{1}{2}\partial_t B_{ij} + c\partial_i E_j\right) + \mathbf{d}x^k \wedge \mathbf{d}x^i \wedge \mathbf{d}x^j (\tfrac{1}{2}\partial_k B_{ij}) \tag{1.63}$$

$$\mathbf{d}F = 0 \quad \Longleftrightarrow \quad \partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla}\vec{B} = 0 \tag{1.64}$$

Similarly

$$\mathbf{d}\star F = 0 \iff \partial_t \vec{E} - \vec{\nabla} \times \vec{B} = 0, \quad \vec{\nabla}\vec{E} = 0 \tag{1.65}$$

- Noether identities: (see later)

$$0 = D_\mu (D_\nu F^{\mu\nu}) = [F_{\mu\nu}, F^{\mu\nu}] = 0 \tag{1.66}$$

- Remark on "gauging": fermion action with global invariance -> gauge field action,

$$S[\psi, \bar{\psi}] = \int d^4x \quad \bar{\psi}(\Gamma^\mu \partial_\mu - m)\psi \tag{1.67}$$

$$\text{or } S[\phi, \phi*] = \int d^4x \quad \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*\phi) \tag{1.68}$$

Invariant under global U(1) transformations

$$\tilde{\psi} = e^{i\theta}\psi, \quad \theta = \text{const}, \qquad \delta\psi = i\theta\psi \tag{1.69}$$

$$\text{or } \tilde{\phi} = e^{i\theta}\phi \tag{1.70}$$

Not invariant under local transformation, because of derivative. Introducing gauge field A_μ ("minimal coupling") leads to locally invariant theory:

$$S[\psi, \bar{\psi}, A] = \int d^4x \quad \bar{\psi}(\Gamma^\mu \underbrace{(\partial_\mu - A_\mu)}_{D_\mu} - m)\psi + \mathcal{L}_{MW} \tag{1.71}$$

$$S[\phi, \phi^*, A] = \int d^4x \quad D_\mu \phi^* D^\mu \phi - V(\phi^*\phi) + \mathcal{L}_{MW} \tag{1.72}$$

Similarly: S is invariant under global Poincaré-transformations (translation+$SO(1,3)$). Can make the local by introducing metric and vielbeins (local diffeo-invariance+local Lorentz invariance). Adding some kinetic term for metric $\Rightarrow$general relativity coupled to fermions+Maxwell;
Note that A_μ is coupled to the current $j^\mu = \bar{\psi}\Gamma^\mu\psi$. This is the starting point of the "Noether procedure" to "gauge" a global symmetry.

1.5 Noether Theorem and Noether identities

Consider a quite general action functional of the form

$$S[\phi^{\mathcal{I}}_{\text{all}}] \equiv \overbrace{\int_\Sigma \underbrace{d^d\sigma}_{\mu}}^{\int_{\mathbb{R}} d\tau \int_{\Sigma^{(d-1)}} d^{(d-1)}\sigma} \qquad \mathcal{L}(\phi^{\mathcal{I}}_{\text{all}}, \partial_\mu \phi^{\mathcal{I}}_{\text{all}}, \partial_{\mu_1}\partial_{\mu_2}\phi^{\mathcal{I}}_{\text{all}}, \ldots) \tag{1.73}$$

Theorem 1.1 (Noether)**.** *To every transformation $\delta_{(\rho)}\phi^{\mathcal{I}}_{all}$ which leaves the action S invariant, i.e. transforms the Lagrangian as*

$$\delta_{(\rho)}\mathcal{L} \stackrel{!}{=} \partial_\mu \mathcal{K}^\mu_{(\rho)} \qquad \text{with} \;\; n_\mu \mathcal{K}^\mu_{(\rho)}\Big|_{\partial\Sigma} = 0 \tag{1.74}$$

symmetry-cond

there is an on-shell divergence-free current $j^\mu_{(\rho)}$ whose explicit form can be chosen to be

$$j^\mu_{(\rho)} \equiv \delta_{(\rho)}\phi^{\mathcal{I}}_{all} \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{\mathcal{I}}_{all})} + \sum_{k\geq 1}\sum_{i=0}^{k}(-)^i \partial_{\nu_1}\ldots\partial_{\nu_{k-i}}\delta\phi^{\mathcal{I}}_{all}\cdot\partial_{\nu_{k-i+1}}\ldots\partial_{\nu_k}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\nu_1}\ldots\partial_{\nu_k}\phi^{\mathcal{I}}_{all})} - \mathcal{K}^\mu_{(\rho)} \tag{1.75}$$

Its off-shell divergence is given by

$$\boxed{\partial_\mu j^\mu_{(\rho)} = -\delta_{(\rho)}\phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}}} \tag{1.76}$$

eq:noet:curre

The such defined Noether current is unique up to trivially conserved terms of the form $\partial_\nu S^{[\nu\mu]}$.
In turn, for any given on-shell divergence-free current $\tilde{j}^\mu$ with

$$\partial_\mu \tilde{j}^\mu = -y^{\mathcal{I}}_{(0)}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} - y^{\mathcal{I}\mu_1}_{(1)}\partial_{\mu_1}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} - \ldots - y^{\mathcal{I}\mu_N\ldots\mu_1}_{(N)}\partial_{\mu_1}\ldots\partial_{\mu_N}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} \tag{1.77}$$

which is furthermore itself on-shell neither vanishing nor trivial, there is a corresponding nonzero symmetry transformation $\delta\phi^{\mathcal{I}}_{all}$ of the form

$$\delta\phi^{\mathcal{I}}_{all} \equiv \sum_{k=0}^{N}(-)^k\partial_{\mu_1}\ldots\partial_{\mu_k}y^{\mathcal{I}\mu_1\ldots\mu_k}_{(k)} \tag{1.78}$$

The simple form of the off-shell divergence given in (1.76) can be recovered upon redefining

$$j^\mu \equiv \tilde{j}^\mu + \sum_{k=1}^{N}\sum_{i=0}^{k-1}(-)^i\partial_{\mu_1}\ldots\partial_{\mu_i}y^{\mathcal{I}\mu\,\mu_1\ldots\mu_{k-1}}_{(k)}\cdot\partial_{\mu_{i+1}}\ldots\partial_{\mu_{k-1}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} \tag{1.79}$$

Proof. See [thesis, p.182]. Let's prove only the mechanics-case $\phi^{\mathcal{I}}_{\text{all}}(\sigma) \to q^i(t)$ and even there further restrict to Lagrangians of the form $L(q,\dot{q})$ without higher derivatives. We start from the symmetry requirement (1.74)

$$\dot{K}_{(\rho)} \stackrel{!}{=} \delta_{(\rho)}L = \tag{1.80}$$

$$= \delta_{(\rho)}q^i\frac{\partial}{\partial q^i}L + \delta_{(\rho)}\dot{q}^i\frac{\partial}{\partial\dot{q}^i}L = \tag{1.81}$$

$$= \delta_{(\rho)}q^i\underbrace{\left(\frac{\partial}{\partial q^i}L - \frac{d}{dt}\frac{\partial}{\partial\dot{q}^i}L\right)}_{\frac{\delta}{\delta q^i(t)}S} + \frac{d}{dt}\left(\delta_{(\rho)}q^i\frac{\partial}{\partial\dot{q}^i}L\right) \tag{1.82}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\delta_{(\rho)}q^i\frac{\partial}{\partial\dot{q}^i}L - K_{(\rho)}\right) = -\delta_{(\rho)}q^i\frac{\delta}{\delta q^i(t)}S \tag{1.83}$$

The main difference for higher derivatives is that one needs to use a generalized formula for partial integration which is of the form

$$\partial^k a\cdot b = \partial\left[\sum_{i=0}^{k-1}(-)^i\partial^{k-1-i}a\cdot\partial^i b\right] + (-)^k a\cdot\partial^k b \tag{1.84}$$

Instead for $L(q,\dot{q})$ we just needed simple partial integration. The inverse direction is left as an exercise... □

Fact. *Every Noether current with on-shell vanishing divergence leads to a conserved* ***Noether charge*** *:*

$$Q \equiv \int_{\Sigma^{(d-1)}} d^{d-1}\sigma\, j^0 \tag{1.85}$$

$$\frac{\partial}{\partial t}Q = \int_{\Sigma^{(d-1)}} d^{d-1}\sigma\,\partial_0 j^0 = -\int_{\Sigma^{(d-1)}} d^{d-1}\sigma\,\partial_i j^i \stackrel{\text{no bdry}}{=} 0 \tag{1.86}$$

Theorem 1.2 (2nd Noether theorem)**.** *If $\delta_\rho \phi^{\mathcal{I}}_{all}$ is a symmetry transformation for arbitrary local (gauge)parameters $\rho^a(\sigma)$ of the form*

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{all} \equiv \underbrace{\rho^a \delta_a \phi^{\mathcal{I}}_{all}}_{\delta^0_{(\rho)}\phi^{\mathcal{I}}_{all}} + \underbrace{\partial_\mu \rho^a \delta^\mu_a \phi^{\mathcal{I}}_{all}}_{\delta^1_{(\rho)}\phi^{\mathcal{I}}_{all}} + \underbrace{\partial_{\mu_1}\partial_{\mu_2}\rho^a \delta^{\mu_1\mu_2}_a \phi^{\mathcal{I}}_{all}}_{\delta^2_{(\rho)}\phi^{\mathcal{I}}_{all}} + \dots \tag{1.87}$$

then the following ***Noether identities*** *hold and are equivalent to (1.76)*

$$\delta_a \phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta \phi^{\mathcal{I}}_{all}} - \partial_{\mu_1}\left(\delta^{\mu_1}_a \phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta \phi^{\mathcal{I}}_{all}}\right) + \dots + (-)^{N+1}\partial_{\mu_1}\dots\partial_{\mu_{N+1}}\left(\delta^{\mu_{N+1}\dots\mu_1}_a \phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta \phi^{\mathcal{I}}_{all}}\right) = 0 \tag{1.88}$$

Proof. Plugging (1.87) into

$$0 \overset{!}{=} \int d^d\sigma \quad \delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}(\sigma)\frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}S[\phi_{\text{all}}] = \tag{1.89}$$

$$= \int d^d\sigma \quad \left(\rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}} + \partial_\mu\rho^a\delta^\mu_a\phi^{\mathcal{I}}_{\text{all}} + \partial_{\mu_1}\partial_{\mu_2}\rho^a\delta^{\mu_1\mu_2}_a\phi^{\mathcal{I}}_{\text{all}} + \dots\right)\frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}S[\phi_{\text{all}}] \quad \forall\rho \tag{1.90}$$

This shows the Noether identities.

Similar to $\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$, we can expand also $j^\mu_{(\rho)}$:

$$j^\mu_{(\rho)} \equiv \rho^a j^\mu_a + \partial_{\mu_1}\rho^a j^{\mu\,\mu_1}_a + \dots + \partial_{\mu_1}\dots\partial_{\mu_{N-1}}\rho^a j^{\mu\,\mu_1\dots\mu_{N-1}}_a \tag{1.91}$$

Plugging (1.87) and the expension of the current (1.91) into (1.76), one can show the equivalence of the Noether identities to (1.76). □

Example: Noether identity for Yang Mills: The equations of motion where

$$\frac{\delta S}{\delta A^c_\mu} \propto D_\nu F^{\nu\mu}_c \tag{1.92}$$

The variation of the gauge field was:

$$\begin{aligned} \delta A_\mu &= [\alpha, A_\mu] - \partial_\mu\alpha = & (1.93)\\ &= \alpha^a[T_a, A_\mu] - \partial_\nu a^a \delta^\nu_\mu T_a & (1.94)\\ \delta A^c_\mu &= \alpha^a f_{ab}{}^c A^b_\mu - \partial_\nu a^a \delta^\nu_\mu \delta^c_a & (1.95)\end{aligned}$$

Compare to

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \underbrace{\rho^a \delta_a \phi^{\mathcal{I}}_{\text{all}}}_{\delta^0_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}} + \underbrace{\partial_\mu \rho^a \delta^\mu_a \phi^{\mathcal{I}}_{\text{all}}}_{\delta^1_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}} + \underbrace{\partial_{\mu_1}\partial_{\mu_2}\rho^a \delta^{\mu_1\mu_2}_a \phi^{\mathcal{I}}_{\text{all}}}_{\delta^2_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}} + \dots \tag{1.96}$$

The Noether identities are in general

$$\delta_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \partial_{\mu_1}\left(\delta^{\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\right) + \dots = 0 \tag{1.97}$$

So this translates for Yang Mills into

$$\begin{aligned} 0 &\overset{?}{=} f_{ab}{}^c A^b_\mu \frac{\delta S}{\delta A^c_\mu} + \partial_\nu\left(\delta^\nu_\mu\delta^c_a\frac{\delta S}{\delta A^c_\mu}\right) = & (1.98)\\ &\propto f_{ab}{}^c A^b_\mu D_\nu F^{\nu\mu}_c + \partial_\mu\left(D_\nu F^{\nu\mu}_a\right) = & (1.99)\\ &= f_{ab}{}^c A^b_\mu\left(\partial_\nu F^{\nu\mu}_c + A^d_\nu f_{dec}F^{\nu\mu e}\right) + \partial_\mu\left(\partial_\nu F^{\nu\mu}_a + A^c_\nu f_{cba}F^{\nu\mu b}\right) = & (1.100)\\ &= f_{ab}{}^c A^b_\mu\partial_\nu F^{\nu\mu}_c + f_{ab}{}^c A^b_\mu A^d_\nu f_{dec}F^{\nu\mu e} + \partial_\mu A^c_\nu f_{cba}F^{\nu\mu}_b + A^c_\nu f_{cba}\partial_\mu F^{\nu\mu b} = & (1.101)\\ &= \underbrace{f_{ab}{}^c A^b_\mu\partial_\nu F^{\nu\mu}_c - A^b_\mu f_{bca}\partial_\nu F^{\nu\mu c}}_{=0} + \left(\partial_\mu A^c_\nu f_{cea} - A^b_\mu A^d_\nu \underbrace{f_{ab}{}^c f_{ced}}_{-\frac{1}{2}f_{ea}{}^c f_{cbd}}\right)F^{\nu\mu e} = & (1.102)\\ &= F^c_{\mu\nu}f_{cea}F^{\nu\mu e} = 0 \quad \surd & (1.103)\end{aligned}$$

Probably boils down to (exercise: check)

$$0 = D_\mu\left(D_\nu F^{\mu\nu}\right) = [F_{\mu\nu}, F^{\mu\nu}] = 0 \tag{1.104}$$

shellZero **Proposition 1.1.** *: The Noether current of a gauge symmetry vanishes on-shell up to trivially conserved terms*

$$j^\mu_{(\rho)} = \sum_{k=0}^{N} \lambda^{\mu\mathcal{I}\mu_1\ldots\mu_k}_{(\rho)} \partial_{\mu_1} \ldots \partial_{\mu_k} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{all}} + \partial_\nu S^{[\mu\nu]}_{(\rho)} \tag{1.105}$$

In turn, if a given global symmetry transformation has an on-shell vanishing current (compare [Henneaux, p.95])

$$j^\mu_a = \sum_{k=0}^{N} \lambda^{\mu\mathcal{I}\mu_1\ldots\mu_k}_{a} \partial_{\mu_1} \ldots \partial_{\mu_k} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{all}} \tag{1.106}$$

then one can extend the transformation to a local one

$$\delta_{(\rho)} \phi^{\mathcal{I}}_{all} \equiv \rho^a \delta_a \phi^{\mathcal{I}}_{all} - \partial_\mu \rho^a \lambda^{\mu\mathcal{I}}_a + \sum_{k=1}^{N} (-)^{k+1} \partial_{\mu_1} \ldots \partial_{\mu_k} \left(\partial_\nu \rho^a \lambda^{\nu\mathcal{I}\,\mu_1\ldots\mu_k}_a \right) \tag{1.107}$$

Proof. Quite technical to prove. We'll skip that as well as the proof of the next statement. □

shellZero **Theorem 1.3.** *Every on-shell vanishing symmetry transformation is a* ***trivial gauge transformation*** *as defined below:*

$$\delta\phi^{\mathcal{I}}_{all} \overset{\text{on-shell}}{=} 0, \quad \delta S = 0 \quad \Rightarrow \quad \delta\phi^{\mathcal{I}}_{all} = \int d^d\sigma \quad \mathcal{A}^{\mathcal{I}\mathcal{J}}(\sigma,\sigma') \frac{\delta S}{\delta \phi^{\mathcal{I}}_{all}(\sigma')} \quad \textit{with}\, \mathcal{A}^{\mathcal{I}\mathcal{J}}(\sigma,\sigma') = -\mathcal{A}^{\mathcal{J}\mathcal{I}}(\sigma',\sigma) \tag{1.108}$$

Proof. See in [Henneaux] (theorem 17.3 on page 414 or theorem 3.1 on page 70 - see also proof(p.229) of thm 10.1(p.209)) for a proof of this theorem. See [Henneaux] p.69 for a discussion of trivial gauge transformations. □

Remark

- Shortcut to calculate Noether current:

$$\delta_{(\rho,\eta)} \phi^{\mathcal{I}}_{\text{all}} \equiv \eta(\sigma) \cdot \delta_\rho \phi^{\mathcal{I}}_{\text{all}} \tag{1.109}$$

$$\delta_{\rho,\sigma} = \int d^d\sigma \quad \partial_\mu \eta \cdot j^\mu_{(\rho)} \tag{1.110}$$

  Where $j^\mu_{(\rho)}$ vanishes on-shell. Can instead also use $\delta^{(0)}_{(\rho)} \phi^{\mathcal{I}}_{\text{all}} \equiv \rho^a \delta^{(0)}_a \phi^{\mathcal{I}}_{\text{all}}$.

- Conserved Noether Charges = Integrals of Motion

Chapter 2

Constrained Hamiltonian/Lagrangian Systems – Classical

2.1 Formulations with brackets (without gauge symmetry)

2.1.1 Legendre Transform without constraints

- The Lagrangian is a function on TM (assuming that it depends maximally on first time derivatives of q)[1]:

$$\begin{aligned} L: \quad TM &\rightarrow \mathbb{R} && (2.1)\\ (q^m, v^m) &\mapsto L(q,v) && (2.2)\end{aligned}$$

- The Hamiltonian instead is a function on T^*M:

$$\begin{aligned} H: \quad T^*M &\rightarrow \mathbb{R} && (2.3)\\ (q^m, p_m) &\mapsto H(q,p) && (2.4)\end{aligned}$$

- The Legendre transformation brings one from one to the other by assigning a momentum (cotangent vector) p for a given q and v as follows

$$p_m \equiv \frac{\partial L(q,v)}{\partial v^m} \tag{2.5}$$

Assuming that the resulting relation between p_m and v^m is invertible, i.e.

$$\det\left(\frac{\partial^2 L(q,v)}{\partial v^m \partial v^n}\right) \neq 0 \tag{2.6}$$

one can build the following function on the cotangent bundle as follows (**Legendre transform**)

$$H(q,p) = p_m v^m(q,p) - L(q, v(q,p)) \tag{2.7}$$

- One can calculate the partial derivatives of H without knowing the explicit form of the inverse transformation $v^m(q,p)$, because $\delta v^m(q,p)$ drops from the variation of H:

$$\begin{aligned}\delta H(q,p) &= \delta p_m v^m(q,p) + p_m \delta v^m(q,p) - \delta q^m \frac{\partial}{\partial q^m} L(q, v(q,p)) - \delta v^m \underbrace{\frac{\partial}{\partial v^m} L(q,v(q,p))}_{p_m} = && (2.8)\\ &= \delta p_m v^m(q,p) - \delta q^m \frac{\partial}{\partial q^m} L(q,v(q,p)) && (2.9)\end{aligned}$$

[1]For Lagrangians that depend on higher derivatives of q^m, see either the comment at the end of the subsequent subsection on first order Lagrangians (suggesting to iteratively replace $\dot{q}$ by momenta and build a "first order Lagrangian" which might not be first order yet, but can be used again to define momenta and so on) , or see [Henneaux, p.47, exercise 1.26]: simply introduce one new variable for each of $\dot{q}, \ddot{q}$ and so on:

$$\begin{aligned} p^{(1)} &\equiv \frac{\partial L}{\partial \dot{q}}, \quad p^{(2)} \equiv \frac{\partial L}{\partial \ddot{q}}, \ldots \\ H &\equiv p^{(1)}\dot{q} + p^{(2)}\ddot{q} - L(q, \dot{q}(q,p), \ddot{q}(q,p), \ldots) \qquad \diamond \end{aligned}$$

The partial derivatives of the two functions thus read:

$$\frac{\partial H}{\partial q^m} = -\frac{\partial L}{\partial q^m} \tag{2.10}$$

$$\frac{\partial H}{\partial p_m} = v^m \quad , \quad p_m = \frac{\partial L}{\partial v^m} \tag{2.11}$$

2.1.2 Hamiltonian eom's with Poisson bracket

- From the Lagrangian eom's

$$\frac{\partial L}{\partial q^m} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = 0 \tag{2.12}$$

one can easily obtain the Hamiltonian ones (using the above relations):

$$\dot{q}^m = \frac{\partial H}{\partial p_m} = \{H, q\} \tag{2.13}$$

$$\dot{p}_m = -\frac{\partial H}{\partial q^m} = \{H, p\} \tag{2.14}$$

The Hamiltonian thus generates the time evolution via the Poisson bracket.

$$\{F, G\} \equiv \partial F/\partial p_m \frac{\partial}{\partial q^m} G - (-)^{FG} \partial G/\partial p_m \frac{\partial}{\partial q^m} F = \tag{2.15}$$

$$= \partial F/\partial p_m \frac{\partial}{\partial q^m} G - \partial F/\partial q^m \frac{\partial}{\partial p_m} G \tag{2.16}$$

$$\{F, G\} = -(-)^{FG}\{G, F\} \tag{2.17}$$

$$\{p_m, q^n\} = \delta_m{}^n \tag{2.18}$$

- Hamiltonian vector field

$$X_H \equiv \{H, -\} = \frac{\partial H}{\partial p_m}\frac{\partial}{\partial q^m} - \frac{\partial H}{\partial q^m}\frac{\partial}{\partial p_m} = \frac{\partial H}{\partial y^M} P^{MN} \frac{\partial}{\partial y^N} \tag{2.19}$$

or equivalently

$$\mathbf{d}H = \imath_{X_H}\omega \tag{2.20}$$

$$\partial_M H = X_H^N \omega_{NM} \tag{2.21}$$

So

$$\dot{y}^M = X_H y \tag{2.22}$$

- The Hamiltonian equations of motion can also be obtained from an action principle (first order action)

$$\tilde{L}: \quad T(T^*M) \rightarrow \mathbb{R} \tag{2.23}$$

$$(q, p, \dot{q}, \dot{p}) \mapsto \tilde{L}(q, p, \dot{q}, \dot{p}) \equiv \dot{q}^m p_m - H(q, p) \tag{2.24}$$

$$\frac{\partial \tilde{L}}{\partial q^m} = \frac{\partial L}{\partial q^m} \tag{2.25}$$

$$\frac{\partial \tilde{L}}{\partial \dot{q}^m} = p_m = \frac{\partial L}{\partial \dot{q}^m} \tag{2.26}$$

$$\frac{\partial \tilde{L}}{\partial p_m} = \dot{q}^m - \frac{\partial H}{\partial p_m} \tag{2.27}$$

- If the original Lagrangian depends on *higher derivatives* of q, say $\ddot{q}$, then $\tilde{L}$ will also depend on $\dot{p}$. But not on $\ddot{q}$ any more. Then one can do the procedure of defining momenta again. If one has even higher derivatives, one does this iteratively.

