TECHNISCHE UNIVERSITÄT WIEN
VIENNA UNIVERSITY OF TECHNOLOGY

Dissertation: Superstrings in General Backgrounds

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines
Doktors der technischen Wissenschaften

unter der Leitung von

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Wien, am 16. August 2007

The present copy of the thesis is an improved version. Some sections have been added, in particular to the first part about superspace conventions. They contain mostly considerations from my PhD time that were not yet written in a nice form at the submission date. Apart from that, the presentation has been smoothed at some points and the supergravity transformation of the gravitino is now reformulated in a way that it is comparable to the literature. Also the appearance of the dilaton has been clarified and a few additional constraints are extracted from the Bianchi identities. The original version of the thesis can be found in the on-line dissertation database of the TU Wien which is currently – and hopefully in future as well – placed at http://www.ub.tuwien.ac.at/diss/AC05035309.pdf.
In spite of the changes, the character has remained that of the original thesis and the text refers to the situation at the submission date. Let me therefore acknowledge at this place the hospitality of George Savvidy and the Institute of Nuclear Physics at the Demokritos research institute in Athens, where part of the improvements were implemented.
Note finally that the address in Vienna given above is of course out of date and was left untouched simply for sentimental reasons. Athens, **July 31, 2008**

A mistake in the argument for calculating the (correct) nilpotency constraints (sections 5.9 and 5.10) has been corrected and the related appendix section E.4 has been added in this new version (arXiv:0807.4968v2). The presentation of the SUSY transformation for the dilatino has been slightly improved. Page numbers may have changed with respect to the previous arXiv version, but equation and section numbers remained the same. Possible future corrections or comments will only be added as errata or addendum. Thanks to the people of the theory department at Turin university, in particular M. A. Rajabpour, for useful discussions. Torino, **March 11, 2009**

Kurzfassung der Dissertation

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Zweitprüfer:	Prof. Ruben Minasian

Superstrings in General Backgrounds

In der vorliegenden Arbeit werden einige Aspekte des Superstrings im allgemeinen Hintergrund betrachtet. Die Arbeit besteht im Wesentlichen aus drei Teilen: Der erste studiert die Vorraussetzungen, unter denen man bosonische Strukturgleichungen in graduierte (z.B. im Superraum) übertragen kann und formuliert diese in einem Satz. Auf diesen Betrachtungen basierend werden Konventionen verwendet, die graduierungsabhängige Vorzeichen absorbieren und die als Grundlage der Rechnungen des zweiten Teils dienen.

Der zweite Teil beschreibt den Typ II Superstring mithilfe von Berkovits' "pure spinor" Formalismus. Die darin u.a. enthaltene Einbettung in einen Target-Superraum ermöglicht im Gegensatz zum üblichen Ramond-Neveu-Schwarz Formalismus eine direkte Kopplung des Strings an Ramond-Ramod-Felder. Er eignet sich damit gut für ein Studium des Superstrings in allgemeinen Hintergründen. In der Arbeit wird die Herleitung der "Supergravity Constraints" aus der klassischen BRST-Invarianz sorgfältig rekapituliert. Die Herangehensweise unterscheidet sich dabei in einigen Punkten von der ursprünglichen Herleitung von Berkovits und Howe. So bleibt die Betrachtung im Unterschied zu deren Rechnung vollständig im Lagrange Formalismus und zur besseren Strukturierung der Variationsrechung wird ein kovariantes Variationsprinzip eingesetzt. Hinzu kommt die Anwendung des im ersten Teil formulierten Satzes. Auch die Reihenfolge, in der die Constraints erzielt werden, weicht von Berkovits und Howe ab. Als neues Resultat werden die BRST Transformationen aller Weltflächen-Felder hergeleitet, die bisher nur für den heterotischen Fall bekannt waren. Ein entscheidender weiterer Schritt ist schließlich die Herleitung der lokalen Supersymmetrie-Transformation der fermionischen Targetraum-Komponenten-Felder.

Dies liefert den Übergang zur sogenannten verallgemeinerten komplexen Geometrie (GCG), die Bestandteil des letzten Teiles der Arbeit ist. Die vierdimensionale effektive Supersymmetrie innerhalb einer zehndimensionalen Typ-II Supergravitation bedingt eine "verallgemeinerte Calabi Yau Mannigfaltigkeit" als Kompaktifizierungsraum, welche wiederum mit Methoden der GCG beschrieben werden kann. In der vorliegenden Arbeit wird gezeigt, dass Poisson- oder Antiklammern in Sigmamodellen auf natürliche Weise sogenannte "derived brackets" im Targetraum induzieren, darunter auch die Courant Klammer der GCG. Weiters wird gezeigt, dass der verallgemeinerte Nijenhuis Tensor der GCG bis auf einen de-Rham geschlossenen Term mit der "derived bracket" der verallgemeinerten Struktur mit sich selbst übereinstimmt, und eine neuartige Koordinatenform dieses Tensors wird präsentiert. Der Nutzen der gewonnenen Erkenntnisse wird dann anhand von zwei Anwendungen zur Integrabilität verallgemeinerter komplexer Strukturen demonstriert.

Der Anhang der Arbeit enthält eine Einführung in einige Aspekte von GCG und "derived brackets". Desweiteren werden u.a. das Noether Theorem, Bianchi Identitäten, WZ-Eichung und Γ-Matrizen in zehn Dimensionen besprochen.

Abstract

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First referee / supervisor:	Prof. Maximilian Kreuzer
Second referee:	Prof. Ruben Minasian

Superstrings in General Backgrounds

In the present thesis, some aspects of superstrings in general backgrounds are studied. The thesis divides into three parts. The first is devoted to a careful study of very convenient superspace conventions which are a basic tool for the second part. We will formulate a theorem that gives a clear statement about when the signs of a superspace calculation can be omitted. The second part describes the type II superstring using Berkovits' pure spinor formalism. Being effectively an embedding into superspace, target space supersymmetry is manifest in the formulation and coupling to general backgrounds (including Ramond-Ramond fields) is treatable. We will present a detailed derivation of the supergravity constraints as it was given already by Berkovits and Howe some years ago. The derivation will at several points differ from the original one and will use new techniques like a covariant variation principle. In addition, we will stay throughout in the Lagrangian formalism in contrast to Berkovits and Howe. Also the order in which we obtain the constraints and at some points the logic will differ. As a new result we present the explicit form of the BRST transformation of the worldsheet fields, which was before given only for the heterotic case[1]. Having obtained all the constraints, we go one step further and derive the form of local supersymmetry transformations of the fermionic fields. This provides a contact point of the Berkovits string in general background to those supergravity calculations which derive generalized Calabi Yau conditions from effective four-dimensional supersymmetry. The mathematical background for this setting is the so-called generalized complex geometry (GCG) which is in turn the motivation for the last part.

The third and last part is based on the author's paper on derived brackets from sigma models which was motivated by the study of GCG. It is shown in there, how derived brackets naturally arise in sigma-models via Poisson- or antibrackets, generalizing an observation by Alekseev and Strobl. On the way to a precise formulation of this relation, an explicit coordinate expression for the derived bracket is obtained. The generalized Nijenhuis tensor of generalized complex geometry is shown to coincide up to a de-Rham closed term with the derived bracket of the structure with itself and a new coordinate expression for this tensor is presented. The insight is applied to two-dimensional sigma models in a background with generalized complex structure.

The appendix contains introductions to geometric brackets and to aspects of generalized complex geometry. It further contains detailed reviews on aspects of Noether's theorem, on the Bianchi identities (including Dragon's theorem), on supergauge transformations and the WZ gauge and on important relations for Γ-matrices (especially in ten dimensions). A further appendix is devoted to the determination of the (super)connection starting from different torsion- or invariance constraints.

[1] These transformations were presented already in the original version of August 16, 2007. In the meantime another paper [1] independently presented BRST transformations for the type IIA string, although in a very different setting, based on free differential algebras. Note also another interesting paper on the pure spinor string in general background [2] which has appeared in the meantime and takes into account recent developments in Berkovits' formalism. ◇

Contents

Some remarks in advance

- The part about the superspace conventions is interesting in itself and was a significant part of my research work. This is why it was not put into the appendix. However, you can read the other parts without this one. Only if you want to follow some calculations in detail, you might miss some signs. Latest at this point you should study the part about the superspace conventions before you assume that you have found a mistake.
- Capital indices M in the part about derived brackets and generalized geometry contain tangent and cotangent indices, while in the context of superspace they contain bosonic and fermionic indices. In the latter case we have $M = \{m, \boldsymbol{\mu}, \hat{\boldsymbol{\mu}}\}$. The two fermionic indices are sometimes collected in a capital curly index $\boldsymbol{\mathcal{M}} = \{\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}\}$.
- The thesis-index at the end contains also a list of most of the used symbols. So in case you start somewhere in the middle of the document and would like to know, where some symbols or notations were introduced, have a try to look at the index.
- There are a couple of propositions contained in this thesis. They simply contain more or less clear statements that one could have given in the continuous text as well. In particular, their formulations and proofs are mostly not of the same rigorousness as one would expect it in mathematical literature. In addition, there is no clear rule which statements are given as proposition and which are only given in the text. The ones in propositions are important, but the ones in the text can also be ...
- Everything in this thesis has to be understood as graded. Graded antisymmetrization will just be called 'antisymmetrization' and the square brackets $[\ldots]$ will be used to denote this, no matter if the graded antisymmetrized objects are bosonic or fermionic. Likewise, the supervielbein will often just be called 'vielbein'. Only at some points the terms 'graded' or 'super' will be explicitly used.
- It is a somewhat strange habit to desperately avoid the word "I" in articles, in order to express ones own modesty. Writing instead "the author" seems unnecessary long and writing instead "we" resembles the *pluralis majestatis*, and I don't see how this can possibly express modesty (although one then calls it *pluralis auctoris* or even *pluralis modestiae*). In spite of this, I got used myself to use frequently (and without thinking) the word "we". Understanding it as *pluralis modestiae* is probably only possible if one can replace "we" with "the reader and myself", for example in "we will see in the following ...". However, you, the reader, would probably loudly protest when I write things like "we think ..." or "we have no idea why..." and claim that the reader is included. Nevertheless, I am afraid that sentences like this will appear quite frequently and in order to avoid inconsistencies, they have to be understood as the *pluralis majestatis ...*
- The symbol $\diamond$ marks the end of a footnote. If this mark is missing, it means that the footnote is continued on the next page or that I simply forgot to put it . (This remark was simply copied from my diploma thesis, but at least I have changed the footnote symbol and the language)
- This document was created with LyX which is based on LaTeX.

Acknowledgements

First of all I would like to thank my supervisor Maximilian Kreuzer for all what he taught me, for his patience and for the chocolate that he sometimes distributes in the coffee room. His keen mind and challenging courses certainly had a big impact. Nevertheless I enjoyed the freedom to work at whatever I was interested in. Only sometimes I was told, not to waste my time on conventions (although inspired by him) but to do some physics instead ;-)

There were a lot of people accompanying my last few years at the Institute for Theoretical Physics of the TU Vienna. In the beginning mainly Emanuel Scheidegger, Ulrich Theis and Erwin Riegler had to suffer under my numerous questions. With Erwin I shared the interest in many recreational activities which were important to gather new energy for our minds. I enjoyed this time very much.

Next I would like to thank Gernot Pauschenwein (who unfortunately decided to change the group) and especially Johanna Knapp for collaboration in the early phase of this work. Johanna is a truly pleasant colleague and friend and I missed her collaboration when she moved on to CERN.

I had the pleasure to share an office in Vienna with Robert Wimmer, Andreas Ipp, Andreas Gerhold, Paul Romatschke, Robert Schöfbeck, Urko Reinosa and Christoph Mayrhofer. With all of them I had many discussions and share several experiences. Andreas Ipp helped me with the algebraic computer program Mathematica and wrote "decision programs" for me (climbing or working?), Andreas Gerhold for me was the shining example concerning "efficiency". I will never make it to this level. Paul accompanied me to unfortunately only one ski-tour and with the two Roberts I had numerous discussions on physics. Robert W. was a frequent fellow at very-late-night-work and Christoph equipped me with French songs and moral support when I left to Paris. There is a special relationship to good old Urko, as he is probably still grateful for the construction site which I had organized for him as an "apartment". ;-) Instead of being angry, he, and also Diego Arbo and Maria-Jose Fernaud from "the other side" of the tenth floor, truely enriched my stay in Vienna.

With Christoph, accompanied by Michal Michalcik, Rashid Ahmad, Maria Schimpf, Viktor Atalla and finally Nils-Ole Walliser, the new generation in our string group took over. Again all of them are more than just colleagues and it is a pity to leave this group. In addition, Stefan Stricker, Max Attems and recently arrived Aleksi Vuorinen in the group of Anton Rebhan build an enriching counterpart with much overlap. I would like to thank Max for help with the computer, Rashid and Michal for checking some of my calculations and Viktor for being one of the few people appreciating my conventions.

With Radoslav Rashkov, I have by now a new pleasant office mate, and I am happy to have access to his inexhaustible knowledge. I also should not forget to mention Karl-Georg Schlesinger, who never hesitates to share his deep mathematical insight. There were quite a lot of other people at the ITP who made my stay enjoyable. I am grateful to Herbert Balasin and Anton Rebhan for entertaining coffee-breaks, discussions and lunch breaks and to Elfriede Mössmer and Franz Hochfellner for helping with real-world-problems.

During my PhD I also spent (in 2006) almost a year at the "Service de Physique Théorique" of CEA in Saclay/Paris as a guest of the string group consisting at that time of Mariana Graña, Ruben Minasian, Pierre Vanhove, Michael Chesterman, Kazuo Hazomichi and Yann Michel, who warmly integrated me and took their time for all my questions on pure spinors and generalized geometry. I am truly grateful to Ruben who never hesitated to accept organizational efforts, in order to enable that truly pleasant stay for me. He was rewarded by becoming my second referee which implied yet additional work for him. Similarly, Pierre offered any help and also did not hesitate to share his ideas and thoughts. In particular I learned some gamma-matrix gymnastics from him. Numerous other people deserve to be acknowledged, like all the (not always) string-related PhD-students from Paris, who regularly met for a very interesting and pedagogical seminar. In particular I want to thank my former office-mate Marc Thormeier and also Michele Frigerio for their cordiality during my stay.

I also want to thank my long-time physics colleagues Marco Baumgartl, Josef Dorfmeister, Steffen Metzger and Marcus Müller for keeping contact and sharing experiences. Special thanks to Steffen for offering a place to stay in Paris even in the final phase of his PhD thesis. I am also grateful to Carlos Mafra for repeated encouragement concerning my research work, even the one about conventions. A person who played an important role in the early stage of my scientific career is Dierk Schleicher, whom I want to thank for his believe in my abilities and his enthusiasm about mathematics and physics. He enabled the stay in Stony Brook during my diploma studies and was a reliable mentor during that time. There are many other people who contributed at some point, like Claus Jeschek, Frederik Witt, Stefan Antusch, ...

I also appreciate very much Peter van Nieuwenhuizen's help to understand his and his collaborator's approach to the covariant quantization at the early stage of my PhD studies. It is a pity that the work on this subject came to an (intermediate?) end, just after the exchange of some insight had started.

Am Ende möchte ich meinen Eltern für ihren wertvollen Rückhalt und ihre stete große Unterstützung in jeder Hinsicht danken. Ebenso meinem Bruder, der sich in manch schwierigen Phasen wie kaum ein anderer in mich hineinversetzen konnte. Auch aus der weiteren Verwandtschaft kam viel Unterstützung und Zuspruch. Besonders bedanken möchte ich mich in diesem Zusammenhang bei Brigitta Abele-Zöllner und Peter Abele. Auch alle meine persönlichen Freunde, die sich regelmäßig nach dem Fortgang meiner Arbeit erkundigt haben, und hier namentlich unerwähnt bleiben, werden sich hoffentlich angesprochen fühlen. Zu guter Letzt möchte ich Dir, Katharina, für all Deine Unterstützung, Geduld und Verzicht in der Endphase dieser Arbeit danken.

Introduction

This thesis is devoted to superstrings in general backgrounds, but it will of course restrict to only some aspects, leaving out many important areas.

Apart from a few other simple cases, the quantized superstring is well understood only in a flat background where the worldsheet fields have basically free-field equations of motion. The physical spectrum of a string in flat background, however, contains itself fluctuations around this background. A huge number of strings therefore can sum up to a non-vanishing mean background field, for example a curved metric or even Ramond-Ramond bispinor-fields. The worldsheet dynamics for the individual strings then has to be adjusted. In other words, it is very natural to study the superstring in the most general background. Consistency conditions from the worldsheet point of view implement constraints and/or equations of motion on the background fields. On the worldsheet level, the form of the consistency conditions depends very much on the formalism one is using to describe the superstring. In general, the gauge symmetries or alternatively BRST symmetries of the action in flat background should be present in some form also for the deformed action (string in general background), especially after quantization. For the Ramond-Neveu-Schwarz (RNS) string, with worldsheet fermions, this boils down to the quantum Weyl invariance of the action, which also yields the critical dimension. For the Green Schwarz (GS) string and for the Berkovits pure spinor string (to be explained later), there are instead additional conditions. For the Green Schwarz string, the so called κ gauge symmetry has to be preserved, while for the Berkovits pure spinor string one has to guarantee the existence of a BRST operator which has the form $Q = \oint dz\, \boldsymbol{\lambda}^{\boldsymbol{\alpha}} d_{z\boldsymbol{\alpha}}$ in the flat case. In fact, in the latter two cases, the BRST symmetry and the κ-symmetry are already strong enough to implement the background field equations of motion at lowest order in α', i.e. supergravity, such that quantum Weyl invariance does not give additional constraints at this order.

There are of course backgrounds which are more interesting than others for phenomenological reasons. First of all, as we are observing four spacetime dimensions, we expect to live in a solution to the background field equations where 6 of the 10 dimensions are compactified on a small radius, such that they are effectively not visible. This compactification has to be compatible with the supergravity equations, but without restrictive boundary conditions there are infinitely many possibilities. For a long time, people were hoping that there is a dynamical mechanism, preferring precisely the compactification (or 'vacuum') that corresponds to our world. By now it seems more and more likely that there is no such mechanism or at least not such a strong one. Instead, the picture might be that we are simply sitting in a huge 'landscape' of possible vacua, where some of them are more probable than others. As there is such a huge number of effective four dimensional theories, it seems improbable that 'our world' is not contained in them. Of course, being able to derive the real world from string theory is a necessary requirement, if this theory is supposed to be more than just interesting mathematics. By now there exists a huge model building machinery. People are considering orbi- and orientifolds and are putting intersecting D-branes into the compactification manifold. The number of possibilities is huge. Quite a lot of models come reasonably close to the standard model, but none of them really matches. But even if there might be a lot of justified criticism to string theory, this particular problem of finding the real world is rather a matter of time. So far, only a very tiny, mathematically treatable subset of solutions has been studied and it would have been a lucky coincidence to find a suitable vacuum in a simple setting. The bigger problem might show up only after finding a vacuum which effectively reproduces the standard model: there might be a still big number of different models which likewise reproduce the standard model. Without knowing all of them and their common properties, one cannot really make predictions about so far unknown physics. This is, however, not an argument against string theory. If there is another theory, unrelated to string theory, which also describes correctly the standard model and gravity, then this model simply has to be added to the set of all models which describe the so far observable physics consistently. There is no reason to throw out the ones that might have been obtained from string theory. Any approach that can consistently describe the so far observable physics is of course admissible.

It is not the immediate aim of this thesis, however, to describe observable physics, but to study the string in a general background in ten dimensions. As argued above, one can be optimistic that someone will find real physics within string theory. But sometimes it is easier to recognize simplifying structures in the general setting and not in some particular cases. Moreover, considerations like this should survive changes in the communities opinion of what is an interesting model to look at. This was the idea, but in the end, not everything in this thesis is as general as it should be. First of all, mainly classical closed strings in a type II background are considered. At some places we keep boundary terms for later studies of open strings. Secondly a whole part of the thesis is inspired by generalized complex geometry. This in turn is related to a not very special but still special type of compactifications. Let us recall this in the following lines:

Again for phenomenological reasons, in particular the hierarchy problem, it is reasonable to expect that the four dimensional effective theory resulting from compactification is $N = 1$ supersymmetric. For that reason, Candelas, Horowitz, Strominger and Witten introduced in 1985 [3] Calabi Yau manifolds into string theory. These manifolds are Ricci flat and obey therefore the Einstein field equations in vacuum. The supersymmetry constraint then corresponds to the existence of a covariantly conserved (w.r.t. Levi Civita) $Spin(6)$-spinor. Soon after, Strominger realized in [4] that a background B-field, in combination with a non-constant dilaton, is also consistent with supersymmetric compactification. Nevertheless, there has been very little activity on this more general case while the Calabi-Yau case was intensively studied. This intensive study lead to invaluable insights concerning dualities and the form of the landscape in the Calabi-Yau case.

Only quite recently the importance of the general case including fluxes was properly noticed. It was realized that the Calabi-Yau condition gets replaced by a "generalized Calabi-Yau" condition, which brings the so-called generalized complex geometry into the game. See the introduction to part III on page 117 for the relevant references. The derivation of this is mainly based on supergravity calculations. Starting from ten dimensional type II supergravity one demands effective $N = 1$ supersymmetry in four dimensions after compactification [5, 6]. The results could in general be modified by string corrections. In order to study this, one has to set up the problem in the worldsheet language. In other words, the superstring has to be placed into a general type II background.

The first striking fact is that there is so far no treatable way to couple the RNS string to Ramond-Ramond fields. Ramond-Ramond fields can be either seen as bispinors (fields with two spinorial indices) or equivalently (expanding in Γ-matrices) as a collection of differential p-forms. Pullbacks of p-forms with p bigger than two vanish on the worldsheet. Likewise we do not have elementary fields with spacetime spinor indices in the RNS description. This is in short the reason why coupling to the RR-fields is an open issue in the RNS formalism. The natural alternative is the GS string which is basically an embedding of the string into a target superspace. The fermionic superspace coordinates or their momenta provide natural candidates for the coupling to the RR-bispinor-fields. This formalism, however, happens to have a fermionic gauge symmetry whose constraints are infinitely reducible and would require an infinite tower of ghosts for ghosts in the standard BRST covariant quantization procedure. It can be quantized in flat space in the light cone gauge and shown to be equivalent to RNS, but higher loop calculations are difficult because of the lack of manifest covariance.

The problem of covariant quantization of the GS superstring was bothering people for many painful years without real progress until Berkovits came up in 2000 with an alternative formalism [7], based on commuting pure spinor ghost variables, which can be covariantly quantized in the flat background. It is similar to the GS string in that the target space is a supermanifold, but the origin of the pure spinor ghost is still a bit mysterious. This ghost field and the corresponding BRST operator are related to the κ-symmetry of the GS string, but the relation is not very transparent. In addition, the pure spinor condition is a quadratic constraint on the spinorial ghosts, which seemed in the beginning not very attractive. For this reason there were several attempts to get rid of this constraint or at least to explain its occurrence. The beginning of my PhD research was devoted to a promising approach by Grassi, Porrati, Policastro and van Nieuwenhuizen[8, 9, 10, 11] and I will give a few remarks about this at a later point. By now the need for an alternative formalism has decreased, as Berkovits managed to give a consistent multiloop picture in [12]. In any case the pure spinor formalism seems to provide the adequate tool to study the superstring in curved background. On the classical level this has already been done in [13]. It was shown that classical BRST invariance of the pure spinor string in general background already implies the supergravity constraints on the background fields.

One major subject of the thesis is to rederive this important result with different techniques. All steps will be carefully motivated and the calculations given in detail. Most importantly the calculation given in this thesis can be seen as an independent check, as it is done entirely in the Lagrangian formalism in contrast to [13]. Moreover, a covariant variational principle will be established and used to calculate the worldsheet equations of motion. Some results are obtained in a different order but match in the end. One new result is the explicit form for the BRST transformations of the worldsheet fields of the type II string in general background, which were so far only presented for the heterotic string in [14]. After the derivation of the constraints, we go one step further and derive the supergravity transformations of the fermionic fields. The transformations are in principle well known, but the idea is to obtain them in the parametrization of the fields in which they enter the pure spinor string. The supersymmetry transformations of the fermionic fields are the starting point for the derivation of the generalized complex Calabi-Yau conditions for supersymmetric compactifications. Having a closed logical line from the pure spinor string to generalized geometry hopefully opens the door for the study of quantum or string corrections to this geometry. There is still a part missing in this line from the Berkovits string to generalized complex geometry, as we will end with the presentation of the supergravity transformations and not proceed with the derivation of the generalized Calabi-Yau conditions. Again, this calculation would not deliver new results (following [5, 6]), but it would be important to have everything in the same setting and with the same conventions. One might expect in addition that the superspace formulation will give additional insight to the geometrical role of the RR-fields. They are so far only spectators in generalized geometry. A bispinor is from the superspace point of view just a part of a rank two tensor, and it seems natural to include it into geometry by establishing some version of generalized supergeometry. See also in the conclusions for other possible extensions.

Another new feature of the re-derivation of the supergravity constraints from the pure spinor string is the rigorous (and in some sense very unusual) application of some powerful superspace conventions. To be more precise, we are going to use conventions where all the signs which depend on the grading are absorbed via the use of a graded summation convention and a graded equal sign. This a not a completely new idea and northwest-southeast conventions (NW) or northeast-southwest conventions (NE) already reflect this philosophy. Nevertheless most of the authors still write the signs and take the rules of NW and NE only as a check. Only in [15], I have found an example where the signs were likewise absorbed. However, a careful study, under which circumstances this is possible seemed to be missing. This is the subject of part I on page 6. This part is more than just the declaration of the used conventions. The upshot is the formulation of a theorem about when the grading dependent signs may be dropped. The application to supermatrices shows that the underlying ideas lead to slightly different definitions of e.g. supertraces or some matrix operations. Using these definitions, all equations take exactly the form they have for bosonic matrices. In particular the equation for the superdeterminant reduces to an equation which holds in the very same form for purely bosonic matrices.