2.1.3 Schouten-Nijenhuis bracket on $\Gamma(\Lambda^\bullet TM)$

- Generalization of Lie-bracket of vector fields and of Lie-derivative
- Remember for vector fields $\boldsymbol{v} = v^m \boldsymbol{\partial}_m$ the **Cartan formulae**:

$$
\begin{aligned}
[\imath_{\boldsymbol{v}}, \imath_{\boldsymbol{w}}] &= 0 \quad (\equiv \imath_{[v,w]_{alg}}) && (2.28)\\
[\mathbf{d}, \mathbf{d}] &= 0 && (2.29)\\
\boldsymbol{\mathcal{L}}_{\boldsymbol{v}} &\equiv [\imath_{\boldsymbol{v}}, \boldsymbol{d}] && (2.30)\\
[\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}, \mathbf{d}] &= 0 && (2.31)\\
[\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{w}}] &= \boldsymbol{\mathcal{L}}_{[\boldsymbol{v},\boldsymbol{w}]} && (2.32)\\
[\underbrace{[\imath_{\boldsymbol{v}}, \mathbf{d}]}_{\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}}, \imath_{\boldsymbol{w}}] &= \imath_{[\boldsymbol{v},\boldsymbol{w}]} && (2.33)
\end{aligned}
$$

- Last line shows that the Lie bracket is a derived bracket. Inherits the Jacobi-property.
- Take now multivectors

$$
\begin{aligned}
v^{(p)} &\equiv \tfrac{1}{p!} v^{m_1 \ldots m_p} \boldsymbol{\partial}_{m_1} \wedge \ldots \wedge \boldsymbol{\partial}_{m_p} && (2.34)\\
\imath_{v^{(p)}} \rho^{(r)} &\equiv \tfrac{1}{p!(r-p)!} v^{k_1 \ldots k_p} \rho_{k_p \ldots k_1 m_1 \ldots m_{r-p}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p}} && (2.35)
\end{aligned}
$$

- Define Lie derivative in the same way

$$
\begin{aligned}
\boldsymbol{\mathcal{L}}_{v^{(p)}} \rho^{(r)} &\equiv [\imath_{v^{(p)}}, \boldsymbol{d}] \rho^{(r)} = && (2.36)\\
&= \frac{1}{(p-1)!(r-p+1)!} v^{k_1 \ldots k_p} \partial_{k_p} \rho_{k_{p-1} \ldots k_1 m_1 \ldots m_{r-p+1}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p+1}} + \\
&\quad -(-)^p \frac{1}{p!(r-p)!} \partial_{m_1} v^{k_1 \ldots k_p} \rho_{k_p \ldots k_1 m_1 \ldots m_{r-p}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p}} && (2.37)
\end{aligned}
$$

- The Cartan formulae then hold exactly in the same way and define a generalization of the Lie-bracket of vector fields, namely the **Schouten-Nijenhuis-bracket** of multivector fields

$$
\begin{aligned}
[\underbrace{[\imath_{v^{(p)}}, \mathbf{d}]}_{\boldsymbol{\mathcal{L}}_{v^{(p)}}}, \imath_{w^{(p)}}] &\equiv \imath_{[v^{(p)}, w^{(p)}]} && (2.38)\\
[v^{(p)}, w^{(q)}] &= \Big(\tfrac{1}{(p-1)!q!} v^{[m_1 \ldots m_{p-1}|k} \partial_k w^{|m_p \ldots m_{p+q-1}]} + \\
&\quad -(-)^{(p-1)(q-1)} \tfrac{1}{(q-1)!p!} w^{[m_1 \ldots m_{q-1}|k} \partial_k v^{|m_q \ldots m_{p+q-1}]} \Big) \boldsymbol{\partial}_{m_1} \ldots \boldsymbol{\partial}_{m_{p+q-1}} && (2.39)
\end{aligned}
$$

- It is a **Lie bracket of degree -1** :

$$
\begin{aligned}
\deg[v^{(p)}, w^{(q)}] &= p+q-1 && (2.40)\\
[v^{(p)}, w^{(q)}] &= -(-)^{(p-1)(q-1)} (w^{(q)}, v^{(p)}) && (2.41)\\
[v, [w, u]] &= ((v, w), u) + (-)^{(v-1)(w-1)} (w, (v, u)) && (2.42)\\
[v, w \wedge u] &= [v, w] \wedge u + (-)^{(v-1)w} w \wedge [v, u] && (2.43)
\end{aligned}
$$

- Alternative approach: Start by identifying the bracket of a vector v with a general tensor T with the Lie-derivative of the tensor T with respect to the vector:

$$[v, T] \equiv \mathcal{L}_v T \qquad (2.44)$$

  In particular for a scalar ϕ or a vector w we have

$$[v, \phi] \equiv \mathcal{L}_v \phi \;=\; v^m \partial_m \phi, \quad [v, w] \equiv \mathcal{L}_v w = (v^k \partial_k w^m - w^k \partial_k v^m) \boldsymbol{\partial}_m \qquad (2.45)$$

  Note that this definition implies a Leibniz rule for tensor products in the right argument of the bracket

$$[v, (w \otimes y)] = [v, w] \otimes y + w \otimes [v, y] \qquad (2.46)$$

Next one can try to generalize also the lefthand side of the bracket $[v, T]$ to tensors, by demanding some Leibniz rule for tensor products $[v \otimes w, T]$. For some reason (which I cannot reproduce at the moment) this turns out not to be possible for general tensors and for the general tensor product $v \otimes w$. However, if one restricts T to multivectors $y^{(p)}$ and demands the graded Leibniz rule

$$[v \wedge w, y] = v \wedge [w, y] + (-)^{(y-1)v}[v, y] \wedge w \tag{2.47}$$

when acting on the wedge product of two other multivectors, then this works and one obtains precisely the SN-bracket. The above Leibniz rule is of course equivalent to (2.43) when taking into account the graded antisymmetry of the bracket.

- Application: the SN-bracket appears in the criterion for integrability of a Poisson-structure:

$$0 = [P, P] = P^{[m_1|k}\partial_k P^{|m_2 m_3]}\boldsymbol{\partial}_{m_1}\boldsymbol{\partial}_{m_2}\boldsymbol{\partial}_{m_3} \iff \text{Jacobi of Poisson-bracket} \tag{2.48}$$

If the Poisson bivector P is non-degenerate, then this condition is equivalent to closure of its inverse ω which is a symplectic structure

$$\overset{\text{if P invertible}}{\iff} \mathbf{d}x^m \mathbf{d}x^n \mathbf{d}x^k \partial_m \omega_{nk} = \mathbf{d}\omega = 0 \tag{2.49}$$

2.1.4 Lagrangian eom's with antibracket

- Note first the formal isomorphism

$$\Gamma(\Lambda^\bullet TM) \cong \mathcal{F}(\Pi T^* M) \tag{2.50}$$

where

$$\Pi T^*M: \quad T^*M \text{ with "parity-inversed fiber"}: \mathbb{R}^N \to \Lambda^{N(odd)}, \quad \mathcal{F}: \text{into } \Lambda^N \tag{2.51}$$

Take coordinates $(q^m, \boldsymbol{q}^+_m)$ on ΠT^*M and $f \in \mathcal{F}(\Pi T^*M)$. It can be expanded as

$$f(q, \boldsymbol{q}^+) = \sum_{k=0}^{D} \tfrac{1}{k!} f^{m_1 \ldots m_k}(q) \boldsymbol{q}^+_{m_1} \cdots \boldsymbol{q}^+_{m_k} \tag{2.52}$$

Each of the expansion coefficients corresponds to a section of $\Lambda^k TM$.

- The Schouten-Nijenhuis bracket of above now corresponds to a bracket $[\boldsymbol{q}^+_m, q^n] = \delta^n_m$. However, as we are actually interested in paths on M ($\in \mathcal{P}M$) we will actually need to extend this bracket from $\mathcal{F}(\Pi T^*M)$ to $\mathcal{F}(\Pi T^*\mathcal{P}M)$ (i.e. functionals) which will be denoted by round brackets and defined by simply setting

$$(\boldsymbol{q}^+_m(\tau), q^n(\tau')) = \delta^n_m \delta(\tau - \tau') \tag{2.53}$$

This bracket is called the **antibracket**. The $\boldsymbol{q}^+_m$ are known as **antifields**.

- For general functionals on $\Pi T^*\mathcal{P}M$ the antibracket thus reads

$$\begin{aligned}
(F, G) &\equiv \int d\tau\, \delta F/\delta \boldsymbol{q}^+_m(\tau) \frac{\delta}{\delta q^m(\tau)} G - (-)^{(F-1)(G-1)} \delta G/\delta \boldsymbol{q}^+_m(\tau) \frac{\delta}{\delta q^m(\tau)} F && (2.54)\\
(\boldsymbol{q}^+_m(\tau), G) &= \frac{\delta}{\delta q^m(\tau)} G && (2.55)\\
(q^m(\tau), G) &= -\frac{\delta}{\delta \boldsymbol{q}^+_m(\tau)} G && (2.56)\\
(F, G) &= -(-)^{(F-1)(G-1)}(G, F) && (2.57)\\
(F, (G, H)) &= ((F, G), H) + (-)^{(F-1)(G-1)}(G, (F, H)) \quad \text{(graded Jacobi)} && (2.58)
\end{aligned}$$

It is a graded Lie-bracket of degree -1.

- Using this bracket, the equations of motion appear in the bracket of the action with the antifields:

$$(\boldsymbol{q}^+_m(\tau), S) = \frac{\delta}{\delta q_m(\tau)} S \tag{2.59}$$

This can be either seen as $\boldsymbol{q}^+_m$ acting on the action functional S or as S acting on $\boldsymbol{q}^+_m$. The latter interpretation is interesting, because

$$\mathbf{s} \equiv (S, -) \tag{2.60}$$

defines a differential (of degree -1) which (as a differential should) squares to zero

$$\begin{aligned}
\mathbf{s}^2 &= (S, (S, -)) \overset{\text{Jac}}{=} (\underbrace{(S, S)}_{=0}, -) - (S, (S, -)) && (2.61)\\
\Rightarrow \mathbf{s}^2 &= 0 && (2.62)
\end{aligned}$$

$\mathbf{s}$ can thus be used to build a homology (as it reduces the multivector degree). In this homology the equations of motion $\frac{\delta}{\delta q_m(\tau)} S$ are apparently $\mathbf{s}$-exact and are thus implemented homologically

$$\frac{\delta}{\delta q_m(\tau)} S = -\mathbf{s} \boldsymbol{q}^+_m(\tau) \tag{2.63}$$

In other words the functionals on the physical subspace of $\Pi T^*\mathcal{P}M$ (consisting of those paths $\mathcal{P}_{phys}M$ on M that obey the equations of motion) is given by the zero-degree homology

$$\mathcal{F}(\mathcal{P}_{phys}M) = H_0(\mathbf{s}|\mathcal{F}(\Pi T^*\mathcal{P}M)) \equiv \left(\frac{\mathrm{Ker}(\mathbf{s})}{\mathrm{Im}(\mathbf{s})}\right)\Bigg|_{\deg 0} \tag{2.64}$$

- At degree 1 we have[2] we have vector fields which we can identify with symmetry transformations along these vectors

$$U[q,\boldsymbol{q}^+] \equiv \int d\tau' \quad V^m[q](\tau')\boldsymbol{q}^+_m(\tau') \quad (\delta_{(\varepsilon)}q^m(\tau) \equiv V^m_{(\varepsilon)}[q](\tau)) \tag{2.65}$$

  And indeed the requirement of $\mathbf{s}$-invariance (BRST invariance) of the vector field is equivalent to invariance of the action under the corresponding transformation

$$\mathbf{s}U[q,\boldsymbol{q}^+] \;=\; -\int d\tau' \quad V^m[q](\tau')\frac{\delta S}{\delta q^m(\tau')} \stackrel{!}{=} 0 \quad (\iff \delta_{(\varepsilon)}S = 0) \tag{2.66}$$

  So it's the symmetry transformations which are BRST invariant. Instead the exact ones are obtained from

$$\mathbf{s}\int d\tau' \int d\tau \tfrac{1}{2}\Omega^{mn}(\tau,\tau')\boldsymbol{q}^+_m\boldsymbol{q}^+_n(\tau') \;=\; -\int d\tau' \underbrace{\int d\tau \Omega^{[mn]}[q](\tau,\tau')\frac{\delta S}{\delta q^m(\tau)}}_{-\delta V^m[q](\tau')}{}^{+}\boldsymbol{q}^+_n(\tau') \tag{2.67}$$

  So the exact functionals contain just the trivial gauge transformations.

- If we have a local symmetry, then the Noether identities imply that there are also non-integrated BRST-invariant (and non-exact) vertices

$$0 \stackrel{NI}{=} \delta^{(0)}_a q^m \frac{\delta}{\delta q^m}S - \frac{d}{dt}\left(\delta^{(1)}_a q^m \frac{\delta}{\delta q^m}S\right) + \ldots = \tag{2.68}$$

$$= -\mathbf{s}\left(\delta^{(0)}_a q^m \boldsymbol{q}^+_m - \frac{d}{dt}\left(\delta^{(1)}_a q^m \boldsymbol{q}^+_m\right) + \ldots\right) \tag{2.69}$$

  Trivial gauge transformations can also be written in an unintegrated form

$$\mathbf{s}\left(\sum_{r,s\geq 0} \tfrac{1}{2}\Omega^{mn}_{(r)(s)}[q]\boldsymbol{q}^{(r)+}_m\boldsymbol{q}^{(s)+}_n\right) \;=\; -\sum_{r,s\geq 0}\tfrac{1}{2}\Omega^{mn}_{(r)(s)}[q]\left(\left(\frac{d}{dt}\right)^r \frac{\delta S}{\delta q^m}\boldsymbol{q}^{(s)+}_n - \left(\frac{d}{dt}\right)^s \frac{\delta S}{\delta q^n}\boldsymbol{q}^{(r)+}_m\right) = \tag{2.70}$$

$$= -\sum_{r,s\geq 0}\tfrac{1}{2}\left(\Omega^{mn}_{(r)(s)}[q] - \Omega^{nm}_{(s)(r)}[q]\right)\left(\frac{d}{dt}\right)^r \frac{\delta S}{\delta q^m}\boldsymbol{q}^{(s)+}_n \tag{2.71}$$

- Noether current

$$\partial_\mu j^\mu(\tau) = -\delta_{(\varepsilon)}q^m(\tau)\frac{\delta S}{\delta q^m(\tau)} = \mathbf{s}\big(V^m[q](\tau)\boldsymbol{q}^+_m(\tau)\big) \tag{2.72}$$

  For a typical global symmetry, only the integrated vertex $U = \int V$ is $\mathbf{s}$-invariant. The integrand V instead induces descent equations

$$\mathbf{s}V \;=\; \mathbf{d}j \tag{2.73}$$

$$\mathbf{s}j \;=\; 0 \tag{2.74}$$

[2] If one sticks to the finite dimensional manifold M (instead of the space of paths $\mathcal{P}M$) together with the Schouten-Nijenhuis-bracket on $\Gamma(\Lambda^\bullet TM) \cong \mathcal{F}(\Pi T^*\mathcal{P}M)$, then we have

$$[v,w] \;=\; \partial v/\partial(\boldsymbol{\partial}_m)\frac{\partial}{\partial q^m}w - (-)^{(v-1)(w-1)}\partial w/\partial(\boldsymbol{\partial}_m)\frac{\partial}{\partial q^m}v, \quad [\boldsymbol{\partial}_m, q^n] = \delta^n_m$$

In the same way as in the infinite dimensional case let us define a functional $\mathbf{s}$ generated by a 0-vector (a function on M) $S(q)$ via

$$\mathbf{s} \equiv [S, -]$$

Then at 0-grade all 0-vectors (functions) are in the kernel. They are $\mathbf{s}$-exact (the image of some 1-vector) if they are the Lie-derivative (directional derivative) of the function S along that vector:

$$\mathbf{s}f = 0 \quad \forall f \in \mathcal{F}(M), \quad f(q) \sim f(q) + \mathbf{s}(\xi^m\boldsymbol{\partial}_m) = f(q) - \underbrace{\xi^m\partial_m S}_{\imath_\xi \mathbf{d}S}(q)$$

So

$$H_0(\mathbf{s}|\Gamma(\Lambda^\bullet TM)) \equiv \left.\left(\frac{\mathrm{Ker}(\mathbf{s})}{\mathrm{Im}(\mathbf{s})}\right)\right|_{\deg 0} = \mathcal{F}(\{q \in M | \partial_m S(q) = 0\})$$

At grade 1, for a vector to be in the kernel it has to be a symmetry-direction (flat direction) of the function S

$$\begin{aligned} v &\equiv v^m(q)\boldsymbol{\partial}_m \\ \mathbf{s}v &= -v^m(q)\partial_m S = -\mathcal{L}_v S = [S,v] \\ v &\sim v + \mathbf{s}(\tfrac{1}{2}\xi^{mn}(q)\boldsymbol{\partial}_m\boldsymbol{\partial}_n) = (v^n - \xi^{mn}(q)\partial_m S(q))\,\boldsymbol{\partial}_n = v + \xi(\mathbf{d}S, -) \quad \diamond \end{aligned}$$

The current is thus the non-integrated $\mathbf{s}$-invariant vertex corresponding to the symmetry transformation V^min the integrated vertex.

- We can also generate the transformation itself with the antibracket

$$\delta q^m[q](\tau) \quad = \quad (\int d\tau' \delta q^k[q](\tau')\boldsymbol{q}_k^+(\tau'), q^m(\tau)) = (\boldsymbol{U}, q^m(\tau)) \tag{2.75}$$

What do the non-integrated vertices induce? E.g. $(j, F[q]) = 0$.

Remarks

- Future: BRST-differential

$$\mathbf{s} = \boldsymbol{\delta} + \mathbf{d}_l + \ldots \tag{2.76}$$

  - δ:Koszul-Tate-differential: homology puts you on the constraint surface (Hamiltonian formalism=BRST-formalism) or on the equations of motion (Antifield-formalism=BV=BFV). (Different δ's!)
  - $\mathbf{d}_l$: longitudinal exterior derivative: cohomology restricts to gauge invariant objects
  - combining them to $\mathbf{s}$ is known as **homological perturbation theory**

$$H^\bullet(\mathbf{s}|\ldots) = H^\bullet(\mathbf{d}|H_\bullet(\boldsymbol{\delta}|\ldots)) \tag{2.77}$$

  - in the above antifield-discussion the differential $\mathbf{s} = (S, \ldots)$ did not yet take care of any gauge symmetry. So we had just $\mathbf{s} = \boldsymbol{\delta}$. Indeed we observed that $\mathbf{s}$ just put us on the equations of motion.

- Manifold: M^D looks locally like $\mathbb{R}^D$ (charts map to $\mathbb{R}^D$)
Supermanifold: $M^{D_c|D_a}$ looks locally like $\mathbb{R}_c^{D_c} \times \mathbb{R}_a^{D_a}$ where $\mathbb{R}_c$ and $\mathbb{R}_a$ are commuting/anticommuting supernumbers respectively, where the **supernumbers** $\Lambda_\infty \equiv \mathbb{R}_c \oplus \mathbb{R}_a$ are the formal limit of a Grassmann algebra Λ_N with ∞ generators: see next item. It can be expanded as follows:

$$\begin{aligned}
z &= \sum_{k=0}^{\infty} \tfrac{1}{k!} z_{i_1\ldots i_k} \boldsymbol{\eta}^{i_1} \cdots \boldsymbol{\eta}^{i_k} && (2.78)\\
&\equiv \underbrace{z_B}_{\text{body}} + \underbrace{\sum_{k=1}^{\infty} \tfrac{1}{k!} z_{i_1\ldots i_k} \boldsymbol{\eta}^{i_1} \cdots \boldsymbol{\eta}^{i_k}}_{\equiv z_S \text{ (soul)}} \qquad (z_B, z_{i_1\ldots i_k} \in \mathbb{R}) && (2.79)\\
&\equiv \underbrace{z_{even}}_{\in \mathbb{R}_c} + \underbrace{z_{odd}}_{\in \mathbb{R}_a} && (2.80)
\end{aligned}$$

Transition functions for the supermanifold have to be **superanalytic functions** $\Lambda_\infty \to \Lambda_\infty$.

- **Grassmann algebra:** Λ_N generated by $\boldsymbol{\eta}^i, \quad i \in \{1, \ldots, N\}$ with

$$\boldsymbol{\eta}^i \boldsymbol{\eta}^j = -\boldsymbol{\eta}^j \boldsymbol{\eta}^i, \quad \not\Sigma (\boldsymbol{\eta}^i)^2 = 0 \quad \forall i \tag{2.81}$$

$\sum_{k=0}^N \binom{N}{k} = 2^N$ dimensional (e.g. real or complex) vector space. (think of exterior algebra of differential forms $\boldsymbol{\eta}^i = \mathbf{d}x^i$).

- Cotangent bundle with parity reversed fiber ΠT^*M

$$\begin{aligned}
T^*M &\overset{local}{=} M^D \times \mathbb{R}^D, \quad \&\text{transition functions} && (2.82)\\
\Pi T^*M &\overset{local}{=} M^D \times \mathbb{R}_a^D, \quad \&\text{same transition functions} && (2.83)
\end{aligned}$$

- Schouten-Nijenhuis-bracket (partial derivative)

$$\begin{aligned}
[-,-]: \quad \Gamma(\Lambda^\bullet TM) \times \Gamma(\Lambda^\bullet TM) &\to \Gamma(\Lambda^\bullet TM) && (2.84)\\
\mathcal{F}(\Pi T^*M) \times \mathcal{F}(\Pi T^*M) &\to \mathcal{F}(\Pi T^*M) && (2.85)\\
[\boldsymbol{q}_m^+, q^n] &= \delta_m^n && (2.86)
\end{aligned}$$

$$q^m \in M, \quad (q^m, \boldsymbol{q}_m^+) \in \Pi T^*M \tag{2.87}$$

<-> antibracket (variational derivative):

$$\begin{aligned}\Gamma(\Lambda^\bullet T\mathcal{P}M)\times\Gamma(\Lambda^\bullet T\mathcal{P}M) &\rightarrow \mathcal{P}\Gamma(\Lambda^\bullet T\mathcal{P}M) &(2.88)\\ \mathcal{F}(\Pi T^*\mathcal{P}M)\times\mathcal{F}(\Pi T^*\mathcal{P}M) &\rightarrow \mathcal{F}(\Pi T^*\mathcal{P}M) &(2.89)\end{aligned}$$

$$q^m\in\mathcal{P}M,\quad (q^m,\boldsymbol{q}^+_m)\in\mathcal{P}\Pi T^*M \tag{2.90}$$

$$q^m(\tau)\in M,\quad (q^m(\tau),\boldsymbol{q}^+_m(\tau))\in\Pi T^*M \tag{2.91}$$

- graded commutator

$$\begin{aligned}[\imath_v,\imath_w] &\equiv \imath_v\imath_w+(-)^{vw}\imath_w\imath_v \quad \mid\imath_v\mid=\mid v\mid &(2.92)\\ \boldsymbol{\mathcal{L}}_v &\equiv [\mathbf{d},\imath_v] \quad \mid\boldsymbol{\mathcal{L}}_v\mid=\mid v\mid+1 &(2.93)\end{aligned}$$

2.1.5 More about symmetries

- A symmetry $\delta q(q,\dot q,\ddot q,\dddot q,...,t)$ of a Langrangian $L(q,\dot q)$ can always be split into a symmetry $\delta q(q,\dot q,t)$ plus a trivial symmetry! (Henneaux, Exercise 3.8, page 96). This allows to write symmetries in the Hamiltonian formalism as functions of q and p (and maybe t) only.
- Similarly for the Noether charge, see exercise 3.28, page 100

Moment map

- Noether theorem $\leftrightarrow$ moment map? [Silva,p.131]
  - **Action** ψ of a Lie group G on a (symplectic) manifold M

$$\begin{aligned}\psi:\quad G &\rightarrow Diff(M)\\ g &\mapsto \psi_g\end{aligned}\tag{2.94}$$

    Similar to embedding (group homomorphism, so a representation? right-action: ψ is an antihomomorphism). The associated **evaluation map** is

$$\begin{aligned}\mathrm{ev}_\psi: M\times G &\rightarrow M\\ (p,g) &\mapsto \psi_g(p)\end{aligned}\tag{2.95}$$

    Action is called smooth, if ev_ψ is smooth.
  - **symplectic action**

$$\psi:\quad G\rightarrow Sympl(M,\omega)\subset Diff(M) \tag{2.96}$$

  - a symplectic action ψ of S^1 (or $\mathbb{R}$) on (M,ω) is **hamiltonian** if the vector field generated by ψ is hamiltonian ($\exists H$ s.t. $X=\{H,-\}$ or $\mathbf{d}H=\imath_\omega X$)
  - a symplectic action ψ of G on (M,ω) is a **hamiltonian action** if there exists a map

$$\mu: M\rightarrow\mathfrak{g}^* \tag{2.97}$$

    satisfying:
    * $\forall X\in\mathfrak{g}$ let
      $\mu^X: M\rightarrow\mathbb{R}, \mu^X(p):=\langle\mu(p),X\rangle$ be the component of μ along X.
      $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX|t\in\mathbb{R}\}\subseteq G$
      Then

$$\mathbf{d}\mu^X=\imath_{X^\#}\omega \tag{2.98}$$

      i.e. μ^X is a hamiltonian function for the vector field $X^\#$.
    * μ is equivariant with respect to the given action ψ of G on M and the coadjoint action Ad^* of G on $\mathfrak{g}^*$:

$$\mu\circ\psi_g=Ad^*_g\circ\mu\quad\forall g\in G \tag{2.99}$$

      The tuple (M,ω,G,μ) is then called a **hamiltonian G-space** and μ is a **moment map**.
  - It would be nice to translate the last general definition of a hamiltonian action of a nonabelian group into the Poisson-bracket language. For me it would be natural to call such an action hamiltonian, if there exist generators G_a such that $\delta_\varepsilon=\varepsilon^a\{G_a,-\}$.

2.2 Hamiltonian system with constraints

2.2.1 Primary phase space constraints

- Remember that the Legendre transformation

$$H(q,p) = p_m v^m(q,p) - L(q, v(q,p)) \tag{2.100}$$

with the assignment of a momentum (cotangent vector) p for a given q and v via

$$p_m \quad \equiv \quad \frac{\partial L(q,v)}{\partial v^m} \tag{2.101}$$

needs a nondegenerate Hessian (for the relation between p_m and v^m to be invertible), i.e.

$$\det\left(\frac{\partial^2 L(q,v)}{\partial v^m \partial v^n}\right) \neq 0 \tag{2.102}$$

- If instead $\det\left(\frac{\partial^2 L(q,v)}{\partial v^m \partial v^n}\right) = 0$, it means that the relation between p_m and v^m is not invertible and therefore that the p_m are not all independent. So the map from TM to T^*M is not surjective but will instead map only onto a constraint surface Σ_{prim} within T^*M (phase space). They will be defined by some functions

$$\phi_a(q,p) = 0 \tag{2.103}$$

- Examples:

  –

$$\begin{aligned} L(q_1, q_2, \dot q_1, \dot q_2) &= \tfrac{1}{2}\dot q_1^2 - V(q_1, q_2) && (2.104)\\ p_1 &= \dot q_1, \quad p_2 = 0 && (2.105)\\ \Rightarrow \phi(q_1, q_2, p_1, p_2) &= p_2 && (2.106)\end{aligned}$$

  –

$$\begin{aligned} L &= q_1 \dot q_2 && (2.107)\\ p_1 &= 0, \quad p_2 = q_1 && (2.108)\\ \phi_1 = p_1 & \quad\;\; \phi_2 = p_2 - q_1 && (2.109)\end{aligned}$$

- **Def:** A function $F : T^*M \to \mathbb{R}$ is called **first class** if

$$\{F, \phi_a\} = 0 \quad \forall a \tag{2.110}$$

otherwise it is called **second class**.

-

$$\{\phi_a, \phi_b\} = C_{ab} \begin{array}{l} \stackrel{?}{\approx} 0 \\ \stackrel{?}{\approx} \det \neq 0 \end{array} \tag{2.111}$$

- All constraint functions are second class $\iff$

$$\{\phi_a, \phi_b\} = C_{ab} \quad \text{is nondegen} \tag{2.112}$$

- All constraint functions are first class

$$\{G_a, G_b\} = f_{ab}{}^c G_c \approx 0 \tag{2.113}$$

Constraints generate gauge transformation. Each constraint removes 2dof's!

- The constraints so far are called **primary** as they were obtained just from the definition $p_m = \frac{\partial L}{\partial \dot q^m}$ without taking into account the equations of motion which can imply further (**secondary**) constraints. For their treatment, this distinction is not at all essential.

- Constraints are called **reducible**, if they are not (linearly) independent

2.2.2 Total Hamiltonian

In the presence of constraints, the map $v^m \mapsto p_m = \frac{\partial L(q,v)}{\partial v^m}$ is not invertible. In order to better understand what happens, let us split the *Legendre transformation* into
- building the difference $vp - L(q,v)$
- and only then identify (parts of) p and v via $p = \frac{\partial L(q,v)}{\partial v}$.

So a priori we will obtain a function H which depends on q^m, v^m and p_m. Geometrically one could say it is defined on the fiberwise direct product of TM with T^*M, which we will denote simply by $TM \times T^*M$).

$$H(q,p,v) \equiv (v^m p_m - L(q,v)) \tag{2.114}$$

As noted already earlier, the variation of v^m drops completely if we restrict to the subspace of $\{(q,p,v)\}$ where $p = \frac{\partial L(q,v)}{\partial v}$. This is true even if $p(q,v)$ is not invertible:

$$\delta\, H(q,p,v)|_{p=\frac{\partial L(q,v)}{\partial v}} = \left(v^m \delta p_m + \delta v^m \left(p_m - \frac{\partial L(q,v)}{\partial v^m}\right) - \delta q^m \frac{\partial L(q,v)}{\partial q^m}\right)\Bigg|_{p=\frac{\partial L(q,v)}{\partial v}} = \tag{2.115}$$

$$= v^m \delta p_m - \delta q^m \frac{\partial L(q,v)}{\partial q^m} \tag{2.116}$$

This means that $H(q,p,v)$ does not depend on v when we restrict to a subspace $\Xi \subset TM \times T^*M$ defined via $p = \frac{\partial L(q,v)}{\partial v}$. Remember that the same equation $p = \frac{\partial L(q,v)}{\partial v}$ implies also the primary constraints $\Sigma_{\rm prim} \subset T^*M$ in phase space. This means that the we can define a well-defined q,p-dependent Hamiltonian on $\Sigma_{\rm prim} \subset T^*M$ by restricting[3] $H(q,p,v)$ to Ξ:

$$H(q,p)|_{\Sigma_{\rm prim}} \equiv H(q,p,v)|_{p=\frac{\partial L(q,v)}{\partial v}} \tag{2.117}$$

As $H(q,p)$ coincides only on $\Sigma_{\rm prim}$ with $H(q,p,v)$, also their p and q-derivatives coincide only when they are along the surface. So we can say something about the variation of $H(q,p)$ if we restrict in (2.116) not only the variables to the surface, but also the variations to be along Σ. Any such constrained variation δ_Σ has to obey $\delta_{\Sigma_{\rm prim}}\phi_a = 0$ or any linear combination thereof::

$$0 \stackrel{!}{=} u^a \delta_{\Sigma_{\rm prim}} \phi_a(q,p) = \tag{2.118}$$

$$= \delta_{\Sigma_{\rm prim}} q^m \cdot u^a \frac{\partial \phi_a(q,p)}{\partial q^m} + \delta_{\Sigma_{\rm prim}} p_m \cdot u^a \frac{\partial \phi_a(q,p)}{\partial p_m} \tag{2.119}$$

Therefore to a variation of $H(q,p)|_\Sigma$ which stays on the surface one can always add (or for later convenience subtract) these vanishing terms and obtains in general a variation

$$\delta_{\Sigma_{\rm prim}} H(q,p)\big|_{\Sigma_{\rm prim}} \stackrel{(2.116)}{=} \delta_{\Sigma_{\rm prim}} p_m v^m - \delta_{\Sigma_{\rm prim}} q^m \frac{\partial L(q,v)}{\partial q^m} = \tag{2.120}$$

$$\stackrel{(2.119)}{=} \delta_{\Sigma_{\rm prim}} p_m \left(v^m - u^a \frac{\partial \phi_a(q,p)}{\partial p_m}\right) + \delta_{\Sigma_{\rm prim}} q^m \left(-u^a \frac{\partial \phi_a(q,p)}{\partial q^m} - \frac{\partial L(q,v)}{\partial q^m}\right) \tag{2.121}$$

This tells us that independent from how we extend H off the surface $\Sigma_{\rm prim}$, the partial derivatives on the surface

[3] A different approach might be to work immediately with $H(q,p,v)$, instead of first restricting to the constraint surface and then extending again with the help of Lagrange multipliers. The function $H(q,p,v)$ already provides the partial derivatives that one wants, in order to reformulate the Lagrangian equations of motion in terms of Poisson brackets:

$$\frac{\partial H}{\partial q^m} = -\frac{\partial L(q,v)}{\partial q^m}, \quad \frac{\partial H}{\partial p_m} = v^m, \quad \frac{\partial H}{\partial v^m} = p_m - \frac{\partial L(q,v)}{\partial v^m}$$

I.e. Lagrangian equations of motion $\frac{\partial L}{\partial q^m} - \frac{d}{dt}\frac{\partial L}{\partial v} = 0$ (with $v^m \equiv \dot q^m$) are equivalent to

$$\begin{aligned} p_m &= \frac{\partial L}{\partial v^m} &&\Longleftrightarrow \frac{\partial H}{\partial v^m} = 0 \\ \dot p_m &= \frac{\partial L}{\partial q^m} = -\frac{\partial H}{\partial q^m} &&\Longleftrightarrow \dot p_m = \{H, p_m\} \\ v^m &= \frac{\partial H}{\partial p_m} &&\Longleftrightarrow \dot q^m = \{H, q^m\} \quad \diamond \end{aligned}$$

will be of the above form[4]

$$\left.\frac{\partial H(q,p)}{\partial q^m}\right|_{\Sigma_{\text{prim}}} = -u^a \frac{\partial \phi_a(q,p)}{\partial q^m} - \frac{\partial L(q,v)}{\partial q^m} \tag{2.122}$$

$$\left.\frac{\partial H(q,p)}{\partial p_m}\right|_{\Sigma_{\text{prim}}} = v^m - u^a \frac{\partial \phi_a(q,p)}{\partial p_m} \quad \text{for some } u^a \tag{2.123}$$

It should be clear that the first naive extension $H(q,p)$ can be redefined by arbitrary linear combinations of the constraints $\phi_a(q,p)$ as this does not change $H(q,p)|_\Sigma$. In particular we can define an extension for which the partial derivatives reduce to the ones we are used to from the non-degenerate case. Remember that the coefficients u^a can be completely arbitrary. So if we act on them with derivatives, we might allow also a q or p dependence, but we might also treat them as new independent variables. The two different points of view are[5]

$$H_{tot}(q,p) \equiv H(q,p) + u^a(q,p)\phi_a(q,p) \quad (i) \tag{2.124}$$

$$\text{or } H_{tot}(q,p,u) \equiv H(q,p) + u^a\phi_a(q,p) \quad (ii) \tag{2.125}$$

The variation of this so-called **total Hamiltonian** becomes for both approaches (remember $\approx$ means on the constraint surface, so for $\phi_a = 0$)

$$\delta H_{tot}(q,p \underbrace{,u}_{(ii)}) = \Big(\frac{\partial H}{\partial p_m} + u^a\frac{\partial \phi_a}{\partial p_m} \underbrace{+ \frac{\partial}{\partial p_m} u^a \phi_a}_{(i)}\Big)\delta p_m - \delta q^m \Big(-\frac{\partial H}{\partial q^m} - u^a \frac{\partial \phi_a}{\partial q^m} \underbrace{- \frac{\partial u^a}{\partial q^m}\phi_a}_{(i)}\Big) \underbrace{+\delta u^a \phi_a}_{(ii)} \tag{2.126}$$

$$\overset{(2.122)(2.123)}{\approx} v^m \delta p_m - \delta q^m \frac{\partial L}{\partial q^m} \tag{2.127}$$

So the definition of the total Hamiltonian is such that on the constraint surface it has precisely the above simple partial derivatives $\frac{\partial}{\partial p_m} H_{tot} = v^m$ and $\frac{\partial}{\partial q^m} H_{tot} = -\frac{\partial}{\partial q^m} L$. Now we can translate the original Lagrangian equations into Hamiltonian language:

$$0 = \frac{\partial L(q,v)}{\partial q^m} - \frac{d}{dt}\underbrace{\frac{\partial L(q,v)}{\partial \dot q^m}}_{p_m} = -\left.\frac{\partial H_{tot}(q,p)}{\partial q^m}\right|_{\Sigma_{\text{prim}}} - \dot p_m \tag{2.128}$$

$$(\dot q^m \equiv) v^m = \left.\frac{\partial H_{tot}(q,p)}{\partial p_m}\right|_{\Sigma_{\text{prim}}} \tag{2.129}$$

Or in terms of brackets and explicitly demanding the constraints:

$$\dot p_m = \{H_{tot}, p_m\}, \quad \dot q^m = \{H_{tot}, q^m\} \tag{2.130}$$

$$\phi_a(q,p) = 0 \tag{2.131}$$

This can be obtained from a first order action, containing u^a a priori as independent Lagrange multipliers which force the constraints $\phi_a = 0$.

$$\tilde S[q,p,u] \equiv \int dt \quad \dot q^m p_m - H(q,p) - u^a \phi_a(q,p) \tag{2.132}$$

Indeed the variation yields

$$\frac{\delta S}{\delta q^m} = -\dot p_m - \frac{\partial H}{\partial q^m} - u^a \frac{\partial \phi_a}{\partial q^m} \tag{2.133}$$

$$\frac{\delta S}{\delta p_m} = \dot q^m - \frac{\partial H}{\partial p_m} - u^a\frac{\partial \phi_a}{\partial p_m} \tag{2.134}$$

$$\frac{\delta S}{\delta u^a} = \phi_a(q,p) \tag{2.135}$$

---

[4]The appearance of v on the righthand side of ((2.122))-((2.123)) seems to contradict the statement that $H(q,p)$ is a function of q and p only. v on the righthand side has to be understood as inverting $p = \frac{\partial L}{\partial v}$ as much as possible and removing the remaining v's by choosing $u^a(q,p)$ appropriately. So in fact one might want to give the coefficients u^a an explicit v-dependence. Take for example the extreme case where $L(q,v) = L(q)$ and thus $p_m = \frac{\partial L}{\partial v^m} = 0$. In this case the v-dependence in the first line drops explicitly, while in the second we have $\left.\frac{\partial H(q,p)}{\partial p_m}\right|_\Sigma = v^m + u^n \frac{\partial p_n}{\partial p_m} = v^m + u^m$ for some u^m which in fact is just $u^m = -v^m$. ◇

[5]Note that the two ansatzs i) and ii) for H_{tot} both don't mean any loss of generality. If we have $u^a(q,p)$, it contains also the possibility of taking u^a to be a constant. Then the definition of H_{tot} depends on the choice of this constant, and we are basically at ii). Instead if we start from ii), we can always plug a q,p-dependent function u(q,p) as third argument. This would mean restricting the (q,p,u) space to a subspace where $u = u(q,p)$. Then we are back at i). The difference is thus more a notational one and the decision if a total derivative (or variation) should contain the derivative (variation) with respect to u or not. ◇

The equation $\dot{q}^m = \frac{\partial H}{\partial p_m} + u^a \frac{\partial \phi_a}{\partial p_m}$ should be the "inverse" of $p_m = \frac{\partial L}{\partial \dot{q}^m}$. If the latter is not invertible, the u^a have to make up for it.

This indicates already what we will see in the discussion of the secondary constraints: some of the Lagrange multipliers can be integrated out via their equations of motions (those corresponding to 2nd class constraints), while others cannot and correspond to the gauge symmetries induced by 1st class constraints.

Altogether we have from the Lagrangian to the Hamiltonian formalism a coordinate transformation $(q^m, v_m) \mapsto (q^m, p_m, u^a)$ with

$$q^m = q^m \tag{2.136}$$

$$p_m = \frac{\partial L(q,v)}{\partial v^m} \tag{2.137}$$

$$\left.\frac{\partial H(q,p)}{\partial p_m}\right|_{\Sigma_{\rm prim}} + u^a \frac{\partial \phi_a(q,p)}{\partial p_m} = v^m \tag{2.138}$$

2.2.3 Secondary Constraints

- Secondary constraints are obtained as consistency conditions on the equations of motion. Namely the time evolution should stay on the constraint surface:

$$\begin{aligned} 0 &\stackrel{!}{\approx} \dot{\phi}_b = && (2.139)\\ &= \{H_{tot}, \phi_b\} = && (2.140)\\ &\approx \{H, \phi_b\} + u^a C_{ab} && (2.141) \end{aligned}$$

  Whenever possible, we use the freedom in the functions $u^a(q,p)$ to impose this equality.

- Sometimes it is not possible and leads to new **secondary constraints** on the phase space which we add to the set $\{\phi_a\}$ of primary constraints (while in the total Hamiltonian we keep a priori only the primary constraints). These new constraints again must obey the same consistency condition which can again lead to new constraints (still called secondary). And so on.

- Example for secondary constraints

$$L = L_0 + \lambda f(q) \tag{2.142}$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 \quad \phi_1 = p_\lambda \tag{2.143}$$

$$H_{tot} = p\dot{q} - L_0 - \lambda f(q) + u p_\lambda \tag{2.144}$$

$$\dot{p}_\lambda = \{H_{tot}, p_\lambda\} \approx -f(q) \stackrel{!}{\approx} 0 \Rightarrow \boxed{\phi_2 \equiv f(q)} \quad \text{(secondary)} \tag{2.145}$$

- Assume now that $\{\phi_A\}$ are all constraints (not just the primary ones} that we obtain in this way and define a submanifold Σ. Let us now focus on the constraints that we get on the functions $u^a(q,p)$:

$$0 \stackrel{!}{\approx} \{H, \phi_B\} + u^a C_{aB} \tag{2.146}$$

- If all $\{\phi_A\} = \{G_A\}$ are all first class, then $C_{aB} \approx 0$, so u^a drops from the equation and we would get a new constraint on q,p, but by assumption all these constraints are already part of $\{G_A\}$. This means in the first class case there is <u>no condition on the u^a's</u>! They are thus completely free parameters in the time evolution and thus correspond to <u>gauge symmetries</u> of the system. The time evolution using the total Hamiltonian makes only part of these gauge symmetries manifest. If one wants all of them manifest one can add also the remaining ones to the the total Hamiltonian and obtain the so-called **extended Hamiltonian**

$$H_{ext} \equiv H + u^A G_A \tag{2.147}$$

- Instead if all $\{\phi_A\} = \{\chi_A\}$ are second class we can solve explicitly (at least on the constraint surface) for u^a:

$$u^a \approx -\{H, \chi_B\}(C^{-1})^{Ba} \tag{2.148}$$

  However, for the remaining indicex-values of A, where we don't have a corresponding u^a, we obtain also consistency relations of the form

$$(u^\alpha \equiv) 0 \approx -\{H, \chi_B\}(C^{-1})^{B\alpha} \tag{2.149}$$

  These should be automatically fulfilled, if the original Lagrangian was well-defined (as the two systems of equations were equivalent).

2.2.4 Second class constraints and Dirac bracket

- For purely second class constraints

$$\{\chi_a, \chi_b\} = C_{ab} \quad \det C \neq 0 \tag{2.150}$$

one can define a bracket which is compatible with the constraints. It is called the **Dirac bracket**

$$\{f, g\}_D = \{f, g\}_{PB} - \{f, \chi_a\} C^{-1\, ab} \{\chi_b, g\} \tag{2.151}$$

Indeed the bracket of a constraint with anything yields a vanishing result:

$$\{\chi_c, g\}_D = \{\chi_c, g\}_{PB} - \{\chi_c, \chi_a\} C^{-1\, ab} \{\chi_b, g\} = 0 \tag{2.152}$$

- The Dirac bracket is defined also off the constraint surface Σ, but what actually matters is only what happens on the surface. The claims are that
  - second class constraints define a surface on which the symplectic form ω of T^*M induces a symplectic form (nondegenerate) on the constraint surface
  - and that the Dirac bracket restricted to the surface agrees with the the Poisson bracket defined with the induced symplectic form ([Henneaux],p.57):

Proof: Take the coordinates y^M in M such that we extend the coordinates on Σ to M and take the constraint-functions χ_a as orthogonal coordinates

$$y^M = (\sigma^{\mathcal{M}}, \chi_a) \tag{2.153}$$

in such a way that

$$\{\chi_a, \sigma^{\mathcal{M}}\} \approx 0 \quad (0 \text{ on } \Sigma = \{\chi_a = 0\}) \tag{2.154}$$

If a first naive choice $\tilde{\sigma}^{\mathcal{M}}$ leads to $\{\tilde{\sigma}^{\mathcal{M}}, \chi_b\} \not\approx 0$ then redefine

$$\sigma^{\mathcal{M}} = \tilde{\sigma}^{\mathcal{M}} - \{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca} \chi_a \tag{2.155}$$

which leads to the desired relation:

$$\begin{aligned} \{\sigma^{\mathcal{M}}, \chi_b\} &= \{\tilde{\sigma}^{\mathcal{M}} - \{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca} \chi_a, \chi_b\} && (2.156) \\ &= \underbrace{\{\tilde{\sigma}^{\mathcal{M}}, \chi_b\} - \{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca} C_{ab}}_{=0} \underbrace{- \{\{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca}, \chi_b\} \chi_a}_{\approx 0} && (2.157) \end{aligned}$$

The remaining Poisson brackets are

$$\begin{aligned} \{\chi_a, \chi_b\} &= C_{ab} \quad rank\, 2N \leq D \text{ on } \Sigma && (2.158) \\ \{\sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}\} &= (\omega^{-1})^{\mathcal{M}\mathcal{N}}\, rank\, 2(D - N) && (2.159) \end{aligned}$$

The fact that we were able to implement $\{\chi_a, \sigma^{\mathcal{M}}\} \approx 0$ together with $\{y^M, y^N\}$ and $\{C_a, C_b\}$ having full rank, implies that also $(\omega^{-1})^{\mathcal{M}\mathcal{N}}$ has full rank on Σ as indicated above. This proves the first claim.
Our choice of coordinates immediately provide the explicit embedding function of Σ into M:

$$X^M: \quad \sigma^{\mathcal{M}} \mapsto X^M(\sigma) = (\sigma^{\mathcal{M}}, 0) \tag{2.160}$$

The pullback of ω onto the constraint surface then reads

$$(X^*\omega)_{\mathcal{M}\mathcal{N}} = \partial_{\mathcal{M}} X^M \omega_{MN} \partial_{\mathcal{N}} X^N = \omega_{\mathcal{M}\mathcal{N}} \tag{2.161}$$

Now build the Poisson bracket that is built using the inverse $(\omega^{-1})^{\mathcal{M}\mathcal{N}}$ of the induced 2-form $\omega_{\mathcal{M}\mathcal{N}}$ and rewrite it in terms of the Poisson bracket in the ambient phase space

$$\begin{aligned} \{X^*F, X^*G\}_D &\stackrel{?}{=} \partial_{\mathcal{M}} F\, (\omega^{-1})^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} G\big|_\Sigma = && (2.162) \\ &= \partial_M F\, (\omega^{-1})^{MN} \partial_N G\big|_\Sigma - \partial^a F\, C_{ab} \partial^b G\big|_\Sigma = && (2.163) \\ &= \{F, G\}|_\Sigma - \{F, \chi_c\} (C^{-1})^{ca} C_{ab} (C^{-1})^{bd} \{\chi_d, G\}\big|_\Sigma = && (2.164) \\ &= X^*\{F, G\} - X^*\{F, \chi_c\} (C^{-1})^{cd} X^*\{\chi_d, G\} \qquad \surd && (2.165) \end{aligned}$$

This indeed coincides with the Dirac bracket as we defined it.

- Second class can always be seen as gauge fixed first class and vice versa ([Henneaux,p.31, p.46)
- regularity condition for constraint (example of 1st-class=2nd class):

$$\begin{aligned}\{\chi_a, \chi_b\} &= C_{ab} \quad \text{(assume 2nd class)} &(2.166)\\ \{\chi_a\chi_b, \chi_c\chi_d\} &= C_{ac}\chi_b\chi_d + C_{ad}\chi_b\chi_c + C_{bc}\chi_a\chi_d + C_{bd}\chi_a\chi_c \quad \text{(looks 1st class)} &(2.167)\end{aligned}$$

  Condition: Constraints χ_a should define local coordinates around the constraint manifold.