Applying this philosophy to the Berkovits string calculation has some strange effects. Most importantly, the commuting pure spinor ghosts are treated as anticommuting objects. And likewise confusing, the chiral blocks $\gamma^c_{\boldsymbol{\alpha\beta}}$ of the 10-dimensional Γ-matrices are treated as antisymmetric objects although they are in fact symmetric. This nevertheless makes perfect sense and the confusion is not, because the conventions themselves are confusing, but because of the difference to what one is used to. It is therefore a very nice confirmation of the consistency of the conventions that the quite lengthy calculation with the pure spinor string in general background went through and led to the same results as the original calculation. No single grading dependent sign had to be used. The part about the superspace conventions – although very interesting in itself – is not needed to understand the basic steps and ideas of the other parts. Finally it should be mentioned that the appendix about Γ-matrices in ten dimensions is written in ordinary conventions for 'historical reasons'. It is, however, simple to translate the equations to the other convention where needed.

There is finally part III on page 117 of the thesis, which is dealing basically with so called derived brackets and how they arise in sigma models. This part is based on my paper [16]. The efforts to understand some aspects of the integrability of generalized complex structures have led to the observation that super Poisson brackets and super anti-brackets of worldsheet-supersymmetric or topological sigma models induce quite naturally derived brackets in the target space. A more detailed introduction and motivation for this part is given at its beginning.

The structure of the thesis is as follows: We start in part I on page 6 with the discussion of the superspace conventions. In part II on page 39 we will consider Berkovits pure spinor string. After a short motivation for the formalism – coming from the Green Schwarz string – the derivation of the supergravity constraints will be given and the supergravity transformations of the fermionic fields will be derived. In part III on page 117 the appearance of derived brackets in sigma models and the relation to integrability of generalized complex structures is discussed. All parts contain their own small introduction. After the Conclusions on page 143 there are a number of more or less useful appendices. It starts with notations and conventions in appendix A on page 145. This appendix does of course not contain the superspace conventions which are treated in part I. Note also that there is an index at the end of the thesis (page 233) which should contain most of the used symbols. Appendices B on page 148 and C on page 159 give introductions to some aspects of generalized complex geometry and derived brackets, respectively. Appendix D on page 167 summarizes some important facts and equations for Γ-matrices with an emphasis on the ten-dimensional case. In particular the explicit representation is given and the Fierz identities for the chiral submatrices are derived. Appendix E on page 181 presents the Lagrangian version of the Noether theorem and the Noether identities. Additional statements which are important for our BRST invariance calculations of the pure spinor string are likewise given. Appendix F on page 189 recalls the general definitions of torsion, curvature and H-field (valid as well in superspace) . It likewise recalls the derivation of the Bianchi identities and gives the proof for a slightly modified version of Dragon's theorem [15] about the relation of second and first Bianchi identities. Appendix G on page 199 contains a general discussion on how the connection is determined by invariance conditions and certain constraints on torsion components. The simplest example is of course the Levi Civita connection which is given by invariance of the metric and vanishing torsion. In ten dimensional superspace there is no canonically given superspace metric. In this appendix it will be discussed how the connection is reconstructed from more general constraints, like a given non-metricity or preserved structure constants. In addition the Levi Civita connection will be extracted from a given general superspace connection. And finally, in appendix H on page 206, the Wess Zumino gauge will be reviewed in a general setting. This gauge is useful and natural to eliminate auxiliary gauge degrees of freedom. By fixing part of the superdiffeomorphism invariance, one recovers ordinary diffeomorphism invariance and local supersymmetry. This will be used in part II on page 39 to determine the supergravity transformations of the fermionic background fields of the pure spinor string.

Part I

Convenient Superspace Conventions

Chapter 1

The general idea and setting

Most bosonic definitions or equations have a natural generalization to superspace. There are, however, always sign ambiguities in the super-extensions of the definitions. For this reason, bosonic structural equations only hold up to signs in the superspace or graded case. The information that they hold up to signs is already a useful qualitative statement, but it can be very cumbersome to determine the correct signs. Rules like northwest-southeast or northeast-southwest were introduced to fix the sign ambiguities. These rules in principle allow to reconstruct the grading dependent signs from the structure of the equation. It is then a natural step to drop all the signs during the calculations and reintroduce them only at the very end. Or in other words, simply take over the results from a bosonic calculation and decorate it with the appropriate signs. But as usual, there exist some subtle cases in which a strict application of the sign rules compromises some other philosophy or is simply not possible. For this reason a large majority of people working in that field prefer to carry along all the signs and leave them away only in intermediate steps where it is obvious that no problems will occur. A paper by Dragon [15] is the only example I know, where the parity-dependent signs are left away completely. Nevertheless a precise formulation of the conditions under which this is possible still seems to be missing. Statements like "everything works basically the same in the fermionic case, but one has to be careful with the signs" are used frequently in talks. This is the reason, why we want to find out the precise form of the above conditions. In addition, this idea can probably be applied to many more situations than it was done so far. In this first part of the thesis, we try to fill part of this gap.

1.1 Leading principle, graded Einstein summation convention

The leading principle of our conventions is that every abstract calculation looks formally exactly the same as in the bosonic case. All modifications (signs etc) which are due to the fact that there are anticommuting variables involved should be assigned only in the very end, to the result of a purely bosonic calculation.

The conventions will be based on either northwest-southeast (NW for short) or northeast-southwest (NE for short) conventions, which we will explain a bit below. The NW convention is used for example in some standard references as [17, 18] while in B. DeWitt's book on supermanifolds [19] the NE convention is used (although this is not immediately obvious, due to his notation with some indices on the left). It is important, however, that we will in the end have a formalism which looks exactly the same for NW and NE.

Our considerations will mainly treat objects with indices, for example - but not necessarily - coordinates or tensor components. We assume that there is an associative product among the objects being distributive over a likewise present abelian group structure (the sum). Sometimes we have even several of such products (tensor product or wedge product, product of components, ...), which all will be treated in the same way. The described setting simply forms a general associative algebra. But let us start with the motivating example.

Let x^M be the coordinates in a local patch of a supermanifold. Assume that the first components are bosonic and the following are fermionic (anticommuting).

$$x^M \equiv (x^m, x^{\boldsymbol{\mathcal{M}}}) \equiv (x^m, \boldsymbol{\theta}^{\mathcal{M}}) \tag{1.1}$$

The somewhat unusual choice of a curley capital letter for the fermionic indices will be convenient for part II on page 39. There we have two different spinorial indices that we combine in the capital curled one: $x^{\boldsymbol{\mathcal{M}}} \equiv (x^{\boldsymbol{\mu}}, x^{\hat{\boldsymbol{\mu}}})$. As usual, we assign a grading to the indices according to the split into bosonic and fermionic variables.

$$\mid x^M \mid \equiv \mid M \mid \equiv \begin{cases} 0 \text{ for } M = m \\ 1 \text{ for } M = \boldsymbol{\mathcal{M}} \end{cases} \tag{1.2}$$

For grading-dependent signs we use the shorthand notation

$$(-)^M \equiv (-1)^{|M|} \tag{1.3}$$
$$(-)^{K(M+N)} \equiv (-1)^{|K|(|M|+|N|)} \tag{1.4}$$

A general object of interest is an object with r_u upper and r_l lower indices (e.g. a rank (r_u, r_l)-tensor, but our conventions should also extend to non-tensorial objects like connection-coefficients). The overall grading of such an object is

$$| T^{M_1 \ldots M_u}{}_{N_1 \ldots N_l} | \quad \equiv \quad | T | + | M_1 | + \ldots + | M_u | + | N_1 | + \ldots + | N_l | \tag{1.5}$$

where a nonvanishing grading $| T |$ of the "body" of the object (let us call it the **rumpf**, in order not to mix it up with the body of a supernumber) makes sense when there are *ghosts* involved, i.e. objects, with the same index-structure as the coordinates, but opposite grading.

$$| \boldsymbol{c}^M | = | \boldsymbol{c} | + | M | \stackrel{\boldsymbol{c} \text{ is a ghost}}{=} 1 + | M | = \begin{cases} 1 \text{ for } M = m \\ 0 \text{ for } M = \mu \end{cases} \tag{1.6}$$

Also for differential forms we will have in general a grading that differs from their index-grading. E.g. for the cotangent basis elements, we will assign the grading $| \mathbf{d}x^M | = | \mathbf{d} | + | M | = 1 + | M |$.

Superspace coordinates x^M, the element $\mathbf{d}x^M$ of the exterior algebra and the classical ghost field $\boldsymbol{c}^M$ are examples of graded commuting objects which are the main motivation for the following discussion. Let us therefore give the definition:

$$a, b \text{ are } \textbf{graded commuting} \; :\Longleftrightarrow \; ab = (-)^{ab} ba \tag{1.7}$$

For objects where part of the grading is assigned to the indices, this simply becomes

$$a^M, b^N \text{ are } \textbf{graded commuting} \; :\Longleftrightarrow \; a^M b^N = (-)^{(a+M)(b+N)} b^N a^M \tag{1.8}$$

Before we come to our conventions, let us quickly remind the existing ones which already have the basic idea inherent. The generalization of definitions from the commuting (bosonic) case to the graded commuting case is not unique. A very simple example is the interior product which has in local coordinates the form $\imath_v \omega = \sum_m v^m \omega_m = \sum_m \omega_m v^m$. If one wants to extend this definition to vectors and forms that have graded components as well, the order makes a difference. In the **northwest-southeast convention** (**NW** for short) the extension is chosen in such a way that there is no additional sign if the contraction of the indices is from the upper left (northwest) to the lower right (southeast), i.e. $\imath_v \omega \equiv \sum_M v^M \omega_M = \sum_M (-)^M \omega_M v^M$. Within the **northeast-southwest convention** (**NE** for short) instead, there is no sign when contracting from the lower left to the upper right: $\imath_v \omega \equiv \sum_M \omega_M v^M = \sum_M (-)^M v^M \omega_M$.

It is also possible and sometimes very convenient to use a **mixed convention** with different summation conventions for different index subsets. One could for example define $\imath_v \omega \equiv \sum_m \left(v^m \omega_m + v^{\boldsymbol{\mu}} \omega_{\boldsymbol{\mu}} + (-)^{\hat{\boldsymbol{\mu}}} v^{\hat{\boldsymbol{\mu}}} \omega_{\hat{\boldsymbol{\mu}}} \right)$. We will come back to this below.

The above definitions are 'definitions by examples'. There will be additional examples in what follows. In any case, the philosophy of NW and NE is that for every new definition, possible ambiguities are fixed by the contraction directions. This should give a unique way of generalizing bosonic equations and already implies the possibility that one can calculate in a purely bosonic manner and reconstruct the signs at the very end, at least under certain conditions.

In our convention, we will completely omit those signs which are encoded in the structure of the terms. NW, NE or mixed conventions then formally look the same, and there is no reason to decide a priori for one of them. During the derivation and motivation we will always give the signs for NW and only in important cases for NE.

One of the main ingredients of our conventions will be what we call the **graded Einstein summation convention**: repeated indices in opposite positions (upper-lower) are summed over their complete range, taking into account additional signs corresponding to either NW, NE or mixed conventions.

$$a^M b_M \quad \equiv \quad \begin{cases} \sum_M (-)^{bM} a^M b_M \text{ for NW} \\ \sum_M (-)^{bM+M} a^M b_M \text{ for NE} \end{cases} \qquad b_M a^M \equiv \begin{cases} \sum_M (-)^{aM+M} b_M a^M \text{ for NW} \\ \sum_M (-)^{aM} b_M a^M \text{ for NE} \end{cases} \tag{1.9}$$

The factor $(-)^M$ appears always in the "wrong" contraction direction (i.e. in a NE contraction in NW conventions and vice verse). The factors $(-)^{aM}$ and $(-)^{bM}$ bring the contracted indices next to each other. This definition of the graded summation convention guarantees (in both cases, NW and NE) the following important properties:

- All signs which depend on the grading of the dummy-indices, disappear in the equation for graded commutativity. If a^M and b_M are graded commuting objects with $a^M b_N = (-)^{(a+M)(b+N)} b_N a^M$ then the definition (1.9) simply implies for their contraction

$$a^M b_M = (-)^{ab} b_M a^M \tag{1.10}$$

- In an associative algebra it is important that the definition of the graded sum is compatible with associativity. Taking a third algebra element c (which may or may not have an index) and multiplying from left, we have

$$c(a^M b_M) \quad = \quad (ca)^M b_M \tag{1.11}$$

This is kind of trivial, because the grading of the first rumpf-symbol in the sum in (1.9) does not enter the definition. The other way round, however, we learn that the above property forces the definition of the graded sum to avoid the grading of the first element.

In fact one can see the above properties as the defining properties of the graded summation convention. We could have made a more general ansatz with a sign depending on the rumpfs a, b, the index M and the contraction direction $\searrow$ or $\nearrow$:

$$a^M b_M \equiv \sum_M (-)^{\phi(a,b,M,\searrow)} a^M b_M, \quad b_M a^M \equiv \sum_M (-)^{\phi(b,a,M,\nearrow)} b_M a^M \tag{1.12}$$

Demanding the associativity property (1.11) implies that $\phi(a,b,M,\searrow) = \phi(b,M,\searrow)$, $\phi(b,a,M,\nearrow) = \phi(a,M,\nearrow)$. The graded commutativity property (1.10) then puts an additional restriction

$$(-)^{\phi(b,M,\searrow)+bM+M} = (-)^{\phi(a,M,\nearrow)+aM} \tag{1.13}$$

This fixes the a and b dependency of $(-)^\phi$ completely, namely $(-)^{\phi(b,M,\searrow)} = (-)^{\phi_0(M,\searrow)+bM}$ and $(-)^{\phi(a,M,\nearrow)} = (-)^{\phi_0(M,\nearrow)+aM}$. In addition we have $(-)^{\phi_0(M,\searrow)} = (-)^M(-)^{\phi_0(M,\nearrow)}$ with some ϕ_0. The most general definition of the graded summation convention which has the above properties (1.10) and (1.11) therefore reads[1]

$$a^M b_M \equiv \sum_M (-)^{bM}(-)^{\phi_0(M)} a^M b_M, \quad b_M a^M \equiv \sum_M (-)^{M+aM}(-)^{\phi_0(M)} b_M a^M \tag{1.14}$$

For $\phi_0(M) = 0$, we arrive at NW-conventions, while for $\phi_0(M) = \mid M \mid$ we are in NE. In general the function $\phi_0(M)$ may depend arbitrarily on the index M. A natural condition is of course that for M being a bosonic index, the summation should reduce to the ordinary one, so that we require $\phi_0(M) = 0$ for $\mid M \mid = 0$. For the fermionic indices, we could in principle define the sign differently for every single index. In superspace applications, however, the result would then in general not be Lorentz invariant and therefore not very useful. But as mentioned already with the introductory example of the interior product, it is consistent e.g. in extended superspace to switch the sign between different subsets, each corresponding to a representation of the Lorentz group. A mixed convention is also useful in phase space considerations, where we combine configuration space coordinates x^M and momenta p_M to Darboux coordinates $z^{\underline{M}} \equiv (x^M, p_M)$. The definition of the graded summation convention for the combined indices $\underline{M}$ will then change by $(-)^M$ when the index range goes from the coordinate index to the momentum index.

By now we have defined in (1.9) or (1.14) only an index contraction between two graded commuting objects. The first generalization is to allow $a^M$and b_M to be not necessarily graded commuting. The definitions (1.9) or (1.14) make still sense and (1.11) is still fulfilled, if a^M and b_M are elements of an associative algebra. There is no good argument to modify the definition in this more general case. Finally, we go one step further and assume that b in $a^M b_M$ is not necessarily an algebra element, but simply a placeholder for either indices or rumpfs which can carry gradings. Likewise a will also be allowed to contain indices in addition to one or more rumpfs. I.e., we could replace b by an index $b \to {}_N$, to get a definition for $a^M{}_{NM}$. We could even remove b completely $b \to \{\}$ to obtain $a^M{}_M$, or replace both by s.th. more complicated: $a \to A_{KL}$, $b \to {}^{PQ}B_R$ yields the definition for $A_{KL}{}^M{}_{PQ}\,B_{RM}$. This allows to define almost all possible contractions. Unfortunately, we are in this way restricted to expressions which end with the dummy index M. To close this gap we can introduce a third placeholder and define $a^M b_M c \equiv \sum_M (-)^{bM}(-)^{\phi_0(M)} a^M b_M c$ and $b_M a^M c \equiv \sum_M (-)^{M+aM}(-)^{\phi_0(M)} b_M a^M c$. Similar to a, c is just a spectator and does not enter the signs in the sums. We should now check that with this general definition the graded sum is well defined, in particular when two index pairs are contracted.

- The graded summation for more than one index pair is **well-defined** in the sense that the contraction-operations commute.

In order to verify this statement, let a, b, c, d and e be placeholders in the above sense. In the following two examples of index contractions over M and N we will first start with the M-contraction followed by the N-

[1] Some people prefer to have not one single $\mathbb{Z}_2$-grading which governs the signs in a graded commutative algebra, but to have several distinct $\mathbb{Z}_2$-gradings. For example one can distinguish between the $\mathbb{Z}_2$ grading $\mid \ldots \mid_d$ of differential forms (even and odd) and the fermion grading $\mid \ldots \mid_f$ (fermion or boson). The graded summation convention can then be extended to $a^M b_M \equiv \sum_M (-)^{|b|_d |M|_d}(-)^{|b|_f |M|_f}(-)^{\phi_d(M)}(-)^{\phi_f(M)} a^M b_M$. One could even introduce a seperate grading for ghost fields $\mid \ldots \mid_g$. Although the present discussion uses only a single $\mathbb{Z}_2$ grading, basically everything works the same for distinct gradings. As the summation convention swallows all the grading dependent signs anyway, one can even decide only at the end, which picture one prefers. $\diamond$

contraction and then reverse the order. The simple case is when one contraction encloses the other:

$$\begin{aligned} a^M b_N c^N d_M &\overset{\text{first } M}{=} \sum_M (-)^{\phi_0(M)} (-)^{M(b+N+c+N+d)} a^M b_N c^N d_M = && (1.15)\\ &= \sum_{M,N} (-)^{\phi_0(M)+\phi_0(N)} (-)^{M(b+N+c+N+d)} (-)^{Nc+N} a^M b_N c^N d_M && (1.16)\\ &\overset{\text{first } N}{=} \sum_M (-)^{\phi_0(N)} (-)^{cN+N} a^M b_N c^N d_M = && (1.17)\\ &= \sum_{M,N} (-)^{\phi_0(N)+\phi_0(M)} (-)^{Nc+N} (-)^{M(b+c+d)} a^M b_N c^N d_M && (1.18) \end{aligned}$$

There is certainly no problem with the above case. But also for the case where the contractions intersect, everything goes fine if indices which are already contracted are not taken into account in the second contraction:

$$\begin{aligned} a_N b^M c^N d_M &\overset{\text{first } M}{=} \sum_M (-)^{\phi_0(M)} (-)^{M(c+N+d)} a_N b^M c^N d_M = && (1.19)\\ &= \sum_{M,N} (-)^{\phi_0(M)+\phi_0(N)} (-)^{M(c+N+d)} (-)^{N(b+c)+N} a_N b^M c^N d_M && (1.20)\\ &\overset{\text{first } N}{=} \sum_M (-)^{\phi_0(N)} (-)^{N(b+M+c)+N} a_N b^M c^N d_M = && (1.21)\\ &= \sum_{M,N} (-)^{\phi_0(N)+\phi_0(M)} (-)^{N(b+M+c)+N} (-)^{M(c+d)} a_N b^M c^N d_M && (1.22) \end{aligned}$$

Let us give one last example in (NW) (upper line) and (NE) (lower line)to clarify the general treatment:

$$A^{M_1}{}_{KN_1N_2}{}^{M_2}{}_{N_3} B^{N_3N_1}{}_{M_1M_2}{}^{LN_2} \equiv \qquad (1.23)$$

$$\equiv \begin{cases} \sum_{M_1,M_2,N_1,N_2,N_3} (-)^{M_1(K+N_2+M_2+B)+M_2(B+N_1)+N_1(1+N_2+B)+N_2(1+B+L)+N_3(1+B)} A^{M_1}{}_{KN_1N_2}{}^{M_2}{}_{N_3} B^{N_3N_1}{}_{M_1M_2}{}^{LN_2} \\ \sum_{M_1,M_2,N_1,N_2,N_3} (-)^{M_1(1+K+N_2+M_2+B)+M_2(1+B+N_1)+N_1(N_2+B)+N_2(B+L)+N_3B} A^{M_1}{}_{KN_1N_2}{}^{M_2}{}_{N_3} B^{N_3N_1}{}_{M_1M_2}{}^{LN_2} \end{cases}$$

The terrible signs in the lower lines of (1.23) are exactly those which we want to omit during calculations. We thus will define every calculational operation in such a way that it is consistent with this graded summation convention, s.th. one can calculate only with expressions as in the upper line of (1.23) and assign the signs only in the end of all the calculations.

By definition all the signs which depend on dummy indices are swallowed by the definition of the graded summation. As mentioned, the equation $a^M b_N = (-)^{(a+M)(b+N)} b_N a^M$ for graded commuting algebra elements reduces in a sum to $a^M b_M = (-)^{ab} b_M a^M$. The same simplification occurs for terms with several contracted indices, like in (1.23). Assuming that the objects there are graded commuting as well, we get

$$A^{M_1}{}_{KN_1N_2}{}^{M_2}{}_{N_3} B^{N_3N_1}{}_{M_1M_2}{}^{LN_2} = (-)^{(A+K)(B+L)} B^{N_3N_1}{}_{M_1M_2}{}^{LN_2} A^{M_1}{}_{KN_1N_2}{}^{M_2}{}_{N_3} \qquad (1.24)$$

Although there are still signs depending on the naked indices, this is far better than without the graded summation convention, where we would have obtained instead the full sign factor

$$(-)^{(A+M_1+K+N_1+N_2+M_2+N_3)(B+N_3+N_1+M_1+M_2+L+N_2)} \qquad (1.25)$$

1.2 Graded equal sign

The graded summation convention takes care of all dummy indices. But we can still be left with naked indices and/or graded rumpfs, which likewise produce inconvenient signs. Also the summation convention on its own might be dangerous. To show this, look at the following example: Consider graded commutative variables a^M, b^M, c^M and d^M with bosonic rumpfs. Then the following equations, which are obviously correct (using our graded summation convention)

$$\begin{aligned} a^M b^N c_N d_M - a^M b^N d_M c_N &= 0 \quad \forall \text{graded comm. } a^M, b^N, d_M, c_N && (1.26)\\ \Rightarrow a^M b^N (c_N d_M - d_M c_N) &= 0 \quad \forall \text{graded comm. } a^M, b^N, d_M, c_N && (1.27) \end{aligned}$$

could lead to the – in general – wrong assumption

$$c_N d_M - d_M c_N = 0 \quad \forall \text{graded comm. } d_M, c_N \text{ (not true in general!)} \qquad (1.28)$$

We therefore introduce a **graded equal sign** $=_g$, which states that the equality holds if for each term a mismatch in some common ordering of the indices is taken care of by an appropriate sign factor:

$$c_N d_M - d_M c_N =_g 0 \quad :\Longleftrightarrow \quad c_N d_M - (-)^{MN} d_M c_N = 0 \qquad (1.29)$$

If we imagine objects like in (1.23), the graded equal sign allows one to write down quickly correct equations without bothering all the involved signs. And it will also lead as a guiding line for all definitions of new objects, which should all be writable in terms of the graded equal sign, in order to make them compatible with the graded summation convention.

The idea of how to define the graded equal sign should be clear from (1.29), but in order to be able to write down a definition for the general case, we have to be a little more careful. For practical purposes it should be enough to have a look at the examples following the general definition, to convince yourself that everything is very natural and intuitive.

Let us introduce the graded equal-sign for the most general case in two steps. At first we look at equations with only bosonic rumpfs, like in (1.26).