- Use of Dirac-bracket: Let us demonstrate that the Dirac-bracket allows to calculate with H instead of H_{tot}. To this end, let us again split the phase space coordinates $y^M = (q^m, p_m)$ into Σ-coordinates $\sigma^{\mathcal{M}}$ (coordinates of the constraint surface) and the constraint functions χ_a themselves as orthogonal coordinates. As shown above, the Poisson-brackets can be chosen to be

$$\begin{aligned}\{\chi_a, \sigma^{\mathcal{M}}\} &\approx 0 \quad (0 \text{ on } \Sigma = \{\chi_a = 0\}) &(2.168)\\ \{\chi_a, \chi_b\} &= C_{ab} \quad rank\, 2N \leq D \text{ on } \Sigma &(2.169)\\ \{\sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}\} &= (\omega^{-1})^{\mathcal{M}\mathcal{N}}\, rank\, 2(D-N) &(2.170)\end{aligned}$$

  The time evolution of a general phase space function generated by the total Hamiltonian reads

$$\begin{aligned}\dot{F}(q,p) &= \{H_{tot}, F(\sigma,\chi)\} = &(2.171)\\ &\approx \{H, F\} + u^a\{\chi_a, F(\sigma,\chi)\} &(2.172)\end{aligned}$$

  Now we can use the previous result $u^a \approx -\{H, \chi_B\}(C^{-1})^{Ba}$ to obtain

$$\dot{F}(q,p) \approx \{H, F\} - \{H, \chi_B\}(C^{-1})^{Ba}\{\chi_a, F(\sigma,\chi)\} \qquad (2.173)$$

  Using $0 \approx -\{H, \chi_B\}(C^{-1})^{B\alpha}$, we can extend the sum over a (primary constraints) to a sum over A (all constraints) and obtain precisely the Dirac bracket

$$\dot{F}(q,p) \approx \{H, F\}_D \qquad (2.174)$$

  Now $H(q,p)$ was a priori only one naive extension off the constraint surface and therefore quite ambiguous. But any other Hamiltonian, differing just by a linear combination of the constraint functions would work as well, in particular the total Hamiltonian: $\{H, F\}_D = \{H_{tot}, F\}_D = \{H_{tot}, F\} \neq \{H, F\}$.

Main 2nd-class example: Dirac bracket directly from first order action

See also [Henneaux, p.59]

Dirac bracket for a general first order action

Consider now a first order Lagrangian of the form

$$S[y^I] = \int \mathbf{d}t\, \left(\dot{y}^I A_I(y) + B(y)\right) \qquad (2.175)$$

(We call the variables y^I instead of q^i , because a typical case is $y^I = (q^i, p_i)$.)
Its momenta are all constrained

$$\begin{aligned}\pi_I &:= \dot{\partial}_I L(y, \dot{y}) = A_I(y) &(2.176)\\ \Rightarrow \qquad \Phi_I &= \pi_I - A_I(y) &(2.177)\\ C_{IJ} &:= \{\Phi_I, \Phi_J\}_{PB} = \{\pi_I - A_I(y), \pi_J - A_J(y)\}_{PB} = &(2.178)\\ &= \partial_I A_J(y) - \partial_J A_I(y) &(2.179)\end{aligned}$$

If we define

$$\begin{aligned}\boldsymbol{A} &:= A_I \mathbf{d}y^I &(2.180)\\ C &:= \frac{1}{2} C_{IJ} \mathbf{d}x^I \mathbf{d}x^J\end{aligned}$$

then A is a **symplectic (pre)potential** [6] in the sense

$$\mathbf{d}A = \frac{1}{2}\left(\partial_I A_J - (-)^{IJ}\partial_J A_I\right)\mathbf{d}x^I\mathbf{d}x^J = C \tag{2.181}$$

The Dirac bracket belonging to a first order action of the upper type implements a symplectic structure within the coordinate-space (so actually the coordinate space is already a phase space):

$$\begin{aligned}
\{F(y), G(y)\}_D &:= \underbrace{\{F,G\}_{PB}}_{0} - \{F,\Phi_I\}_{PB}\, C^{IJ}\,\{\Phi_J, G\}_{PB} = && (2.182)\\
&= -\Big(\underbrace{\partial^K\Phi_I}_{\delta_I^K}\partial_K F - \underbrace{\partial^K F}_{0}\partial_K\Phi_I\Big)C^{IJ}\Big(\underbrace{\partial^K G}_{0}\partial_K\Phi_J - \underbrace{\partial^K\Phi_J}_{\delta_J^K}\partial_K G\Big) = && (2.183)\\
&= \partial_I F C^{IJ}\partial_J G && (2.184)
\end{aligned}$$

Poisson bracket <-> Dirac bracket of first order Lagrangian

If the first order action is just the first order formalism of a second order action, then the Dirac-bracket on this "extended phase space" coincides with the Poisson bracket of the original phase space:

$$\begin{aligned}
\left(\tilde{S}[q^i, p_i] \equiv\right) \quad S[y^I] &= \int \dot{q}^i p_i - H(q,p) && (2.185)\\
\Rightarrow \quad A_I &= (A_i, A^{\underline{i}}) = (p_i, 0), \quad \partial_I = (\partial_i, \partial^{\underline{i}}) && (2.186)\\
C_{IJ} &= \partial_I A_J(y) - \partial_J A_I(y) = && (2.187)\\
&= \begin{pmatrix} 0 & -\partial^{\underline{j}} p_i \\ \partial^{\underline{i}} p_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\delta_i{}^{\underline{j}} \\ \delta^{\underline{i}}{}_j & 0 \end{pmatrix} && (2.188)
\end{aligned}$$

C_{IJ} thus coincides with the symplectic two form ω_{IJ} of the Poisson bracket of the original action!

Calculation of the symplectic 2-form via variation of the action (Witten's method)

Consider now the variation of a first-order action

$$\begin{aligned}
\left(\tilde{S}[q^i, p_i] \equiv\right) \quad S[y^I] &= \int \mathbf{d}t\,\left(\dot{y}^J A_J(y) + B(y)\right) && (2.189)\\
\delta S[y^I] &= \int \mathbf{d}t\,\left(\delta\dot{y}^J A_J + \dot{y}^J\delta y^I\partial_I A_J + \delta y^J\partial_J B\right) = && (2.190)\\
&\overset{p.I.}{=} \int \mathbf{d}t\Big(\dot{y}^J\delta y^I\underbrace{(\partial_I A_J - \partial_J A_I)}_{C_{IJ}} + \delta y^J\partial_J B\Big) && (2.191)
\end{aligned}$$

One can thus read off the symplectic two form of the variation of the first order action. In the point particle case there seems no advantage over just reading off A_J and then calculating its exterior derivative. However, in field theory we enter the ∞-dimensional case with continous index $\vec{\sigma}$ over which is integrated. The method of partial integration then becomes quite convenient. In particular in the 2-dimensional WZW-sigma model which has the special property that part of it can only be locally written via a 2-dimensional integral, for a global description instead requires a 3-dimensional one. This mixture of dimensions makes it quite challenging to read of $A_J(y)$, while the above method to determine C_{IJ} still works. (see Wittens "nonabelian bosonization"-paper).

[6] The symplectic prepotential of the canonical phase space symplectic 2-form $\omega = \mathbf{d}q^i\mathbf{d}p_i$ is $\boldsymbol{A} := -\mathbf{d}q^i p_i$ for which clearly

$$\omega = \mathbf{d}\boldsymbol{A} \qquad \diamond$$

2.2.5 First Class Constraints and longitudinal exterior derivative

2.2.5.1 First class constraints generate gauge symmetries

- Claim: The following transformations generated by the first class constraints

$$\delta_\varepsilon \equiv \varepsilon^a\{G_a, -\} = \varepsilon^a X_a = \varepsilon^a \partial_K G_a \omega^{KN}\partial_N \tag{2.192}$$

$$\delta_\varepsilon q^m \equiv \varepsilon^a\{G_a, q^m\} = \varepsilon^a \frac{\partial G_a}{\partial p_m} \tag{2.193}$$

$$\delta_\varepsilon p_m \equiv \varepsilon^a\{G_a, p_m\} = -\varepsilon^a \frac{\partial G_a}{\partial q_m} \tag{2.194}$$

  are gauge transformations!

- (higher derivative gauge trafos can be written as canonical(as above)+trivial)

- To see that they are gauge transformations, let us show that they are a symmetry of the first order action

$$S[q,p,u] = \int d\tau\ (\dot{q}^m p_m - H(q,p) - u^c G_c) \tag{2.195}$$

  The generators G_a act on the various terms as follows:

$$\{G_a, H\} = \nu_a{}^b G_b \approx 0 \quad \text{(this was the cons-cond leading to 2ndary constr's)} \tag{2.196}$$

$$\{G_a, u^c G_c\} = u^c f_{ac}{}^b G_b \tag{2.197}$$

$$\{G_a, \dot{q}^m p_m\} = \{G_a, \dot{q}^m\}p_m + \dot{q}^m\{G_a, p_m\} = \tag{2.198}$$

$$= \frac{d}{d\tau}\frac{\partial G_a}{\partial p_m} p_m - \dot{q}^m \frac{\partial G_a}{\partial q_m} = \tag{2.199}$$

$$= \frac{d}{d\tau}\left(\frac{\partial G_a}{\partial p_m} p_m\right) - \frac{\partial G_a}{\partial p_m}\dot{p}_m - \dot{q}^m \frac{\partial G_a}{\partial q_m} = \tag{2.200}$$

$$= \frac{d}{d\tau}\underbrace{\left(\frac{\partial G_a}{\partial p_m} p_m - G_a\right)}_{K_a} \tag{2.201}$$

  Need to transform also Lagrange-multipliers

$$\delta_a u^c = -u^b f_{ab}{}^c - \nu_a{}^c \tag{2.202}$$

  The corresponding Noether charge is indeed G_a:

$$Q_a = \delta_a q^m \frac{\partial L}{\partial \dot{q}^m} + \delta_a p^m \frac{\partial L}{\partial \dot{p}^m} - K_a = \tag{2.203}$$

$$= \{G_a, q^m\}\frac{\partial L}{\partial \dot{q}^m} - K_a = \tag{2.204}$$

$$= G_a \quad \surd \tag{2.205}$$

- Global symmetry with Noether current G_a which is on-shell vanishing. Has a local extension.

$$\delta_\varepsilon S[q,p,u] = \int d\tau\, \dot{\varepsilon}^a G_a$$

  As u^a is coupling to the current, it is the natural candidate of a gauge field

$$\delta_\varepsilon u^c = \dot{\varepsilon}^c - \varepsilon^a\left(u^b f_{ab}{}^c + \nu_a{}^c\right) \tag{2.206}$$

  and indeed with this choice we obtain $\delta_\varepsilon S[q,p,u] = 0$. At least for $\nu_a{}^c = 0$ the above transformation looks indeed like a transformation of a gauge field.

2.2.5.2 Gauge orbits with Hamiltonian vector fields as frame

- **Remark on notation:** Remember we used coordinates y^M on the phase space T^*M. In Darboux-coordinates, these are just

$$y^M = (q^m, p_m) \tag{2.207}$$

As we will quite often deal with the exterior algebra over $T(T^*M)$ and $T^*(T^*M)$ (where basis elements are multiplied with the antisymmetric wedge product $\wedge$), I will treat the 1-form basis elements $\mathbf{d}y^M$ as well as the vector basis elements $\boldsymbol{\partial}_M$ as anticommuting variables (and print them boldface). So also vector fields are denoted boldface

$$\boldsymbol{X} = X^M \boldsymbol{\partial}_M \tag{2.208}$$

The interior product with a vector field reduces the degree of a p-form by 1 and is thus an odd operation which is still stressed by printing the vector (but not the symbol of the interior product) boldface

$$\deg(\imath_{\boldsymbol{X}}) = \deg(\boldsymbol{X}) \tag{2.209}$$

Instead the Lie derivative with respect to a vector field does not change the grading of a tensor (it maps vectors to vectors, 1-forms to 1-forms and so on). This is because the grading of the vector field is compensated by the grading of the exterior derivative in the definition of the Lie derivative

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{X}} \quad \equiv \quad [\imath_{\boldsymbol{X}}, \mathbf{d}] \tag{2.210}$$

When the Lie derivative acts on a scalar field F, it reduces to a directional derivative which looks precisely like the vector field, but it doesn't carry a grading. It will thus be denoted simply by X

$$X \equiv X^M \partial_M, \quad XF \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} F = X^M \partial_M F \tag{2.211}$$

- The relation between Poisson bracket and **Hamiltonian vector fields** has to be understood in the above sense:

$$\begin{aligned} X_f \quad &\equiv \quad \{f, -\} \text{ (actually } \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_f}) && (2.212)\\ \Longleftrightarrow X_f^M \quad &= \quad \partial_N f (\omega^{-1})^{NM} && (2.213)\end{aligned}$$

Multiplying from the right with the matrix ω_{MN} yields the version that is more common in symplectic geometry literature

$$\imath_{\boldsymbol{X}_f} \omega = \mathbf{d}f \tag{2.214}$$

The vector field $\boldsymbol{X}_f$ is called "Hamiltonian", because the time evolution in Hamiltonian mechanics is given by the Hamiltonian vector field with respect to the *Hamiltonian* H:

$$\dot{F} = \{H, F\} = X_H F \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_H} F \tag{2.215}$$

- Hamiltonian vector fields obey the following **product rule** and/or linearity

$$\begin{aligned} X_{fg+h} \quad &= \quad \{fg + h, -\} = && (2.216)\\ &= \quad gX_f + fX_g + X_h && (2.217)\end{aligned}$$

Note that if we simply identify X_f with the Lie derivative $\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_f}$ (which is true for scalar fields) this would read

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_{fg+h}} \quad \overset{?}{=} \quad g\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_f} + f\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_g} + \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_h} \tag{2.218}$$

However, we have to be more careful for Lie derivatives acting on general tensor fields where we also get derivative terms on the vector field and thus on any scalar field multiplying the vector field. In particular when we act on general forms, we obtain[7]

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_{fg+h}} \quad &= \quad \boldsymbol{\mathcal{L}}_{g\boldsymbol{X}_f + f\boldsymbol{X}_g + h} = && (2.219)\\ &\overset{\text{on forms}}{=} \quad g\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_f} + f\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_g} + \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_h} + \mathbf{d}g \wedge \imath_{\boldsymbol{X}_f} + \mathbf{d}f \wedge \imath_{\boldsymbol{X}_g} && (2.220)\end{aligned}$$

erivative

[7]Remember, the **Lie derivative** $\boldsymbol{\mathcal{L}}_{\boldsymbol{X}}$ of a general tensor with components $t_{M_1 \ldots M_p}^{N_1 \ldots N_q}$ with respect to a vector field $\boldsymbol{X} = X^K \boldsymbol{\partial}_K$, can be written in terms of partial ([thesis,p.159]) or covariant derivatives ([thesis, p.208]). For the latter, one gets additional

- Lie brackets between Hamiltonian vector fields are in 1:1 correspondence with the Poisson-brackets of their defining functions[8]

$$
\begin{aligned}
X_{\{f,g\}}F &= \{\{f,g\},F\} = && (2.221)\\
&= \{\{f,F\},g\}+\{f,\{g,F\}\} = && (2.222)\\
&= \{X_f F,g\}+\{f,X_g F\} = && (2.223)\\
&= [X_f,X_g]F && (2.224)
\end{aligned}
$$

This holds for all F, so we obtain

$$[\boldsymbol{X}_f,\boldsymbol{X}_g] = \boldsymbol{X}_{\{f,g\}} \tag{2.225}$$

- Let us denote the Hamiltonian vector fields which correspond to the gauge constraint functions G_a by $\boldsymbol{X}_a$ and the corresponding Lie-derivative on scalar fields by X_a :

$$X_a \equiv X_{G_a} = \{G_a,-\} = \partial_M G_a \omega^{MN}\partial_N \quad (X_a F \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_a}F) \tag{2.226}$$

Then the first class constraint algebra $\{G_a,G_b\} = f_{ab}{}^c G_c$ translates into

$$[\boldsymbol{X}_a,\boldsymbol{X}_b] = \boldsymbol{X}_{f_{ab}{}^c G_c} \tag{2.227}$$

According to (2.217) we have

$$\boldsymbol{X}_{f_{ab}{}^c G_c} = f_{ab}{}^c \boldsymbol{X}_c + G_c \boldsymbol{X}_{f_{ab}{}^c} \approx f_{ab}{}^c \boldsymbol{X}_c \tag{2.228}$$

---

torsion-terms:

$$
\begin{aligned}
\boldsymbol{\mathcal{L}}_{\boldsymbol{X}} t_{M_1\ldots M_p}^{N_1\ldots N_q} &= X^K\partial_K t_{M_1\ldots M_p}^{N_1\ldots N_q} - \sum_i \partial_K X^{N_i} t_{M_1\ldots M_p}^{N_1\ldots N_{i-1}KN_{i+1}\ldots N_q} + \sum_i \partial_{M_i} X^K t^{N_1\ldots N_q}_{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p} = \\
&= X^K\nabla_K t_{M_1\ldots M_p}^{N_1\ldots N_q} - \sum_i \Big(\nabla_K X^{N_i} + X^L \underbrace{(\Gamma_{LK}{}^{N_i} - \Gamma_{KL}{}^{N_i})}_{T_{LK}{}^{N_i}}\Big) t_{M_1\ldots M_p}^{N_1\ldots N_{i-1}KN_{i+1}\ldots N_q} + \\
&\quad + \sum_i \Big(\nabla_{M_i} X^K + \underbrace{(\Gamma_{LM_i}{}^K - \Gamma_{M_iL}{}^K)}_{T_{LM_i}{}^K} X^L\Big) t^{N_1\ldots N_q}_{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p}
\end{aligned}
$$

From the first version we can read off a product rule if $\boldsymbol{X}$ is multiplied with a scalar function α

$$\boldsymbol{\mathcal{L}}_{\alpha\boldsymbol{X}} t_{M_1\ldots M_p}^{N_1\ldots N_q} = \alpha \boldsymbol{\mathcal{L}}_{\boldsymbol{X}} t_{M_1\ldots M_p}^{N_1\ldots N_q} - \sum_i \partial_K \alpha X^{N_i} t_{M_1\ldots M_p}^{N_1\ldots N_{i-1}KN_{i+1}\ldots N_q} + \sum_i \partial_{M_i}\alpha X^K t^{N_1\ldots N_q}_{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p}$$

This can be written in a nice coordinate independent form when the tensor t is a differential p-form $\omega^{(p)}$

$$\boldsymbol{\mathcal{L}}_{\alpha\boldsymbol{X}}\omega^{(p)} = \alpha\omega^{(p)} + \mathbf{d}\alpha \wedge \imath_{\boldsymbol{X}}\omega^{(p)}$$

If $\boldsymbol{X} = \boldsymbol{X}_{(R)}$ is the r-th basis vector of the coordinate frame, so with constant (not covariantly constant) components $X^K_{(R)} = \delta^K_R$, the Lie derivative $\mathcal{L}_R \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_{(R)}}$ reads simply

$$\mathcal{L}_R t_{M_1\ldots M_p}^{N_1\ldots N_q} = \partial_R t_{M_1\ldots M_p}^{N_1\ldots N_q}$$

If $\boldsymbol{X} = \boldsymbol{E}_{(A)}$ is the A-th basis vector of a local frame with components $E^K_{(A)} = E^K_A$, the Lie derivative $\mathcal{L}_A \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{E}_{(A)}}$ in terms of partial derivatives reads

$$\mathcal{L}_A t_{M_1\ldots M_p}^{N_1\ldots N_q} = E^K_A \partial_K t_{M_1\ldots M_p}^{N_1\ldots N_q} - \sum_i \partial_K E^{N_i}_A t_{M_1\ldots M_p}^{N_1\ldots N_{i-1}KN_{i+1}\ldots N_q} + \sum_i \partial_{M_i} E^K_A t^{N_1\ldots N_q}_{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p}$$

For objects with "flat index" A we will use the convention where the covariant derivative ∇_K includes (in addition to the Christoffel Symbols $\Gamma_{KM}{}^N$) also a structure group connection $\Omega_{KA}{}^B$ acting on the flat index. With respect to this covariant derivative, $E_A{}^M$ is covariantly constant, but the above general tensor formula gets modified by the connection terms

$$
\begin{aligned}
\mathcal{L}_A t_{M_1\ldots M_p}^{N_1\ldots N_q} &= E^K_A \nabla_K t_{M_1\ldots M_p}^{N_1\ldots N_q} - \sum_i \Big(\underbrace{\nabla_K E^{N_i}_A}_{=0} + \Omega_{KA}{}^B E^{N_i}_B + E^L_A T_{LK}{}^{N_i}\Big) t_{M_1\ldots M_p}^{N_1\ldots N_{i-1}KN_{i+1}\ldots N_q} + \\
&\quad + \sum_i \Big(\underbrace{\nabla_{M_i} E^K_A}_{=0} + \Omega_{M_iA}{}^B E_B{}^K + T_{LM_i}{}^K E^L_A\Big) t^{N_1\ldots N_q}_{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p} \quad \diamond
\end{aligned}
$$

[8]Note that the vector fields act on functions via the Lie derivative, i.e. $X_f F \equiv \boldsymbol{\mathcal{L}}_{\boldsymbol{X}_f} F$. The commutator of vector fields acting on a function therefore actually means the commutator of Lie derivatives, which as we remember from the Schouten Nijenhuis bracket discussion, is the Lie derivative with respect to the Lie bracket of the vector fields.

$$[X_f,X_g]F = \Big[\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_f},\boldsymbol{\mathcal{L}}_{\boldsymbol{X}_g}\Big]F = \boldsymbol{\mathcal{L}}_{[\boldsymbol{X}_f,\boldsymbol{X}_g]}F = [X_f,X_g]F$$

So in (2.221) we started with $[X_f,X_g]$ meaning the commutator of the corresponding Lie derivatives and in (2.224) we ended up with the Lie bracket of vector fields which acts as a Lie derivative on F. As they coincide, it is not necessary to distinguish in notation, but one should keep the conceptional difference in mind. $\diamond$

From this follows that

$$[\boldsymbol{X}_a,\boldsymbol{X}_b] \approx f_{ab}{}^c \boldsymbol{X}_c \tag{2.229}$$

This corresponds to **Frobenius integrability** , i.e. the vector fields are surface forming and generate the so-called **gauge orbits**. In this form they cannot be integrated to coordinates though, because then we would need commutativity of the vector fields at least on the constraint surface (this differs from the Poisson-algebra of G_a's which commutes on the surface).

- The Hamiltonian vector fields are $\|$ to the constraint surface Σ. In order to see this, assume $y \in \Sigma$ and make an infinitesimal shift in the direction of $\boldsymbol{X}_a$. Is the new y still in Σ?

$$G_a(y + \varepsilon^b \boldsymbol{X}_b) = \underbrace{G_a(y)}_{0} + \varepsilon^b X_b G_a(y) = \tag{2.230}$$

$$= \varepsilon^b \{G_b, G_a\} \approx 0 \quad \surd \tag{2.231}$$

Or a more sophisticated way to argue: Remember first that Σ is the zero locus of G_a

$$G_a : \quad M \supset \Sigma \to \{0\} \subset \mathbb{R}^n \cong \mathfrak{g}^*, \quad \Sigma = G_a^{-1}(0) \tag{2.232}$$

and therefore $T\Sigma$ is the zero locus of the push-forward-map G_{a*} (the Jacobian)

$$G_{a*} : \quad TM \supset T\Sigma \;\to\; \{0\} \cong T\{0\} \subset T\mathbb{R}^n \cong T\mathfrak{g}^* \tag{2.233}$$

$$\dot{y}^M(t) \;\mapsto\; \frac{d}{dt} G_a(y(t)) = \frac{\partial G_a}{\partial y^K} \dot{y}^K \overset{y\in\Sigma}{=} 0 \tag{2.234}$$

So $T\Sigma$ is the Kernel of G_{a*}:

$$T\Sigma = \mathrm{Ker}(G_{a*}) \tag{2.235}$$

Now we just need to show that indeed $\boldsymbol{X}_b$ is in the kernel:

$$\frac{\partial G_a}{\partial y^K} X_b^K = \frac{\partial G_a}{\partial y^K} \{G_b, y^K\} = \{G_b, G_a\} \approx 0 \tag{2.236}$$

- Off the constraint surface in general no integrability: gauge orbits exist only on-shell.

2.2.5.3 Dual frame, ghosts and longitudinal exterior derivative

- In other words, $\{\boldsymbol{X}_a\}$ build a local frame of half of $T\Sigma$ (namely of the gauge orbits). Let us define dual 1-forms $\boldsymbol{c}^a = \mathbf{d}y^M c^a_M \in T^*\Sigma$ via

$$\underbrace{\boldsymbol{c}^a(\boldsymbol{X}_b)}_{=\imath_{\boldsymbol{X}_b}\boldsymbol{c}^a = X_b^M c^a_M} \equiv \delta^a_b \tag{2.237}$$

In physics language they are called **ghosts**.

$$\underbrace{\mathbf{d}\imath_{\boldsymbol{X}_b}\boldsymbol{c}^a}_{\mathbf{d}y^M\left(\partial_K c^a_M X_b^M + c^a_M \partial_K X_b^M\right)} = 0 \tag{2.238}$$

They are a subset of the frames of TM and T^*M:

$$\{\boldsymbol{X}_a\} \subset \{\boldsymbol{E}_A\} \tag{2.239}$$

$$\{\boldsymbol{c}^a\} \subset \{\boldsymbol{e}^A\} \tag{2.240}$$

$$\text{where } \imath_{\boldsymbol{E}_A}\boldsymbol{e}^B = \delta_A^B \qquad \big(\mathbf{d}(\imath_{\boldsymbol{E}_A}\boldsymbol{e}^B) = 0\big) \tag{2.241}$$

The interior products $\imath_{\boldsymbol{E}_A}$ and $\imath_{\boldsymbol{X}^a}$ act like a derivative with respect to the dual 1-forms

$$\imath_{\boldsymbol{E}_A} = \frac{\partial}{\partial \boldsymbol{e}^A}, \qquad \imath_{\boldsymbol{X}^a} = \frac{\partial}{\partial \boldsymbol{c}^a} \tag{2.242}$$

This allows to build a counting oberator that counts the number of $\boldsymbol{e}^A$'s (counts form degree) or in particular of $\boldsymbol{c}^a$'s (counts the so-called **pure ghost number**, i.e. the longitudinal form-degree)

$$\boldsymbol{e}^A \imath_{\boldsymbol{E}_A} = \boldsymbol{e}^A \frac{\partial}{\partial \boldsymbol{e}^A} = \text{"}\boldsymbol{e}^A\text{-counting operator" (form-degree)} \tag{2.243}$$

$$\boldsymbol{c}^a \imath_{\boldsymbol{X}_a} = \boldsymbol{c}^a \frac{\partial}{\partial \boldsymbol{c}^a} = \text{"}\boldsymbol{c}^a\text{-counting operator" (pure ghost number)} \tag{2.244}$$

- The Lie bracket between two basis vectors $\boldsymbol{E}_A$ is again a tangent vector which can be written again as a linear combination of the basis vectors

$$[\boldsymbol{E}_A, \boldsymbol{E}_B] = f_{AB}{}^C(y)\boldsymbol{E}_C \tag{2.245}$$

with some y-dependent coefficients $f_{AB}{}^C$. It is a well-known fact for local frames (see e.g. [Lee, p.311]) that then the corresponding dual frame obeys

$$\mathbf{d}\boldsymbol{e}^A \quad = \quad -\tfrac{1}{2} f_{AB}{}^C \boldsymbol{e}^A \boldsymbol{e}^B \tag{2.246}$$

Although well know, let us still prove this formula in three different ways, in order to get a good feeling for it:

  - direct proof in coordinates

$$\begin{aligned} E_A^K \partial_K E_B^L - E_B^K \partial_K E_A^L &= f_{AB}{}^C E_C^L \quad | e_L^D & (2.247) \\ E_A^K \underbrace{\partial_K E_B^L e_L^D}_{-E_B^L \partial_K e_L^D} - E_B^K \underbrace{\partial_K E_A^L \cdot e_L^D}_{-E_A^M \partial_K e_M^D} &= f_{AB}{}^D \quad | \cdot (-e_{[M}^A e_{N]}^B) & (2.248) \\ \partial_M e_N^D - \partial_N e_M^D &= -f_{AB}{}^D e_{[M}^A e_{N]}^B \quad \surd & (2.249) \end{aligned}$$

  - or using Cartan formulae:

$$\begin{aligned} \imath_{[\boldsymbol{E}_A, \boldsymbol{E}_B]} &= f_{AB}{}^C \imath_{\boldsymbol{E}_C} & (2.250) \\ [[\imath_{\boldsymbol{E}_A}, \mathbf{d}], \imath_{\boldsymbol{E}_B}] &= f_{AB}{}^C \imath_{\boldsymbol{E}_C} \quad | \boldsymbol{e}^A \boldsymbol{e}^B (\ldots) \boldsymbol{e}^D & (2.251) \\ \boldsymbol{e}^A \boldsymbol{e}^B \underbrace{[[\imath_{\boldsymbol{E}_A}, \mathbf{d}], \imath_{\boldsymbol{E}_B}] \boldsymbol{e}^D}_{-\imath_{\boldsymbol{E}_B} \imath_{\boldsymbol{E}_A} \mathbf{d}\boldsymbol{e}^D} &= \boldsymbol{e}^A \boldsymbol{e}^B f_{AB}{}^C \underbrace{\imath_{\boldsymbol{E}_C} \boldsymbol{e}^D}_{\delta_C^D} & (2.252) \end{aligned}$$

    Now we note that $\boldsymbol{e}^B \imath_{\boldsymbol{E}_B}$ is a counting operator that counts the form degree 1 of $\imath_{\boldsymbol{E}_A} \mathbf{d}\boldsymbol{e}^D$. So we obtain on the lefthand side $-\boldsymbol{e}^A \imath_{\boldsymbol{E}_A} \mathbf{d}\boldsymbol{e}^D$. Now the counting operator $\boldsymbol{e}^A \imath_{\boldsymbol{E}_A}$ counts the form degree 2 of $\mathbf{d}\boldsymbol{e}^D$ so that indeed we precisely obtain the claimed result.

  - finally, using footnote 7, we can relate the Lie bracket of vector fields (or better the commutator of their corresponding Lie derivatives) to the commutator of covariant derivatives. Because of the previously mentioned convention that the covariant derivative ∇_K contains also an action of a structure group connection $\Omega_{KA}{}^B$ on "flat indices". In particular when one acts twice with a Lie derivative e.g. on a scalar field, i.e. $\mathcal{L}_A \mathcal{L}_B \phi$, then the second action also sees the index B. Any covariant derivative on $\mathcal{L}_B \phi$ would then act also on B, which was not taken into account in footnote 7 and has to be removed manually:

$$\begin{aligned} [\mathcal{L}_A, \mathcal{L}_B]\phi &= 2E_{[A}^M \nabla_M (E_{B]}^N \nabla_N \phi) + 2E_{[A}^M \Omega_{MB}{}^C (E_{C]}^N \nabla_N \phi) = & (2.253) \\ &= 2\nabla_{[A} \nabla_{B]} \phi + 2E_{[A}^M \Omega_{MB]}{}^C \nabla_C \phi & (2.254) \end{aligned}$$

    Using that the commutator on covariant derivatives([thesis,p.190]) acting on scalars reads

$$[\nabla_A, \nabla_B]\phi = -T_{AB}{}^C \nabla_C \phi \tag{2.255}$$

    the commutator of Lie derivatives becomes

$$[\mathcal{L}_A, \mathcal{L}_B]\phi = \underbrace{\left(2\Omega_{[AB]}{}^C - T_{AB}{}^C\right)}_{\equiv f_{AB}{}^C} \nabla_C \phi \tag{2.256}$$

    Now we compare to the "definition" of torsion (at least in the vielbein-formalism, this is how torsion is defined)

$$\mathbf{d}\boldsymbol{e}^A = T^A - \boldsymbol{\Omega}_C{}^A \boldsymbol{e}^C = \underbrace{\left(\tfrac{1}{2} T_{BC}{}^A - \Omega_{[BC]}{}^A\right)}_{-\frac{1}{2} f_{AB}{}^C} \boldsymbol{e}^B \boldsymbol{e}^C \tag{2.257}$$

    This completes the last proof.

- The exterior derivative on a general p-form $\omega^{(p)} \equiv \frac{1}{p!}\omega_{A_1\ldots A_p}(y)\boldsymbol{e}^{A_1}\cdots\boldsymbol{e}^{A_p}$ can be completely defined by giving its action on the on the coefficient-functions $\omega_{A_1\ldots A_p}(y)$ (or equivalently just on the coordinate y^M) together with the above action on the cotangent basis $\boldsymbol{e}^A$.[9]

$$\begin{aligned} \mathbf{d}y^M &= \boldsymbol{e}^A E_A{}^M = \boldsymbol{e}^A \mathcal{L}_A y^M &(2.258)\\ \mathbf{d}\omega_{A_1\ldots A_p}(y) &= \boldsymbol{e}^A E_A{}^M \partial_M \omega_{A_1\ldots A_p}(y) = \boldsymbol{e}^A \mathcal{L}_A \omega_{A_1\ldots A_p}(y) &(2.259)\\ \mathbf{d}\boldsymbol{e}^A &= T^A - \boldsymbol{\Omega}_C{}^A \boldsymbol{e}^C = \underbrace{\left(\tfrac{1}{2}T_{BC}{}^A - \Omega_{[BC]}{}^A\right)}_{-\frac{1}{2}f_{AB}{}^C} \boldsymbol{e}^B \boldsymbol{e}^C &(2.260) \end{aligned}$$

Let us check that this indeed reproduces the correct exterior derivative (repeated boldface indices at the same level stand in the following simply for antisymmetrized indices):

$$\begin{aligned} \mathbf{d}\omega^{(p)} &= \tfrac{1}{p!}\omega^{(p)}_{\boldsymbol{A}\ldots\boldsymbol{A}}(\boldsymbol{e}^{\boldsymbol{A}})^p = &(2.261)\\ &\overset{\text{Leibniz}}{=} \tfrac{1}{p!}E_{\boldsymbol{A}}{}^M \partial_M \omega^{(p)}_{\boldsymbol{A}\ldots\boldsymbol{A}}(\boldsymbol{e}^{\boldsymbol{A}})^{p+1} + \tfrac{1}{(p-1)!}\omega^{(p)}_{D\boldsymbol{A}\ldots\boldsymbol{A}}\left(T^D - \boldsymbol{\Omega}_C{}^D \boldsymbol{e}^C\right)(\boldsymbol{e}^{\boldsymbol{A}})^{p-1} = &(2.262)\\ &= \tfrac{1}{p!}\nabla_{\boldsymbol{A}}\omega^{(p)}_{\boldsymbol{A}\ldots\boldsymbol{A}}(\boldsymbol{e}^{\boldsymbol{A}})^{p+1} + \tfrac{1}{(p-1)!}T^D\omega^{(p)}_{D\boldsymbol{A}\ldots\boldsymbol{A}}(\boldsymbol{e}^{\boldsymbol{A}})^{p-1} &(2.263) \end{aligned}$$

Compare with coordinate basis:

$$\begin{aligned} \mathbf{d}\omega^{(p)} &= \tfrac{1}{p!}\partial_{\boldsymbol{M}}\omega^{(p)}_{\boldsymbol{M}\ldots\boldsymbol{M}}(\mathbf{d}y^{\boldsymbol{M}})^{p+1} = &(2.264)\\ &= \tfrac{1}{p!}\nabla_{\boldsymbol{M}}\omega^{(p)}_{\boldsymbol{M}\ldots\boldsymbol{M}}(\mathbf{d}y^{\boldsymbol{M}})^{p+1} + \tfrac{1}{(p-1)!}\underbrace{\Gamma_{\boldsymbol{M}\boldsymbol{M}}{}^K \mathbf{d}y^{\boldsymbol{M}}\mathbf{d}y^{\boldsymbol{M}}}_{T^K}\omega^{(p)}_{K\boldsymbol{M}\ldots\boldsymbol{M}}(\mathbf{d}y^{\boldsymbol{M}})^{p-1} \quad \checkmark &(2.265) \end{aligned}$$

- Because of the integrability of the Hamiltonian vector fields $\boldsymbol{X}_a$, we can use the same formulas to define a differential, which acts only in the directions of the gauge orbit, the so-called **longitudinal exterior derivative $\mathbf{d}_{(L)}$**:

$$\begin{aligned} [\boldsymbol{X}_a, \boldsymbol{X}_b] &\approx f_{ab}{}^c \boldsymbol{X}_c &(2.266)\\ \mathbf{d}_{(L)}\boldsymbol{c}^c &\equiv -\tfrac{1}{2}f_{ab}{}^c \boldsymbol{c}^a \boldsymbol{c}^b &(2.267) \end{aligned}$$

It is clear that on the gauge-orbit (smaller than the constraint surface Σ ! but inside) this is just the ordinary exterior differential, because the $\boldsymbol{X}_a$ build a local frame on it and $\boldsymbol{c}^a$ are their duals. However, in the second line we have already extended its action on $\boldsymbol{c}^c$ off the surface. In addition, the action on y-dependent coefficient functions of differential forms finally will be given by the action on the coordinate y^M:

$$\mathbf{d}_{(L)}y^M \equiv X_a y^M = X_a{}^M \qquad (2.268)$$

- This is enough to uniquely determine the action of $\mathbf{d}_{(L)}$ on **longitudinal differential forms**, i.e. forms which are monomials in $\boldsymbol{c}^a$ with coefficient functions that depend on the coordinates y^M:

$$\boxed{\mathbf{d}_{(L)} \equiv \boldsymbol{c}^a \underbrace{X_a^K \partial_K}_{\{G_a,-\}} - \tfrac{1}{2}f_{ab}{}^c \boldsymbol{c}^a \boldsymbol{c}^b \frac{\partial}{\partial \boldsymbol{c}^c}} \qquad (2.269)$$

[9] Note that $\mathbf{d} \neq \boldsymbol{e}^A\mathcal{L}_A$, in contrast to what I had claimed during the lecture. However, almost:

$$\begin{aligned} \mathbf{d}\omega^{(p)} &= \tfrac{1}{p}\mathbf{d}\Big(\boldsymbol{e}^A \boldsymbol{\imath}_A \omega^{(p)}\Big) = \\ &= \tfrac{1}{p}(\mathbf{d}\boldsymbol{e}^A)\boldsymbol{\imath}_A\omega^{(p)} - \tfrac{1}{p}\boldsymbol{e}^A \underbrace{\mathbf{d}\boldsymbol{\imath}_A}_{-\boldsymbol{\imath}_A\mathbf{d}+\mathcal{L}_A} \omega^{(p)} = \\ &= \tfrac{1}{p}(\mathbf{d}\boldsymbol{e}^A)\boldsymbol{\imath}_A\omega^{(p)} + \tfrac{1}{p}\underbrace{\boldsymbol{e}^A\boldsymbol{\imath}_A\mathbf{d}\omega^{(p)}}_{(p+1)\mathbf{d}\omega^{(p)}} - \tfrac{1}{p}\boldsymbol{e}^A\mathcal{L}_A\omega^{(p)} \end{aligned}$$

$$\mathbf{d}\omega^{(p)} = \boldsymbol{e}^A\mathcal{L}_A\omega^{(p)} - (\mathbf{d}\boldsymbol{e}^A)\boldsymbol{\imath}_A\omega^{(p)}$$

For the case where $\omega^{(p)}$ itself is just a basis vector $\boldsymbol{e}^B$, this implies

$$\begin{aligned} \mathbf{d}\boldsymbol{e}^B &= \boldsymbol{e}^A\mathcal{L}_A\boldsymbol{e}^B - (\mathbf{d}\boldsymbol{e}^A)\underbrace{\boldsymbol{\imath}_A\boldsymbol{e}^B}_{\delta^B_A} \\ \Rightarrow \mathbf{d}\boldsymbol{e}^B &= \tfrac{1}{2}\boldsymbol{e}^A\mathcal{L}_A\boldsymbol{e}^B \qquad \diamond \end{aligned}$$

This will be enough for our purposes, but in principle one can define the action of $\mathbf{d}_{(L)}$ also on more general differential forms.[10]

- Let me stress again that the coefficients $f_{AB}{}^C$ are not constant in general, but depend on y^M (the coordinates on T^*M). This implies that the **Jacobi identity** as an identity for the coefficients gets slightly modified to the ususal $f_{[AB|}{}^D f_{D|C]} = 0$ (repeated boldface indices at the same level stand in the following simply for antisymmetrized indices):

$$\begin{aligned} 0 &\stackrel{!}{=} [\boldsymbol{E_A},[\boldsymbol{E_A},\boldsymbol{E_A}]] = && (2.270)\\ &= [\boldsymbol{E_A}, f_{\boldsymbol{AA}}{}^C \boldsymbol{E}_C] = && (2.271)\\ &= -f_{\boldsymbol{AA}}{}^C f_{C\boldsymbol{A}}{}^D \boldsymbol{E}_D + (\mathcal{L}_{\boldsymbol{A}} f_{\boldsymbol{AA}}{}^D)\boldsymbol{E}_D && (2.272)\end{aligned}$$

So

$$E_{\boldsymbol{A}}{}^M \partial_M f_{\boldsymbol{AA}}{}^D - f_{\boldsymbol{AA}}{}^C f_{C\boldsymbol{A}}{}^D = 0 \quad \text{(Jacobi)} \qquad (2.273)$$

For the subframe $\{X_a\}$ the situation is slightly more subtle, at least when leaving the constraint surface. Let us first have a look at the Poisson-Jacobi-identity:

$$\begin{aligned} 0 &\stackrel{!}{=} \{G_{\boldsymbol{a}},\{G_{\boldsymbol{a}},G_{\boldsymbol{a}}\}\} = && (2.274)\\ &= \{G_{\boldsymbol{a}}, f_{\boldsymbol{aa}}{}^c G_c\} = && (2.275)\\ &= \left(-f_{\boldsymbol{aa}}{}^c f_{c\boldsymbol{a}}{}^d + \{G_{\boldsymbol{a}}, f_{\boldsymbol{aa}}{}^d\}\right) G_d \approx 0 && (2.276)\end{aligned}$$

This yields a condition for f only off the surface but not on the surface! And off the surface the bracket does not need to vanish by itself, but instead the general solution is of the form

$$-f_{\boldsymbol{aa}}{}^c f_{c\boldsymbol{a}}{}^d + \{G_{\boldsymbol{a}}, f_{\boldsymbol{aa}}{}^d\} \propto f^{(2)}_{\boldsymbol{aaa}}{}^{bc} G_c \qquad (2.277)$$

Instead for the corresponding Hamiltonian vector fields we obtain a condition on the surface:

$$\begin{aligned} 0 &\stackrel{!}{=} [\boldsymbol{X_a},[\boldsymbol{X_a},\boldsymbol{X_a}]] = && (2.278)\\ &\approx [\boldsymbol{X_a}, f_{\boldsymbol{aa}}{}^c \boldsymbol{X}_c] = && (2.279)\\ &\approx \left(-f_{\boldsymbol{aa}}{}^c f_{c\boldsymbol{a}}{}^d + (X_{\boldsymbol{a}} f_{\boldsymbol{aa}}{}^d)\right) \boldsymbol{X}_d && (2.280)\end{aligned}$$

So we get on the constraint surface a condition that is in agreement with the condition that we had obtained from the Poisson bracket only off the surface:

$$-f_{\boldsymbol{aa}}{}^c f_{c\boldsymbol{a}}{}^d + (X_{\boldsymbol{a}} f_{\boldsymbol{aa}}{}^d) \approx 0 \qquad (2.281)$$

- It is <u>nilpotent</u>, but only on functions restricted to the constraint surface:

$$\begin{aligned} \mathbf{d}^2_{(L)} y^M &= \mathbf{d}_{(L)}(\boldsymbol{c}^a X_a{}^K) = && (2.282)\\ &\approx -\tfrac{1}{2} f_{bc}{}^a \boldsymbol{c}^b \boldsymbol{c}^c X_a{}^K + \boldsymbol{c}^b \boldsymbol{c}^a \underbrace{X_b{}^L \partial_L X_a{}^K}_{\approx \frac{1}{2} f_{ba}{}^c X_c{}^K} = && (2.283)\\ &\approx 0 && (2.284)\end{aligned}$$

[10] Choose a parametrization in which close to the constraint surface Σ the coordinates y^M split into coordinates y^m of the gauge orbit ($\subset \Sigma$) and remaining coordinates y^μ. Then the definition of the longitudinal exterior derivative on the constraint surface Σ is simply such that

$$\mathbf{d}_{(L)} \equiv \mathbf{d} y^m \partial_m$$

Introducing the local orbit-frames $\boldsymbol{c}^a = \mathbf{d}y^m c^a_m$ leads to (2.267) with the dual vector field $\boldsymbol{X}_a = X^M_a \partial_M = X^m_a \partial_m$. If now the local orbit frame is completed to a frame $\{\boldsymbol{e}^C\} = \{\boldsymbol{c}^c, \boldsymbol{e}^\gamma\}$ of the whole cotangent bundle of phase space with $\boldsymbol{e}^\gamma = \mathbf{d}y^M e^\gamma_M$, then the action of $\mathbf{d}_{(L)}$ on $\boldsymbol{e}^\gamma$ depends very much on the choice of the coefficients e^γ_M, so on the way in which $\{\boldsymbol{c}^a\}$ is completed to $\{\boldsymbol{e}^C\} = \{\boldsymbol{c}^c, \boldsymbol{e}^\gamma\}$. In general one just obtains

$$\begin{aligned} \mathbf{d}_{(L)} \boldsymbol{e}^\gamma &= -\mathbf{d}y^M \boldsymbol{c}^b X^n_b \partial_n e^\gamma_M = \\ &= -E^M_A X^n_b \partial_n e^\gamma_M \boldsymbol{e}^A \boldsymbol{c}^b \end{aligned}$$

For a suitable choice of $\boldsymbol{e}^\gamma$ with $\boldsymbol{e}^\gamma = \mathbf{d}y^M e^\gamma_M = \mathbf{d}y^\mu e^\gamma_\mu$ (so with $e^\gamma_m = 0$), this would reduce to

$$\mathbf{d}_{(L)} \boldsymbol{e}^\gamma = \underbrace{-E^\mu_\alpha X^n_b \partial_n e^\gamma_\mu}_{\equiv -f_{\alpha b}{}^\gamma \boldsymbol{e}^\alpha \boldsymbol{c}^b} \boldsymbol{e}^A \boldsymbol{c}^b$$

Only in such a basis one can consistently assign ghost number 0 $\boldsymbol{e}^\alpha$ while keeping ghost number 1 for $\boldsymbol{c}^a$ and $\mathbf{d}_{(L)}$. $\diamond$

The action on $\boldsymbol{c}^c$ is likewise nilpotent only on the surface:

$$
\begin{aligned}
\mathbf{d}_{(L)}^2 \boldsymbol{c}^c &= \mathbf{d}_{(L)} \left(-\tfrac{1}{2} f_{ab}{}^c \boldsymbol{c}^a \boldsymbol{c}^b\right) && (2.285)\\
&= \tfrac{1}{2} f_{ab}{}^c f_{de}{}^a \boldsymbol{c}^d \boldsymbol{c}^e \boldsymbol{c}^b - \tfrac{1}{2} \boldsymbol{c}^d \underbrace{X_d{}^K \partial_K f_{ab}{}^c}_{\{G_d, f_{ab}{}^c\}} \boldsymbol{c}^a \boldsymbol{c}^b = && (2.286)\\
&= \tfrac{1}{2} \boldsymbol{c}^d \left(f_{eb}{}^c f_{da}{}^e - \{G_d, f_{ab}{}^c\}\right) \boldsymbol{c}^a \boldsymbol{c}^b = && (2.287)\\
&\overset{\text{Jac}}{\approx} 0 && (2.288)
\end{aligned}
$$

- Note that the longitudinal exterior derivative on phase space functions $F(y)$ is generated by $\boldsymbol{c}^a G_a$:

$$\mathbf{d}_{(L)} = \{\boldsymbol{c}^a G_a, -\} \tag{2.289}$$

Compare BRST (Becchi, Rouet, Stora and Tyutin)-nilpotence $\boldsymbol{c}^a G_a - \frac{1}{2} f_{ab}{}^c \boldsymbol{c}^a \boldsymbol{c}^b \boldsymbol{b}_c$

$$
\begin{aligned}
\mathbf{s}\boldsymbol{c}^a &= -\tfrac{1}{2} f^a{}_{bc} \boldsymbol{c}^b \boldsymbol{c}^c && (2.290)\\
\mathbf{s}y^M &= \boldsymbol{c}^a X_a^M && (2.291)
\end{aligned}
$$

- Consider the 1-forms $\boldsymbol{c}^a = c^a_M(y)\mathbf{d}y^M$ as independent Grassman variables (not y-dependent). In other words, consider it to be an anticommuting vector

$$(y, \boldsymbol{c}) \in \Pi T(T^* M) \tag{2.292}$$

Then functions in y and $\boldsymbol{c}$ are in 1:1 correspondence with longitudinal forms $\subset \Omega^\bullet(T^* M)$

$$\underbrace{\mathcal{F}(\overbrace{T^* M}^{y^M}) \otimes \mathbb{R}[\boldsymbol{c}]}_{\mathcal{F}(\Pi T(T^* M))} \cong \Omega^\bullet(T^*_{(L)} M) \tag{2.293}$$

- Pure ghost-number operator:

$$\boldsymbol{c}^a \frac{\partial}{\partial \boldsymbol{c}^a} \tag{2.294}$$

Eigenvalues are called the **pure ghost number** and correspond to the longitudinal form degree of a form.