Graded equal sign for bosonic rumpfs

Any term $T_{(i)}$ of the equation (which can be a product of a lot of objects with indices) has some nonnegative integer number k of naked indices (the vertical position of the indices does not play a role for this definition, so we write them all upstairs, but the very same definition holds for any position). We take the first term in the equation, call it ${T_{(1)}}^{M_1\ldots M_k}$, as reference term. Any other term $T_{(i)}$ in the equation has to have the same index set but perhaps with a different order or permutation $P_{(i)}$ of the indices. A permutation of an index set $\{M_1,\ldots,M_k\}$ is defined via a permutation of the set $\{1,\ldots,k\}$

$$P_{(i)}(M_1,\ldots,M_k) \quad\equiv\quad (M_{P_{(i)}(1)},\ldots,M_{P_{(i)}(k)}),\quad P_{(1)}\equiv \mathbb{1} \tag{1.30}$$

In order to assign the appropriate signs to the terms, we introduce for any of the k indices M_i an auxiliary graded commutative object o^{M_i} which carries the grading of the index

$$o^{M_i}o^{M_j} = (-)^{M_iM_j}o^{M_j}o^{M_i} \tag{1.31}$$

If M_i are just supercoordinate-indices, then the supercoordinates x^M themselves can be taken instead of defining new variables o^M. Let us now define something which we call a **grading structure** for a given term, namely a product of those objects o with as many factors as the term has naked indices:

$$\mathrm{gs}(T_{(1)}^{M_1\ldots M_k}) \quad\equiv\quad o^{M_1}\cdots o^{M_k} \tag{1.32}$$

In the grading structures of different terms, we can rearrange the objects until all the naked indices have some common order. For example for two terms with 3 naked indices we have

$$\mathrm{gs}(T_{(1)}^{M_1M_2M_3}) \quad=\quad o^{M_1}o^{M_2}o^{M_3} \tag{1.33}$$

$$\mathrm{gs}(T_{(2)}^{M_3M_2M_1}) \quad=\quad o^{M_3}o^{M_2}o^{M_1} = (-)^{M_1(M_2+M_3)+M_2M_3}o^{M_1}o^{M_2}o^{M_3} \tag{1.34}$$

We call the resulting sign the **relative sign of the grading structures**

$$\mathrm{gs}\big({T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}}\big) \quad\equiv\quad \mathrm{sign}^g_{T_{(1)}^{M_1\ldots M_k}}\big({T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}}\big)\cdot \mathrm{gs}\big(T_{(1)}^{M_1\ldots M_k}\big) \tag{1.35}$$

As the rumpfs carry no grading so far, it is notationally more convenient to replace $\mathrm{sign}^g_{T_{(1)}^{M_1\ldots M_k}}\big({T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}}\big)$ by[2] $\mathrm{sign}^g_{M_1\ldots M_k}\big(M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}\big)$. For the above two terms with three naked indices we thus have

$$\mathrm{sign}^g_{T_{(1)}^{M_1M_2M_3}}\left(T_{(2)}^{M_3M_2M_1}\right) \quad=\quad \mathrm{sign}^g_{M_1M_2M_3}\left(M_3M_2M_1\right) = (-)^{M_1(M_2+M_3)+M_2M_3} \tag{1.36}$$

Using this definition of the relative sign of grading structures, we can now define the graded equal sign for an equation with general terms (but still bosonic rumpfs) as

$$\boxed{\sum_i {T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}} =_g 0 \quad:\Longleftrightarrow\quad \sum_i \mathrm{sign}^g_{T_{(1)}^{M_1\ldots M_k}}({T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}})\cdot {T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}} = 0} \tag{1.37}$$

This definition does not depend on the choice of the reference term (above it is $T_{(1)}^{M_1\ldots M_k}$), because only the relative sign is relevant. One can replace $\mathrm{sign}^g_{T_{(1)}^{M_1\ldots M_k}}({T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}})$ by $\mathrm{sign}^g_{T_{(j)}^{M_{P_{(j)}(1)}\ldots M_{P_{(j)}(k)}}}({T_{(i)}}^{M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)}})$ for any j. As mentioned above we can also replace it by simply $\mathrm{sign}^g_{M_1\ldots M_k}(M_{P_{(i)}(1)}\ldots M_{P_{(i)}(k)})$.

[2] Note that this sign does not in general coincide with the signature of a permutation. The relative sign $\mathrm{sign}^g_{M_1\ldots M_k}\left(P_{(i)}(M_1,\ldots,M_k)\right)$ coincides with the signature of the permutation $P_{(i)}$ (which is given by minus one to the number of switches one needs to build the permutation) only if all indices carry an odd grading. ◇

In the following sections we will always give definitions and important equations with the graded equal sign and with the ordinary one. The somewhat long-winded definition of above should therefore become more transparent in numerous examples later on. But let us first complete our definition to the case involving graded rumpfs. One could get rid of all graded rumpfs by shifting the grading to the indices (if present), or create a new index with only one possible value. As this would be notationally not very nice, we stay with graded rumpfs, but we keep in mind that a graded rumpf is similar to a naked index. Problems for including the rumpfs in the definition of the graded equal sign appear, when the same rumpf appears several times in one term, which is thus similar to to having coinciding naked indices:

Problem of coinciding indices:

The graded equal sign above (1.37) is only well defined if all naked indices can be distinguished. In general calculations one usually uses different letters for each index, even if they are allowed to coincide, and then there is no problem. What, however, if one looks at some special case with two coinciding indices? Consider the following relations (which simply apply the definition of the graded equal sign):

$$(a)\quad T_{(1)}{}^{MN} =_g T_{(2)}{}^{NM} \quad \iff \quad T_{(1)}{}^{MN} = (-)^{NM} T_{(2)}{}^{NM} \tag{1.38}$$
$$(b)\quad T_{(1)}{}^{MN} =_g T_{(2)}{}^{MN} \quad \iff \quad T_{(1)}{}^{MN} = T_{(2)}{}^{MN} \tag{1.39}$$

For $M = N$ (no sum) this reads

$$(a)\quad T_{(1)}{}^{MM} =_g T_{(2)}{}^{MM} \quad \iff \quad T_{(1)}{}^{MM} = (-)^{M} T_{(2)}{}^{MM} \quad \text{no sum over } M \tag{1.40}$$
$$(b)\quad T_{(1)}{}^{MM} =_g T_{(2)}{}^{MM} \quad \iff \quad T_{(1)}{}^{MM} = T_{(2)}{}^{MM} \quad \text{no sum over } M \tag{1.41}$$

Now (a) and (b) obviously contradict themselves and the graded equal sign is therefore ill-defined. There are two options to solve this notational problem. The first is to *always rewrite the equation with an ordinary equal sign before looking at any special case.* The second is to make apparent the original name of the index in the following way (this is also useful to suppress summation over repeated indices if it is not wanted)

$$(a)\quad T_{(1)}{}^{M(N=M)} =_g T_{(2)}{}^{(N=M)M} \quad \iff \quad T_{(1)}{}^{M(N=M)} = (-)^{M} T_{(2)}{}^{(N=M)M} \tag{1.42}$$
$$(b)\quad T_{(1)}{}^{M(N=M)} =_g T_{(2)}{}^{M(N=M)} \quad \iff \quad T_{(1)}{}^{M(N=M)} = T_{(2)}{}^{M(N=M)} \tag{1.43}$$

Graded rumpfs

A grading of a rumpf is like a naked index grading at the position of the rumpf. The lesson from above is, that we can only include the rumpfs completely into the definition of the graded equal sign, if in each term every rumpf appears exactly once. As we can't rely that this is the case in all equations of interest, we will include the rumpfs only partially in the definition of the graded equal sign. Namely, the graded equal sign will not compare the order of the rumpfs, but the position of the indices with respect to the rumpfs. This is again necessary to stay consistent with the graded summation convention. Consider therefore the same trivial example as in (1.26), however, now with graded rumpfs

$$a^M b^N c_N d_M - (-)^{cd} a^M b^N d_M c_N \;=\; 0 \quad \forall \text{graded comm. } a^M, b^N, d_M, c_N \tag{1.44}$$
$$\Rightarrow a^M b^N \left(c_N d_M - (-)^{cd} d_M c_N\right) \;=\; 0 \quad \forall \text{graded comm. } a^M, b^N, d_M, c_N \tag{1.45}$$

We now want to simply read off

$$c_N d_M - (-)^{cd} d_M c_N \;=_g\; 0 \quad \forall \text{graded comm. } d_M, c_N \tag{1.46}$$

In order for this to be correct, we have to extend the definition of $=_g$ appropriately to the case of graded rumpfs. Let us therefore write out the summation convention in (1.45) explicitely (in NW-conventions):

$$\sum_{M,N} a^M b^N \left((-)^{M(b+c+d)+Nc} c_N d_M - (-)^{M(b+N+d)+N(d+c)} (-)^{cd} d_M c_N\right) \;=\; 0 \tag{1.47}$$
$$\Rightarrow (-)^{Mc} c_N d_M - (-)^{MN+Nd} (-)^{cd} d_M c_N \;=\; 0 \tag{1.48}$$
$$\Rightarrow (-)^{Nd} c_N d_M - (-)^{MN+Mc} (-)^{cd} d_M c_N \;=\; 0 \tag{1.49}$$

Comparing the last line with (1.46) we get

$$c_N d_M - (-)^{cd} d_M c_N =_g 0 \quad :\iff \quad (-)^{Nd} c_N d_M - (-)^{MN+Mc} (-)^{cd} d_M c_N = 0 \tag{1.50}$$

The graded equal sign therefore takes care of the order of the naked indices via $(-)^{MN}$ and of the order of the naked indices with respect to the rumpfs, i.e. it puts their grading to the very right of all rumpfs via $(-)^{Nd}$ and

$(-)^{Mc}$. Only the order of the rumpfs among themselves is taken care of by hand via $(-)^{cd}$. As stated before, the correct order of the rumpfs cannot a posteriori be figured out, when some of them coincide. E.g. for $d = c$, the above equivalence would become

$$c_N c_M - (-)^c c_M c_N =_g 0 \quad \Longleftrightarrow \quad (-)^{Nd} c_N c_M - (-)^{MN+Mc}(-)^c c_M c_N = 0 \tag{1.51}$$

There is no way to deduce the sign $(-)^c$ from the structure of the equation itself, if one doesn't see it as a special case of (1.50). The relative order of the rumpfs is not visible in (1.51). For that reason we did not a priori include the order of the rumpfs into the definition of the graded equal sign, as it can be ill-defined in such situations. Nevertheless we will make a suggestion a bit later, how to include the rumpfs to some extent into a graded equal sign. The nice observation so far is that we got rid of **all** index-dependent signs! The use of the graded equal is in particular useful to define **composite objects** of the form

$$A^{MN} \equiv_g B^{NK} C_K{}^M \quad \Longleftrightarrow \quad A^{MN} \equiv (-)^{CN+MN} B^{NK} C_K{}^M \overset{NW}{=} (-)^{CN+MN} \sum_K (-)^{KC} B^{NK} C_K{}^M \tag{1.52}$$

This makes sure that the notation A^{MN} is consistent with the position of the gradings. This is again necessary to guarantee consistency with the graded summation convention. I.e. for every D_{MN} we have (ordinary equal sign, all indices contracted)

$$A^{MN} D_{MN} \quad = \quad B^{NK} C_K{}^M D_{MN} \tag{1.53}$$

which would not be true for the definition $A^{MN} \equiv B^{NK} C_K{}^M$ without the graded equal sign or the appropriate signs in front.

For a more general definition of the graded equal sign in the case of graded rumpfs, we can again introduce auxiliary graded commuting objects o and extend our previous definition of the **grading structure**, i.e. the product of these objects o with as many factors as there are naked indices and rumpfs in a given term. For every rumpf which appears twice in a term we have to introduce a second graded commuting object (call it o'), because sticking to only one object would lead to $o^c o^c = 0$ for $\mid c \mid= 1$. Instead of giving a general definition, let us give two examples:

$$\mathrm{gs}(c^M c^N T^{KL} x^P) \quad \equiv \quad o^c o^M o'^c o^N o^T o^K o^L o^x o^P \tag{1.54}$$

$$\mathrm{gs}(x^K A^{MPN} c^L) \quad \equiv \quad o^x o^K o^A o^M o^P o^N o^c o^L \tag{1.55}$$

In the grading structure, we can now rearrange the objects until all the rumpfs are in the front (with unchanged relative position) and the naked indices have some common order. E.g.

$$\mathrm{gs}(c^M c^N T^{KL} x^P) \quad = \quad (-)^{cM+T(M+N)+x(M+N+K+L)} o^c o'^c o^T o^x \cdot o^M o^N o^K o^L o^P \tag{1.56}$$

$$\mathrm{gs}(x^K A^{MPN} c^L) \quad = \quad (-)^{AK+c(K+M+P+N)} o^x o^A o^c \cdot o^K o^M o^P o^N o^L = \tag{1.57}$$

$$= \quad (-)^{AK+c(K+M+P+N)} (-)^{MK+N(K+P)+LP} o^x o^A o^c \cdot o^M o^N o^K o^L o^P \tag{1.58}$$

We call the resulting sign the **relative sign of the grading structures**

$$\mathrm{sign}^g_{c^M c^N T^{KL} x^P}\left(x^K A^{MPN} c^L\right) = (-)^{cM+T(M+N)+x(M+N+K+L)} (-)^{AK+c(K+M+P+N)} (-)^{MK+N(K+P)+LP} \tag{1.59}$$

This definition of the relative sign reduces to (1.35) in the case of bosonic rumpfs. In order to write down the general definition for the graded equal sign, let us replace the terms of an equation (like $c^M c^N T^{KL} x^P$ and $x^K A^{MPN} c^L$ above) by placeholders $T_{(i)}$ (where i just labels the different terms). In the same way as for the bosonic rumpfs in (1.37) we can finally give the definition for the graded equal sign in the general case:

Definition 1 (graded equal sign '$=_g$')

$$\boxed{\sum_i T_{(i)} =_g 0 \quad :\Longleftrightarrow \quad \sum_i sign^g_{T_{(1)}}(T_{(i)}) \cdot T_{(i)} = 0} \tag{1.60}$$

Sometimes we call '$=_g$' also the "small graded equal sign".

In our example of above, this reads

$$c^M c^N T^{KL} x^P - x^K A^{MPN} c^L =_g 0 \quad :\Longleftrightarrow \quad c^M c^N T^{KL} x^P - \mathrm{sign}_{c^M c^N T^{KL} x^P}\left(x^K A^{MPN} c^L\right) \cdot x^K A^{MPN} c^L = 0 \tag{1.61}$$

Proposition 1 (Equivalence relation) *The such defined graded equal sign obeys transitivity ($X =_g Y$, $Y =_g Z \Rightarrow X =_g Z$) as well as reflexivity ($X =_g X$) and symmetry ($X =_g Y \Rightarrow Y =_g X$) and is therefore an equivalence relation.*

Proof: Reflexivity: If the expression X is a sum of terms $T_{(i)}$, i.e. $X = \sum_i T_{(i)}$ then the claim that $\sum_i T_{(i)} =_g \sum_i T_{(i)}$ is equivalent to $\sum_i \mathrm{sign}^g_{T_{(1)}}(T_{(i)}) \cdot T_{(i)} = \sum_i \mathrm{sign}^g_{T_{(1)}}(T_{(i)}) \cdot T_{(i)}$ which is obviously true. The symmetry is induced by the fact that $\mathrm{sign}_{T_{(i)}} T_{(j)} = \mathrm{sign}_{T_{(j)}} T_{(i)}$. Transitivity finally is seen as follows: Assume that we have $\sum_i T_{(i)} =_g \sum_i \tilde{T}_{(i)}$ (equivalent to $\sum_i \mathrm{sign}^g_{T_{(1)}}(T_{(i)}) \cdot T_{(i)} = \sum_i \mathrm{sign}^g_{T_{(1)}}(\tilde{T}_{(i)}) \cdot \tilde{T}_{(i)}$) and $\sum_i \tilde{T}_{(i)} =_g \sum_i \tilde{\tilde{T}}_{(i)}$ (equivalent to $\sum_i \mathrm{sign}^g_{\tilde{T}_{(1)}}(\tilde{T}_{(i)})\tilde{T}_{(i)} = \sum_i \mathrm{sign}^g_{\tilde{T}_{(1)}}(\tilde{\tilde{T}}_{(i)})\tilde{\tilde{T}}_{(i)}$). Then it follows (using transitivity of the ordinary equal sign) that $\sum_i \mathrm{sign}^g_{T_{(1)}}(T_{(i)}) \cdot T_{(i)} = \mathrm{sign}^g_{T_{(1)}}(\tilde{T}_{(1)}) \sum_i \mathrm{sign}^g_{\tilde{T}_{(1)}}(\tilde{\tilde{T}}_{(i)})\tilde{\tilde{T}}_{(i)} = \sum_i \mathrm{sign}^g_{T_{(1)}}(\tilde{\tilde{T}}_{(i)})\tilde{\tilde{T}}_{(i)}$ which is in turn equivalent to $\sum_i T_{(i)} =_g \sum_{(i)} \tilde{\tilde{T}}_{(i)}$. □

Remark: In part II, beginning with chapter 5 on page 43, we will throughout use the graded summation convention (based on NW) and the graded equal sign $=_g$. The latter will then simply be denoted with an ordinary equal sign $=$, in order to keep the notations simple.

Next we go one step further and define a **big graded equal sign** $=_G$ which also takes care of the order of as many rumpfs as possible. Let us give some simple examples:

$$(AB)^T =_G B^T A^T \quad :\Longleftrightarrow \quad (AB)^T = (-)^{AB} B^T A^T \tag{1.62}$$
$$(AB)^\dagger =_G B^\dagger A^\dagger \quad :\Longleftrightarrow \quad (AB)^\dagger = (-)^{AB} B^\dagger A^\dagger \tag{1.63}$$
$$(ab)^* =_G a^* b^* \quad :\Longleftrightarrow \quad (ab)^* = a^* b^* \tag{1.64}$$
$$D_M(AB) =_G (D_M A)B + A(D_M B) \quad :\Longleftrightarrow \quad D_M(AB) = (D_M A)B + (-)^{(D+M)A} A(D_M B) \tag{1.65}$$

The above examples are well-designed. Every rumpf or naked index appears in every term exactly once and a comparison of the order in each term is possible.

- In more general situations, the **big graded equal sign** $=_G$ will be defined by first adding the signs corresponding to the use of the small graded equal sign $=_g$ and then taking care of a maximum of common (to all terms) and distinguishable (among themselves) rumpf-symbols. For all remaining rumpf-symbols, a sign will be included that assumes that their standard position is to the very left (not changing their relative order).

Writing down a more formal definition of this idea in general would probably be lengthy and not very illuminating, so let us again consider some examples (which are not necessarily meaningful in real calculations):

$$ABAC =_G CB \quad :\Longleftrightarrow \quad (-)^{BA} ABAC = (-)^{CB} CB \tag{1.66}$$

The maximum set of symbols common to each term is $\{B, C\}$. Their relative order is different in the two terms, so that we get the factor $(-)^{CB}$, while the factor $(-)^{BA}$ is the sign that compares to the structure where all A's (which do not belong to the common set) are to the very left. Another example (with explanation right afterwards):

$$\begin{aligned} 0 =_G \quad & A_M B_N A_K C_L + B_M A_N A_L C_K + A_N B_M C_K A_L :\Longleftrightarrow \\ 0 \quad = \quad & (-)^{BM+A(M+N)+C(M+N+K)} (-)^{BA} A_M B_N A_K C_L + \\ & +(-)^{AM+A(M+N)+C(M+N+L)} (-)^{LK} B_M A_N A_L C_K + \\ & +(-)^{BN+C(N+M)+A(N+M+K)} (-)^{NM} (-)^{(B+C)A} A_N B_M C_K A_L \end{aligned} \tag{1.67}$$

In a first step we have applied the small graded equal sign, which includes moving all rumpf-gradings to the very left without changing their relative order. This leads to the sign $(-)^{BM+A(M+N)+C(M+N+K)}$ for the first, $(-)^{AM+A(M+N)+C(M+N+L)}$ for the second and $(-)^{BN+C(N+M)+A(N+M+K)}$ for the third term. The small graded equal sign also takes care of the relative order of the naked indices in all terms. If we take the first term as reference term, this yields the factors $(-)^{LK}$ for the second and $(-)^{NM}$ for the third term. The additional contribution from the big graded equal sign is obtained as follows: This time the set of all rumpf-symbols $\{B, C, A\}$ is common to all terms, but A appears in two indistinguishable copies. The maximum set of common (to all terms) and distinguishable (among themselves) rumpf-symbols is thus again $\{B, C\}$. The gradings of the remaining A's are put to the very left, which yields a factor $(-)^{BA}$ for the first term, $(-)^{B(A+A)} = 1$ for the second and $(-)^{(B+C)A}$ for the third term. Finally the relative order of B and C in each term is compared which gives no extra factor in this example.

Note that the naked index in (1.65) was treated on equal footing with the rumpfs. The big graded equal sign simply compared the relative order of all involved symbols, no matter if they were rumpf or naked index. In this case, where all rumpfs appear in each term exactly once, this is equivalent to applying our more general definition (given below (1.65)), where we first apply the small graded equal sign, which moves all the rumpf-gradings to the very left. Indeed the example (1.65) can equivalently be written as

$$D_M(AB) =_G (D_M A)B + A(D_M B) :\Longleftrightarrow (-)^{(A+B)M} D_M(AB) = (-)^{(A+B)M} (D_M A)B + (-)^{BM} (-)^{DA} A(D_M B) \tag{1.68}$$

There is a serious drawback of the so far given definition of the big graded equal sign: it does not in general obey transitivity. We will below modify the definition such that transitivity is guaranteed, but let us first give examples where it is violated. If one defines composite objects, like $A \equiv_G ba$, using the big graded equal sign, it does not have any effect. The maximum set of symbols common to all terms is empty. The symbol 'A' on the lefthand side doesn't appear in the term on the righthand side, and the symbols 'a' and 'b' do not appear in the term on the lefthand side. The same reasoning holds for $B \equiv_G ab$:

$$A \equiv_G ba \iff A \equiv ba, \qquad B \equiv_G ab \iff B \equiv ab \tag{1.69}$$

Assume that we have $A =_G B$ (which is equivalent to $A = B$, i.e. $ba = ab$). Transitivity would then imply that $ba =_G ab$ which is equivalent to $ba = (-)^{ba} ab$ and does in general not agree with the starting point $A = B$. A way out is to define the big graded equal sign not for a single equation, but for the whole system of equations under consideration.

Definition 2 (big graded equal sign '$=_G$') *Given a system of equations, we first determine for each equation i the set $\mathbb{M}_i$ of rumpf-symbols which appear either exactly once in each term or not at all in the given equation. Call the intersection of these sets $\mathbb{M} \equiv \cap_i \mathbb{M}_i$. The **big graded equal sign** '$=_G$' in a system of equations is now defined by first applying the sign rules corresponding to the small graded equal sign '$=_g$' and then adding a sign that compares the relative order of all rumpf-symbols which are in the set $\mathbb{M}$. For all remaining rumpf-symbols, a sign will be included that assumes that their standard position is to the very left (not changing their relative order).*

In the previous example this works as follows: The equations under consideration are $A =_G ba$, $B =_G ab$ and $ab =_G ba$. The symbol 'A' in the first equation appears once in the term on the lefthand side, but not at all in the term on the righthand side. It is thus not in the set $\mathbb{M}_1$. The same is true for the rumpf symbols 'B' in the second equation and for 'a' and 'b' in the first and second equation. We thus have $\mathbb{M}_1 = \{\}$, $\mathbb{M}_2 = \{\}$. Only for the last equation the rumpf symbols 'a' and 'b' appear exactly once in each term so that $\mathbb{M}_3 = \{a, b\}$. The intersection, however, is still empty $\mathbb{M} = \mathbb{M}_1 \cap \mathbb{M}_2 \cap \mathbb{M}_3 = \{\}$. The big graded equal sign compares only the relative order of the symbols in $\mathbb{M}$. In this case it therefore reduces to an ordinary equal sign and transitivity is trivially preserved.

Proposition 2 (Transitivity) *In addition to symmetry and reflexivity, the above defined big graded equal sign $=_G$ obeys transitivity within the given set of equations that was used for its definition and is therefore an equivalence relation within this set.*

Proof: Under the conditions of the definition (all rumpf symbols appear for any given equation either exactly once in each term or not at all in this equation) one can replace every rumpf by a bosonic rumpf with an auxiliary naked index which carries the grading. The big graded equal sign then reduces to the small graded equal sign whose transitivity we have seen already. □

1.3 Calculating with fermions as with bosons - a theorem

Now we are equipped with the main tools that are necessary to turn bosonic structural equations into graded structural equations. The set $\mathbb{M}$ in the definition of the big graded equal sign contains all symbols whose relative positions in a system of equations can be uniquely determined. This is precisely the property that allows to assign a grading to such a symbol and therefore deserves its own definition.

Definition 3 (Gradifiable) *We call a naked index or rumpf of an algebra element gradifiable in a given equation iff it either appears in every term of this equation exactly once or it does not appear in the equation at all. We call it gradifiable in a system of equations iff it is gradifiable in each of them. In addition, every dummy index (one which appears in a single term twice, once in upper and once in lower position) is also called a gradifiable index.*

Example In the equation $a^M b_N = b_N a^M$ all indices $\{M, N\}$ and all rumpfs $\{a, b\}$ are gradifiable, because they appear in every term exactly once. However in the set of equations $a^M b_N = b_N a^M$, $A^M{}_N = a^M b_N$ only the indices $\{M, N\}$ are gradifiable, while the rumpfs $\{A, a, b\}$ are not gradifiable any longer, as they all appear in the second equation, but not exactly once in each term. The same set of equations, with the second one written as $A(a,b)^M{}_N = a^M b_N$, however, has gradifiable rumpf-symbols a and b. The notion 'gradifiable' therefore depends on the way how objects are denoted.

Definition 4 (Gradification) *The gradification of an index 'K' or rumpf 'a' assigns an undetermined parity $\mid K \mid$ or $\mid a \mid$ to it, which will enter the graded summation convention and the graded equal sign. The gradification of a given set of algebraic equations is defined to be a new set of equations with all gradifiable objects gradified, the equal sign replaced by the big graded equal sign and the sum over dummy indices replaced by the graded sum (using an arbitrary but well-defined sign rule like NW or NE) over graded dummy indices.*

More or less by definition, the following theorem holds:

Theorem 1 *If a set of algebraic equations implies (perhaps via some intermediate equations) a second set of algebraic equations, then the same holds true for the gradification of the whole system.*

Remark: According to the definition of 'gradifiable in a system of equations' only those indices and rumpfs which are gradifiable in each equation (even the intermediate ones) are gradifiable in the whole system.