- Want to reduce functions on phase space $T^* M$ to the physical constraint surface via the homology of a differential, namely the **Koszul-Tate differential $\boldsymbol{\delta}$**.

$$\boldsymbol{\delta}(\underbrace{\boldsymbol{b}_a}_{\hat{=}\mathbf{d}G_a \text{ or } \boldsymbol{X}_a?}) \equiv G_a \tag{2.295}$$

$$\boldsymbol{\delta}(\tfrac{1}{2}\Omega^{ab}\mathbf{d}G_a \mathbf{d}G_b) = \Omega^{ab} G_a \mathbf{d}G_b \tag{2.296}$$

Compare BRST

$$\mathbf{s}\boldsymbol{b}_a = G_a + \dots \tag{2.297}$$

2.2.5.4 Reducibility / ghosts for ghosts

If the constraints (and thus the corresponding Hamiltonian vector fields) are not independent

$$Z^{a_0}_{a_1} G_{a_0} = 0 \tag{2.298}$$

introducing as many ghosts as constraints leads to constraints between the ghosts. These constraints can be treated in the same way as before and thus lead to new "ghosts for ghosts". ....

For example in the Lagrangian formalism the constraints are the equations of motion. If there are gauge symmetries, one has the Noether identities, which show that the eom's are not independent. This will require ghosts for ghosts. We will most probably come back to this.

2.3 Homological Perturbation Theory

2.3.1 Resolution

- N-graded ($\mathbb{Z}_{N+1}$-graded) algebra $\bar{A} = \bar{A}_0 \oplus \bar{A}_1 \oplus \ldots \oplus \bar{A}_{N-1}$, $a_{(i)}a_{(j)} = a_{(i+j)}$,$\bar{A}_i\bar{A}_j \subseteq \bar{A}_{i+j}$. An element of $\bar{A}$ that is entirely in one of the spaces $\bar{A}_i$ can be assigned a grading

$$| a | \equiv \deg(a) = i \quad \text{for } a \in A_i \tag{2.299}$$

Induced $\mathbb{Z}_2$-grading:

$$\epsilon_a = \begin{matrix} 0 & \text{for } \deg(a) even \\ 0 & \text{for } \deg(a) odd \end{matrix} \tag{2.300}$$

- **Differential:** odd nilpotent (of order two) derivation $\boldsymbol{\delta}$

$$\boldsymbol{\delta}^2 = \tfrac{1}{2}[\boldsymbol{\delta}, \boldsymbol{\delta}] = 0, \quad \epsilon(\boldsymbol{\delta}) = 1 \tag{2.301}$$

Definition 2.1. A **homological resolution** of an algebra A is a N-graded differential algebra $\bar{A} = \bar{A}_0 \oplus \bar{A}_1 \oplus \ldots \oplus \bar{A}_N$ with differential $\boldsymbol{\delta}$

$$r(\boldsymbol{\delta}) \equiv \deg(\boldsymbol{\delta}) \quad = \quad -1, \quad \boldsymbol{\delta}(x) = 0 \forall x \in \bar{A}_0 \tag{2.302}$$

such that the **homology**

$$H_\bullet(\boldsymbol{\delta}) \equiv H_\bullet(\boldsymbol{\delta}|\bar{A}) \equiv \frac{\mathrm{Ker}(\boldsymbol{\delta})}{\mathrm{Im}(\boldsymbol{\delta})} \tag{2.303}$$

is nontrivial only for degree 0 and is isomorphic to the original algebra A

$$H_r(\boldsymbol{\delta}) \quad = \quad 0 \quad \forall r > 0 \tag{2.304}$$

$$H_0(\boldsymbol{\delta}) \quad = \quad A \tag{2.305}$$

$$\Rightarrow H_\bullet(\boldsymbol{\delta}) \equiv \bigoplus_r H_r \quad = \quad H_0(\boldsymbol{\delta}) = A \tag{2.306}$$

The grading r of the homology space $H_\bullet(\boldsymbol{\delta}) \equiv \bigoplus_r H_r$ and of the underlying graded algebra $\bar{A} = \bigoplus_{r=0}^N \bar{A}_r$ is called **resolution degree** .

Remark The grading of the algebra $\bar{A}$ induces also a grading on the space of endomorphisms mapping $\bar{A}$ onto itself, depending on how it changes the grading of the element it acts on. (the following is a bit sloppy, as one would need to first split x into its components of definite grading and then also $\Xi(x)$ into its components. This induces a splitting of Ξ. But the essence should be clear from below):

$$r(\Xi) \equiv \deg(\Xi) \equiv | \Xi |: \quad | \Xi(x) | \equiv | \Xi | + | x | \quad \text{for } \Xi \in \mathrm{End}(\bar{A}), \quad x \in \bar{A} \tag{2.307}$$

In contrast to $\bar{A}$, this grading extends also to negative values (because endomorphisms can lower the grading):

$$\mathrm{End}(\bar{A}) = \bigoplus_{r=-N}^{N} \mathrm{End}(\bar{A})_r \tag{2.308}$$

Together with the composition of endomorphisms as algebra-product (matrix-product), it becomes a graded algebra. . Based on this grading of $\mathrm{End}(\bar{A})$ we can define the subspace of **graded derivations** $\mathrm{Der}(\bar{A})$ on $\bar{A}$ which are given by those endomorphisms which obey a graded Leibniz-rule.

$$\mathrm{End}(\bar{A}) \supset \mathrm{Der}(\bar{A}) \ni \sigma : \quad \sigma(xy) = \sigma x \cdot y + (-)^{\sigma x} x \sigma y \quad \forall x, y \in \bar{A} \tag{2.309}$$

Derivations are not closed under ordinary composition, but using instead the commutator as algebra product, $\mathrm{Der}(\bar{A})$ becomes also a graded algebra. For a given differential $\boldsymbol{\delta}$ of degree ∓ 1 acting on $\bar{A}$ ($\boldsymbol{\delta} \in \mathrm{Der}(\bar{A})_{\mp 1}$). Due to the Jacobi-identity of the commutator, every element of $\mathrm{Der}(\bar{A})$ acts like a derivation on $\mathrm{Der}(\bar{A})$ via the commutator:

$$[\sigma_1, [\sigma_2, \sigma_3]] = [[\sigma_1, \sigma_2], \sigma_3] + (-)^{\sigma_1 \sigma_2} [\sigma_2, [\sigma_1, \sigma_3]] \tag{2.310}$$

(Also on $\mathrm{End}(\bar{A})$ with the composition product, the commutator acts like a derivation). This property implies that any differential $\boldsymbol{\delta}$ on $\bar{A}$ induces a differential $[\boldsymbol{\delta}, -]$ acting on $\mathrm{Der}(\bar{A})$ (and on $\mathrm{End}(\bar{A})$). That it's a derivation has already been shown in the previous equation. But it is also nilpotent:

$$[\boldsymbol{\delta}, [\boldsymbol{\delta}, \sigma]] = [\underbrace{[\boldsymbol{\delta}, \boldsymbol{\delta}]}_{=0}, \sigma] - [\boldsymbol{\delta}, [\boldsymbol{\delta}, \sigma]] \tag{2.311}$$

$$\Rightarrow [\boldsymbol{\delta}, [\boldsymbol{\delta}, \sigma]] = 0 \tag{2.312}$$

This allows to define also a **homology on the algebra of derivations**

$$\mathcal{H}_\bullet(\boldsymbol{\delta}) \equiv H_\bullet([\boldsymbol{\delta}, -]|\mathrm{Der}(\bar{A})) \equiv \frac{\mathrm{Ker}([\boldsymbol{\delta}, -])}{\mathrm{Im}([\boldsymbol{\delta}, -])} \tag{2.313}$$

Definition 2.2. Let $\Lambda \in \mathrm{End}(\bar{A})$ be a diagonalizable linear operator

$$\bar{A} = \bigoplus_\lambda \tilde{A}_\lambda \quad (= \bigoplus_r \bar{A}_r) \tag{2.314}$$

such that

$$\tilde{A}_0 \subset \bar{A}_0 \tag{2.315}$$

A (odd) linear operator $\boldsymbol{\sigma} \in \mathrm{End}(\bar{A})$ is called a **contracting homotopy** for Λ w.r.t. the differential $\boldsymbol{\delta}$ iff

$$[\boldsymbol{\delta}, \boldsymbol{\sigma}] = \Lambda \tag{2.316}$$

This means that Λ is $\boldsymbol{\delta}$-exact.

- Compare: in topology, curves are homotopic (equivalent) if they can be continously be deformed into each other (corresponds to the difference being exact). The deformation is called a homotopy (this would rather Λ make the homotopy than $\boldsymbol{\sigma}$!?)

Theorem 2.1. *If $\exists$ a contracting homotopy $\boldsymbol{\sigma} \Rightarrow$ the differential $\boldsymbol{\delta}$ is acyclic in degree $r > 0$, i.e. $H_r(\boldsymbol{\delta}) = 0$. If $\boldsymbol{\sigma}$ is a derivation, also $[\boldsymbol{\delta}, -]$ acting on the algebra of derivations is acyclic in degrees $r \neq 0$.*

Proof. The fact that Λ is $\boldsymbol{\delta}$-exact implies that it is also closed, i.e. commutes with $\boldsymbol{\delta}$

$$[\boldsymbol{\delta}, \Lambda] = 0 \tag{2.317}$$

Therefore $\boldsymbol{\delta}$ stays in the eigenspaces A_λ. So in particular if $a \in \bar{A}$ is decomposed as $a = \sum_\lambda a_\lambda$, then assuming that it is closed

$$\boldsymbol{\delta} a = 0 \quad (assump) \tag{2.318}$$

implies that

$$\boldsymbol{\delta} a_\lambda = 0 \quad \forall \lambda \tag{2.319}$$

Assuming further that

$$\lambda \neq 0 \quad (assump) \tag{2.320}$$

we can use the eigenvalue-equation $\Lambda a_\lambda = \lambda a_\lambda$ to write a_λ as

$$a_\lambda = \frac{1}{\lambda} \Lambda a_\lambda \overset{[\boldsymbol{\delta},\boldsymbol{\sigma}]=\Lambda}{=} \frac{1}{\lambda} [\boldsymbol{\delta}, \boldsymbol{\sigma}] a_\lambda \overset{\boldsymbol{\delta} a_\lambda = 0}{=} \boldsymbol{\delta} \left(\frac{1}{\lambda} \boldsymbol{\sigma} a_\lambda \right) \tag{2.321}$$

This means that every closed a_λ with $\lambda \neq 0$ is also exact. This means that any nontrivial cycles can lie only in $\tilde{A}_0 \subset \bar{A}_0$.

If $\boldsymbol{\sigma}$ is a derivation, then also Λ will be a derivation on $\bar{A}$ ($\boldsymbol{\delta}$ is in any case) and they will induce corresponding derivations on $\mathrm{Der}(\bar{A})$, namely $[\boldsymbol{\sigma}, -]$, $[\Lambda, -]$ and $[\boldsymbol{\delta}, -]$. And indeed these obey the defining properties of an ordinary contracting homotopy: First $[\Lambda, -]$ is diagonalizable with the zero eigenvalue-space contained in $\mathrm{Der}(\bar{A})_0$ (make more explicit!) and also

$$\text{"}[[\boldsymbol{\delta}, -], [\boldsymbol{\sigma}, -]]\text{"} = [\boldsymbol{\delta}, [\boldsymbol{\sigma}, -]] + [\boldsymbol{\sigma}, [\boldsymbol{\delta}, -]] = [[\boldsymbol{\delta}, \boldsymbol{\sigma}], -] = [\Lambda, -] \tag{2.322}$$

Now the proof given before for $\bar{A}$ holds the same for $\mathrm{Der}(\bar{A})$. $\square$

Hamiltonian picture:

The algebra A is the algebra of functions on the constraint surface $\Sigma \subset T^*M$

$$A = \mathcal{F}(\Sigma) \tag{2.323}$$

Its homological resolution is

$$\bar{A} = \mathcal{F}(T^*M) \otimes \mathbb{R}[\boldsymbol{b}_a] \subset \mathcal{F}(T^*\Pi TM) \tag{2.324}$$

-
$$\boldsymbol{\delta b}_a \quad = \quad G_a(y) \quad \left(= "p_a - \frac{\partial L}{\partial \dot{q}^a}"\right), \quad \boldsymbol{\delta} y^M = 0. \quad (y^M = (q^m, p_m)) \tag{2.325}$$

$$\boldsymbol{\delta} = G_a \frac{\partial}{\partial \boldsymbol{b}_a} \tag{2.326}$$

- resolution degree: **antighost number**, i.e. eigenvalues of

$$\boldsymbol{b}_a \frac{\partial}{\partial \boldsymbol{b}_a} \tag{2.327}$$

- Split phase space coordinates y^M into those $\sigma^{\mathcal{M}}$ parallel to the surface[11] and those G_a perpendicular to it:

$$y^M \quad = \quad (\sigma^{\mathcal{M}}, G_a) \tag{2.328}$$
$$\boldsymbol{\delta} \quad = \quad G_a \frac{\partial}{\partial \boldsymbol{b}_a} \tag{2.329}$$

Locally analytic functions on Σ correspond (Taylor-expansion) to the polynomial algebra $\mathbb{R}[\sigma^{\mathcal{M}}]$.
Claim: The differential algebra $(\mathbb{R}[\sigma^{\mathcal{M}}, G_a, \boldsymbol{b}_a], \boldsymbol{\delta})$ provides a resolution of $\mathbb{R}[\sigma^{\mathcal{M}}]$

$$H_0(\boldsymbol{\delta}) \quad = \quad \mathbb{R}[\sigma^{\mathcal{M}}] \tag{2.330}$$
$$H_r(\boldsymbol{\delta}) \quad = \quad 0 \quad \forall r > 0 \tag{2.331}$$

Furthermore:

$$\mathcal{H}_0(\boldsymbol{\delta}) \quad = \quad \text{Derivations on } \mathbb{R}[\sigma^{\mathcal{M}}] \tag{2.332}$$
$$\mathcal{H}_r(\boldsymbol{\delta}) \quad = \quad 0 \quad \forall r \neq 0 \tag{2.333}$$

(always when a contracting homotopy is a derivation. Note that the derivation homology $\mathcal{H}_\bullet$ in contrast to $H_\bullet$ also has negative degrees)
Proof: of the first ones

- H_0 :
  - $\text{Ker}_0(\mathbb{R}[\sigma^{\mathcal{M}}, G_a, \boldsymbol{b}_a], \boldsymbol{\delta}) = \mathbb{R}[\sigma^{\mathcal{M}}, G_a, \boldsymbol{b}_a]_0 = \mathbb{R}[\sigma^{\mathcal{M}}, G_a]$
  - Im_0:

$$G_a \frac{\partial}{\partial \boldsymbol{b}_a} (\lambda^c(\sigma, G)\boldsymbol{b}_c) \quad = \quad \lambda^a(\sigma, G) G_a \tag{2.334}$$

  -
$$H_0 = \frac{\text{Ker}_0}{\text{Im}_0} \cong \mathbb{R}[\sigma^{\mathcal{M}}] \tag{2.335}$$

[11]The choice of the capital calligraphic character $\mathcal{M}$ seems a bit weird, but like for $y^M = (q^m, p_m)$, it would be useful to have the possibility to split also the coordinates on the constraint surface into configuration space variables and momenta, and then this notation is conventient:

$$\sigma^{\mathcal{M}} = (q^\mu, p_\mu)$$

Not sure, however, if we will ever use this split... $\diamond$

– $H_k = 0 \forall k > 0$: It suffices to show the existence of a contracting homotopy. As diagonalizable operator Λ, we can choose a counting operator which counts the number of $\boldsymbol{b}_a$'s and G_a's:

$$\Lambda \equiv \boldsymbol{b}_a \frac{\partial}{\partial \boldsymbol{b}_a} + G_a \frac{\partial}{\partial G_a} \tag{2.336}$$

$$\Lambda \boldsymbol{b}_a = \boldsymbol{b}_a, \quad \Lambda G_a = G_a, \quad \Lambda \sigma^{\mathcal{M}} = 0 \tag{2.337}$$

This is clearly chosen in such a way that $\tilde{A}_0 = \mathbb{R}[\sigma^{\mathcal{M}}] \subset A_0 = \mathbb{R}[\sigma^{\mathcal{M}}, G_a]$ and that it is exact

$$\Lambda = [\boldsymbol{\delta}, \boldsymbol{\sigma}] \tag{2.338}$$

$$\text{with}\, \boldsymbol{\sigma} = \boldsymbol{b}_a \frac{\partial}{\partial G_a} \tag{2.339}$$

$\boldsymbol{\sigma}$ is almost the inverse of $\boldsymbol{\delta}$. □

Lagrangian picture:

The algebra A is the algebra of functionals on the paths on M

$$A = \mathcal{F}(\underbrace{\mathcal{P}_{\text{phys}} M}_{\equiv \Sigma \equiv \{q_{\text{phys}} | \frac{\delta S[q_{\text{phys}}]}{\delta q^m} = 0\}}) \tag{2.340}$$

while its homological resolution is

$$\bar{A} = \text{"}\mathcal{F}(\mathcal{P}M) \otimes \mathbb{R}[\boldsymbol{q}^+_m(\tau)]\text{"} = \mathcal{F}(\Pi T^* \mathcal{P} M) \tag{2.341}$$

- $$\boldsymbol{\delta} \boldsymbol{q}^+_m = (S, \boldsymbol{q}^+_m) = -\frac{\delta S}{\delta q^m}, \quad \boldsymbol{\delta} q^m = 0 \tag{2.342}$$

- resolution degree: **antifield number**, i.e. eigenvalues of

$$\boldsymbol{q}^+_m \frac{\partial}{\partial \boldsymbol{q}^+_m} \tag{2.343}$$

  Corresponds to **multivector-degree.** $\boldsymbol{q}^+_m$ has ghost number -1.

- $$H_0(\boldsymbol{\delta}) \cong \mathcal{F}(\underbrace{\mathcal{P}_{\text{phys}} M}_{\frac{\delta S[q_{\text{phys}}]}{\delta q^m} = 0})$$

- ghosts $\mathbf{d}_{(L)} q^m = \boldsymbol{c}^a \delta_a q^m$. (same as in Hamiltonian!). Come with antifields c^+_a.

$$\mathbf{d}_{(L)} F = \int \int \boldsymbol{c}^a(t) \delta_a q^m(t, t') \frac{\delta F}{\delta q^m(t')} \tag{2.344}$$

$$\mathbf{d}_{(L)} \boldsymbol{c}^a = \int \int \tfrac{1}{2} f^a{}_{bc}(t, t') \boldsymbol{c}^b(t) \boldsymbol{c}^c(t') \tag{2.345}$$

- Constraints are not independent (Noether-identities). Reducibility! $\Rightarrow c^+_a$.

2.3.2 Relative cohomology & extension of the Poisson-bracket

- differential $\boldsymbol{d}_{(L)}$ modulo $\boldsymbol{\delta}$:

$$\mathbf{d}_{(L)} = \boldsymbol{c}^a \partial_M G_a \omega^{MN} \partial_N - \tfrac{1}{2} f_{ab}{}^c \boldsymbol{c}^a \boldsymbol{c}^b \frac{\partial}{\partial \boldsymbol{c}^c} \tag{2.346}$$

$$\mathbf{d}^2_{(L)} = \boldsymbol{\delta}(\ldots) = -[\boldsymbol{\delta}, \mathbf{s}^{(1)}], \quad r(\mathbf{d}_{(L)}) = 0, \tag{2.347}$$

$$x \in H^k(\mathbf{d}): \quad \mathbf{d}x = \delta y, \quad x \sim x + \mathbf{d}z + \delta z' \tag{2.348}$$

$$r(x) = 0 = r(z), \quad r(y) = 1 = r(z') \tag{2.349}$$

- Example

$$\mathbf{s} = \delta + \mathbf{d} + \ldots \tag{2.350}$$
$$\mathbf{s}^2 = \underbrace{\delta^2}_{0} + \underbrace{[\delta, \mathbf{d}]}_{0} + \underbrace{\mathbf{d}^2 + [\delta, \mathbf{s}^{(1)}]}_{0} + \ldots \tag{2.351}$$

- total ghost number

$$\mathrm{gh}(x) = \underbrace{\deg(x)}_{pure\,gh=form\,deg} - r(x) \tag{2.352}$$

$$\mathrm{gh}(\mathbf{d}) = \mathrm{gh}(\boldsymbol{\delta}) = 1 \tag{2.353}$$
$$\mathrm{antgh}(\mathbf{d}) = 0, \quad \mathrm{antgh}(\boldsymbol{\delta}) = -1 \tag{2.354}$$

- ghost number operator

$$J = \boldsymbol{c}^a \frac{\partial}{\partial \boldsymbol{c}^a} - \boldsymbol{b}_a \frac{\partial}{\partial \boldsymbol{b}_a} \equiv \{\boldsymbol{c}^a \boldsymbol{b}_a, -\} \tag{2.355}$$

- Geometrically it makes sense to identify the antighosts $\boldsymbol{b}_a$ with the Hamiltonian vector fields $\boldsymbol{X}_a$ (with the wedge product as anticommuting product between them).
- Functions of y^M, $\boldsymbol{c}^a$ and $\boldsymbol{b}_a$ are then formal sums of (particular) multivector valued differential forms:

$$f(y, \boldsymbol{c}^a, \boldsymbol{b}_a) = f^{(0)}(y) + \sum_{p,q\geq 1} \tfrac{1}{p!q!} f^{(p,q)\boldsymbol{b}\ldots\boldsymbol{b}}_{\boldsymbol{c}\ldots\boldsymbol{c}}(y) \boldsymbol{c}^{\boldsymbol{c}} \ldots \boldsymbol{c}^{\boldsymbol{c}} \boldsymbol{b}_{\boldsymbol{b}} \cdots \boldsymbol{b}_{\boldsymbol{b}} \tag{2.356}$$

- A natural generalization of the interior product of vectors acting on differential forms is the following interior product of multivector valued forms acting on forms:

$$\imath_{\frac{1}{p!q!} f^{(p,q)\boldsymbol{b}\ldots\boldsymbol{b}}_{\boldsymbol{c}\ldots\boldsymbol{c}}(y)\boldsymbol{c}^{\boldsymbol{c}}\ldots\boldsymbol{c}^{\boldsymbol{c}}\boldsymbol{b}_{\boldsymbol{b}}\cdots\boldsymbol{b}_{\boldsymbol{b}}} \equiv \tfrac{1}{p!q!} f^{(p,q)\boldsymbol{b}\ldots\boldsymbol{b}}_{\boldsymbol{c}\ldots\boldsymbol{c}}(y) \overbrace{\boldsymbol{c}^{\boldsymbol{c}} \wedge \ldots \boldsymbol{c}^{\boldsymbol{c}}}^{p} \wedge \overbrace{\underbrace{\imath_{\boldsymbol{b}_{\boldsymbol{b}}}}_{\equiv \imath_{\boldsymbol{X}_{\boldsymbol{b}}} = \frac{\partial}{\partial \boldsymbol{c}^{\boldsymbol{b}}}} \cdots \imath_{\boldsymbol{b}_{\boldsymbol{b}}}}^{q} \tag{2.357}$$

The map from the multivector valued form $f(y, \boldsymbol{c}^a, \boldsymbol{b}_a)$ to this operator can be seen is the **quantization** (in Schrödinger representation) of the ghost-variables $\boldsymbol{c}^a, \boldsymbol{b}_a$.

- The commutator of two such operators induces an algebraic bracket between multivector valued forms

$$[\imath_K, \imath_L] \equiv \imath_{[K,L]^\Delta} \tag{2.358}$$
$$\text{with} \quad [K, L]^\Delta = \sum_{p\geq 1} \left(\tfrac{1}{p!} K \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_{\boldsymbol{a}}} \right)^p \left(\frac{\partial}{\partial \boldsymbol{c}^{\boldsymbol{a}}} \right)^p L - (-)^{(k-k')(l-l')} K \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_{\boldsymbol{a}}} \right)^p \left(\frac{\partial}{\partial \boldsymbol{c}^{\boldsymbol{a}}} \right)^p L \right) \tag{2.359}$$

Each pair $\frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_{\boldsymbol{a}}} \frac{\partial}{\partial \boldsymbol{c}^{\boldsymbol{a}}}$ has pure ghost number -1, antighost-number -1 and total ghost number 0. The lowest term with just one derivative (which therefore has the highest antighost number) obviously has the form of a Poisson-bracket

$$[K, L]^{(1)} = K \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_{\boldsymbol{a}}} \right) \left(\frac{\partial}{\partial \boldsymbol{c}^{\boldsymbol{a}}} \right) L - (-)^{(k-k')(l-l')} K \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_{\boldsymbol{a}}} \right) \left(\frac{\partial}{\partial \boldsymbol{c}^{\boldsymbol{a}}} \right) L \tag{2.360}$$

It yields to the contraction of one index pair between the multivector valued forms. The full algebraic bracket $[-,-]^\Delta$ is thus the quantum version of the Poisson bracket!