Comment on the proof: All definitions were chosen precisely with having in mind that the theorem should hold. Therefore it seems that there is nothing to prove and the theorem just holds by definition. Nevertheless, any attempts of mine to make this statement more rigorous, failed so far. One might therefore insist on calling the above theorem a 'conjecture' only. Calling it a conjecture, however, would somehow implement that the proof is difficult. But as argued above, I suspect that it is rather a triviality as soon as an appropriate setting is used. A naive idea for a proof would be that the gradification provides an isomorphism from one algebra to another. However, the gradification map is not in general invertible. For example a commutative but otherwise freely generated algebra is mapped to a graded commutative (and otherwise freely generated) one. For odd generators, the square is zero and therefore the gradification has less basis elements than the original algebra, if the number of generators is the same. What is mapped one to one is therefore not the algebra itself, but a certain (sub)set of equations which characterize the algebra, namely the gradifiable ones. $\square$

Further remarks:

- The example given after the definition of 'gradifiable' demonstrates that the power of the theorem depends on how the original equations are written. If one introduces auxiliary variables for composite objects (like $A^M{}_N \equiv a^M b_N$), the number of gradifiable objects may reduce, if the elementary variables are not denoted as an argument (like in $A(a,b)^M{}_N$). The theorem gives no statement about the best notation to use. It rather gives a statement which holds for any notation, but the notation has an influence on the number of gradifiable objects. Sometimes rumpf-symbols can be turned gradifiable by a change of notation but sometimes this seems impossible. It would be useful to characterize the 'best notation' which makes as many symbols as possible gradifiable.
- This theorem provides the possibility to use existing bosonic tensor manipulation packages for Mathematica or other computer algebra systems also for the graded case!
- It is not excluded a priori that the original set of equations contains fermionic variables which are then made bosonic (or are assigned an undetermined grading). However, one has to make sure that equations like

  $$\boldsymbol{\theta} \cdot \boldsymbol{\theta} \quad = \quad 0 \tag{1.70}$$

  are not contained in the set of equations that were needed to derive something. In the above equation, $\boldsymbol{\theta}$ obviously appears twice in one term and is thus not gradifiable. This is also the reason why anticommuting variables cannot be replaced completely by commuting ones. In particular the sum of two nilpotent objects is not necessarily nilpotent any longer in the commuting case. A recent paper [20] studies the properties of nilpotent commuting variables where some further differences (e.g. in the Leibniz rule) appear w.r.t. the anticommuting case.

Counterexamples

In the rest of this part of the thesis we will give a lot of examples and applications of the theorem. There will, however, also be some rather subtle examples which seem to be counterexamples at first sight. One of those "counterexamples" is the graded inverse of a matrix with graded rumpf, treated in subsection 2.4 on page 22. Another "counterexample" is the derivative with respect to Grassmann variables: the bosonic equation

$$\frac{\partial}{\partial x} x = 1 \tag{1.71}$$

suggests to define

$$\frac{\partial}{\partial \theta} \theta \stackrel{?}{=} 1 \tag{1.72}$$

for fermionic variables. This definition makes perfect sense, but results using this derivative cannot be derived via the theorem from the bosonic case, as the rumpf theta does not appear excatly once in every term. This problem can be omitted, if one introduces a new index and puts the grading into the index. We discuss such derivatives in subsection 3.1 on page 28.

Finally, a quite disturbing counterexample, which demonstrates that intermediate equations have to be taken into account in the process of gradification, is discussed on page 30.

Chapter 2

Graded matrices (supermatrices) and graded matrix operations

Supermatrices are the perfect objects to study the effects of our considerations. We will drop the word 'super' or 'graded' in every definition, since everything in this part has to be understood as graded. The equations of this section will all be written in two ways: once in the left column with the help of the (small) graded equal sign and the implicit graded summation conventions and once on the righthand side with ordinary equal sign, and the sum written out explicitely (in NW conventions), in order to make the reader familiar with the new conventions.

Within this chapter, we will always consider four different kinds of matrices, which differ in their index-positions:

$$A^{MN}, B^M{}_N, C_M{}^N, D_{MN} \tag{2.1}$$

Remark: In case that we have several matrices of one type, e.g. type B, we will denote them by B_1, B_2,... . It is important to have in mind that we consider B_1 as a rumpf by itself and not as a rumpf B together with an index '1'.

2.1 Transpose and hermitean conjugate

Let us start with the definition of a transposed matrix and a hermitean conjugate matrix in each of the four cases. The simple rule is to take the bosonic definition and replace the equal sign by the big graded one (which reduces to the small graded one in the below cases):

$$\begin{aligned}
(A^T)^{MN} &\equiv_g A^{NM} & (A^T)^{MN} &\equiv (-)^{MN} A^{NM} && (2.2)\\
(B^T)_M{}^N &\equiv_g B^N{}_M & (B^T)_M{}^N &\equiv (-)^{MN} B^N{}_M && (2.3)\\
(C^T)^M{}_N &\equiv_g C_N{}^M & (C^T)^M{}_N &\equiv (-)^{MN} C_N{}^M && (2.4)\\
(D^T)_{MN} &\equiv_g D_{NM} & (D^T)_{MN} &\equiv (-)^{MN} D_{NM} && (2.5)
\end{aligned}$$

$$\begin{aligned}
(A^\dagger)^{MN} &\equiv_g (A^{NM})^* & (A^\dagger)^{MN} &\equiv (-)^{MN} (A^{NM})^* && (2.6)\\
(B^\dagger)_M{}^N &\equiv_g (B^N{}_M)^* & (B^\dagger)_M{}^N &\equiv (-)^{MN} (B^N{}_M)^* && (2.7)\\
(C^\dagger)^M{}_N &\equiv_g (C_N{}^M)^* & (C^\dagger)^M{}_N &\equiv (-)^{MN} (C_N{}^M)^* && (2.8)\\
(D^\dagger)_{MN} &\equiv_g (D_{NM})^* & (D^\dagger)_{MN} &\equiv (-)^{MN} (D_{NM})^* && (2.9)
\end{aligned}$$

Clearly we have

$$(M^T)^T = M \tag{2.10}$$

$$(M^\dagger)^\dagger = M \tag{2.11}$$

for all matrices M, which is a first simple confirmation of the theorem.

2.2 Matrix multiplication

We meet a first deviation from usual definitions when we consider matrix multiplications.[1] The definition of the matrix multiplication will depend on the index structure of the matrix. Both, graded equal sign and the

[1] Although they seem to agree with the definitions in [19], when one moves there all indices which are to the left of a rumpf to the right with the corresponding sign according to that reference. ◇

graded summation convention have an influence now:

$$
\begin{aligned}
(AC)^{MN} &\equiv_g A^{MK}C_K{}^N & (AC)^{MN} &\equiv (-)^{MC}A^{MK}C_K{}^N = \\
& & &\stackrel{NW}{=} (-)^{MC}\sum_K(-)^{KC}A^{MK}C_K{}^N && (2.12)\\
(AD)^M{}_N &\equiv_g A^{MK}D_{KN} & (AD)^M{}_N &\equiv (-)^{MD}A^{MK}D_{KN} = \\
& & &\stackrel{NW}{=} (-)^{MD}\sum_K(-)^{KD}A^{MK}D_{KN} && (2.13)\\
(AB^T)^{MN} &\equiv_g A^{MK}(B^T)_K{}^N & (AB^T)^{MN} &\equiv (-)^{MB}A^{MK}(B^T)_K{}^N = \\
&= A^{MK}B^N{}_K & &= (-)^{MB}A^{MK}B^N{}_K = \\
& & &\stackrel{NW}{=} (-)^{MB}\sum_K(-)^{K(B+N)}A^{MK}B^N{}_K && (2.14)\\
(BA)^{MN} &\equiv_g B^M{}_KA^{KN} & (BA)^{MN} &\equiv (-)^{MA}B^M{}_KA^{KN} = \\
& & &\stackrel{NW}{=} (-)^{MA}\sum_K(-)^{K+KA}B^M{}_KA^{KN} && (2.15)\\
(B_1B_2)^M{}_N &\equiv_g B_1{}^M{}_KB_2{}^K{}_N & (B_1B_2)^M{}_N &\equiv (-)^{MB_2}B_1{}^M{}_KB_2{}^K{}_N = \\
& & &= (-)^{MB_2}\sum_K(-)^{K+KB_2}B_1{}^M{}_KB_2{}^K{}_N && (2.16)\\
&\ddots & &\ddots
\end{aligned}
$$

Associativity

Up to now, we have used the graded equality and summation mainly for definitions (apart from (2.10) and (2.11)). Now we can apply our theorem by stating that the (graded) matrix multiplication as defined above is associative

$$((B_1B_2)B_3)^M{}_N = B_1(B_2B_3)^M{}_N \tag{2.17}$$
$$((C_1C_2)C_3)_M{}^N = C_1(C_2C_3)_M{}^N \tag{2.18}$$

The graded equal sign has no effect in these equation. Associativity is guaranteed by theorem 1. The full reasoning in the B-case would be the following:

In the bosonic case we have

$$(B_1B_2)^M{}_N \equiv B_1{}^M{}_KB_2{}^K{}_N \Rightarrow ((B_1B_2)B_3)^P{}_Q = B_1(B_2B_3)^P{}_Q \tag{2.19}$$

The dummy indices are by definition gradifiable. Each of the naked indices M and N appears in every term of the first equation exactly once and not at all in the second and is therefore gradifiable. One could have written the second equation also with the same indices M and N and they still would be gradifiable. The same reasoning holds for P and Q. Finally, B_1 and B_2 each appear in every term of the first as well as of the second equation exactly once, while B_3 does not appear in the first at all, but it appears in the second in every term exactly once. All the rumpfs B_1,B_2 and B_3 are thus gradifiable in this system of two equations. The gradification of the whole system then reads

$$(B_1B_2)^M{}_N \equiv_G B_1{}^M{}_KB_2{}^K{}_N \Rightarrow ((B_1B_2)B_3)^P{}_Q =_G B_1(B_2B_3)^P{}_Q \tag{2.20}$$

where B_1, B_2 , B_3, M, N , P and Q have been assigned an undetermined grading, the sum over dummy indicies now has to be understood as the graded sum and the equal signs were replaced by the big graded equal sign (which reduces to the small graded equal sign in the first equation and to the ordinary equal sign in the second).

For this example it is still quite simple to check the validity of the statement explicitly, e.g. in NW

$$
\begin{aligned}
&(-)^{MB_3}\sum_L(-)^{LB_3+L}\left((-)^{MB_2}\sum_K(-)^{KB_2+K}B_1{}^M{}_KB_2{}^K{}_L\right)B_3{}^L{}_N = \\
&= (-)^{M(B_2+B_3)}\sum_K(-)^{K(B_2+B_3)+K}B_1{}^M{}_K\left((-)^{KB_3}\sum_L(-)^{LB_3+L}B_2{}^K{}_LB_3{}^L{}_N\right) \qquad (2.21)
\end{aligned}
$$

Unit matrix

The definition of the unit matrix is

$$M\mathbb{1} \;=\; M \quad \forall M \tag{2.22}$$

which implies via associativity (for the matrices of type B and C) that $M(\mathbb{1}N) = (M\mathbb{1})N = MN\,\forall M, N$ and thus

$$\mathbb{1}N \;=\; N \quad \forall N \tag{2.23}$$

For the different types of matricies A, B, C and D, we have in fact different types of unit matrices:

$$(A\mathbb{1})^{MN} \equiv \quad A^{MK}\delta_K{}^N \;\stackrel{!}{=}\; A^{MN} \qquad (A\mathbb{1})^{MN} \stackrel{NW}{\equiv} \sum_K A^{MK}\delta_K{}^N \;\stackrel{!}{=}\; A^{MN} \tag{2.24}$$

$$(B\mathbb{1})^M{}_N \equiv \quad B^M{}_K\delta^K{}_N \;\stackrel{!}{=}\; B^M{}_N \qquad (B\mathbb{1})^M{}_N \stackrel{NW}{\equiv} \sum_K (-)^K B^M{}_K\delta^K{}_N \;\stackrel{!}{=}\; B^M{}_N \tag{2.25}$$

$$(C\mathbb{1})_M{}^N \equiv \quad C_M{}^K\delta_K{}^N \;\stackrel{!}{=}\; C_M{}^N \qquad (C\mathbb{1})_M{}^N \stackrel{NW}{\equiv} \sum_K C_M{}^K\delta_K{}^N \;\stackrel{!}{=}\; C_M{}^N \tag{2.26}$$

$$(D\mathbb{1})_{MN} \equiv \quad D_{MK}\delta^K{}_N \;\stackrel{!}{=}\; D_{MN} \qquad (D\mathbb{1})_{MN} \stackrel{NW}{\equiv} \sum_K (-)^K D_{MK}\delta^K{}_N \;\stackrel{!}{=}\; D_{MN} \tag{2.27}$$

From the righthand side we can see

$$\delta_M{}^N \;=\; \begin{cases} \delta_M^N \text{ for NW} \\ (-)^{MN}\delta_M^N \text{ for NE} \end{cases} \tag{2.28}$$

with δ_M^N being the numerical Kronecker delta, and

$$\delta^M{}_N \;=_g\; \delta_N{}^M \qquad\qquad \delta^M{}_N \;=\; (-)^{MN}\delta_N{}^M \tag{2.29}$$

This graded Kronecker (the lefthand side shows that both versions are graded equal anyway) of course also fullfils its task for vectors and arbitrary rank tensors:[2]

$$a^M\delta_M{}^N \;=\; a^N \tag{2.30}$$

$$T_{M_1\ldots M_{r-1}K}\delta^K{}_N \;=\; T_{M_1\ldots M_{r-1}N} \tag{2.31}$$

2.3 Conjugations of matrix products – hermitean scalar product

Other simple applications of theorem 1 are statements about the transpose and the hermitean conjugate of a matrix product. Both, transposition and hermitean conjugation, were defined as gradifications of the bosonic versions and thus the equations for their action on matrix products will simply be the gradification of the corresponding bosonic equation. We will start with the transposition. The hermitean conjugation will follow a bit later after the discussion of complex conjugation and hermitean scalar product.

2.3.1 Transpose of matrix products

The transpose of a matrix product in terms of the big graded equal sign has the familiar bosonic behaviour.

$$\left((AC)^T\right)^{MN} =_G \left(C^TA^T\right)^{MN} \qquad \left((AC)^T\right)^{MN} = (-)^{AC}\left(C^TA^T\right)^{MN} \tag{2.32}$$

$$\left((AD)^T\right)^M{}_N =_G (D^TA^T)^M{}_N \qquad \left((AD)^T\right)^M{}_N = (-)^{AD}(D^TA^T)^M{}_N \tag{2.33}$$

$$\left((BA)^T\right)^{MN} =_G (A^TB^T)^{MN} \qquad \left((BA)^T\right)^{MN} = (-)^{AB}(A^TB^T)^{MN} \tag{2.34}$$

$$\ddots \qquad\qquad \ddots$$

[2] If the capital index combines two subsets of (small) indices with different position, we might insist on NW (or any other convention) for the small indices which leads to different definitions for the Kronecker delta:

$$\begin{aligned} a^M &= (a^m, a_\mu) \\ a^M\delta_M{}^N &= a^m\delta_m{}^N + a_\mu\delta^{\mu N} = \\ &\stackrel{\text{mixed conv.}}{\equiv} \sum_m a^m\delta_m{}^N + \sum_\mu (-)^\mu a_\mu\delta^{\mu N} \stackrel{!}{=} a^N \\ \delta_m{}^N &= \delta_m^N \\ \delta^{\mu N} &= (-)^\mu\delta^N_\mu \qquad \diamond \end{aligned}$$

Let us again verify explicitly that this is indeed true for e.g. the first line (in NW conventions):

$$\begin{aligned}
\left((AC)^T\right)^{MN} &= (-)^{MN}(AC)^{NM} = \\
&= (-)^{MN}(-)^{NC}\sum_K (-)^{CK} A^{NK} C_K{}^M = \\
&= (-)^{MN+NC}\sum_K (-)^{CK+(C+K+M)(A+N+K)} C_K{}^M A^{NK} = \\
&= \sum_K (-)^{CA+KA+KN+K+MA+MK} C_K{}^M A^{NK} = \\
&= (-)^{AC}(-)^{MA}\sum_K (-)^{KA+K} (C^T)^M{}_K (A^T)^{KN} = \\
&= (-)^{AC}(-)^{MA}(C^T)^M{}_K (A^T)^{KN} = \\
&= (-)^{AC}\left(C^T A^T\right)^{MN} \qquad (2.35)
\end{aligned}$$

2.3.2 Complex conjugation of products of (graded) commuting variables

Before we come to the discussion of hermitean scalar products and hermitean conjugation of matrix products, we will have a short look at complex conjugation of graded commuting variables (we will often call them graded numbers, or just numbers) and products of them. The reason to do so, is that the complex conjugate of a product of two Grassmann variables is often defined differently to our way, and we therefore want to motivate it carefully.

Complex conjugation of usual complex numbers is just what it is. For a (graded commuting) algebra based on a complex vector space one usually defines some basis to be real, so that the complex conjugation acts only on the expansion coefficients. Different definitions of the action on the basis elements are possible and simply a matter of convenience. However, the definition of the conjugation of the basis vectors should at least obey the conjugation property $(\,)^{**} = (\,)$. For an algebra whose vector-basis is generated by some generating set, the reality properties of the composite objects are determined by the reality properties of the generating set and the action of the complex conjugation on the product of elements. It is natural to define $(ab)^* = a^*b^*$, but using the opposite sign $(ab)^* = -a^*b^*$ for vectors a, b would also be consistent. Indeed, in the case of an anticommuting algebra this definition is very common because it can then be written as $(ab)^* = b^*a^*$ and resembles the bosonic version of hermitean conjugation where the order of objects is interchanged. Although there is thus good reason to make this choice, we want to convince the reader in the following that there is even better reason not to make this choice. For a graded commuting algebra, where a and b are of arbitrary grading, the choice

$$(ab)^* \equiv a^*b^* \qquad (2.36)$$

is certainly the one which fits into our philosophy, as it is the gradification of the usual choice for (bosonic) commuting algebras. This choice implies that the product of real objects is real again and the real elements thus form a subalgebra. Indeed the above conjugation rule can be derived from this reality condition. We could thus go the other way round and define the complex conjugation simply by saying that the product of two real products is always real. To derive the above conjugation rule from that condition, consider the (graded) commuting variable a and decompose it into its real part $\Re(a)$ and its imaginary part $\Im(a)$, defined by (use of a graded equal sign makes no difference here)

$$\Re(a) \equiv \frac{a+a^*}{2} \qquad (2.37)$$

$$\Im(a) \equiv \frac{a-a^*}{2i} \qquad (2.38)$$

Both are real because $a**=a$

$$\Re(a)^* = \Re(a), \qquad \Im(a)^* = \Im(a) \qquad (2.39)$$

and we have

$$a = \Re(a) + i\Im(a) \qquad (2.40)$$

$$a^* = \Re(a) - i\Im(a) \qquad (2.41)$$

We thus can seperate any number into a real and imaginary part, and complex conjugation flips (as usual) the sign of the imaginary part. Consider now the complex conjugation of the product of two graded numbers

$$\begin{aligned}
(ab)^* &= [(\Re(a)\Re(b) - \Im(a)\Im(b)) + i(\Re(a)\Im(b) + \Im(a)\Re(b))]^* = \\
&= (\Re(a)\Re(b) - \Im(a)\Im(b)) - i(\Re(a)\Im(b) + \Im(a)\Re(b)) && (2.42)\\
a^*b^* &= (\Re(a) - i\Im(b))\,(\Re(a) - i\Im(b)) = \\
&= (\Re(a)\Re(b) - \Im(a)\Im(b)) - i(\Re(a)\Im(b) + \Im(a)\Re(b)) && (2.43)\\
\Rightarrow (ab)^* &= a^*b^* && (2.44)
\end{aligned}$$

From the first to the second line we have used that the product of two real variables is real again. From our definitions of real and imaginary part in (2.37) and (2.38), which are just graded versions of the bosonic case, we could have deduced (2.44) as well via our theorem. We just want to stress that in our context this is the only natural complex conjugation. In order to allow a comparison with the 'usual' definition[3], let us for the moment denote the alternative version of complex conjugation by $(\ldots)^{\tilde{*}}$.

$$(ab)^{\tilde{*}} = b^{\tilde{*}}a^{\tilde{*}} = (-)^{ab}a^{\tilde{*}}b^{\tilde{*}} \qquad (2.45)$$

As mentioned, this behaviour would not at all fit into our philosophy. The same is true for the hermitean conjugation of the product of graded matrices in the next but one subsection (as well as of graded operators in the infinite dimensional case). How can we easily switch in applications from one definition to the other? Instead of redefining the complex conjugation itself, the switch of the behaviour from (2.44) to (2.45) can also be achieved by redefining the algebra product appropriately:

$$\begin{aligned}
a \circ b &\equiv i^{\epsilon_a\epsilon_b} a \cdot b && (2.46)\\
\Rightarrow (a \circ b)^* &= (-i)^{\epsilon_a\epsilon_b} a^*b^* = (-)^{ab}a^* \circ b^* && (2.47)
\end{aligned}$$

We used here the symbol ϵ_a to denote the parity, in order to emphasize that the exponent of $'i'$ really should take only values 0 and 1, while for our usual prefactors $(-)^{ab} \equiv (-)^{|a||b|}$, the grading $\mid a \mid$ does not need to be a $\mathbb{Z}_2$ grading. The parity is given by $\epsilon_a \equiv \mid a \mid \mod 2$.

2.3.3 Hermitean scalar product

Using our above definition of complex conjugation also fixes the behaviour of the graded version of a Hermitean scalar product. We use the index notation $(v^*)^{\bar{M}} \equiv (v^M)^*$. The scalar product (in a finite dimensional vector space for the beginning) then will be defined as

$$\langle v \mid w\rangle \equiv_G \underbrace{(v^*)^{\bar{M}}}_{(v^M)^*} H_{\bar{M}N}w^N \quad \text{with } (H_{\bar{M}N})^* =_G H_{\bar{N}M} \qquad \langle v \mid w\rangle \stackrel{NW}{\equiv} \sum_{\bar{M},N} (-)^{N+wN} \underbrace{(v^*)^{\bar{M}}}_{(v^M)^*} H_{\bar{M}N}w^N \quad \text{with } (H_{\bar{M}N})^* = (-)^{MN}H_{\bar{N}M} \qquad (2.48)$$

where H is a matrix of type 'D' which is (graded) hermitean. Strictly speaking, the rumpf H appears only on the righthand side and is therefore not gradifiable. However, if we identified on the lefthand side the vertical line '|' as a placeholder for the H-rumpf and also identify their grading, then it would be fine to even gradify the rumpf H. For the following considerations we will nevertheless stick to a bosonic rumpf H, i.e. $H_{\bar{M}N}$ should be considered as a bosonic supermatrix. The resulting scalar product is (graded) sesquilinear in the sense

$$\begin{aligned}
&\langle \lambda v_1 + v_2 \mid \mu w_1 + w_2\rangle =_G \\
&=_G \lambda^*\mu\langle v_1 \mid w_1\rangle + \lambda^*\langle v_1 \mid w_2\rangle + \\
&\quad +\mu\langle v_2 \mid w_1\rangle + \langle v_2 \mid w_2\rangle
\end{aligned}
\qquad
\begin{aligned}
&\langle \lambda v_1 + v_2 \mid \mu w_1 + w_2\rangle = \\
&= (-)^{\mu v_1}\lambda^*\mu\langle v_1 \mid w_1\rangle + \lambda^*\langle v_1 \mid w_2\rangle + \\
&\quad +(-)^{\mu v_2}\mu\langle v_2 \mid w_1\rangle + \langle v_2 \mid w_2\rangle
\end{aligned}
\qquad (2.49)$$

for λ and μ being complex supernumbers. It is furthermore (graded) hermitean, i.e.

$$\langle v \mid w\rangle =_G \langle w \mid v\rangle^* \qquad \langle v \mid w\rangle = (-)^{vw}\langle w \mid v\rangle^* \qquad (2.50)$$

The last equation implies that a scalar product of a vector with itself obeys

$$\langle v \mid v\rangle = (-)^v\langle v \mid v\rangle^* \qquad (2.51)$$

and is therefore real only for even vectors and purely imaginary for odd vectors. Note that a scalar product $\langle\,|\,\rangle_0$ which obeys $\langle v \mid w\rangle_0 = \langle w \mid v\rangle^*$ is obtained by either replacing $*$ by $\tilde{*}$ of the previous subsection or by defining $\langle v \mid w\rangle_0 \equiv (-i)^{\epsilon_v\epsilon_w}\langle v \mid w\rangle$.

The adjoint $B^\dagger$ of a matrix B with respect to our scalar product is defined as

[3] It seems that in the last decade, the definition $(ab)^* = a^*b^*$ has already become more popular (see for example [21]), while in [19] it was still defined with the opposite order. Another discussion of complex conjugation can be found in [22]. ◇

$$\langle v \mid Bw\rangle =_G \langle B^\dagger v \mid w\rangle \qquad\qquad \langle v \mid Bw\rangle = (-)^{Bv}\langle B^\dagger v \mid w\rangle \tag{2.52}$$

Assume that the hermitean matrix is non-degenerate in the sense that it has an inverse

$$H_{\bar{M}K}H^{K\bar{N}} = \delta_{\bar{M}}{}^{\bar{N}} \quad , \quad H^{M\bar{K}}H_{\bar{K}N} = \delta^M{}_N \tag{2.53}$$

Although it is more common to use only the symmetric part of a scalar product to pull indices up and down, we will in this section use $H_{\bar{M}N}$ and $H^{M\bar{N}}$ to pull indices. For a vector v^M we thus have the following additional possibilities of index-position and form:

$$v_{\bar{M}} \equiv H_{\bar{M}N}v^N \tag{2.54}$$
$$(v^*)_M \equiv (v_{\bar{M}})^* \tag{2.55}$$
$$(v^*)^{\bar{M}} \equiv (v^M)^* = (v^*)_N H^{N\bar{M}} \quad \left(= (H_{\bar{N}K}v^K)^* H^{N\bar{M}} =_g \underbrace{H_{\bar{K}N}H^{N\bar{M}}}_{\delta_{\bar{K}}{}^{\bar{M}}}(v^K)^*\right) \tag{2.56}$$

Using the inverse matrix $H^{M\bar{N}}$, we can now give an explicit expression for the adjoint matrix of B:
$\langle v \mid Bw\rangle = (v^*)^{\bar{M}}H_{\bar{M}N}(B^N{}_K w^K) = (v^*)^{\bar{M}}\underbrace{(H_{\bar{M}N}B^N{}_L H^{L\bar{P}})}_{\equiv B_{\bar{M}}{}^{\bar{P}}}H_{\bar{P}K}w^K =_G \big(\underbrace{(B^*)_M{}^P}_{\equiv (B_{\bar{M}}{}^{\bar{P}})^*} v^M\big)^* H_{\bar{P}K}w^K \stackrel{!}{=} \langle B^\dagger v \mid w\rangle.$

From this calculation we can read off

$$(B^\dagger)^P{}_M =_g (B_{\bar{M}}{}^{\bar{P}})^* = \left(H_{\bar{M}N}B^N{}_L H^{L\bar{P}}\right)^* =_g H^{P\bar{L}}\underbrace{(B^N{}_L)^*}_{(B^\dagger)_{\bar{L}}{}^{\bar{N}}}H_{\bar{N}M} \tag{2.57}$$

Up to pulling indices with H this agrees with our earlier definition of the hermitean conjugate of a matrix $(B^\dagger)_{\bar{L}}{}^{\bar{N}} =_g (B^N{}_L)^*$.

Having used indices all the time, we have implicitely chosen some **basis**

$$|e_M> \equiv |_M> \tag{2.58}$$

Every vector $|v>$ of definite grading can be written as a linear combination

$$|v>= v^M|_M> \tag{2.59}$$

The complex conjugate basis is $|_{\bar{M}}>\equiv |_M>^*$, so that $|v^*>\equiv |v>^*= (v^*)^{\bar{M}}|_{\bar{M}}>$. Bra-vectors involve a complex conjugation. Because of $<v^M e_M| = (v^*)^{\bar{M}} <e_M|$ it is convenient to denote

$$<e_M| \equiv <_{\bar{M}}| \tag{2.60}$$

such that

$$<v| = (v^*)^{\bar{M}} <_{\bar{M}}| \quad \text{and} \quad \langle_{\bar{M}},{}_N\rangle = H_{\bar{M}N} \tag{2.61}$$

The **dual basis** will be denoted by $<^M|$ and it is defined via

$$\langle^M \mid {}_N\rangle = \delta^M{}_N \tag{2.62}$$

After pulling down one index with H one arrives at the equation $\langle_{\bar{M}} \mid {}_N\rangle = H_{\bar{M}N}$ which we just had before and which is in turn consistent with $\langle v \mid w\rangle = (v^*)^{\bar{M}}H_{\bar{M}N}w^N$. The dual basis $<^M|$ thus agrees with the "hermitean conjugate" $<_{\bar{M}}|$ of $|_M>$ up to raising the index with $H^{M\bar{N}}$.

Clifford vacuum The above recall of some basic linear algebra will help us to understand the graded version of creation and annihilation operators acting on some Clifford vacuum. Let us denote just for this paragraph the index of the creation operators by $k, l, m, \ldots$, although we used those indices before for bosonic indices, while now we still assume them to be graded and not purely bosonic. The creation operators generate a complete basis from the Clifford vacuum, s.th. the indices $k, l, m, ...$ are just a subset of the basis-indeces $M, N, \ldots$. Let us denote the annihilation and creation operators by a^k and $(a^\dagger)_k$ respectively and the corresponding vectors or states by

$$|_k> \equiv (a^\dagger)_k|0>, \quad |_{k_1k_2}>\equiv (a^\dagger)_{k_1}(a^\dagger)_{k_2}|0>, \quad |_{k_1k_2k_3}>\equiv (a^\dagger)_{k_1}(a^\dagger)_{k_2}(a^\dagger)_{k_3}|0>, \ldots \tag{2.63}$$

The basis is then given by

$$|_K>\in \{|0>, |_k>, |_{k_1k_2}>, |_{k_1k_2k_3}>, \ldots\} \tag{2.64}$$

Finally we need the annihilation property of a^k and their commutator with the creation operators:

$$a^k|0>= 0, \qquad \left[a^k, (a^\dagger)_l\right] =_g \delta^k{}_l \tag{2.65}$$

Assume that the Clifford vacuum is bosonic, so that we can normalize it to one

$$\langle 0 \mid 0\rangle = 1 \tag{2.66}$$

This equation is not gradifiable, which is the reason why a bosonic vacuum is preferrable. The dual basis is then given by the dual vacuum $< 0|$ and its descendents

$$\begin{aligned}
<{}^k| &\equiv <0|a^k, \quad \tfrac{1}{2}<{}^{k_1k_2}| \equiv \tfrac{1}{2}<0|a^{k_1}a^{k_2}, \quad \tfrac{1}{3!}<{}^{k_1k_2k_3}| \equiv \tfrac{1}{3!}<0|a^{k_1}a^{k_2}a^{k_3}, \quad \ldots & (2.67)\\
\langle{}^k \mid {}_l\rangle &= <0|a^k(a^\dagger)_l|0>= \delta^k{}_l & (2.68)\\
\tfrac{1}{2}\langle{}^{k_1k_2} \mid {}_{l_2l_1}\rangle &= \tfrac{1}{2}<{}^{k_1}|a^{k_2}(a^\dagger)_{l_2}|_{l_1}>=_g \tfrac{1}{2}<{}^{k_1}|(a^\dagger)_{l_2}a^{k_2}|_{l_1}> + \tfrac{1}{2}<{}^{k_1}|\delta^{k_2}{}_{l_2}|_{l_1}>=_g \delta^{k_1}{}_{(l_2}\delta^{k_2}{}_{l_1)} & (2.69)\\
\tfrac{1}{3!}\langle{}^{k_1k_2k_3} \mid {}_{l_3l_2l_1}\rangle &=_g \delta^{k_1}{}_{(l_1}\delta^{k_2}{}_{l_2}\delta^{k_3}{}_{l_3)} & (2.70)\\
&\ddots &\\
<{}^K| &\in \left\{<0|, <{}^k|, \tfrac{1}{2}<{}^{k_1k_2}|, \tfrac{1}{3!}<{}^{k_1k_2k_3}|, \ldots\right\} & (2.71)
\end{aligned}$$

In the literature the indices of creation and annihilation operators are usually put at the same vertical position, and the corresponding states are normalized to be $\langle_k \mid {}_l\rangle = \delta_{kl}$. The Kronecker delta on the righthand side corresponds to a special choice of the scalar product and should in our context be replaced by

$$\langle_{\bar{k}} \mid {}_l\rangle = H_{\bar{k}l} \tag{2.72}$$

which agrees with (2.68) after pulling one index with H.

Note that the definition of a **norm** induced by the scalar product will not be possible under the conditions of theorem 1. The bosonic definition $\parallel v \parallel \equiv \langle v, v\rangle$ has the rumpf v appearing twice on the righthand side which is therefore not gradifiable. Still it makes sense to define a norm, but it will not simply have gradified properties of the bosonic one. In order to get a real norm, (while $\langle v \mid v\rangle$ is imaginary for odd v), we have to include an imaginary factor in the fermionic case and fix the arbitrary overall sign: E.g.

$$\parallel v \parallel \equiv \tfrac{1}{i^{\epsilon_v}}\langle v, v\rangle \tag{2.73}$$

Only at this point (choosing an appropriate $H_{\bar{M}N}$) we make contact to the usual definitions in the literature. Physical observables and probabilities should of course not depend on the conventions in the end. In the same way as above, the definition of the probability of some transition (which contains an absolute value square and is therefore also not gradifiable) has to include an appropriate complex factor. We are not going to work with Hilbert spaces in the second part of this thesis anyway and therefore leave the details for further studies. The leading thought was just to keep the idea of gradification as long as possible and break it only in the last step, in the definition of the norm and of probabilities.

2.3.4 Hermitean conjugate of matrix products

From our definition of a hermitean conjugate and of complex conjugation of products of numbers, we get via the theorem the natural rules for complex conjugation of (graded) matrix products:

$$\begin{aligned}
\left((AC)^\dagger\right)^{MN} &=_G \left(C^\dagger A^\dagger\right)^{MN} & \left((AC)^\dagger\right)^{MN} &= (-)^{AC}\left(C^\dagger A^\dagger\right)^{MN} & (2.74)\\
\left((AD)^\dagger\right)^M{}_N &=_G (D^\dagger A^\dagger)^M{}_N & \left((AD)^\dagger\right)^M{}_N &= (-)^{AD}(D^\dagger A^\dagger)^M{}_N & (2.75)\\
\left((BA)^\dagger\right)^{MN} &=_G (A^\dagger B^\dagger)^{MN} & \left((BA)^\dagger\right)^{MN} &= (-)^{AB}(A^\dagger B^\dagger)^{MN} & (2.76)\\
&\ddots & &\ddots &
\end{aligned}$$

Similarly we expect for operators in the infinite dimensional case

$$(\hat{A}\hat{B})^\dagger =_G \hat{B}^\dagger\hat{A}^\dagger \qquad\qquad (\hat{A}\hat{B})^\dagger = (-)^{AB}\hat{B}^\dagger\hat{A}^\dagger \tag{2.77}$$

As mentioned in the context of complex conjugation, it is simply a matter of redefining the operator product with a factor $(-i)^{\epsilon_A\epsilon_B}$ if one wants to make contact to the usual definition without sign.

2.4 Graded inverse - a nice "counterexample" to the theorem

Consider for the beginning matrices with even rumpf only

$$| A |=| B |=| C |=| D |= 0 \tag{2.78}$$

We say A is the **(graded) inverse** of D, B_2 the inverse of B_1 and C^2 the inverse of C^1 iff

$$\begin{aligned} D_{MK}A^{KN} &= \delta_M{}^N & (2.79)\\ A^{MK}D_{KN} &= \delta^M{}_N & (2.80)\\ B_1^M{}_K B_2^K{}_N &= \delta^M{}_N & (2.81)\\ C^1_M{}^K C^2_K{}^N &= \delta_M{}^N & (2.82)\end{aligned}$$

with

$$\delta_M{}^N = (-)^{MN}\delta^N{}_M \qquad (2.83)$$

The so defined inverses in general do not coincide with the naive inverses.[4]

From our theorem we can e.g. deduce that for matrices M N of any type (with even rumpf) we have

$$\begin{aligned} & (MN)^{-1} =_G (N^{-1}M^{-1}) & (2.84)\\ \overset{|M|=|N|=0}{\Rightarrow}\ & (MN)^{-1} = (N^{-1}M^{-1}) & (2.85)\end{aligned}$$

This is easily directly verified using associativity of our graded matrix multiplication.

Counterexample

If we take the rumpfs arbitrarily graded and still define an inverse via $M^{-1}M = \mathbb{1}$, then we still have[5]

$$\begin{aligned} (MN)(N^{-1}M^{-1}) &\overset{\text{assoz}}{=} M(NN^{-1})M^{-1} = \mathbb{1} & (2.86)\\ \Rightarrow\ (MN)^{-1} &= (N^{-1}M^{-1}), \quad \text{for any } |M| \text{and } |N| & (2.87)\end{aligned}$$

There is no expected prefactor $(-)^{MN}$ in the lower line! This looks strange in terms of the big graded equal sign, which should swallow the rumpf-dependend signs, but produces one here:

$$(MN)^{-1} =_G (-)^{MN}(N^{-1}M^{-1}) \qquad (2.88)$$

The theorem thus is not applicable here! What went wrong? Our definition of the inverse

$$(MM^{-1}) = \mathbb{1} \qquad (2.89)$$

is a non-valid gradification of the bosonic one: The theorem allows us to assign a grading only to rumpfs which appear exactly once in each term. The rumpf M appears twice on the lefthand side and not at all on the righthand side. Thus, the theorem does not allow to give M a grading. If we do so nevertheless, we can't derive known rules from the bosonic case. The definition itself is of course ok, but in order to stress that it is not simply a gradification of a bosonic definition, we should better give it a new name, like **special graded inverse**.

The naked indices in (2.79) to (2.82) appear excactly once in each term and can therefore be generalized to graded indices.

---

[4]To verify this statement, write out the equations (2.79)-(2.82) in NW-conventions, using $\delta_M{}^N = \delta^N_M$:

$$\begin{aligned} \sum D_{MK}(-)^K A^{KN} &= \delta^N_M\\ \sum A^{MK}D_{KN}(-)^N &= \delta^M_N\\ \sum B_1^M{}_K(-)^{K+N}B_2^K{}_N &= \delta^M_N\\ \sum C^1_M{}^K C^2_K{}^N &= \delta^N_M\end{aligned}$$

Only in the last case C^2 is the naive inverse of C^1. ⋄

[5]Note that although a Grassmann-variable has no inverse, a matrix with fermionic rumpf can have an inverse. Take e.g. $x, y \neq 0$ bosonic and c fermionic, then we have

$$\begin{pmatrix} c & x\\ y & 0\end{pmatrix}\begin{pmatrix} 0 & \frac{1}{y}\\ \frac{1}{x} & -\frac{c}{xy}\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \quad (\#)$$

The matrix multiplication above, however, is not according to our graded matrix multiplication rules, which are

$$\begin{aligned} (CC^{-1})_M{}^N &\equiv_g C_M{}^K(C^{-1})_K{}^N =_g \delta_M{}^N\\ \Rightarrow (CC^{-1})_M{}^N &\overset{NW}{=} \sum_K (-)^{KA+MA}C_M{}^K(C^{-1})_K{}^N = \delta_M{}^N\end{aligned}$$

The following choice of matrices therefore correspond to the equation (#):

$$C = \begin{pmatrix} c & -x\\ -y & 0\end{pmatrix} \quad C^{-1} = \begin{pmatrix} 0 & \frac{1}{y}\\ \frac{1}{x} & -\frac{c}{xy}\end{pmatrix} \qquad \diamond$$

2.5 (Super) trace

We now come to another important deviation from usual supermatrix-definitions which will enter an interesting result for superdeterminants. The trace is the sum of the diagonal entries and makes sense for matrices of type C and B only (matrices with one upper and one lower index, i.e. endomorphisms)

$$\mathrm{tr}\, B \equiv B^M{}_M = \begin{cases} \sum_M B^M{}_M & NW \\ \sum_M (-)^M B^M{}_M & NE \end{cases} \tag{2.90}$$

$$\mathrm{tr}\, C \equiv C_M{}^M = \begin{cases} \sum_M (-)^M C_M{}^M & NW \\ \sum_M C_M{}^M & NE \end{cases} \tag{2.91}$$

The $(-)^M$ is familiar from usual definitions. We have it here, however, either only for NW for matrices of type C or for NE for matrices of type B while the other cases do not have the familiar $(-)^M$ in the trace-definition. The reason is that e.g. for B-type matrices in NW (where the trace has no sign factor) the $(-)^M$ instead is hidden in the matrix multiplication of two matrices. Thus, either the matrix multpilication contains an extra $(-)^M$ and the trace doesn't, or the other way round. In any case, the graded cyclicity property of the trace holds:

$$\mathrm{tr}\, B_1 B_2 = B_1^M{}_K B_2{}^K{}_M = (-)^{B_2 B_1} \mathrm{tr}\, B_2 B_1 \tag{2.92}$$

$$\Longleftrightarrow \quad \mathrm{tr}\, [B_1, B_2] = 0 \tag{2.93}$$

For matrices of type A and D, we need a metric, in order to define a meaningful trace:

$$\mathrm{tr}\, A \equiv A^{MN} G_{MN} \tag{2.94}$$

$$\mathrm{tr}\, D \equiv D_{MN} G^{MN} \tag{2.95}$$

2.6 (Super) determinant

We finally come to the so far most interesting demonstration of the use of our conventions. Namely the definition of the superdeterminant. As usual, we start from the definition via the exponential:

$$\det C \equiv e^{\mathrm{tr}\, \ln C}, \quad \det B \equiv e^{\mathrm{tr}\, \ln B}, \tag{2.96}$$

Remember that in NW-conventions for a matrix of type B, the definition of the trace matches the bosonic definition, while the definition of the matrix product differs. For NE or for matrices of type C the situation is just the other way round. In both cases the above definition thus differs from the bosonic one, even if the matrix is purely bosonic (but having two fermionic indices). Let us derive this in detail.

Consider the decomposition of B in bosonic and fermionic blocks:

$$(B^M{}_N) \equiv \begin{pmatrix} B^m{}_n & B^m{}_\nu \\ B^\mu{}_n & B^\mu{}_\nu \end{pmatrix} \equiv \begin{pmatrix} a^m{}_n & b^m{}_\nu \\ c^\mu{}_n & d^\mu{}_\nu \end{pmatrix}, \quad |\, m \,|= 0, |\, \mu \,|= 1 \tag{2.97}$$

Assuming that the matrix (a) is invertible (which implies that a (and thus the rumpf of B) is bosonic, because a matrix with purely fermionic entries cannot be inverted), one can seperate C in a product of two block-triangular matrices

$$B = B_1 B_2 \tag{2.98}$$

$$B_1 = \begin{pmatrix} a & 0 \\ c & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & (a^{-1}b) \\ 0 & d - ca^{-1}b \end{pmatrix} \tag{2.99}$$

Now we will use two facts. One is that the trace of the logarithm factorizes:

$$e^F e^G \overset{BCH}{=} e^{F+G+\frac{1}{2}[F,G]+\dots} \tag{2.100}$$

$$B_1 B_2 = e^{\ln B_1 + \ln B_2 + \frac{1}{2}[\ln B_1, \ln B_2]+\dots} \tag{2.101}$$

$$\Rightarrow \mathrm{tr}\ \ln(B_1 B_2) \overset{(2.93)}{=} \mathrm{tr}\, \ln B_1 + \mathrm{tr}\, \ln B_2 \tag{2.102}$$

And the other fact is that an arbitrary power of a block-triangular matrix stays a blocktriangular matrix with the powers of the diagonal blocks in the block diagonal:

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ * & c^n \end{pmatrix} \tag{2.103}$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^n = \begin{pmatrix} a^n & * \\ 0 & d^n \end{pmatrix} \quad \forall a, b, c, d \tag{2.104}$$

In particular

$$
\begin{aligned}
(B_1 - \mathbb{1})^n &= \begin{pmatrix} (a-\mathbb{1})^n & 0 \\ * & 0 \end{pmatrix} && (2.105)\\
(B_2 - \mathbb{1})^n &= \begin{pmatrix} 0 & 0 \\ * & (d-ca^{-1}b-\mathbb{1})^n \end{pmatrix} && (2.106)
\end{aligned}
$$

Now we use the power series for the logarithm

$$
\begin{aligned}
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{1}{n!} \ln^{(n)}(1) x^n = \sum_{n=1}^{\infty} (-)^{n-1} \frac{x^n}{n} && (2.107)\\
\mathrm{tr}\ \ln(B_1) &= \sum_{n=1}^{\infty} (-)^{n-1} \frac{\mathrm{tr}\,(B_1-\mathbb{1})^n}{n} = && (2.108)\\
&= \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} \mathrm{tr} \begin{pmatrix} (a-\mathbb{1})^n & 0 \\ * & 0 \end{pmatrix} = && (2.109)\\
&= \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} \mathrm{tr}\,(a-\mathbb{1})^n = && (2.110)\\
&= \mathrm{tr}\ \ln a && (2.111)\\
\mathrm{tr}\ \ln(B_1) &= \mathrm{tr}\ \ln(d - ca^{-1}b) && (2.112)
\end{aligned}
$$

We thus get

$$
\begin{aligned}
\det B &= \det B_1 \cdot \det B_2 = && (2.113)\\
&= \det a \cdot \det(d - ca^{-1}b) && (2.114)
\end{aligned}
$$

This result is true for every block-decomposition. a, d do not necessarily have to be bosonic as well as b and c do not have to be fermionic. At first sight this seems to contradict the expression that one usually finds in the literature, namely $\mathrm{sdet} B = \det a / \det(d - ca^{-1}b)$.

The reason for this mismatch lies simply in the graded definition of the matrix multiplication (or the trace) and thus of the determinant of a bosonic matrix with two fermionic indices. For NE-conventions, the trace of the type-B submatrix ($d^{\mu}{}_{\nu}$) gives an extra minus w.r.t. its naive bosonic trace. Its determinant defined via the exponential and the graded trace is thus equal to "$1/\det(d)$", where now the determinant is the naive bosonic one, built with the naive trace. The same is true, if we consider the corresponding submatrices of a matrix of type C in NW-conventions. For the determinant of a matrix of type B in NW (or likewise type C in NE), however, the comparison between our and the usual convention is a bit more subtle. In the following we write terms in the usual convention in quotation marks. At first, let us define the **dimension** of a square matrix (or of the vector space it is acting on) as the trace of the corresponding unit-matrix:

$$
\begin{aligned}
\dim(B) &\equiv \delta^M{}_M = \text{"}\dim(a) - \dim(d)\text{"} && (2.115)\\
\dim(d) &= \text{"} -\dim(d)\text{"} && (2.116)
\end{aligned}
$$

I.e., fermionic dimensions are negative dimensions![6] The logarithm in the definition of the determinant has to be understood as a power series, so that we first should look at simple powers of the block d:

$$
\begin{aligned}
d^{2\mu}{}_{\nu} &= d^{\mu}{}_{\lambda} d^{\lambda}{}_{\nu} = && (2.117)\\
&\overset{NW}{=} \sum_{\lambda} d^{\mu}{}_{\lambda} d^{\lambda}{}_{\nu} (-)^{\lambda} && (2.118)\\
\Rightarrow d^n &= \text{"}(-1)^{n-1} d^n = -(-d)^n\text{"} \quad \text{naive matrix mult in quot} && (2.119)
\end{aligned}
$$

Logarithm and determinant of $d^{\mu}{}_{\nu}$ can thus be written as

$$
\begin{aligned}
\ln(d) &= \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} (d-\mathbb{1})^n \underset{\text{and } (2.119)}{\overset{\mathbb{1}=\text{"}-\mathbb{1}\text{"}}{=}} && (2.120)\\
&\underset{\text{and } (2.119)}{\overset{\mathbb{1}=\text{"}-\mathbb{1}\text{"}}{=}} \text{"} -\sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} (-d-\mathbb{1})^n\text{"} && (2.121)\\
&= \text{"} -\ln(-d)\text{"} \quad \text{naive matrix mult in quot} && (2.122)\\
\det(d) &= \exp \mathrm{tr}\ \ln d = \text{"}1/\det(-d) = (-1)^{\dim(d)} 1/\det d\text{"} && (2.123)
\end{aligned}
$$

[6] The observation that fermionic dimensions can be considered to be negative dimensions has been made in literature at several places and with several arguments. From the group theoretic point of view, this has been studied in [23, 24]. ⋄

The sub-matrix $(d - ca^{-1}b)$ is of the same type as d, so that we finally get

$$\det(d - ca^{-1}b) \overset{a^{-1}="a^{-1}"}{\underset{ca^{-1}b="ca^{-1}b"}{=}} "(-1)^{\dim(d)} 1/\det(d - ca^{-1}b)" \tag{2.124}$$

$$\det B = "(-1)^{\dim(d)} \det a/\det(d - ca^{-1}b)" \quad \text{naive matrix mult in quot} \tag{2.125}$$

For matrices of type C in NW-convention, the situation is the same as for matrices of type B in NE-convention:$d^n = "d^n"$, $\mathbb{1}_d = "\mathbb{1}_d"$, $\ln d = "\ln d"$, $\operatorname{tr} \ln d = " - \operatorname{tr} \ln d"$. We thus get

$$\boxed{\det B = \det a \cdot \det(d - ca^{-1}b) = \begin{cases} "(-1)^{\dim(d)} \det a/\det(d - ca^{-1}b)" & \text{NW} \\ " \det a/\det(d - ca^{-1}b)" & \text{NE} \end{cases}} \tag{2.126}$$

$$\text{for } B^M{}_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^M{}_N \tag{2.127}$$

and

$$\boxed{\det C = \det a \cdot \det(d - ca^{-1}b) = \begin{cases} " \det a/\det(d - ca^{-1}b)" & \text{NW} \\ "(-1)^{\dim(d)} \det a/\det(d - ca^{-1}b)" & \text{NE} \end{cases}} \tag{2.128}$$

$$\text{for } C_M{}^N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_M{}^N \tag{2.129}$$

As a check, let us take $C = B^T = \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} = " \begin{pmatrix} a^T & c^T \\ b^T & -d^T \end{pmatrix} "$. Then we expect, following our theorem:

$$\det B = \det B^T \tag{2.130}$$

Indeed, in NW-conventions this becomes in naive matrix-notations:

$$"(-1)^{\dim(d)} \det(d - ca^{-1}b)" \overset{!}{=} " \det(-d^T - b^T(a^{-1})^T c^T)" = \tag{2.131}$$

$$= " \det\left(-d^T - (-)^{cb} ca^{-1}b\right)^T " = \tag{2.132}$$

$$= " \det\left(-d + ca^{-1}b\right) " = \tag{2.133}$$

$$= "(-1)^{\dim(d)} \det(d - ca^{-1}b)" \quad \surd \tag{2.134}$$

2.7 Graded gamma-matrices

Gamma matrices and some of their properties are discussed in appendix D on page 167. Usually, they are considered to be ordinary bosonic matrices with the anticommutator relation

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1} \tag{2.135}$$

There are two ways how a grading can be introduced into the gamma-matrix algebra. Either via the rumpf or via the indices. Let us start with the rumpf.