- In particular for the 1-vectors $\boldsymbol{b}_a \equiv \boldsymbol{X}_a$ and the 1-forms $\boldsymbol{c}^b$ we have $\imath_{\boldsymbol{X}_a} \boldsymbol{c}^b = \delta^b_a$.

$$[\imath_{\boldsymbol{X}_a}, \boldsymbol{c}^b \wedge] = \delta^a_b = [\boldsymbol{X}_a, \boldsymbol{c}^b]^\Delta = \{\boldsymbol{X}_a, \boldsymbol{c}^b\} \tag{2.361}$$

2.3.3 Main Theorem

Theorem 2.2 (main theorem)**.** *(a) if the derivation homology classes $\mathcal{H}_k(\boldsymbol{\delta}) = 0 \quad \forall k \neq 0$ then there exists a differential* $\mathbf{s}$ *that combines* $\boldsymbol{\delta}$ *with* $\mathbf{d}$

$$\mathbf{s} = \overset{(-1)}{\boldsymbol{\delta}} + \overset{(0)}{\mathbf{d}} + \sum_{r\geq 1} \mathbf{s}^{(r)} \tag{2.362}$$

$$\mathbf{s}^2 = 0 \tag{2.363}$$

$$r(\mathbf{s}^{(k)}) = k, \quad \mathrm{gh}(\mathbf{s}^{(k)}) = 1 \quad (\text{form deg} - \textit{multivec deg}) \tag{2.364}$$

(b) any such differential $\mathbf{s}$ *obeys*

$$H^k(\mathbf{s}) = H^k(\mathbf{d}|H_0(\boldsymbol{\delta})) \tag{2.365}$$

$$x \in H^k(\mathbf{d}): \quad \mathbf{d}x = \delta y, \quad x \sim x + \mathbf{d}z + \delta z' \tag{2.366}$$

$$r(x) = 0 = r(z), \quad r(y) = 1 = r(z') \tag{2.367}$$

Proof. i) Existence: Need to show that the equation $0 = \mathbf{s}^2 = \sum_{r\geq -2} (\mathbf{s}^2)^{(r)}$ is solvable for given $\boldsymbol{\delta}, \mathbf{d}$ and $\mathbf{s}^{(1)}$ ($\mathbf{d}^2 = -[\boldsymbol{\delta}, \mathbf{s}^{(1)}]$). We will show this by induction over the resolution degree r (either antighost degree or antifield degree). At the lower degrees we have

$$(\mathbf{s}^2)^{(-2)} = \boldsymbol{\delta}^2 = 0 \quad \checkmark \tag{2.368}$$

$$(\mathbf{s}^2)^{(-1)} = [\boldsymbol{\delta}, \mathbf{d}] = 0 \quad \checkmark \tag{2.369}$$

$$(\mathbf{s}^2)^{(0)} = \mathbf{d}^2 + [\boldsymbol{\delta}, \mathbf{s}^{(1)}] = 0 \quad \checkmark \tag{2.370}$$

At positive degree $r \geq 1$ we have

$$(\mathbf{s}^2)^{(r)} = [\delta, \mathbf{s}^{(r+1)}] + [\mathbf{d}, \mathbf{s}^{(r)}] + \sum_{k=1}^{r-1} \tfrac{1}{2} [\mathbf{s}^{(k)}, \mathbf{s}^{(r-k)}] \overset{!}{=} 0 \tag{2.371}$$

The equation $(\mathbf{s}^2)^{(r-1)} = 0$ involves maximally $\mathbf{s}^{(r)}$. So if we assume for the induction that this degree $r-1$-equation already holds, it does not put any restriction on $\mathbf{s}^{(r+1)}$. We therefore can see the equation at degree r as an equation for $\mathbf{s}^{(r+1)}$, which is solvable if the rest is $\boldsymbol{\delta}$-exact. This is equivalent to showing that the complete $(\mathbf{s}^2)^{(r)}$ is exact. Being at resolution degree $r \geq 1$ with trivial derivative-cohomology classes $\mathcal{H}_r$, it is enough to show that it is $\boldsymbol{\delta}$-closed:

$$[\boldsymbol{\delta}, (\mathbf{s}^2)^{(r)}] = [\mathbf{s}, (\mathbf{s}^2)^{(r)}]^{(r-1)} \tag{2.372}$$

By induction assumption $\mathbf{s}^2$ is zero up to degree $r-1$, i.e. $\mathbf{s}^2 = \frac{1}{2}[\mathbf{s}, \mathbf{s}] = \sum_{k\geq r} (\mathbf{s}^2)^{(r)}$, so that the above equation can be rewritten as

$$[\boldsymbol{\delta}, (\mathbf{s}^2)^{(r)}] = [\mathbf{s}, \underbrace{\mathbf{s}^2}_{\sum_{k\geq r} (\mathbf{s}^2)^{(k)}}]^{(r-1)} = \tag{2.373}$$

$$\overset{\mathrm{Jac}}{=} 0 \quad \checkmark \tag{2.374}$$

This completes the proof of the existence of a solution.

ii) Calculate $H^k(\mathbf{s}) \overset{?}{=} H^k(\mathbf{d}|H_\bullet(\boldsymbol{\delta}))$ (k is the ghost number). In the following $\sim$ means equal up to $\boldsymbol{\delta}$-exact. Take $x \in \mathcal{F}(T^*M) \otimes \mathbb{R}[\boldsymbol{c}^a] \otimes \mathbb{R}[\boldsymbol{b}_a]$ with fixed ghost-number k. It is in general a sum of components of different antighost-number.

$$x = \sum_{r\geq 0} x^{(r)}, \quad \mathrm{gh}(x) = k, \quad r(x^{(r)}) = r \tag{2.375}$$

Define a map

$$\mathcal{F}(T^*M) \otimes \mathbb{R}[\boldsymbol{c}^a] \otimes \mathbb{R}[\boldsymbol{b}_a] \to \mathcal{F}(T^*M) \tag{2.376}$$

$$x \mapsto \pi(x) = x^{(0)} \tag{2.377}$$

From

$$\mathbf{s}x = \overbrace{\mathbf{d}x^{(0)} + \boldsymbol{\delta}x^{(1)}}^{(\mathbf{s}x)^{(0)}} + \text{higher} \tag{2.378}$$

it follows

$$\begin{aligned} \pi \mathbf{s} x &\sim \mathbf{d} x^{(0)} && (2.379) \\ \Rightarrow \pi \mathbf{s} &\sim \mathbf{d} \pi && (2.380) \end{aligned}$$

So π is compatible with the differentials. It thus induces a map between the cohomologies

$$\begin{aligned} \pi : \quad H^k(\mathbf{s}) &\to H^k(\mathbf{d} | H_\bullet(\boldsymbol{\delta})) && (2.381) \\ [x] &\mapsto \pi([x]) \equiv [x^{(0)}] && (2.382) \end{aligned}$$

We want to show that it's an isomorphism:

First π preserves the algebra product

$$\pi(xy) = \pi(x)\pi(y) \tag{2.383}$$

Remains to show that it's surjective and injective.

a) surjective: Take $[x^{(0)}] \in H^k(\mathbf{d}|H_\bullet(\boldsymbol{\delta}))$, i.e.

$$\mathbf{d} x^{(0)} = -\boldsymbol{\delta} x^{(1)}, \quad \boldsymbol{\delta} x^{(0)} = 0 \tag{2.384}$$

We need to show that for any such $x^{(0)}$ there exists an x with $\mathbf{s}x = 0$ and $\pi x = x^0$. So it suffices to find $x^{(r)}$ $(r \geq 1)$ such that $\mathbf{s}\Big(x^{(0)} + \sum_{r\geq 1} x^{(r)}\Big) = 0$:

$$\begin{aligned} 0 &\overset{!}{=} \mathbf{s}\left(x^{(0)} + \sum_{r\geq 1} x^{(r)}\right) = && (2.385) \\ &= \left(\boldsymbol{\delta} + \mathbf{d} + \sum_{\tilde r \geq 1} \mathbf{s}^{(r)}\right)\left(x^{(0)} + \sum_{r\geq 1} x^{(r)}\right) = && (2.386) \\ &= \underbrace{\boldsymbol{\delta} x^{(0)}}_{=0} + \underbrace{(\boldsymbol{\delta} x^{(1)} + \mathbf{d} x^{(0)})}_{=0} + \sum_{r \geq 1}\left(\boldsymbol{\delta} x^{(r+1)} + \mathbf{d} x^{(r)} + \sum_{i=1}^{r} \mathbf{s}^{(i)} x^{(r-i)}\right) && (2.387) \end{aligned}$$

It has to vanish for each resolution degree r (antighost number or antifield number) separately. Assume that we have managed to make all resolution degrees up to $r-1$ vanish, giving equations for $x^{(1)}, \ldots, x^{(r)}$ but leaving undetermined $x^{(r)}, x^{(r+1)}, \ldots$. The resolution degree expansion of $\mathbf{s}x$ thus starts at degree r:

$$\mathbf{s} x = \sum_{\tilde r \geq r} (\mathbf{s} x)^{(r)} \tag{2.388}$$

Nilpotency of the differential $\mathbf{s}$ ($\mathbf{s}^2 x = 0$) then implies that $(\mathbf{s}x)^{(r)}$ is $\boldsymbol{\delta}$-closed:

$$\begin{aligned} 0 &= (\mathbf{s}^2 x)^{(r-1)} = && (2.389) \\ &= \left((\sum_{r' \geq -1} \mathbf{s}^{(r')}) \sum_{\tilde r \geq r} (\mathbf{s} x)^{(r)} \right)^{(r-1)} = && (2.390) \\ &= \boldsymbol{\delta} (\mathbf{s} x)^{(r)} && (2.391) \end{aligned}$$

Triviality of $H_r(\boldsymbol{\delta})$ for nonzero r then implies that $(\mathbf{s}x)^{(r)}$ is not only closed, but also $\boldsymbol{\delta}$-exact. Now the $\mathbf{s}$-invariance equation at degree r reads

$$0 \overset{!}{=} \overbrace{(\mathbf{s} x)^{(r)}}^{\boldsymbol{\delta}-\text{exact}} = \overbrace{\boldsymbol{\delta} x^{(r+1)} + \underbrace{\mathbf{d} x^{(r)} + \sum_{i=1}^{r} \mathbf{s}^{(i)} x^{(r-i)}}_{\Rightarrow \boldsymbol{\delta}-\text{exact}}}^{\boldsymbol{\delta}-\text{exact}} \tag{2.392}$$

and is solvable for $x^{(r+1)}$ because of the $\boldsymbol{\delta}$-exactness of $\mathbf{d}x^{(r)} + \sum_{i=1}^{r} \mathbf{s}^{(i)} x^{(r-i)}$ (if $(\mathbf{s}x)^{(r)}$ is $\boldsymbol{\delta}$-exact, also $(\mathbf{s}x)^{(r)} - \boldsymbol{\delta} x^{(r+1)}$ is $\boldsymbol{\delta}$-exact. This shows that every element in $H^k(\mathbf{d}|H_\bullet(\boldsymbol{\delta}))$ can be written as $\pi[x]$ with $[x] \in H^k(\mathbf{s})$ and completes the proof of surjectivity.

b) For injectivity, we need to show that the constructed x is unique. But as the map π is obviously linear, it is enough to show that the kernel is $[0]$, i.e. that $[0] \in H^\bullet(\mathbf{s})$ is the only element which is mapped to $[0] \in H^\bullet(\mathbf{d}|H_\bullet(\boldsymbol{\delta}))$. So assume $[y] \in H^\bullet(s)$ is mapped to $[0] \in H^\bullet(\mathbf{d}|H_\bullet(\boldsymbol{\delta}))$:

$$
\begin{aligned}
\mathbf{s}y &= 0 && (2.393)\\
\pi y &= y^{(0)} = \mathbf{d}z^{(0)} + \boldsymbol{\delta} z^{(1)}, \quad \boldsymbol{\delta} z^{(0)} = 0 \quad (\pi([y]) = [0]) && (2.394)
\end{aligned}
$$

We need to show that $y = [0]$, i.e. that y is $\mathbf{s}$-exact. Again this will be shown recursively in the resolution degree. So first expand y in the degree:

$$
y = \underbrace{\mathbf{d}z^{(0)} + \boldsymbol{\delta} z^{(1)}}_{\left(\mathbf{s}(z^{(0)}+z^{(1)})\right)^{(0)}} + \sum_{r\geq 1} y^{(r)} \tag{2.395}
$$

At resolution degree 0, we obviously have

$$
y^{(0)} = (\mathbf{s}z)^{(0)} \tag{2.396}
$$

Now define

$$
\begin{aligned}
y' &\equiv y - \mathbf{s}(z^{(0)} + z^{(1)}) = && (2.397)\\
&= \sum_{r\geq 1}\left(y^{(r)} - \left(\mathbf{s}(z^{(0)} + z^{(1)})\right)^{(r)}\right) = && (2.398)\\
&= \left(y^{(1)} - \mathbf{s}^{(1)} z^{(0)} - \mathbf{d}z^{(1)}\right) + \sum_{r\geq 2}\left(y^{(r)} - \mathbf{s}^{(r)} z^{(0)} - \mathbf{s}^{(r-1)} z^{(1)}\right) && (2.399)
\end{aligned}
$$

Certainly y' is $\mathbf{s}$-exact iff y is $\mathbf{s}$-exact. Next $\mathbf{s}$-invariance of y' at lowest resolution degree will give us a condition on $y^{(1)}$ that will allow us to see that $y^{(1)}$ is exact:

$$
\begin{aligned}
0 &= (\mathbf{s}y')^{(0)} = && (2.400)\\
&= \boldsymbol{\delta}\left(y^{(1)} - \mathbf{s}^{(1)} z^{(0)} - \mathbf{d}z^{(1)}\right) && (2.401)
\end{aligned}
$$

This implies (trivial homology) that the bracket is $\boldsymbol{\delta}$-exact, i.e. $\exists z^{(2)}$ such that

$$
\begin{aligned}
y^{(1)} &= \boldsymbol{\delta} z^{(2)} + \mathbf{d}z^{(1)} + \mathbf{s}^{(1)} z^{(0)} = && (2.402)\\
&= \left(\mathbf{s}(z^{(0)} + z^{(1)} + z^{(2)})\right)^{(1)} && (2.403)
\end{aligned}
$$

Now we can define

$$
\begin{aligned}
y'' &\equiv y' - \mathbf{s}(z^{(2)}) = && (2.404)\\
&= \sum_{r\geq 2}\left(y^{(r)} - \left(\mathbf{s}(z^{(0)} + z^{(1)} + z^{(2)})\right)^{(r)}\right) = && (2.405)\\
&= \left(y^{(2)} - \mathbf{s}^{(2)} z^{(0)} - \mathbf{s}^{(1)} z^{(1)} - \mathbf{d}z^{(2)}\right) + \sum_{r\geq 3}\left(y^{(r)} - \mathbf{s}^{(r)} z^{(0)} - \mathbf{s}^{(r-1)} z^{(1)} - \mathbf{s}^{(r-2)} z^{(2)}\right) && (2.406)
\end{aligned}
$$

$\mathbf{s}$-invariance of y'' at lowest resolution degree 1 yields

$$
\begin{aligned}
0 &= (\mathbf{s}y'')^{(1)} && (2.407)\\
&= \boldsymbol{\delta}\left(y^{(2)} - \mathbf{s}^{(2)} z^{(0)} - \mathbf{s}^{(1)} z^{(1)} - \mathbf{d}z^{(2)}\right) && (2.408)
\end{aligned}
$$

which implies (trivial homology) that there is a $z^{(3)}$ with

$$
\begin{aligned}
y^{(2)} &= \boldsymbol{\delta} z^{(3)} + \mathbf{d}z^{(2)} + \mathbf{s}^{(1)} z^{(1)} + \mathbf{s}^{(2)} z^{(0)} = && (2.409)\\
&= \left(\mathbf{s}(z^{(0)} + z^{(1)} + z^{(2)} + z^{(3)})\right)^{(2)} && (2.410)
\end{aligned}
$$

Obviously this can be continued recursively and used to make a complete induction, showing that $y = \mathbf{s}z$ is indeed exact and thus $[y] = [0]$ which in turn shows that the kernel of π is $\{[0]\}$ and therefore π is injective. □

Remarks

- The grading of the $\mathbf{s}$-cohomology is the total ghost-grading. It can either be seen as cohomology, or if one inverts the sign of the grading, as homology. In particular for $\mathbf{d} = 0$ the differential $\mathbf{s}$ coincides with the Koszul-Tate differential $\boldsymbol{\delta}$ and it is then more natural to regard the $\mathbf{s}$-cohomology as a homology. In finite dimensions the grading is in any case bounded from below and above (in the bosonic case), and there is no need for a distinction. In infinite dimensions in general the $\mathbf{d}$-grading is bounded from below and the $\boldsymbol{\delta}$-grading from above, while the $\mathbf{s}$-grading is not bounded. Only when $\mathbf{d} = 0$, it becomes bounded from above.

2.4 BRST formalism classical

If we apply the above homological perturbation theory to the Hamiltonian system, we obtain the so-called BRST (Becchi, Rouet, Stora and Tyutin) differential $\mathbf{s}$, build from the Koszul Tate differential $\boldsymbol{\delta}$ (whose homology restricts phase space functions to the constraint surface) and from the longitudinal exterior derivative $\mathbf{d}_{(L)}$ (whose cohomology restricts to gauge invariant functions).

2.4.1 Mapping the dynamics to extended phase space

- In the proof of the main theorem we have defined the isomorphism

$$\pi: \quad F = F^{(0)} + F^{(1)} + \ldots \mapsto \pi(F) \equiv F^{(0)} \tag{2.411}$$

  between the $\mathbf{d}_{(L)}-$Cohomology (which contains at zero ghost number just the gauge invariant functions on the constraint surface: $0 \overset{!}{\approx} \mathbf{d}_{(L)} F^{(0)} = \boldsymbol{c}^a\{G_a, F^{(0)}\}$) and the $\mathbf{s}$-cohomology (BRST-cohomology) in the extended phase space (which contains BRST-invariant functions on the extended phase space $0 \overset{!}{=} \mathbf{s}F$). The **extended phase space** is parametrized by the variables $(y^M, \boldsymbol{c}^a, \boldsymbol{b}_a)$ where $y^M = (q^m, p_m)$ parametrize the original phase space T^*M which contains also the constraint surface Σ.

- The existence of this isomorphism implies that every gauge invariant function on the constraint surface can be lifted to a $\mathbf{s}$-closed (BRST invariant) function on the extended phase space.

- The isomorphism further respects the bracket structure at ghost number 0. In order to see this, expand two functions F and G on the extended phase space in the resolution degree

$$F(y, \boldsymbol{c}^a, \boldsymbol{b}_a) = \underbrace{F^{(0)}}_{\pi(F)} + \sum_{r\geq 1} \tfrac{1}{r!r!} (\boldsymbol{c}^{\boldsymbol{c}})^r F^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^r \tag{2.412}$$

  Their Poisson bracket reads

$$\begin{aligned}
\{F, G\} &= \left\{ F^{(0)} + \sum_{r\geq 1} \tfrac{1}{r!r!} (\boldsymbol{c}^{\boldsymbol{c}})^r F^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^r, G^{(0)} + \sum_{s\geq 1} \tfrac{1}{s!s!} (\boldsymbol{c}^{\boldsymbol{c}})^s G^{(s)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^s \right\} = &(2.413)\\
&= \{\underbrace{F^{(0)}}_{\pi F}, \underbrace{G^{(0)}}_{\pi G}\} + \sum_{s\geq 1} \tfrac{1}{s!s!} \underbrace{(\boldsymbol{c}^{\boldsymbol{c}})^s \left\{ F^{(0)}, G^{(s)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}} \right\} (\boldsymbol{b}_{\boldsymbol{b}})^s}_{\text{resol deg } s\geq 1} + \sum_{r\geq 1} \tfrac{1}{r!r!} \underbrace{(\boldsymbol{c}^{\boldsymbol{c}})^r \left\{ F^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}, G^{(0)} \right\} (\boldsymbol{b}_{\boldsymbol{b}})^r}_{\text{resol deg } r\geq 1} + \\
&\quad + \sum_{r\geq 1} \sum_{s\geq 1} \tfrac{1}{r!r!} \tfrac{1}{s!s!} \underbrace{\left\{ (\boldsymbol{c}^{\boldsymbol{c}})^r F^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^r, (\boldsymbol{c}^{\boldsymbol{c}})^s G^{(s)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^s \right\}}_{\text{resol deg } \geq r+s-1 \geq 1} &(2.414)
\end{aligned}$$

  Clearly only the first term has resolution degree 0, so that we indeed obtain

$$\pi\{F, G\} = \{\pi F, \pi G\} \tag{2.415}$$

- Remember the consistency conditions $\{H_{tot}, G_a\} \approx 0$ (constraints are conserved in time) which were used to derive secondary constraints. As also previously discussed, the same equation implies that $H_{tot} = H^{(0)} + \lambda^{a_p} G_{a_p}$ (we denote the original H now by $H^{(0)}$ as it has resolution degree 0) is gauge invariant on Σ. Because of $\{G_a, G_b\} \approx 0$ (or simply because $H_{tot} \approx H^{(0)}$), it further implies that the basic **Hamiltonian** $H^{(0)}$ itself is **gauge invariant** on the constraint surface:

$$\{G_a, H^{(0)}\} \approx 0 \tag{2.416}$$

We can therefore use π^{-1} to map $H^{(0)}$ (or equivalently $H_{tot} \approx H^{(0)}$) to its BRST invariant extension H in extended phase space.

$$\mathbf{s}H = 0 \tag{2.417}$$

$$H = \underbrace{H^{(0)}}_{\pi(H)} + H^{(1)} + \dots \tag{2.418}$$

$$\pi H = H^{(0)} \tag{2.419}$$

- In other words we can use the isomorphism π (see proof of the main theorem) to map the whole dynamics of gauge invariant functions on the constraint surface in original phase space to an isomorphic dynamics in the extended phase space:

$$\dot{F} = \{H, F\} \tag{2.420}$$

The BRST cohomology defines the physical space, so the dynamics on $\mathbf{s}$-closed objects is all what matters. Nevertheless it is convenient to extend the dynamics to non-closed functions as well, by simply the same equation of motion as above. This even assigns a dynamics to the ghosts

$$\dot{\boldsymbol{c}}^a = \{H, \boldsymbol{c}^a\}, \quad \boldsymbol{b}_a = \{H, \boldsymbol{b}_a\} \tag{2.421}$$

- The extension H (in extended phase space) is unique as a BRST-equivalence class $[H]$. So in BRST cohomology also time evolution is unique and there is no gauge invariance. On the other hand, in the underlying algebra of functions on the extended phase space, H is not unique, but can be modified by arbitrary BRST-exact terms

$$H \to H + \mathbf{s}\boldsymbol{K} \tag{2.422}$$

For any particular choice of $\boldsymbol{K}$ (choice of a representative of $[H]$) we obtain a unique time evolution without gauge invariance. Therefore $\boldsymbol{K}$ is called the **gauge fixing fermion**. ("Fermion" because it has to be Grassman-odd although it is a scalar). One could recover the gauge ambiguity of the total Hamiltonian by introducing a family of gauge fixing fermions $\boldsymbol{K}_\lambda \equiv \lambda^a \boldsymbol{b}_a$ depending on the Lagrange multipliers λ^a.

$$\mathbf{s}\boldsymbol{K}_\lambda = \mathbf{s}(\lambda^a \boldsymbol{b}_a) = \lambda^a (G_a^{(0)} + \dots) \tag{2.423}$$

Choosing one representative then corresponds to fixing the λ^a's to a certain value, e.g. 0. This would lead also in the original phase space to a unique (gauge fixed) time evolution.

- According to [Henneaux, p.240] there is no geometric interpretation of the ghost dynamics, but I doubt this. Maybe one should change from active to passive transformations or the other way round. I.e., taking a function $F(y)$ whose coordinates y change in time can also be seen as a function which changes in time itself, i.e. $F'(y) \equiv F(y')$. The same for multivector valued forms given as functions on the extended phase space. Then one can think of the time evolution as a time evolution of the section of multivector valued forms.