The anticommutator is for general matrices not a very natural object. It does not automatically have derivative properties or a Jacobi identity like the commutator. However, the gamma matrices can (in even dimensions) be represented by *off-diagonal* matrices. This offers the possibility to regard them as fermionic supermatrices $\mathbf{\Gamma}^a$ whose fermionic diagonal blocks simply vanish. The anticommutator above then simply becomes the graded commutator

$$\left[\mathbf{\Gamma}^a, \mathbf{\Gamma}^b\right] = 2\eta^{ab}\mathbb{1} \tag{2.136}$$

Terms like $\bar{\psi}\mathbf{\Gamma}^a\partial_a\boldsymbol{\psi}$ in a Lagrangian still stay bosonic, because $\bar{\psi} = \boldsymbol{\psi}^\dagger\mathbf{\Gamma}^0$ contains another odd gamma-matrix. This interpretation of a graded algebra appears naturally in the RNS-string, where the spacetime spinors are generated by acting with fermionic creation operators on a Clifford vacuum. Linear combinations of these odd creation operators then correspond to the (odd) gamma matrices.

It is interesting that in the graded picture the chirality matrix plays a different role than the other gamma-matrices, because (as a product of all gamma-matrices in even dimensions) it is an even object $\Gamma^\# \propto \mathbf{\Gamma}^0 \cdots \mathbf{\Gamma}^{d-1}$. The anticommutation of it with the other matrices stays an anticommutation even in the graded picture

$$\{\Gamma^\#, \mathbf{\Gamma}^a\} = 0, \quad \{\Gamma^\#, \Gamma^\#\} = 2\mathbb{1} \tag{2.137}$$

This is actually also a hint that s.th. like the RNS string could not work in the same way in odd (e.g. 11) dimensions, where one of the gamma-matrices (and thus one of the generators acting on the clifford vacuum) needs to be even.

The second possibility to re-distribute the grading, is to consider the fermionic (Dirac) indices of $\mathbf{\Gamma}^{a\,\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\beta}}}$ to carry an odd grading. (The underline simply shall distinguish the Dirac-indices from Weyl indices, which are mainly used later on.) As the fermionic indices come in pairs it does not change the overall grading. We still assume the rumpf to be odd, too. The graded commutator then becomes (in NW-conventions)

$$\left[\mathbf{\Gamma}^a, \mathbf{\Gamma}^b\right]^{\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\beta}}} \equiv \mathbf{\Gamma}^{a\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\gamma}}}\mathbf{\Gamma}^{b\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\beta}}} + \mathbf{\Gamma}^{b\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\gamma}}}\mathbf{\Gamma}^{a\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\beta}}} = \tag{2.138}$$

$$= \sum_{\underline{\boldsymbol{\gamma}}} \underbrace{(-)^{\underline{\boldsymbol{\gamma}}+\underline{\boldsymbol{\gamma}}\mathbf{\Gamma}}}_{1} \left(\mathbf{\Gamma}^{a\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\gamma}}}\mathbf{\Gamma}^{b\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\beta}}} + \mathbf{\Gamma}^{b\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\gamma}}}\mathbf{\Gamma}^{a\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\beta}}}\right) = \tag{2.139}$$

$$= 2\eta^{ab}\underbrace{\left(-\delta^{\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\beta}}}\right)}_{\delta^{\alpha}_{\beta}} \tag{2.140}$$

The algebra thus changes the sign. It would not do so, however, if we would grade only the indices and not the rumpfs. In any case, in appendix D on page 167 we took the conventional point of view of ordinary gamma-matrices with ungraded indices, because people are more familiar with the equations in the conventional picture. For our application to the Berkovits string in the second part of this thesis, it is then necessary to make a grading-shift in the indices to get the correct equations. However, for future applications in superspace it might be more favourable to have all the equations in the graded picture with graded rumpfs and indices. In this picture it would also be more natural (though it was not done in this thesis) to adjust the definition of the antisymmetrized products of gamma matrices according to the graded summation. E.g. $\Gamma^{a_1 a_2\,\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\beta}}} \equiv_g \mathbf{\Gamma}^{[a_1|\,\underline{\boldsymbol{\alpha}}}{}_{\underline{\boldsymbol{\gamma}}}\mathbf{\Gamma}^{|a_2]\,\underline{\boldsymbol{\gamma}}}{}_{\underline{\boldsymbol{\beta}}}$ with the graded summation convention and the graded equal sign instead of the ordinary ones.

Chapter 3

Other Applications and Some Subtleties

3.1 Left and right derivative

Bosonic rumpfs

In the bosonic case we have for a variation of some function

$$\delta f(x) \quad = \quad \delta x^m \frac{\partial}{\partial x^m} f = \underbrace{f \frac{\overleftarrow{\partial}}{\partial x^m}}_{\partial f/\partial x^m} \delta x^m \tag{3.1}$$

There is no difference between left and right derivative here, except that we write it either on the left or on the right of the function.

$$\frac{\partial}{\partial x^m} f \quad = \quad \partial f/\partial x^m \tag{3.2}$$

For the graded case with bosonic rumpfs, the situation is very similar. We define (using graded summation; no need for graded equal in the beginning, as there are no naked indices, but in the third equation it is essential)

$$\delta f(x) \equiv_g \delta x^M \frac{\partial}{\partial x^M} f \quad \equiv_g \quad \partial f/\partial x^M \delta x^M \tag{3.3}$$

$$\Rightarrow \quad 0 \quad =_g \quad \delta x^M \left(\frac{\partial}{\partial x^M} f - \partial f/\partial x^M \right) \tag{3.4}$$

$$\Rightarrow \quad \frac{\partial}{\partial x^M} f \quad =_g \quad \partial f/\partial x^M \quad \Longleftrightarrow \quad \frac{\partial}{\partial x^M} f = (-)^{fM} \partial f/\partial x^M \tag{3.5}$$

For $f = x^M$ we have

$$\delta x^M \quad = \quad \delta x^K \frac{\partial}{\partial x^K} x^M = \partial x^M/\partial x^K \delta x^K \tag{3.6}$$

$$\Rightarrow \frac{\partial}{\partial x^K} x^M \quad = \quad \delta_K{}^M \tag{3.7}$$

$$\partial x^M/\partial x^K \quad = \quad \delta^M{}_K \tag{3.8}$$

In the case of coordinates with bosonic rumpf, we will also use the following symbols for derivatives

$$\partial_M f \quad \equiv \quad \frac{\partial f}{\partial x^M} \equiv \frac{\partial}{\partial x^M} f \tag{3.9}$$

$$T_{MN,K} \quad \equiv \quad T_{MN} \frac{\overleftarrow{\partial}}{\partial x^K} \equiv \partial T_{MN}/\partial x^K = (-)^{K(T+M+N)} \partial_K T_{MN} \tag{3.10}$$

We will not use the notation ∂_M for derivatives with respect to ghosts or other objects with rumpf of odd or undetermined grading, as the rumpf becomes invisible.

Graded rumpfs

For fermionic indices $\boldsymbol{\alpha}$ the above equations imply

$$\frac{\partial}{\partial x^{\boldsymbol{\alpha}}} f \quad = \quad (-)^f \partial f/\partial x^{\boldsymbol{\alpha}} \tag{3.11}$$

$$\frac{\partial}{\partial x^{\boldsymbol{\alpha}}} x^{\boldsymbol{\beta}} \quad = \quad -\partial x^{\boldsymbol{\beta}}/\partial x^{\boldsymbol{\alpha}} = \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{3.12}$$

This would for fermionic objects $\boldsymbol{c}$ without indices also suggest to define left and right derivative such that

$$\frac{\partial}{\partial \boldsymbol{c}}\boldsymbol{c} \overset{?}{\equiv} -\partial\boldsymbol{c}/\partial\boldsymbol{c} \tag{3.13}$$

However, written without indices it is less intuitive and also not common. We thus follow the literature and use the following definition of **left derivative** and **right derivative** (now for c being of undetermined grading $\mid c \mid$)

$$\delta F(c) \equiv \delta c \frac{\partial}{\partial c} F(c) \equiv \partial F(c)/\partial c\, \delta c \tag{3.14}$$

$$\frac{\partial}{\partial c} F(c) = (-)^c (-)^{Fc} \partial F(c)/\partial c \tag{3.15}$$

$$\frac{\partial}{\partial c} c = \partial c/\partial c = 1 \tag{3.16}$$

Although (3.14) and (3.16) seem to be quite intuitive, (3.15) unfortunately is less intuitive. The factor $(-)^{Fc}$ is expected, because we interchange the order of F and the derivative with respect to c. This factor could be absorbed by using the big graded equal sign. The extra factor $(-)^c$, however, stems from the fact that in (3.14) the order of $\partial/\partial c$ and δc is exchanged, and the big graded equal sign cannot figure that out, so that (3.15) becomes $\frac{\partial}{\partial c}F(c) \overset{?}{=}_G (-)^c \partial F(c)/\partial c$. Thus for graded rumpfs, left and right derivative are simply not the same operation (just written in a different order), but they differ by a sign depending on the grading of the rumpf. The above definition is thus not simply a gradifcation of a bosonic one. Indeed the rumpf 'c' was not gradifiable from the beginning. If one wants to use statements derived via the theorem, one has to introduce an extra index which carries the grading, like in (3.11).

The generalization to the case with graded indices, however, is straight-forward again:

$$\frac{\partial}{\partial c^K} F(c) =_g (-)^c (-)^{Fc} \partial F(c)/\partial c^K \qquad \frac{\partial}{\partial c^K} F(c) = (-)^{FK} (-)^{c+cF} \partial F(c)/\partial c^K \tag{3.17}$$

$$\frac{\partial}{\partial c^M} c^N =_g \delta_M{}^N \qquad (-)^{cM} \frac{\partial}{\partial c^M} c^N = \delta_M{}^N \quad \left(\overset{NW}{=} \delta_M^N\right) \tag{3.18}$$

$$\partial c^M/\partial c^N =_g \delta^M{}_N =_g \delta_M{}^N \qquad (-)^{cM} \partial c^M/\partial c^N = \delta^M{}_N \tag{3.19}$$

$$\partial c^M/\partial c^N =_g \frac{\partial}{\partial c^N} c^M \qquad (-)^{cM} \partial c^M/\partial c^N = (-)^{cN+NM} \frac{\partial}{\partial c^N} c^M \tag{3.20}$$

This implies (using as always the graded summation convention)

$$\delta F(c) = \delta c^K \frac{\partial}{\partial c^K} F(c) = \partial F(c)/\partial c^K\, \delta c^K \tag{3.21}$$

3.2 Tensor and wedge product

Let us consider the wedge product

$$\mathbf{d}x^m \mathbf{d}x^n \equiv \mathbf{d}x^m \wedge \mathbf{d}x^n \equiv \frac{1}{2}\left(\mathbf{d}x^m \otimes \mathbf{d}x^n - \mathbf{d}x^n \otimes \mathbf{d}x^m\right) \tag{3.22}$$

(The normalization $\frac{1}{2}$ implies that p-forms are written as $\omega^{(p)} = \omega_{m_1 \ldots m_p} \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_p}$ without the usual prefactor $\frac{1}{p!}$.) The wedge product is antisymmetric if x^m are the coordinates of a bosonic manifold. If one considers $\mathbf{d}x^m$ to be an odd object (w.r.t. the form grading), the wedge product is a graded commuting product. As x^m itself is even, the grading has to sit in '$\mathbf{d}$', and it is therefore printed boldface. The form grading is a priori independent from the Fermion grading but one can consistently combine them to have only a single $\mathbb{Z}_2$ grading, where e.g. an odd differential form which is at the same time Fermionic is considered to be even. We will take exactly this point of view throughout the thesis, although one should keep in mind that it is especially fitted to the exterior algebra of forms. One can certainly define a symmetrized tensor product as well, for which it would be more natural to consider dx^m as an even object. However, it plays a less important role than the wedge product. As argued already in the very beginning, it does not really matter which point of view one takes, as the use of graded equal sign and graded summation convention swallows all of the signs anyway. One can therefore do all of the calculations without fixing this issue and only in the end choose one or another version of graded summation or graded equal sign.

Let us now consider some tensor of rank $(2,1)$:

$$T^{(2,1)} = T_{mn}{}^k \mathbf{d}x^m \otimes \mathbf{d}x^n \otimes \boldsymbol{\partial}_k \tag{3.23}$$

Already before bringing any Fermion-grading into the game, we have a graded equation which should match our philosophy of notations. The grading on both sides is $\mid T^{(2,1)} \mid = \mid T \mid + 2 \mid \mathbf{d} \mid + \mid \boldsymbol{\partial} \mid$. It is therefore essential that we do not denote the tensor simply by T, because then the tensor T is odd while the rumpf T is even which would lead to confusions. The superscript '(2,1)' therefore should carry the grading $2 \mid \mathbf{d} \mid + \mid \boldsymbol{\partial} \mid$ of the basis elements. Although we might not always write this superscript, it is always understood that $\mid T \mid$ is the grading of the rumpf and not of the tensor.

All the indices in the above equation are dummy indices and are thus gradifiable. The rumpf T appears in every term exactly once (with the above explanation) and is thus gradifiable as well. The rumpf x, instead, is not gradifiable. The gradification of the tensor definition reads

$$T^{(2,1)} =_G T_{MN}{}^K \mathbf{d}x^M \otimes \mathbf{d}x^N \otimes \boldsymbol{\partial}_K$$

$$\begin{aligned} T^{(2,1)} &\overset{NW}{=} \sum_{M,N,K} (-)^{M+N}(-)^{M(N+K)+NK}(-)^{M\mathbf{d}+K\boldsymbol{\partial}} \times \\ &\quad \times T_{MN}{}^K \mathbf{d}x^M \otimes \mathbf{d}x^N \otimes \boldsymbol{\partial}_K \end{aligned} \tag{3.24}$$

A two form e.g. takes the following form:

$$\omega^{(2)} \equiv \omega_{MN}\mathbf{d}x^M \wedge \mathbf{d}x^N \overset{NW}{=} \sum_{M,N} (-)^{MN+N}\omega_{MN}\mathbf{d}x^M \wedge \mathbf{d}x^N \tag{3.25}$$

The grading of a p-form $\omega^{(p)}$ is $\mid \omega^{(p)} \mid = \mid \omega \mid +p$ and the graded Leibniz rule for the exterior derivative acting on the wedge product $\omega^{(p)}\eta^{(q)} \equiv \omega^{(p)} \wedge \eta^{(q)}$ thus reads

$$\mathbf{d}(\omega^{(p)}\eta^{(q)}) =_G \mathbf{d}\omega^{(p)}\eta^{(q)} + \omega^{(p)}\mathbf{d}\eta^{(q)}$$

$$\mathbf{d}(\omega^{(p)}\eta^{(q)}) = \mathbf{d}\omega^{(p)}\eta^{(q)} + (-)^{|\omega|+p}\omega^{(p)}\mathbf{d}\eta^{(q)} \tag{3.26}$$

A subtle counterexample to the theorem Gradification of the exterior algebra is subtle, because we start with something anticommuting and turn it in something commuting, which is less restrictive. One of the problems one meets is the observation that there is no gradification of the definition of the epsilon tensor, which provides the volume form in the bosonic case. The more severe problem is the related to the nilpotency of 1-forms:

We start from the gradifiable anticommutativity equation $\mathbf{d}x^{m_1}\mathbf{d}x^{m_2} = -\mathbf{d}x^{m_1}\mathbf{d}x^{m_2}$ (the indices are gradifiable) and the gradifiable definition of the dimension $d \equiv \delta_m{}^m$. In the bosonic case it follows that $\mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_{d+1}} = 0$. Also this last equation is gradifiable in the indices but is wrong in the general graded case and thus seems to contradict our theorem. But the theorem includes also intermediate equations into the gradification. In the above case, the reasoning goes from $\mathbf{d}x^{m_1}\mathbf{d}x^{m_2} = -\mathbf{d}x^{m_1}\mathbf{d}x^{m_2}$ via $\mathbf{d}x^m\mathbf{d}x^m = 0$ (no sum) to the conclusion $\mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_{d+1}} = 0$. In the intermediate equation $\mathbf{d}x^m\mathbf{d}x^m = 0$, the index m is not gradifiable.

Originally there was the hope that intermediate equations are irrelevant. In particular, if all indices are fermionic, the dimension is negative. The condition $\mathbf{d}x^{\mu_1} \cdots \mathbf{d}x^{\mu_{d+1}} = 0$ then simply would not be a restriction and everything is fine. For mixed fermionic and bosonic variables, however, this mechanism breaks down.

It might be that including intermediate equations in the gradification can be omitted by saying that an index is only gradifiable if the number of copies in which it appears does not exceed the dimension. We leave this for future studies.

3.3 Graded Poisson bracket

For bosonic rumpfs 'q' and 'p' of the phase space variables q^M and p_M, the bosonic Poisson bracket is easily generalized to the graded case. The overall sign, i.e. whether one first takes the derivative with respect to the momenta p_M and then with respect to the configuration space variables q^M or the other way round is already an ambiguity at the bosonic level and is only a matter of taste. As it is just an overall sign, it is easily changed if preferred differently. Our choice (p_M first) was made in order to have the Hamiltonian as the generator of time translations on the left of the bracket. We always try to let generators or operators act from the left. In any case the graded Poisson bracket is a simple gradification of the bosonic one:

$$\begin{aligned}
\{F,G\} &\equiv \partial F/\partial p_M \frac{\partial}{\partial q^M}G - \partial F/\partial q^M \frac{\partial}{\partial p_M}G = && (3.27)\\
&= \partial F/\partial p_M \frac{\partial}{\partial q^M}G - (-)^{FG}\partial G/\partial p_M \frac{\partial}{\partial q^M}F = && (3.28)\\
&= \frac{\partial}{\partial p_M}F\frac{\partial}{\partial q^M}G - \frac{\partial}{\partial q^M}F\frac{\partial}{\partial p_M}G && (3.29)\\
\{F,G\} &= -(-)^{FG}\{G,F\} && (3.30)\\
\{p_M, q^N\} &= \delta_M{}^N \overset{NW}{=} \delta_M^N && (3.31)\\
\{q^M, p_N\} &= -\delta^M{}_N \overset{NW}{=} -(-)^M\delta_M^N && (3.32)
\end{aligned}$$

Like always, the sum over the index 'M' has to be understood as graded sum. The left and right-derivative with respect to variables with bosonic rumpfs coincide (w.r.t. the graded equal sign) and the generalization is therefore unique, as soon as the underlying summation convention (NW or NE) is chosen. The sign $(-)^{FG}$ in the second and fourth line of the above equation array would disappear upon the use of the big graded equal sign. The rumpfs 'q' and 'p' are a priori not gradifiable in these equations.

Nevertheless the *case of graded rumpfs* 'q' and 'p' can be covered by just gradifying the indices. Assume for example that we have in addition to q^M and p_M (with bosonic rumpfs) also some ghost variables $\boldsymbol{c}^M$ and $\boldsymbol{b}_M$ with the same indices. In general, the indices of ghost variables would just cover a subset of the index range of the original phase space, but this subtlety does not matter for the present discussion. The rumpfs of the ghost variables carry a grading and it is thus not uniquely fixed how to extend the definition of the Poisson bracket to the ghost variables. A natural way (having in mind the conditions for our theorem) is to introduce some variables with two indices z^{iM} containing q^M as well as $\boldsymbol{c}^M$ and the same for the momenta:

$$z^{iM} \equiv (q^M, \boldsymbol{c}^M), \quad z^{1M} = q^M, \quad z^{\mathbf{2}M} = \boldsymbol{c}^M \tag{3.33}$$

$$\pi_{iM} \equiv (p_M, \boldsymbol{b}_M), \quad \pi_{1M} = p_M, \quad \pi_{\mathbf{2}M} = \boldsymbol{b}_M \tag{3.34}$$

The grading is now sitting in the additional index i, i.e. $\mid i \mid = \begin{cases} 0 \text{ for } i=1 \\ 1 \text{ for } i=2 \end{cases}$. One still has the freedom to decide whether this index should be upstairs or downstairs for z or equivalently whether we choose NW or NE for the graded summation of this index. Choosing the position as above and NW for the summation yields

$$z^{iM}\pi_{iM} = \sum_{i,M}(-)^{iM} z^{iM}\pi_{iM} = \sum_{i,M}\left(q^M p_M + (-)^M \boldsymbol{c}^M \boldsymbol{b}_M\right) = q^M p_M + \boldsymbol{c}^M \boldsymbol{b}_M \tag{3.35}$$

$$\pi_{iM} z^{iM} = \sum_{i,M}(-)^{iM+i+M}\pi_{iM} z^{iM} = \sum_{i,M}\left((-)^M p_M q^M - \boldsymbol{b}_M \boldsymbol{c}^M\right) = p_M q^M - \boldsymbol{b}_M \boldsymbol{c}^M \tag{3.36}$$

Note the sign change of the last term from the first to the second line. Now we can also write down the graded Poisson bracket for this case, which looks in terms of the variables (z^{iM}, π_{iM}) the same as the one before in terms of (q^M, p_M), but contains an additional graded sum over the index i:

$$\{F,G\} \equiv \partial F/\partial\pi_{iM}\frac{\partial}{\partial z^{iM}}G - \partial F/\partial z^{iM}\frac{\partial}{\partial \pi_{iM}}G = \tag{3.37}$$

$$\overset{NW}{=} \sum_{i,M}(-)^{iM}\partial F/\partial\pi_{iM}\frac{\partial}{\partial z^{iM}}G - (-)^{iM+i+M}\partial F/\partial z^{iM}\frac{\partial}{\partial \pi_{iM}}G \tag{3.38}$$

Before we rewrite this Poisson bracket in terms of $q^M, p_M, \boldsymbol{c}^M$ and $\boldsymbol{b}_M$, let us recall the definition of left and right-derivative of page 28. With the graded equal sign, left and right derivative w.r.t. z^{iM} are simply given by $\frac{\partial}{\partial z^{iM}} z^{jN} =_g \delta_i{}^j \delta_M{}^N =_g \partial z^{jN}/\partial z^{iM}$. The same is true for the derivatives w.r.t. π_{iM}. Written with the ordinary equal sign, this reads

$$\frac{\partial}{\partial z^{iM}} z^{jN} = (-)^{jM} \underbrace{\delta_i{}^j}_{NW:\delta_i^j} \delta_M{}^N = (-)^{(j+N)(i+M)}\partial z^{jN}/\partial z^{iM} \tag{3.39}$$

$$\frac{\partial}{\partial \pi_{iM}} \pi_{jN} = (-)^{jM} \underbrace{\delta^i{}_j}_{NW:-\delta_j^i} \delta^M{}_N = (-)^{(j+N)(i+M)}\partial \pi_{jN}/\partial \pi_{iM} \tag{3.40}$$

For $i = j = 1$ this agrees perfectly with the definition of left and right derivative w.r.t. q^M or p_M. For $i = j = 2$ instead (remember $z^{\mathbf{2}M} = \boldsymbol{c}^M$ and $\pi_{\mathbf{2}M} = \boldsymbol{b}_M$), we observe some mismatch (in NW for the right-derivative w.r.t. $\boldsymbol{c}^M$ and for the left-derivative w.r.t. $\boldsymbol{b}_M$, in NE the other way round)

$$\frac{\partial}{\partial \boldsymbol{c}^M}\boldsymbol{c}^N = (-)^M \delta_M{}^N = (-)^{M+N+MN}\partial \boldsymbol{c}^N/\partial \boldsymbol{c}^M \leftrightarrow$$
$$\leftrightarrow \frac{\partial}{\partial z^{\mathbf{2}M}} z^{\mathbf{2}N} = (-)^M \underbrace{\delta_{\mathbf{2}}{}^{\mathbf{2}}}_{NW:1} \delta_M{}^N = -(-)^{N+M+NM}\partial z^{\mathbf{2}N}/\partial z^{\mathbf{2}M} \tag{3.41}$$

$$\frac{\partial}{\partial \boldsymbol{b}_M}\boldsymbol{b}_N = (-)^M \delta^M{}_N = (-)^{M+N+MN}\partial \boldsymbol{b}_N/\partial \boldsymbol{b}_M \leftrightarrow$$
$$\leftrightarrow \frac{\partial}{\partial \pi_{\mathbf{2}M}} \pi_{\mathbf{2}N} = (-)^M \underbrace{\delta^{\mathbf{2}}{}_{\mathbf{2}}}_{NW:-1} \delta^M{}_N = -(-)^{M+N+MN}\partial \pi_{\mathbf{2}N}/\partial \pi_{\mathbf{2}M} \tag{3.42}$$

The definition of left and right derivative therefore depends on the notation we use ($\boldsymbol{c}^M, \boldsymbol{b}_M$ or $z^{\mathbf{2}M}, \pi_{\mathbf{2}M}$). In NW-conventions (for the index i) we have

$$\frac{\partial}{\partial \boldsymbol{c}^M} \overset{NW}{=} \frac{\partial}{\partial z^{\mathbf{2}M}}, \quad \frac{\overleftarrow{\partial}}{\partial \boldsymbol{c}^M} \overset{NW}{=} -\frac{\overleftarrow{\partial}}{\partial z^{\mathbf{2}M}} \tag{3.43}$$

$$\frac{\partial}{\partial \boldsymbol{b}_M} \overset{NW}{=} -\frac{\partial}{\partial \pi_{\mathbf{2}M}}, \quad \frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_M} \overset{NW}{=} \frac{\overleftarrow{\partial}}{\partial \pi_{\mathbf{2}M}} \tag{3.44}$$

In NE conventions (for the index i), we would have the opposite signs. In the Poisson bracket, these signs always cancel (for NW and for NE), because the left derivative w.r.t. $\boldsymbol{b}_M$ comes with the right derivative w.r.t. $\boldsymbol{c}^M$ and vice verse. Looking at (3.38) one can see that the only additional sign which is not absorbed by the graded summation of the index M is the $(-)^i$ in the second term due to the 'wrong' contraction direction. This sign would come with the first term, if we had NE conventions for the index i. The **Poisson bracket** given before in terms of z^{iM} and π_{iM} can therefore be rewritten (in graded summation conventions) as

$$\boxed{\begin{aligned}\{F,G\} &= \partial F/\partial p_M \frac{\partial}{\partial q^M}G - \partial F/\partial q^M \frac{\partial}{\partial p_M}G \pm \left(\partial F/\partial \boldsymbol{b}_M \frac{\partial}{\partial \boldsymbol{c}^M}G + \partial F/\partial \boldsymbol{c}^M \frac{\partial}{\partial \boldsymbol{b}_M}G\right) = &(3.45)\\ &= \partial F/\partial p_M \frac{\partial}{\partial q^M}G - (-)^{FG}\partial G/\partial p_M \frac{\partial}{\partial q^M}F \pm \left(\partial F/\partial \boldsymbol{b}_M \frac{\partial}{\partial \boldsymbol{c}^M}G - (-)^{FG}\partial G/\partial \boldsymbol{b}_M \frac{\partial}{\partial \boldsymbol{c}^M}F\right) &(3.46)\end{aligned}}$$

The upper sign is for the choice of NW-conventions for the index i while the lower sign is for NE. This is in principle independent of the summation convention for the index M. If one prefers overall NE, where the minus in front of the bracket might be annoying, it might be more natural to define the Poisson bracket with an overall minus (or take NW only for the index i). If one wants to apply the gradification theorem in order to derive true statements about the graded Poisson bracket, it is in principle necessary to reintroduce the extra index i which carries the grading and rewrite the result again in terms of the graded rumpfs after having applied the theorem. In practice this is rarely necessary. For example, in order to show the Jacobi identity for the graded Poisson bracket, it is enough to know that one can write it as a gradification of a bosonic Poisson bracket. The Jacobi identity itself does not explicitely contain the variables z^{iM} and therefore has the same form in terms of the variables q^M and $\boldsymbol{c}^M$. The same is true for Leibniz rule when acting on products of phase space functions:

$$\{F,\{G,H\}\} = \{\{F,G\},H\} + (-)^{FG}\{G,\{F,H\}\} \tag{3.47}$$
$$\{F,GH\} = \{F,G\}H + (-)^{FG}G\{F,H\} \tag{3.48}$$

The sign $(-)^{FG}$ would disappear when using the big graded equal sign. Let us now fix the sign-ambiguity in (3.46). We will throughout use the more convenient ***upper sign*** for the definition of the Poisson bracket. This implies

$$\{F,G\} = -(-)^{FG}\{G,F\} \tag{3.49}$$
$$\{\boldsymbol{b}_M,\boldsymbol{c}^N\} =_g \delta_M{}^N, \quad \{p_M,q^M\} =_g \delta_M{}^N \tag{3.50}$$
$$\{\boldsymbol{c}^M,\boldsymbol{b}_N\} =_g \delta^M{}_N, \quad \{q^M,p_N\} =_g -\delta^M{}_N \tag{3.51}$$

Note again that this does not fix the summation convention for the index M. We had only made a convenient choice for the auxiliary index i which is now absent anyway. The above equations further imply

$$\{\boldsymbol{b}_M,\ldots\} = \frac{\partial}{\partial \boldsymbol{c}^M}(\ldots), \quad \{p_M,\ldots\} = \frac{\partial}{\partial q^M}(\ldots) \tag{3.52}$$
$$\{\ldots,\boldsymbol{b}_M\} = \partial(\ldots)/\partial \boldsymbol{c}^M, \quad \{\ldots,p_M\} = -\partial(\ldots)/\partial q^M \tag{3.53}$$

$$\{\boldsymbol{c}^M,\ldots\} = \frac{\partial}{\partial \boldsymbol{b}_M}(\ldots), \quad \{q^M,\ldots\} = -\frac{\partial}{\partial p_M}(\ldots) \tag{3.54}$$
$$\{\ldots,\boldsymbol{c}^M\} = \partial(\ldots)/\partial \boldsymbol{b}_M, \quad \{\ldots,q^M\} = \partial(\ldots)/\partial p_M \tag{3.55}$$

Antibracket A bracket which is closely related to the Poisson bracket is the antibracket. It is defined in an extended configuration space with as many odd variables (antifields) $\boldsymbol{q}^+_M$ as even variables q^M:

$$(F,G) = \partial F/\partial \boldsymbol{q}^+_M \frac{\partial}{\partial q^M}G - (-)^{(F+1)(G+1)}\partial G/\partial \boldsymbol{q}^+_M \frac{\partial}{\partial q^M}F \tag{3.56}$$

Note that this bracket is not simply a gradification of the Poisson bracket. We had discussed before that the rumpfs 'p' and 'q' in the Poisson bracket were not gradifiable but that this problem can be removed by introducing an auxiliary index. However, this implies that still q and p have the same parity, while here they have opposite parity. On the other hand, the above equation can be seen as the gradification of an antibracket defined for purely bosonic rumpfs 'F' and 'G' and bosonic dummy index M. Rewriting it in terms of the big graded equal sign $=_G$, the sign $-(-)^{(F+1)(G+1)}$ would get replaced by a $+$ sign. Writing the antibracket without the big graded equal sign better demonstrates its relation to the Poisson bracket. In a sense, it behaves as if the gradings of 'F' and 'G' were shifted by 1. The antibracket will be further discussed at a later point (see e.g. footnote 13 on page 131 or footnote 1 in the appendix on page 160).

3.4 Lagrangian and Hamiltonian formalism

The structural equations of the Lagrangian or Hamiltonian formalism are good examples for the application of the gradification theorem. Graded versions of the Lagrangian equations of motion will most probably be very familiar to the reader. The intention here is only to carefully demonstrate how at the one hand the choice of the summation convention fixes all ambiguities and how on the other hand this choice need not to be done a priori (apart from the choice for the auxiliary index i to be introduced again below).

Let us consider a Lagrangian $L(q, \boldsymbol{c}, \dot{q}, \dot{\boldsymbol{c}})$ which depends on variables q^M with bosonic rumpf and ghost fields $\boldsymbol{c}^M$ with fermionic rumpf and their time derivatives. The indices of q and $\boldsymbol{c}$ will in general differ, but the assumption of the same index simplifies the presentation. The variation of the action will contain also derivatives w.r.t. $\boldsymbol{c}^M$ and it is thus useful to introduce again the variable $z^{iM} = (z^{1M}, z^{\mathbf{2}M}) = (q^M, \boldsymbol{c}^M)$.

$$\delta S = \int dt \quad \delta z^{iM} \frac{\partial}{\partial z^{iM}} L + \delta \dot{z}^{iM} \frac{\partial}{\partial \dot{z}^{iM}} L = \tag{3.57}$$

$$= \int dt \quad \delta z^{iM} \left(\frac{\partial}{\partial z^{iM}} L - \frac{d}{dt} (\frac{\partial}{\partial \dot{z}^{iM}} L) \right) + \text{bdry terms} \tag{3.58}$$

The **equations of motion** thus have the form

$$\frac{\partial}{\partial z^{iM}} L - \frac{d}{dt} (\frac{\partial}{\partial \dot{z}^{iM}} L) =_g 0 \tag{3.59}$$

where the graded equal sign has no effect here. As discussed earlier, left and right derivative are graded equal and because L is always bosonic (at least in usual examples) they are in fact equal and there is no arbitraryness of choosing left or right derivative. If we have NW conventions for the auxiliary index i, the derivative w.r.t. $z^{\mathbf{2}M}$ becomes the left derivative w.r.t. $\boldsymbol{c}^M$ or minus the right derivative w.r.t. $\boldsymbol{c}^M$, although an overall minus in the equations of motion is of course irrelevant.

In a similar way the definition of the **conjugate momentum** is already fixed by the choice of the summation convention. The definition is simply

$$\pi_{iM} \equiv \frac{\partial}{\partial \dot{z}^{iM}} L = L \frac{\overleftarrow{\partial}}{\partial \dot{z}^{iM}} \tag{3.60}$$

Again, left and right derivative coincide for bosonic rumpf z (when L is bosonic) and their definition is fixed by the choice of the summation convention. If we have NW conventions for the auxiliary index i, this definition becomes

$$p_M \equiv \frac{\partial}{\partial \dot{q}^M} L = L \frac{\overleftarrow{\partial}}{\partial \dot{q}^M} \tag{3.61}$$

$$\boldsymbol{b}_M \equiv \frac{\partial}{\partial \dot{\boldsymbol{c}}^M} L = -L \frac{\overleftarrow{\partial}}{\partial \dot{\boldsymbol{c}}^M} \tag{3.62}$$

For the choice of NE for the index i, the right derivative would be without sign. Remember again that the choice of the summation convention for the index i does not fix the one for the index M.

The **Legendre transformation** to obtain the Hamiltonian is of course also fixed by the summation convention

$$H(z, \pi) \equiv \int dt \quad \dot{z}^{iM} \pi_{iM} - L(z, \dot{z}(z, \pi)) \tag{3.63}$$

Although writing $\dot{z}^{iM}$ at the first position seems to fix NW-conventions, this is not true. The signs are as usual hidden in the summation. We thus have $\dot{z}^{iM} \pi_{iM} = \pi_{iM} \dot{z}^{iM}$ and are still free to decide in the end, which convention will enter the actual summation. As before we have to make a choice for the summation convention of the auxiliary index i, if we want to write this explicitely in terms of q^M and $\boldsymbol{c}^M$ and its momenta:

$$H(q, \boldsymbol{c}, p, \boldsymbol{b}) \overset{\text{NW for } i}{\equiv} \int dt \quad \dot{q}^M p_M + \dot{\boldsymbol{c}}^M \boldsymbol{b}_M - L(q, \boldsymbol{c}, \dot{q}(q, \boldsymbol{c}, p, \boldsymbol{b}), \dot{\boldsymbol{c}}(q, \boldsymbol{c}, p, \boldsymbol{b})) \tag{3.64}$$

The same reasoning is applied for the second Legendre transformation which yields the first order action $\tilde{L}(z, \pi, \dot{z}, \dot{\pi}) \equiv \int dt \quad \dot{z}^{iM} \pi_{iM} - H(z, \pi)$.

We had already mentioned that the summation convention for i could differ from the one for M and that even within M we could have different summation conventions for different index-subsets. Applications where the advantage of such **mixed conventions** becomes obvious, are those where one joins several variable with different index position to one variable, but wants to keep the summation conventions of before. This is the case for example for the introduction of Darboux coordinates to parametrize the phase space. Let us forget for the moment about the ghost variables. We can then define for example

$$Z^{\underline{M}} \equiv (q^M, p_M) \tag{3.65}$$

The Poisson bracket is then written with a mixed summation convention for the index $\underline{M}$ (based on NW for M) as

$$\{F, G\} = F\frac{\overleftarrow{\partial}}{\partial Z^{\underline{M}}}P^{\underline{MN}}\frac{\partial}{\partial Z^{\underline{N}}}G \overset{\text{mixed conv}}{\equiv} \tag{3.66}$$

$$\equiv \sum_{M_1,M_2,N_1,N_2}(-)^M F\frac{\overleftarrow{\partial}}{\partial q^M}P^{MN}\frac{\partial}{\partial q^N}G + (-)^{M+N}F\frac{\overleftarrow{\partial}}{\partial q^M}P^M{}_N\frac{\partial}{\partial p_N}G +$$
$$+F\frac{\overleftarrow{\partial}}{\partial p_M}P_M{}^N\frac{\partial}{\partial q^N}G + (-)^N F\frac{\overleftarrow{\partial}}{\partial p_M}P_{MN}\frac{\partial}{\partial p_N}G \tag{3.67}$$

If we had NW conventions for the indices $\underline{M}$ and $\underline{N}$, the definition of the graded summation would have a $(-)^M$ in front of every of the four terms. For the special choice of coordinates (with split in configuration space coordinates and momenta), the Poisson bivector is simply

$$P^{\underline{MN}} = \begin{pmatrix} 0 & -\delta^M{}_N \\ \delta_M{}^N & 0 \end{pmatrix} \tag{3.68}$$

where the relation of the graded Kronecker deltas in NW-conventions to the numerical δ^N_M is given by $\delta_M{}^N = \delta^N_M = (-)^{MN}\delta^N{}_M$.

3.5 Lie-groups and -algebras

3.5.1 Gradifiable and not gradifiable group definitions

The positive experience with the graded definition of matrix multiplication demands its application to supergroups. The first question arising is, which supergroup definitions have a natural gradification and which do not. Let us just give a few examples to make the idea transparent.

The **general linear group**, i.e. the group of all invertible matrices $GL(n)$ is easily gradifiable, because we know how to gradify the matrix multiplication and we have (for bosonic supermatrices, i.e. matrices with bosonic rumpf) a clear notion of invertability. If the index of the matrix runs over b bosonic and f fermionic indices, the resulting group is denoted by $GL(b|f)$(see e.g. [25, p.90]). Also the definition of the **special linear group** is gradifiable, because the definition of the determinant is gradifiable as we discussed earlier, and the condition $\det M = 1$ thus makes sense in the graded case as well. Because of $\det(M \cdot N) = \det M \cdot \det N$, this condition defines a subgroup which is denoted as $SL(b|f)$.

For bosonic matrices, the **unitary group** is defined via

$$U^\dagger U = \mathbb{1} \tag{3.69}$$

Or with indices

$$(U^\dagger)_m{}^k\delta_{kl}U^l{}_n = \delta_{mn} \tag{3.70}$$

We have a well defined notion of graded hermitean conjugation and also of a graded unity in the sense of a graded Kronecker delta with one lower and one upper index. There is no natural gradification, however, of a Kronecker delta with two indices at the same position. It is strictly speaking a metric and not a unit operator. In even dimensions we could use $\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$ as metric for the fermionic subspace, but this would be an ad-hoc choice. The problem is that there is no characteristic property of δ_{mn} which is gradifiable in our sense to uniquely give its graded version. The characterization that it is a diagonal matrix with only 1's in the diagonal is certainly not suitable for gradification, because for fermionic dimensions the metric should still be graded symmetric (i.e. antisymmetric) and is therefore necessarily off-diagonal. There is thus at first sight no natural gradification of the definition of the unitary group. Note that there exists nevertheless the notion of a unitary supergroup $U(b|f)$ in the literature (see e.g. [25, p.90]) .

The practical meaning of the unitary group is that it leaves the canonical scalar product $\delta_{\bar{m}n}$ in $\mathbb{C}^d$ invariant. Suppose we have a more general scalar product $\langle a, b\rangle = (\bar{a})^{\bar{m}}g_{\bar{m}n}b^n$ and make a basis change. $a^m = U^m{}_k\tilde{a}^k, \quad b^n = U^n{}_l\tilde{b}^l$. Then we obtain $\langle a, b\rangle = (U^m{}_k\tilde{a}^k)^* g_{\bar{m}n}U^n{}_l\tilde{b}^l \overset{!}{=} (\tilde{a}^k)^*\tilde{g}_{\bar{k}l}\tilde{b}^l$. The hermitean scalar product $g_{\bar{m}n}$ therefore transforms like

$$\tilde{g}_{\bar{k}l} = (U^m{}_k)^* g_{\bar{m}n}U^n{}_l = (U^\dagger)_{\bar{k}}{}^{\bar{m}}g_{\bar{m}n}U^n{}_l \tag{3.71}$$

We could define a matrix to be **unitary with respect to** $g_{\bar{m}n}$ iff

$$(U^\dagger)_{\bar{k}}{}^{\bar{m}}g_{\bar{m}n}U^n{}_l = g_{\bar{m}n} \tag{3.72}$$

This is a gradifiable definition, because it is based on some generic $g_{\bar{m}n}$ instead of the specific $\delta_{\bar{m}n}$. As discussed above there is no defining property of $\delta_{\bar{m}n}$ which is gradifiable.

The situation is the same for the **Lorentz group** (or likewise for the orthorgonal group) with

$$(\Lambda^T)_m{}^k \eta_{kl} \Lambda^l{}_n = \eta_{mn} \tag{3.73}$$

where we are again missing a gradification of the definition of η_{mn}.

The situation is a bit different for the **symplectic group**, although its definition is very close to the above two. Symplectic structures need even dimensional spaces. Assigning upper indices ^{k} to the first d dimensions and lower indices $_k$ to the second d dimensions and combine both into one index $\underline{k} \equiv (^k, {}_k)$, then the canonical symplectic form (being the matrix-inverse of the canonical Poisson structure of the previous section) can be written as

$$B_{\underline{kl}} = \begin{pmatrix} 0 & \delta_k{}^l \\ -\delta^k{}_l & 0 \end{pmatrix} \tag{3.74}$$

In contrast to the metrics of before, the symplectic form is gradifiable, because it contains two unit operators in subspaces of which we know the gradification. Elements S of the symplectic group $SP(2d)$ are then given by

$$(S^T)_{\underline{m}}{}^{\underline{k}} B_{\underline{kl}} S^{\underline{l}}{}_{\underline{n}} \quad = \quad B_{\underline{kl}} \tag{3.75}$$

Simply gradifying the indices yields the graded definition of the symplectic group. The body S of the symplectic matrix, however, is not gradifiable, as it appears twice in the term on the left and not at all on the right. If the index k runs over b bosonic and f fermionic indices, the resulting group could be denoted by $SP(2b|2f)$, while in literature it is common to introduce instead the notion of an orthosymplectic group which differs, however, a bit from this group (see e.g. [25, p.90]). The precise form of the group elements $S \in SP(2b|2f)$ depends on the choice of either NW or NE for the definition of the matrix multiplication and of the position of the indices at the matrix (first index up and second down or vice verse).

Having seen the above example, it is obvious that gradification also works for $O(d,d)$ or $SO(d,d)$ based on the metric $\eta_{\underline{mn}} = \begin{pmatrix} 0 & \delta_m{}^n \\ \delta^m{}_n & 0 \end{pmatrix}$. If the indices m, n take d values, this metric has in the bosonic case the signature (d,d). Containing two off-diagonal Kronecker deltas, the graded version of the metric looks just the same. If d splits into b bosonic and f fermionic dimensions, the resulting supergroups could be denoted as $O(b,b|f,f)$ and $SO(b,b|f,f)$. For the fermionic subspace we have $\delta^\mu{}_\nu = -\delta_\nu{}^\mu$, and the corresponding matrix block of the metric is numerically just the matrix of a bosonic symplectic form. In this sense, $O(d,d)$ and $SP(2d)$ interchange their role in the bosonic and fermionic subspaces:

$$O(d,d|0,0) \cong SP(0,0|d,d) \quad \text{and} \quad O(0,0|d,d) \cong SP(d,d|0,0) \tag{3.76}$$

Note finally that all supergroups which cannot be seen as a gradification of a bosonic group, of course still make perfect sense. The message is only that properties of those supergroups must be studied independently and cannot be deduced from the corresponding bosonic groups via the gradification theorem. The main example are groups of fermionic supermatrices. The bosonic definition of a group requires the existence of an inverse matrix. As we discussed already in the chapter on supermatrices, the notion of an inverse matrix can only be gradified in the case of a bosonic supermatrix, while the definition of a 'special graded inverse' of a fermionic supermatrix cannot be used to take advantage of the gradification theorem.

In [23, 24] it was observed that $SO(d)$ can be seen as $SP(-d)$ (with d fermionic, i.e. negative dimensions – see page 25) and that $SP(d)$ can be seen as $SO(-d)$. Understanding $SP(-d) \equiv SP(0|d)$ and $SO(-d) \equiv SO(0|d)$, this does almost but not completely match with our above observation (3.76) which holds only for split signature. This might be due to different definitions of the supergroups and it would be interesting to make the comparison in more detail.

3.5.2 Graded Lie algebra

In the previous subsection we have just discussed a few examples for the gradification of some Lie groups, although a more detailed study would be a very interesting subject. Likewise we are not going to discuss (graded) Lie algebras in any detail in this subsection, but instead want to stress a few minor points, related to the summation convention. In the previous subsection we were only discussing supergroups whose elements are bosonic supermatrices, i.e. graded matrices with bosonic rumpf, because only there we have a natural gradified version of an inverse matrix. Nevertheless, even when the group matrices of a Lie Group are all bosonic, its infinitesimal generators (when based on the module of supernumbers) might well be expanded in a basis T_A that contains fermionic matrices. Each of the T_A's is a supermatrix, and it depends on the index A, whether it is a fermionic or a bosonic one:

$$|\,(T_A)^M{}_N\,| = |\,A\,| + |\,M\,| + |\,N\,| \tag{3.77}$$

Like in the bosonic case, group elements in the connected component of the unity can be parametrized by[1]

$$g(x) = e^{ix^A T_A} \tag{3.78}$$

where x^A are some coordinates whose grading $\mid A \mid$ is the same as the one of the generators T_A, so that the group element is a bosonic supermatrix. For example, for $g(x)$ to be in GL, the exponent can be any (small) supermatrix, while for $g(x)$ to be in SL, it has to be traceless ($\det e^{ix^A T_A} = \exp ix^A \mathrm{tr}\, T_A$). One possible basis of the algebra of all supermatrices consists of the matrices with one entry 1 and zero everywhere else. If the 1 is in one of the diagonal blocks, the corresponding basis matrix T_a is a bosonic one, while if the 1 is in one of the off-diagonal blocks, $T_{\boldsymbol{\mathcal{A}}}$ is considered as a fermionic supermatrix (although it has bosonic entries only). The fermionic supermatrices $T_{\boldsymbol{\mathcal{A}}}$ are contracted with a fermionic parameter $\boldsymbol{\theta}^{\mathcal{A}} \equiv x^{\boldsymbol{\mathcal{A}}}$, so that the resulting group element $g(x)$ is a bosonic supermatrix.

The algebra is determined by providing the **structure constants** for the (graded) commutator

$$[T_A, T_B] \quad =_g \quad if_{AB}{}^C T_C \tag{3.79}$$

The graded equal sign has no effect here again, because the naked indices A and B are in the same order on both sides. If one is dealing naively (see remark in footnote 1) with (graded) hermitean matrices (or operators) $T_A^\dagger = T_A$, then the commutator is always graded antihermitean $[T_A, T_B]^\dagger =_g [T_B^\dagger, T_A^\dagger] =_g [T_B, T_A] =_g -[T_A, T_B]$, no matter whether the indices A and B are bosonic or fermionic. Extracting the imaginary unit 'i' then leads to real structure constants. Note that in most of the literature, fermionic and bosonic operators are treated differently in this issue, because of the different definition of hermitean conjugation. An immediate application of the gradification theorem is the **Jacobi identity** in terms of the structure constants, which has of course the same form as in the bosonic case, but with graded summation and graded antisymmetrization:

$$f_{[AB|}{}^D f_{D|C]}{}^E \quad = \quad 0 \tag{3.80}$$

An **invariant metric**

$$\langle T_A, T_B \rangle \quad \equiv \quad \mathcal{H}_{AB} \tag{3.81}$$

is defined to obey

$$\langle [T_C, T_A], T_B \rangle + \langle T_A, [T_C, T_B] \rangle \quad =_g \quad 0 \tag{3.82}$$

In terms of the structure constants (with $f_{ABC} \equiv f_{AB}{}^D \mathcal{H}_{DC}$), this reads

$$f_{CAB} + f_{CBA} \quad =_g \quad 0 \tag{3.83}$$

which means that the structure constants are also (graded) antisymmetric in the last two indices and therefore in all indices. Indices are pulled up again with the graded inverse of $\mathcal{H}_{AB}$ which is defined by

$$\mathcal{H}_{AC}\mathcal{H}^{CB} = \delta_A{}^B \tag{3.84}$$

or equivalently $\mathcal{H}^{AC}\mathcal{H}_{CB} = \delta^A{}_B$. The graded inverse $\mathcal{H}^{AB}$ differs from the naive (numerical) inverse by a factor $(-)^A$ in NW and by a factor $(-)^B$ in NE.

The defining equation for the structure constants (3.79) seems to suggest that we already have fixed NW conventions, but it can also be rewritten to enfavour NE. To this end we need the fact that in the case of the existence of a group invariant metric to pull up and down the indices A, B and C, the structure constants with all indices down are completely (graded) antisymmetric $f_{ABC} =_g f_{CAB}$. The commutator (3.79) then reads

$$[T_A, T_B] \quad =_g \quad iT_C f^C{}_{AB} \tag{3.85}$$

[1]Note that due to our definition of complex conjugation and hermitean conjugation $x^A T_A$ is hermitean if T_A is hermitean and x^A is real: $(x^A T_A)^\dagger = (x^A)^* T_A^\dagger = x^A T_A$. The group element $e^{ix^A T_A}$ thus would correspond to a unitary group element. This would disagree with the statement before that there is no natural gradification of unitary matrices. In fact, already for the hermiticity we were too sloppy in the above reasoning: A graded hermitean matrix is defined only when both indices are at the same position. If one index is upstairs and the other is downstairs, one needs a metric to define hermiticity, and this is again missing in general in the graded case.

Note further that sometimes it is convenient to parametrize the group element differently, namely by exponentiating seperately the bosonic and the fermionic contributions:

$$g(x) = e^{ix^A T_A} \stackrel{!}{=} e^{iy^a T_a} e^{iy^{\boldsymbol{\alpha}} T_{\boldsymbol{\alpha}}} \equiv g(y)$$

The relation between x and y is obtained by using the graded version of the Baker-Campbell-Hausdorff formula, which is simply the gradification of the bosonic one, i.e. $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\frac{1}{12}[[A,B],B]+\mathcal{O}([.,.]^3)}$. ◇

In both versions of the equation, the actual summation convention has not yet been fixed. Let us finally write down the original form (3.79) of this commutator explicitely in NW-conventions, including the matrix indices:

$$\sum_K \Big\{(-)^{(M+K)B}(-)^K (T_A)^M{}_K (T_B)^K{}_N - (-)^{AB}(-)^{(M+K)A}(-)^K (T_B)^M{}_K (T_A)^K{}_N\Big\} = \sum_C i f_{AB}{}^C (T_C)^M{}_N \tag{3.86}$$

The position of the supermatrix indices (first one upstairs, second downstairs) is more natural for NE conventions, where the sign $(-)^K$ would not appear in the terms on the lefthand side.