2.4.2 BRST differential as a canonical transformation

- Claim: the BRST-differential is a canonical transformation, i.e. can be generated by a **BRST-charge** of the form

$$\boldsymbol{Q} = \boldsymbol{c}^{\boldsymbol{c}} \underbrace{G_{\boldsymbol{c}}}_{f_{\boldsymbol{c}}^{(0)}} - \tfrac{1}{2}\boldsymbol{c}^{\boldsymbol{c}}\boldsymbol{c}^{\boldsymbol{c}} \underbrace{f_{\boldsymbol{cc}}{}^{\boldsymbol{b}}}_{\equiv f_{\boldsymbol{cc}}^{(1)\,\boldsymbol{b}}} \boldsymbol{b}_{\boldsymbol{b}} + \sum_{r\geq 2} \tfrac{(-)^r}{r!(r+1)!} (\boldsymbol{c}^{\boldsymbol{c}})^{r+1} f^{(r)}_{\boldsymbol{c}\dots\boldsymbol{c}}{}^{\boldsymbol{b}\dots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^r \equiv \tag{2.424}$$

$$= \sum_{r\geq 0} \tfrac{(-)^r}{r!(r+1)!} (\boldsymbol{c}^{\boldsymbol{c}})^{r+1} f^{(r)}_{\boldsymbol{c}\dots\boldsymbol{c}}{}^{\boldsymbol{b}\dots\boldsymbol{b}} (\boldsymbol{b}_{\boldsymbol{b}})^r \tag{2.425}$$

The $f^{(r)}_{\boldsymbol{c}\dots\boldsymbol{c}}{}^{\boldsymbol{b}\dots\boldsymbol{b}}$ are called the **higher order structure functions**. A set of constraints is said to have **rank** r if the $f^{(n)}_{\boldsymbol{c}\dots\boldsymbol{c}}{}^{\boldsymbol{b}\dots\boldsymbol{b}}$ vanish for all $n > r$. Examples

  - rank 0: abelian $\{G_a, G_b\} = 0$,

$$\boldsymbol{Q} = \boldsymbol{c}^a G_a \tag{2.426}$$

  - rank 1: group $\{G_a, G_b\} = f_{ab}{}^c G_c$ (with constant $f_{ab}{}^c$!)

$$\boldsymbol{Q} = \boldsymbol{c}^{\boldsymbol{c}} G_{\boldsymbol{c}} - \tfrac{1}{2}\boldsymbol{c}^{\boldsymbol{c}}\boldsymbol{c}^{\boldsymbol{c}} f_{\boldsymbol{cc}}{}^{\boldsymbol{b}}\boldsymbol{b}_{\boldsymbol{b}} \tag{2.427}$$

– The rank is not an intrinsic classification of the constraint surface, as it can change upon choosing different G_a for the same surface.

- In order to prove the claim, it is enough (according to the main theorem) to observe that it generates $\boldsymbol{\delta}+\mathbf{d}_{(L)}$ in the lowest orders and that it can be made square to zero:[12]

$$
\begin{aligned}
0 &\stackrel{!}{=} \{\boldsymbol{Q},\boldsymbol{Q}\} = &(2.428)\\
&= \sum_{r,s\geq 0}(-)^{r+s}\tfrac{1}{r!(r+1)!}\tfrac{1}{s!(s+1)!}\left\{(\boldsymbol{c}^{\boldsymbol{c}})^{r+1}f^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}(\boldsymbol{b}_{\boldsymbol{b}})^r,(\boldsymbol{c}^{\tilde{\boldsymbol{c}}})^{s+1}f^{(s)}_{\tilde{\boldsymbol{c}}\ldots\tilde{\boldsymbol{c}}}{}^{\tilde{\boldsymbol{b}}\ldots\tilde{\boldsymbol{b}}}(\boldsymbol{b}_{\tilde{\boldsymbol{b}}})^s\right\} = &(2.429)\\
&= 2\sum_{r\geq 1,s\geq 0}(-)^{r+s}\tfrac{1}{(r-1)!(r+1)!}\tfrac{1}{s!s!}(\boldsymbol{c}^{\boldsymbol{c}})^{r+1}f^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}a}(\boldsymbol{b}_{\boldsymbol{b}})^{r-1}\left\{\boldsymbol{b}_a,\boldsymbol{c}^d\right\}(\boldsymbol{c}^{\tilde{\boldsymbol{c}}})^s f^{(s)}_{d\tilde{\boldsymbol{c}}\ldots\tilde{\boldsymbol{c}}}{}^{\tilde{\boldsymbol{b}}\ldots\tilde{\boldsymbol{b}}}(\boldsymbol{b}_{\tilde{\boldsymbol{b}}})^s + &\\
&\quad+\sum_{r,s\geq 0}(-)^{r+s}\tfrac{1}{r!(r+1)!}\tfrac{1}{s!(s+1)!}(\boldsymbol{c}^{\boldsymbol{c}})^{r+1}(\boldsymbol{b}_{\boldsymbol{b}})^r\left\{f^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}},f^{(s)}_{\tilde{\boldsymbol{c}}\ldots\tilde{\boldsymbol{c}}}{}^{\tilde{\boldsymbol{b}}\ldots\tilde{\boldsymbol{b}}}\right\}(\boldsymbol{c}^{\tilde{\boldsymbol{c}}})^{s+1}(\boldsymbol{b}_{\tilde{\boldsymbol{b}}})^s = &(2.430)\\
&= 2\sum_{r\geq 1,s\geq 0}(-)^{r+s+(r-1)s}\tfrac{1}{(r-1)!(r+1)!}\tfrac{1}{s!s!}f^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}a}f^{(s)}_{a\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}(\boldsymbol{c}^{\boldsymbol{c}})^{r+s+1}(\boldsymbol{b}_{\boldsymbol{b}})^{r+s-1} + &\\
&\quad+\sum_{r,s\geq 0}(-)^{r+s+r(s+1)}\tfrac{1}{r!(r+1)!}\tfrac{1}{s!(s+1)!}\left\{f^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}},f^{(s)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}\right\}(\boldsymbol{c}^{\boldsymbol{c}})^{r+s+2}(\boldsymbol{b}_{\boldsymbol{b}})^{r+s} &(2.431)
\end{aligned}
$$

Finally we reparametrize the sum, keeping s as a summation variable, but replacing r by the power of $\boldsymbol{b}_{\boldsymbol{b}}$'s, so in the first sum by $R=r+s-1$ ($R\geq 0, 0\leq s\leq R$) and in the second sum by $R=r+s$ ($R\geq 0, 0\leq s\leq R$) and obtain as condition for nilpotency of $\boldsymbol{Q}$

$$
\begin{aligned}
0 \stackrel{!}{=} &\sum_{R\geq 0}\sum_{s=0}^{R}(-)^{Rs}\tfrac{1}{(R-s)!(R-s+2)!}\tfrac{1}{s!(s+1)!}\times\\
&\times\left(-(-)^{R-s}2(s+1)f^{(R-s+1)\boldsymbol{b}\ldots\boldsymbol{b}a}_{\boldsymbol{c}\ldots\boldsymbol{c}}f^{(s)}_{a\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}+(R-s+2)\left\{f^{(R-s)\boldsymbol{b}\ldots\boldsymbol{b}}_{\boldsymbol{c}\ldots\boldsymbol{c}},f^{(s)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}\right\}\right)(\boldsymbol{c}^{\boldsymbol{c}})^{R+2}(\boldsymbol{b}_{\boldsymbol{b}})^R \quad (2.432)
\end{aligned}
$$

Having sorted now by resolution degree, we can read off the equations for each degree R seperately:

$$
\begin{aligned}
0 \stackrel{!}{=} &\sum_{s=0}^{R}(-)^{Rs}\tfrac{1}{(R-s)!(R-s+2)!}\tfrac{1}{s!(s+1)!}\times\\
&\times\left(-(-)^{R-s}2(s+1)f^{(R-s+1)\boldsymbol{b}\ldots\boldsymbol{b}a}_{\boldsymbol{c}\ldots\boldsymbol{c}}f^{(s)}_{a\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}+(R-s+2)\left\{f^{(R-s)\boldsymbol{b}\ldots\boldsymbol{b}}_{\boldsymbol{c}\ldots\boldsymbol{c}},f^{(s)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}\right\}\right)\ \forall R \quad (2.433)
\end{aligned}
$$

For the lowest degrees, this reads

$$
\begin{aligned}
R=0: \quad 0 &\stackrel{!}{=} \tfrac{1}{2}\left(-2f^{(1)a}_{\boldsymbol{cc}}G^{(0)}_a+2\left\{G^{(0)}_{\boldsymbol{c}},G^{(0)}_{\boldsymbol{c}}\right\}\right) \quad \checkmark &(2.434)\\
R=1: \quad 0 &\stackrel{!}{=} \tfrac{1}{3!}\left(2f^{(2)\boldsymbol{b}a}_{\boldsymbol{ccc}}G^{(0)}_a+3\left\{f^{(1)\boldsymbol{b}}_{\boldsymbol{cc}},G^{(0)}_{\boldsymbol{c}}\right\}\right)-\tfrac{1}{2}\tfrac{1}{2}\left(-2\cdot 2f^{(1)a}_{\boldsymbol{cc}}f^{(1)\boldsymbol{b}}_{a\boldsymbol{c}}+2\left\{G^{(0)}_{\boldsymbol{c}},f^{(1)\boldsymbol{b}}_{\boldsymbol{cc}}\right\}\right)= &(2.435)\\
&= \frac{1}{3}f^{(2)\boldsymbol{b}a}_{\boldsymbol{ccc}}G^{(0)}_a+f^{(1)a}_{\boldsymbol{cc}}f^{(1)\boldsymbol{b}}_{a\boldsymbol{c}}-\left\{G^{(0)}_{\boldsymbol{c}},f^{(1)\boldsymbol{b}}_{\boldsymbol{cc}}\right\} \quad \checkmark(\text{Jacobi}) &(2.436)\\
R=2: \quad 0 &\stackrel{!}{=} \tfrac{1}{2!4!}\left(-2\underbrace{f^{(3)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}a}G^{(0)}_a}_{\propto\boldsymbol{\delta}f^{(3)}}+4\left\{f^{(2)\boldsymbol{bb}}_{\boldsymbol{ccc}},G^{(0)}_{\boldsymbol{c}}\right\}\right)+\tfrac{1}{3!}\tfrac{1}{2!}\left(4f^{(2)\boldsymbol{b}a}_{\boldsymbol{ccc}}f^{(1)\boldsymbol{b}}_{a\boldsymbol{c}}+3\left\{f^{(1)\boldsymbol{b}}_{\boldsymbol{cc}},f^{(1)\boldsymbol{b}}_{\boldsymbol{cc}}\right\}\right)+ &\\
&\quad+\tfrac{1}{2!}\tfrac{1}{2!3!}\left(-6f^{(1)a}_{\boldsymbol{cc}}f^{(2)\boldsymbol{bb}}_{a\boldsymbol{cc}}+2\left\{G^{(0)}_{\boldsymbol{c}},f^{(2)\boldsymbol{bb}}_{\boldsymbol{ccc}}\right\}\right) &(2.437)
\end{aligned}
$$

Equations are always of the form $\boldsymbol{\delta}f^{(n)}+(\ldots)=0$ where the $(\ldots)$ need to be shown to be $\boldsymbol{\delta}$-closed which is sufficient to make them exact. So works like previous proofs.

[12]Note that $\boldsymbol{Q}$ is a formal sum of multivector valued forms and that $\{\boldsymbol{Q},\boldsymbol{Q}\}=0$ corresponds to the vanishing of some algebraic and some differential brackets between them. In topological sigmal models this can be used to implement integrability conditions of a Poisson structure or a complex structure or similar in target space via a BRST charge. $\diamond$

- The complete BRST transformations read

$$\mathbf{s}\boldsymbol{c}^c = \underbrace{-\tfrac{1}{2}f^c{}_{ab}\boldsymbol{c}^a\boldsymbol{c}^b}_{\mathbf{d}_{(L)}\boldsymbol{c}^c} + \sum_{r\geq 1}\tfrac{(-)^{r+1}}{r!(r+2)!}(\boldsymbol{c}^{\boldsymbol{c}})^{r+2}f^{(r+1)\boldsymbol{b}\ldots\boldsymbol{b}c}_{\boldsymbol{c}\ldots\boldsymbol{c}}(\boldsymbol{b}_{\boldsymbol{b}})^r \tag{2.438}$$

$$\mathbf{s}\boldsymbol{b}_a = \underbrace{G_a}_{\boldsymbol{\delta b}_a}\underbrace{-\tfrac{1}{2}\boldsymbol{c}^{\boldsymbol{c}}f_{a\boldsymbol{c}}{}^{\boldsymbol{b}}\boldsymbol{b}_{\boldsymbol{b}}}_{\mathbf{d}_{(L)}\boldsymbol{b}_a} + \sum_{r\geq 2}\tfrac{(-)^r}{r!r!}(\boldsymbol{c}^{\boldsymbol{c}})^r f^{(r)}_{a\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}(\boldsymbol{b}_{\boldsymbol{b}})^r \tag{2.439}$$

$$\mathbf{s}y^M = \underbrace{\boldsymbol{c}^a X_a^M}_{\mathbf{d}_{(L)}y^M} - \tfrac{1}{2}\boldsymbol{b}_c\boldsymbol{c}^a\boldsymbol{c}^b\{f^c{}_{ab}, y^M\} + \sum_{r\geq 2}\tfrac{(-)^r}{r!(r+1)!}(\boldsymbol{c}^{\boldsymbol{c}})^{r+1}(\boldsymbol{b}_{\boldsymbol{b}})^r\left\{f^{(r)}_{\boldsymbol{c}\ldots\boldsymbol{c}}{}^{\boldsymbol{b}\ldots\boldsymbol{b}}, y^M\right\} \tag{2.440}$$

- Being a canonical transformation makes the BRST differential (and thus its cohomology) compatible with the Poisson bracket:

$$\mathbf{s}\{F,G\} = \{\mathbf{s}F,G\} + (-)^F\{F,\mathbf{s}G\} \tag{2.441}$$

This means that if F and G are BRST-closed, also $\{F,G\}$ is closed. Furthermore, if $F = \mathbf{s}\boldsymbol{K}$ is BRST exact and G is closed, the result is also exact, as it should be

$$\{\mathbf{s}K,G\} = (-)^K\mathbf{s}\{K,G\} \quad \text{for } \mathbf{s}G = 0 \tag{2.442}$$

2.4.3 BRST differential as a symmetry-transformation

- The gauge fixed action

$$S_{\boldsymbol{K}}[y^M,\boldsymbol{c}^a,\boldsymbol{b}_a] = \int dt(\underbrace{\dot{q}^m p_m}_{\dot{y}^M A_M(y)} + \dot{\boldsymbol{c}}^a\boldsymbol{b}_a - H - \{\boldsymbol{Q},\boldsymbol{K}\}) \tag{2.443}$$

is invariant under the global BRST symmetry

$$\mathbf{s}S_K = 0 \quad \forall\boldsymbol{K} \tag{2.444}$$

Similarly as for a conventional gauge fixing (like just putting $\lambda^a = 0$ in the original first-order Lagrangian), the constraints $G_a = 0$ are not obtained from the equations of motion any longer (although they now hold on the cohomological level).

- If we had started from the beginning with a first order action $S[q,p,\lambda] = \int \dot{q}p - H_0 - \lambda^a G_a$, then (apart form the second class constraints $\pi_p = 0, \pi_q = p$) we would obtain first class constraints $\pi_a \equiv \pi_{\lambda^a} = 0$ on the momenta conjugate to the Lagrange multipliers. They can be dealt with the introduction of corresponding ghost fields $\boldsymbol{\rho}^a$ and conjugate momenta $\bar{\boldsymbol{c}}_a$ together with the BRST transformations

$$\mathbf{s}\bar{\boldsymbol{c}}_a = \pi_a, \quad \mathbf{s}\pi_a = 0 \tag{2.445}$$

$$\mathbf{s}\lambda^a = \boldsymbol{\rho}^a, \quad \mathbf{s}\boldsymbol{\rho}^a = 0 \tag{2.446}$$

They form a so-called **topological quartet** which drops out of the cohomology completely, because each variable is either exact or not invariant. The auxiliary variables $(\lambda^a, \pi_a, \boldsymbol{\rho}^a, \bar{\boldsymbol{c}}_a)$ are known as **non-minimal** sector (one could add more...). So the non-minimal contribution to the BRST charge is

$$\boldsymbol{Q} \rightarrow \boldsymbol{Q}_{min} + \boldsymbol{Q}_{nm} \tag{2.447}$$

$$\boldsymbol{Q}_{nm} = \boldsymbol{\rho}^a\lambda_a \tag{2.448}$$

The gauge fixed action extends to

$$S_{\boldsymbol{K}}[y^M,\boldsymbol{c}^a,\boldsymbol{b}_a,\lambda^a,\pi_a,\boldsymbol{\rho}^a,\bar{\boldsymbol{c}}_a] = \int dt(\underbrace{\dot{q}^m p_m}_{\dot{y}^M A_M(y)} + \dot{\boldsymbol{c}}^a\boldsymbol{b}_a + \dot{\lambda}^a\pi_a + \dot{\boldsymbol{\rho}}^a\bar{\boldsymbol{c}}_a - H - \{\boldsymbol{Q},\boldsymbol{K}\}) \tag{2.449}$$

A very common type of gauge fixing fermion is now

$$\boldsymbol{K} = \chi^a(y)\bar{\boldsymbol{c}}_a + \lambda^a\boldsymbol{b}_a + \overset{\text{some constant}}{\tfrac{\alpha}{2}}\bar{\boldsymbol{c}}_a\underbrace{g^{ab}}_{\text{some } metric}\pi_b \tag{2.450}$$

Let us first ignore the last term (i.e. take $\alpha = 0$). The BRST transformation of the second term will reintroduces the term $\lambda^a G_a$ (which we had in the original first order action) and the first term will effectively induce a derivative gauge for the gauge degrees of freedom λ^a of the form $\dot\lambda^a = \chi^a$:

$$\begin{aligned} S_{\boldsymbol{K}}[y^M, \boldsymbol{c}^a, \boldsymbol{b}_a, \lambda^a, \pi_a, \boldsymbol{\rho}^a, \bar{\boldsymbol{c}}_a] &= \int dt \Big(\underbrace{\dot q^m p_m}_{\dot y^M A_M(y)} + \dot{\boldsymbol{c}}^a \boldsymbol{b}_a + \dot\lambda^a \pi_a + \dot{\boldsymbol{\rho}}^a \bar{\boldsymbol{c}}_a - H + \\ &\quad - \left(\chi^a(y)\pi_a + \lambda^a(G_a + \boldsymbol{c}^b f_{ab}{}^c \boldsymbol{b}_c + \ldots) + \mathbf{s}\chi^a(y)\cdot \bar{\boldsymbol{c}}_a + \boldsymbol{\rho}^a \boldsymbol{b}_a + \tfrac{\alpha}{2}\pi_a g^{ab}\pi_b\right)\Big) \end{aligned} \tag{2.451}$$

- Only if $\alpha \neq 0$, one can use the equations of motion for π_a

$$\frac{\delta S_{\boldsymbol{K}}}{\delta \pi_a} = \dot\lambda^a - \alpha \pi^a \tag{2.452}$$

  to integrate out π_a (replace it in the action by the solution $\pi_a = \frac{1}{\alpha}\dot\lambda^a$). Always if $\boldsymbol{K}$ is such that one can eleminate π_a, it is called a **propagating gauge.**

- If the higher terms ... in $(G_a + \boldsymbol{c}^b f_{ab}{}^c \boldsymbol{b}_c + \ldots)$ are not present (happens when $f_{ab}{}^c$ is constant), the situation simplifies, because one can then integrate out $\boldsymbol{b}_a$ and $\boldsymbol{\rho}^a$ in pairs

$$\frac{\delta S}{\delta \boldsymbol{b}_c} = -\dot{\boldsymbol{c}}^c + \boldsymbol{c}^b f_{ab}{}^c + \boldsymbol{\rho}^c \tag{2.453}$$

$$\frac{\delta S}{\delta \boldsymbol{\rho}^a} = -\dot{\bar{\boldsymbol{c}}}_a - \boldsymbol{b}_a \tag{2.454}$$

  Can eliminate $\boldsymbol{b}_c = -\dot{\bar{\boldsymbol{c}}}_c$ and $\boldsymbol{\rho}^c = \dot{\boldsymbol{c}}^c - \boldsymbol{c}^b f_{ab}{}^c$

$$\begin{aligned} S_{\boldsymbol{K}}[y^M, \boldsymbol{c}^a, \lambda^a, \pi_a, \bar{\boldsymbol{c}}_a] &= \int dt \Big(\underbrace{\dot q^m p_m}_{\dot y^M A_M(y)} - H - \lambda^a \underbrace{\left(G_a - (\boldsymbol{c}^b f_{ab}{}^c \dot{\bar{\boldsymbol{c}}}_c)\right)}_{\mathrm{BRST-inv\ ext\ of\ } G_a} + \\ &\quad + \underbrace{(\ddot{\boldsymbol{c}}^a - \mathbf{s}\chi^a(y))\,\bar{\boldsymbol{c}}_a + \left(\dot\lambda^a - \chi^a(y)\right)\pi_a}_{\mathbf{s}\left(\left(\dot\lambda^a - \chi^a(y)\right)\bar{\boldsymbol{c}}_a\right)} - \tfrac{\alpha}{2}\pi_a g^{ab}\pi_b \Big) \end{aligned} \tag{2.455}$$

  where we used

$$\mathbf{s}\bar{\boldsymbol{c}}_a = \pi_a, \quad \mathbf{s}\pi_a = 0 \tag{2.456}$$

$$\mathbf{s}\lambda^c = \dot{\boldsymbol{c}}^c - \boldsymbol{c}^b f_{ab}{}^c \tag{2.457}$$

$$\mathbf{s}\dot\lambda^c = \ddot{\boldsymbol{c}}^c - \dot{\boldsymbol{c}}^b f_{ab}{}^c \tag{2.458}$$

  The above gauge fixing make contact to the original Faddev-Popov ghosts, where $\boldsymbol{c}^a$ were the ghosts and $\bar{\boldsymbol{c}}_a$ were the antighosts.

- Remark on the Bosonic string: One BRST-exact term is added to the Lagrangian, which at the same time fixes the gauge and introduces a ghost-kinetic term:

$$\mathbf{s}((g^{zz} - \hat g^{zz})\bar{\boldsymbol{c}}_{zz}) = \underbrace{(g^{zz} - \hat g^{zz})\pi_{zz}}_{\mathcal{L}_{\mathrm{gauge-fix}}} + \underbrace{\bar\partial \boldsymbol{c}^z \bar{\boldsymbol{c}}_{zz}}_{\mathcal{L}_{gh}} \tag{2.459}$$

2.4.4 Some more comments

- If the structure "constants" $f_{ab}{}^c$ in the constraint algebra $\{G_a, G_b\} = f_{ab}{}^c G_c$ are not constant, then the corresponding symmetry transformations close only on the constraint surface:

$$[\delta_a, \delta_b] = 2\{G_{[a}, \{G_{b]}, -\}\} = \tag{2.460}$$

$$\stackrel{\mathrm{Jacobi}}{=} \{\{G_{[a}, G_{b]}\}, -\} = \tag{2.461}$$

$$= \{f_{ab}{}^c G_c, -\} = \tag{2.462}$$

$$= f_{ab}{}^c \delta_c + \underbrace{G_c\{f_{ab}{}^c, -\}}_{\approx 0} \tag{2.463}$$

  The last term vanishes in general only on the constraint surface. The algebra is then said to be **off-shell non-closed** or open. Together with the previously mentioned fact that every on-shell vanishing gauge symmetry is a trivial gauge symmetry, this last term has to be a trivial transformation.

- For a closed algebra one can according to [Henneaux, p.47] complete every weekly gauge invariant function to a strictly gauge invariant function: $[G_a, F] \approx 0 \Rightarrow \exists F' \approx F \quad \text{with } [G_a, F] = 0$
- The generator j_0 of a global symmetry is a first class function, i.e. commutes with all constraints. (solutions of eom's are mapped to solutions). j_0 can thus be (via the isomorphism π) extended to be BRST-invariant and then generate a symmetry that commutes with BRST
- The higher ghost number cohomologies $H^1(\mathbf{s}), H^2(\mathbf{s})$ are related to anomalies.
- The BRST generator $\boldsymbol{Q}$ is BRST exact, because it has ghost number 1:

$$\{J_{gh}, \boldsymbol{Q}\} = \boldsymbol{Q} \tag{2.464}$$

$$\text{or } \mathbf{s}J_{gh} = -\boldsymbol{Q} \tag{2.465}$$

- The BRST charge is unique up to canonical transformation in extended phase space $\boldsymbol{Q} \to \boldsymbol{Q} + \{X, \boldsymbol{Q}\}$.