Natural applications of the above considerations appear in the study of WZNW-models based on graded Lie algebras (e.g. in our study [11] of a WZNW-like model [10], where we however not yet rigorously applied the present conventions).

3.6 Remark on the pure spinor ghosts

In part II, we will make frequent use of the presented conventions. In particular, we will always use the graded summation convention and the small graded equal sign without denoting it explicitely! There are some effects that one needs to get used to. The formalism contains among others the variables x^m, $\boldsymbol{\theta}^\mu$, $\hat{\boldsymbol{\theta}}^\mu$ and a commuting ghost variable λ^μ. When we want to describe the first three as just components of a supercoodinate x^M, we have to assign all the grading to the indices: $\boldsymbol{\theta}^\mu \to \theta^{\boldsymbol{\mu}} \equiv x^{\boldsymbol{\mu}}$. We call that a "rumpf-index grading shift". The fermionic variable $\boldsymbol{\theta}^\mu = \theta^{\boldsymbol{\mu}}$ can be treated in both ways, either as odd rumpf with even index or as even rumpf with odd index. The boldface notation should serve as a reminder, which point of view we take. When we are considering the combining object x^M, we have no choice, because all entries share the same rumpf 'x'. Therefore we have to assign the grading to the index and have to do the same for the ghost index, because it simply is the same index:

$$\lambda^\mu \to \boldsymbol{\lambda}^{\boldsymbol{\mu}} \tag{3.87}$$

When we leave away in calculations all index-dependent signs, the pure spinor ghost will effectively be treated as an anticommuting variable, because the rumpf is anticommuting! Another similar effect is the switch of the symmetry properties of bispinors. E.g. the chiral γ-matrices

$$\gamma^c_{(\alpha\beta)} \to \gamma^c_{[\boldsymbol{\alpha\beta}]} \tag{3.88}$$

which are symmetric before the grading shift, become effectively antisymmetric afterwards. As an example, consider the following term

$$(\lambda\gamma^c\partial\lambda) \quad = \quad \lambda^\alpha\gamma^c_{(\alpha\beta)}\partial\lambda^\beta = \partial\lambda^\alpha\gamma^c_{(\alpha\beta)}\lambda^\beta = (\partial\lambda\gamma^c\lambda) \tag{3.89}$$

The calculation goes through in the same way after the shift, because the antisymmetry of the γ-matrix is compensated by the "anticommutativity" of the ghosts.

$$\boldsymbol{\lambda}\gamma^c\partial\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\gamma^c_{[\boldsymbol{\alpha\beta}]}\partial\boldsymbol{\lambda}^{\boldsymbol{\beta}} = \partial\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\gamma^c_{[\boldsymbol{\alpha\beta}]}\boldsymbol{\lambda}^{\boldsymbol{\beta}} = \partial\boldsymbol{\lambda}\gamma^c\boldsymbol{\lambda} \tag{3.90}$$

As one of the summations is over a graded rumpf and another is in the wrong direction, the contraction coincides with the one for ungraded indices. This is not true for θ, where we have a sign change (for NW as well as for NE):

$$\boldsymbol{\lambda}\gamma^c\partial\boldsymbol{\lambda} \quad = \quad \lambda\gamma^c\partial\lambda \tag{3.91}$$

$$\theta\gamma^c\partial\theta \quad = \quad -\boldsymbol{\theta}\gamma^c\partial\boldsymbol{\theta} \tag{3.92}$$

Note finally that the rumpf of γ^c (the off-diagonal block of Γ^c) stays bosonic, even when $\Gamma^c \to \boldsymbol{\Gamma}^c$ is reinterpreted as a fermionic supermatrix as suggested in section 2.7 on page 26.

Part II

Berkovits' Pure Spinor String in General Background

Chapter 4

Motivation of the Pure Spinor String in Flat background

4.1 From Green-Schwarz to Berkovits

The classical type II Green Schwarz (GS) superstring describes the embedding of a string worldsheet into a target type II superspace with coordinates $x^M \equiv (x^m, \boldsymbol{\theta}^\mu, \hat{\boldsymbol{\theta}}^{\hat{\mu}})$. The bosonic coordinates x^m locally parametrize the ten-dimensional spacetime manifold, while the fermionic coordinates $\boldsymbol{\theta}^\mu$ and $\hat{\boldsymbol{\theta}}^{\hat{\mu}}$ have the dimension of Majorana Weyl spinors and thus have each 16 real components. The Lorentz transformation of spinors is from the supermanifold point of view a structure group transformation in the tangent space of the supermanifold. In the flat case, where one can identify the manifold with its tangent space, the $\boldsymbol{\theta}$'s are clearly spinors themselves. In the context of a curved supermanifold that we will treat later on, this will not be the case a priori. The $\boldsymbol{\theta}$'s then only transform under super-diffeomorphisms and not under structure group transformations. However, the supergravity constraints will allow to choose a gauge (WZ-gauge) in which the two transformations are coupled and the $\boldsymbol{\theta}'$s likewise transform under a structure group transformation. This is just a remark on the use of the "curved index" μ. Objects that transform a priori under the structure group carry the flat index A or in particular α.

The cases type IIA and IIB will be treated at the same time via the choice $\hat{\boldsymbol{\theta}}^{\hat{\mu}} \equiv \hat{\boldsymbol{\theta}}_\mu$ for IIA and $\hat{\boldsymbol{\theta}}^{\hat{\mu}} \equiv \hat{\boldsymbol{\theta}}^\mu$ for IIB. The supersymmetry transformation in flat superspace reads

$$\begin{aligned} \delta\boldsymbol{\theta}^\mu &= \varepsilon^\mu, \qquad \delta\hat{\boldsymbol{\theta}}^{\hat{\mu}} = \hat{\varepsilon}^{\hat{\mu}} && (4.1)\\ \delta x^m &= \varepsilon\gamma^m\boldsymbol{\theta} + \hat{\varepsilon}\gamma^m\boldsymbol{\theta} && (4.2)\end{aligned}$$

The small γ-matrices are discussed in the appendix D. In order to build a supersymmetric theory, it is reasonable to consider supersymmetric building blocks, in particular supersymmetric one-forms (vielbeins)

$$E^A \equiv \mathbf{d}x^M E_M{}^A = \big(\underbrace{\mathbf{d}x^a + \mathbf{d}\boldsymbol{\theta}\gamma^a\boldsymbol{\theta} + \mathbf{d}\hat{\boldsymbol{\theta}}\gamma^a\hat{\boldsymbol{\theta}}}_{\Pi^a} \quad , \quad \mathbf{d}\boldsymbol{\theta}^\alpha \quad , \quad \mathbf{d}\hat{\boldsymbol{\theta}}^{\hat{\alpha}}\big) \tag{4.3}$$

Its pullback to the worldsheet will be denoted by

$$\Pi^A_{z/\bar{z}} \equiv \partial_{z/\bar{z}} x^M E_M{}^A \tag{4.4}$$

We do not distinguish notationally between the coordinates of the superspace and the embedding functions. The bosonic components Π^a_z are known as the supersymmetric momentum

$$\Pi^a_{z/\bar{z}} \quad = \quad \partial_{z/\bar{z}}x^a + \partial_{z/\bar{z}}\boldsymbol{\theta}\gamma^a\boldsymbol{\theta} + \partial_{z/\bar{z}}\hat{\boldsymbol{\theta}}\gamma^a\hat{\boldsymbol{\theta}} \tag{4.5}$$

The introduction to the Green Schwarz string and the motivation for the pure spinor formalism will be rather quick and sketchy. We will be much more careful when we start to discuss the pure spinor string in general background.

The classical Green Schwarz superstring in flat background consists of the square of this momentum plus a Wess-Zumino term which establishes a fermionic gauge symmetry. This gauge symmetry, called κ-symmetry, guarantees the matching of the physical fermionic and bosonic degrees of freedom. The GS action has in conformal gauge the following form:

$$\begin{aligned} S_{GS} &= \int d^2z \quad \frac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar{z}} + \mathcal{L}_{WZ} && (4.6)\\ \mathcal{L}_{WZ} &= -\frac{1}{2}\Pi_{zm}\left(\boldsymbol{\theta}\gamma^m\bar{\partial}\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\gamma^m\bar{\partial}\hat{\boldsymbol{\theta}}\right) + \frac{1}{2}(\boldsymbol{\theta}\gamma^m\partial\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}\gamma_m\bar{\partial}\hat{\boldsymbol{\theta}}) - (z \leftrightarrow \bar{z}) && (4.7)\end{aligned}$$

It is covariant and almost manifestly spacetime supersymmetric. In this last feature it differs from the RNS string, where space time supersymmetry only comes in after GSO projection. The problem for the Green Schwarz string on the other hand is that a covariant quantization with the standard BRST procedure does not work. The reason for this misery is a set of 16 mixed first and second class constraints $\boldsymbol{d}_{z\alpha}$ that cannot be split easily into first and second class type in a covariant manner. The conjugate momentum $\boldsymbol{p}_{z\alpha}$ of $\boldsymbol{\theta}^\alpha$ can be entirely expressed in terms of other phase space variables and the corresponding fermionic phase space constraint is just $\boldsymbol{d}_{z\alpha}$. It has the following explicit form (the form of conjugate momentum to x^m was already plugged in)

$$\boldsymbol{d}_{z\alpha} \equiv \boldsymbol{p}_{z\alpha} - (\gamma_a \boldsymbol{\theta})_\alpha \left(\partial x^a - \frac{1}{2}\boldsymbol{\theta}\gamma^a \partial\boldsymbol{\theta} - \frac{1}{2}\hat{\boldsymbol{\theta}}\gamma^a\partial\hat{\boldsymbol{\theta}} \right) \tag{4.8}$$

Half of these constraints are first class and correspond to the above mentioned fermionic κ gauge symmetry. The fact that they have a second-class part can be seen in a non-closure of the Poisson-algebra, which has the following schematica form:

$$\{\boldsymbol{d}_{z\alpha}(\sigma), \boldsymbol{d}_{z\beta}(\sigma')\} \propto 2\gamma^a_{\alpha\beta}\Pi_{za}\delta(\sigma-\sigma') \tag{4.9}$$

Siegel [26] had the idea to make $\boldsymbol{d}_{z\alpha}$ part of a closed algebra by just adding the generators that arise via the Poisson bracket, which leads to a (centrally extended), but otherwise closed algebra

$$\{\boldsymbol{d}_{z\alpha}, \Pi_{za}\} \propto 2\gamma_{a\,\alpha\beta}\partial\theta^\beta\delta(\sigma-\sigma') \tag{4.10}$$

$$\{\Pi_{za}, \Pi_{zb}\} \propto \eta_{ab}\delta'(\sigma-\sigma') \tag{4.11}$$

$$\left\{\boldsymbol{d}_{z\alpha}, \partial\theta^\beta\right\} \propto \delta^\beta_\alpha\delta'(\sigma-\sigma') \tag{4.12}$$

The important observation is now that the same chiral algebra can be obtained from a free-field Lagrangian, where the variable $\boldsymbol{p}_{z\alpha}$ is independent and cannot be integrated out:

$$S_{free} = \int d^2z \;\; \frac{1}{2}\partial x^m \eta_{mn}\bar{\partial}x^n + \bar{\partial}\boldsymbol{\theta}^\alpha \boldsymbol{p}_{z\alpha} + \partial\hat{\boldsymbol{\theta}}^{\hat{\alpha}}\hat{\boldsymbol{p}}_{\bar{z}\hat{\alpha}} = \tag{4.13}$$

$$= \int d^2z \;\; \underbrace{\frac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar{z}} + \mathcal{L}_{WZ}}_{\mathcal{L}_{GS}} + \bar{\partial}\boldsymbol{\theta}^\alpha\boldsymbol{d}_{z\alpha} + \partial\hat{\boldsymbol{\theta}}^{\hat{\alpha}}\hat{\boldsymbol{d}}_{\bar{z}\hat{\alpha}} \tag{4.14}$$

In the second line we have used the original definition (4.8) for $\boldsymbol{d}_{z\alpha}$. Remarkably, this action coincides with the Green Schwarz action for $\boldsymbol{d}_\alpha = \hat{\boldsymbol{d}}_{\hat{\alpha}} = 0$. In the above free theory, however, $\boldsymbol{d}_{z\alpha}$ is a priori not a Hamiltonian constraint, but still a generator of a chiral (not local) symmetry. In any case, the reformulation does not remove the mixed first-second class property of $\boldsymbol{d}_{z\alpha}$, but it provides a simple free-field Lagrangian. Berkovits [7] had the idea to implement the constraints cohomologically with a BRST operator disregarding its non-closure. The corresponding current ($\boldsymbol{Q} = \oint dz\,\boldsymbol{j}_z$) for the left-moving and the right-moving sector take respectively the simple form

$$\boldsymbol{j}_z = \lambda^\alpha\boldsymbol{d}_{z\alpha}, \quad \boldsymbol{j}_{\bar{z}} = 0 \tag{4.15}$$

$$\hat{\boldsymbol{j}}_{\bar{z}} = \hat{\lambda}^\alpha\hat{\boldsymbol{d}}_{\bar{z}\hat{\alpha}}, \quad \hat{\boldsymbol{j}}_z = 0 \tag{4.16}$$

where λ^α is a commuting ghost. For first class constraints the BRST cohomology can be built, because the BRST operator is nilpotent due to the closure of the algebra. For second class constraints, however, the non-closure implies a lack of nilpotency of the BRST operator. To overcome this problem, Berkovits put a constraint on the ghost field λ and $\hat{\lambda}$, the so called pure spinor constraint

$$\lambda\gamma^c\lambda = 0, \qquad \hat{\lambda}\gamma^c\hat{\lambda} = 0 \tag{4.17}$$

This enforces nilpotency of the BRST operator and provides a well-defined theory. The pure spinor constraint and the ghost kinetic term have to be added to the original free action:

$$S_{ps} = \int d^2z \;\; \frac{1}{2}\partial x^m\eta_{mn}\bar{\partial}x^n + \bar{\partial}\boldsymbol{\theta}^\alpha\boldsymbol{p}_{z\alpha} + \partial\hat{\boldsymbol{\theta}}^{\hat{\alpha}}\hat{\boldsymbol{p}}_{\bar{z}\hat{\alpha}} + \mathcal{L}_{gh} \tag{4.18}$$

$$= \int d^2z \;\; \frac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar{z}} + \mathcal{L}_{WZ} + \bar{\partial}\boldsymbol{\theta}^\alpha\boldsymbol{d}_{z\alpha} + \partial\hat{\boldsymbol{\theta}}^{\hat{\alpha}}\hat{\boldsymbol{d}}_{\bar{z}\hat{\alpha}} + \mathcal{L}_{gh} \tag{4.19}$$

$$\Pi^a_z = \partial x^a + \partial\boldsymbol{\theta}\gamma^a\boldsymbol{\theta} + \partial\hat{\boldsymbol{\theta}}\gamma^a\hat{\boldsymbol{\theta}} \tag{4.20}$$

$$\mathbf{d}_{z\alpha} = \boldsymbol{p}_{z\alpha} - (\gamma_m\boldsymbol{\theta})_\alpha\left(\partial x^m - \frac{1}{2}\boldsymbol{\theta}\gamma^m\partial\boldsymbol{\theta} - \frac{1}{2}\hat{\boldsymbol{\theta}}\gamma^m\partial\hat{\boldsymbol{\theta}}\right) \tag{4.21}$$

$$\mathcal{L}_{WZ} = -\frac{1}{2}\Pi_{zm}\left(\boldsymbol{\theta}\gamma^m\bar{\partial}\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\gamma^m\bar{\partial}\hat{\boldsymbol{\theta}}\right) + \frac{1}{2}(\boldsymbol{\theta}\gamma^m\partial\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}\gamma_m\bar{\partial}\hat{\boldsymbol{\theta}}) - (z \leftrightarrow \bar{z}) \tag{4.22}$$

$$\mathcal{L}_{gh} = \bar{\partial}\lambda^\beta\omega_{z\beta} + \partial\hat{\lambda}^{\hat{\beta}}\hat{\omega}_{\bar{z}\hat{\beta}} + \frac{1}{2}L_{z\bar{z}a}(\lambda\gamma^a\lambda) + \frac{1}{2}\hat{L}_{z\bar{z}a}(\hat{\lambda}\gamma^a\hat{\lambda}) \tag{4.23}$$

The pure spinor constraints seem like a replacement of one problem by another. The constraints turn now out to be first class but infinitely reducible. They generate antighost gauge symmetries of the form

$$\delta_{(\mu)}\omega_{z\alpha} = \mu_{za}(\gamma^a\lambda)_\alpha, \qquad \delta_{(\mu)}\hat{\omega}_{\bar{z}\hat{\alpha}} = \hat{\mu}_{\bar{z}a}(\gamma^a\hat{\lambda})_\alpha \tag{4.24}$$

accompanied by some transformation of the Lagrange multipliers. We will discuss this in more detail in the general background-case. In spite of this, the pure spinor constraint can be better handled than the original constraint. One can solve the pure spinor constraint explicitely in a U(5)-parametrization and calculate operator products. Although the U(5) coordinates break manifest ten-dimensional Lorentz-covariance, the resulting gauge-invariant OPE's all have a Lorentz covariant form and the quantization is effectively Lorentz covariant. Berkovits showed in the above cited papers the equivalence to the ordinary string. In [12] he presented a consistent description for the calculation of higher loop amplitudes. There are still many conceptual problems. The pure spinor formalism starts in the conformal gauge and does not have worldsheet diffeomorphism invariance any longer. Attempts to construct a composite b-ghost (as homotopy for the energy momentum tensor) always involved inverse powers of the gost field. In [27], Berkovits recovered a $N = 2$ algebra by the introduction of additional worldsheet fields, which is now known as "non-minimal formalism". Multiloop calculations were described or performed by Berkovits, Mafra, Nekrasov and Stahn in [28, 29, 30, 31] (Since the last version of this thesis new results were obtained. A recent detailed review is provided in [120]). However, there is still a clear picture of the origin of the pure spinor constraint missing. Attempts to relate the pure spinor string to the Green Schwarz string via similarity transformations and redefinitions were successful in [32], but not very enlightening. An additional task is the resolving of the tip-singularity of the pure-spinor-cone. These questions were adressed in [33] and [34].

We should finally mention that the pure spinor approach of Berkovits differs significantly from the hybrid formalism[35], which was developped by the same author and shares only some of the properties of the pure spinor approach. Two recent presentations of this formalism including the numerous relevant references can be found in [36][37].

4.2 Efforts to remove or explain the pure spinor constraint

There were plenty of efforts to get rid of the pure spinor constraint in the years after Berkovits presented his approach the first time. A quite natural ansatz was followed by Chesterman[38, 39], who implemented the first-class pure spinor constraint cohomologically, via a second BRST operator. Due to the infinite reducibility of this constraint, there arises an infinite number of ghost for ghosts. Nevertheless he was able to extract the most important information and avoided solving the pure spinor constraint explicitly.

Somehow related are the considerations of Aisaka and Kazama[40, 41, 42, 43, 44]. They were able to construct a BRST operator with five additional ghost fields and no pure spinor constraint, using however U(5) parametrization and breaking manifest Lorentz invariance. The relation to Chesterman's approach can be established as follows: The infinitely reducible pure spinor constraint can be replaced by an irreducible one in an U(5) parametrization. This constraint can be implemented cohomologically via a second BRST operator in a relative cohomology, and via homological perturbation theory one can replace the two operators by a single one. Within their 'doubled spinor formalism', they provided in [43] a derivation of the pure spinor string from the Green Schwarz String on the quantum level.

Another enlightening approach by Oda, Tonin et al.[45] was the interpretation of the pure spinor formalism as a twisted and gauge fixed version of the superembedding formalism. This led to a slightly modified version of the pure spinior formalism, the Y-formalism, and to new insight about the missing antighost b-field[46, 47, 48, 49].

There was finally yet another approach by Grassi, Policastro, Porrati and van Nieuwenhuizen, at that time most of them in Stony Brook, which we will discuss shortly in a seperate section, as it was subject of my early PhD studies.

4.3 Some more words on the Stony-Brook-approach

In a series of papers [8, 50, 51, 9, 10, 52, 53] Grassi, Policastro, Porrati and van Nieuwenhuizen have removed the pure spinor constraint by adding additional ghost variables. They realized in [10] that their theory has the stucture of a gauged WZNW model with the complete diagonal subgroup gauged. It is based on the chiral algebra above. A current can be set to zero by gauging the corresponding symmetry and thus making it a first class constraint. However, $\boldsymbol{d}_{z\alpha}$ does not form a subalgebra and thus cannot be gauged on its own. So if one starts gauging $\boldsymbol{d}_{z\alpha}$ and tries to make the resulting BRST-operator (4.15) nilpotent by adding further ghosts, one automatically arrives at a BRST operator that corresponds to a theory where also Π_{zm} and $\partial\theta^\alpha$ are gauged (see e.g. [9, p.7] or [10, p.4]; this fact was later also used to describe a topological model in [54]). In the gauged WZNW description this means that the complete diagonal subgroup is gauged. Therefore a grading or filtration had to be introduced, in order to obtain the correct cohomology. In [53] it was argued that for any (simple)

Lie algebra one can in general gauge a coset (in our case the algebra that corresponds to $\boldsymbol{d}_{z\alpha}$, modding out the subalgebra) by gauging the complete algebra and later undo the gauging of the subalgebra by building the relative cohomology with respect to a second BRST operator. This corresponds to the former grading. Despite its elegance there are some puzzling points about the WZNW action:

- For the heterotic string one starts with a chiral algebra and gets from the WZNW model a chiral as well as an antichiral algebra. Somehow one has to get rid of the antichiral one.
- For the type II string one starts with a chiral and antichiral algebra. Both of them double and the Jacobi identity forces one to mix those algebras. Thus it has not been possible yet to produce a WZNW model for the type II string.
- The classical WZNW theory is not a free field theory which might cause problems for calculating OPEs.

For those reasons, we avoided in [11] the WZNW action. Although the cited paper contains the work of the early stage of my PhD, it will not be presented in this thesis in detail. The reason is that it would open yet another field, whereas the presented parts share some common aim. Let me therefore just sketch the results: We started in [11] with the free field action of above, discussed its off-shell symmetry algebra generated by the current $\boldsymbol{d}_{z\alpha}$ and gauged it, in order to turn $\boldsymbol{d}_{z\alpha}$ into a constraint. Before actually gauging the algebra via the Noether procedure, we had to make it close off-shell. To this aim we introduced auxiliary fields P_{zm} and $P_{\bar{z}m}$. There still remained double poles in the current algebra, which caused trouble in the gauging procedure. They were be eliminated by doubling all fields as it was done in [10], in order to establish nilpotent BRST transformations. Gauge fixing leads to the BRST-transformations as they are given in [10].

Finally, we had a closer look at the final BRST operator proposed in [10], which includes diffeomorphism invariance by adding a topological ghost quartet. We came to the conclusion that this operator has to be modified via a second quartett of ghost fields in order to become nilpotent. More details can be found in [11] and [55].

A last major progress was achieved in [56] by establishing an $N = 4$ algebra in this formalism. There exist also independent studies of WZNW models based on supergroups like for example on PSU(1,1|2) in [57] .

Chapter 5

Closed Pure Spinor Superstring in general type II background

The pure spinor string in general background was first studied by Berkovits in [13]. The one-loop conformal invariance of the heterotic version was studied in [58]. The classical worldsheet BRST transformations of the heterotic string in general background were derived in [14]. The one-loop conformal invariance of the type II string finally was shown in [59] where also the derivation of the supergravity constraints was reviewed. Note also [60, 61, 2] for another useful presentation of some aspects of the pure spinor string in general or AdS5xS5 background. In the following we will present again the derivation of the supergravity constraints as it was done in [13],[59] but we will explain in more detail several steps and also we will use a different method to derive the constraints. In particular we will not go to the Hamiltonian formalism in order to derive the BRST transformations as generated via charge and Poisson bracket but we will stay in the Lagrangian formalism and will use what we call "inverse Noether". In addition we will use a spacetime covariant variation in order to derive the classical equations of motion in a spacetime covariant manner and we will present the BRST transformations of all the worldsheet fields for the type II string in general background. This has so far been done only for the heterotic string in [14]. Having derived the supergravity constraints we will finally go to the Wess Zumino gauge and derive the local supersymmetry transformations of at least the fermionic fields in order to make contact to generalized complex geometry.

Note that there was a carefull study in [62] of how to construct type II vertex operators in the pure spinor formalism. This is at least for massless fields directly related to the deformations of the action that we are going to study now. (After the first arXiv-version of this thesis, another thesis by O. Bedoya [121] studying and reviewing many aspects of the pure spinor string in general background has appeared).

5.1 Ansatz for action and BRST operators and some EOM's

In the following we will consider the closed pure spinor string coupled to general background fields. One can either add small perturbations (integrated vertex operators) to the action or simply consider the most general classically conformally invariant action with the given field content and the same antighost gauge symmetry (generated by the pure spinor constraint). The action, however, is not enough to specify the string completely. In addition, we need two (one left-moving and one right-moving) BRST operators in the general background. The existence of two such BRST operators which have to be nilpotent and conserved (holomorphic and antiholomorphic respectively) turns out to be equivalent to supergravity constraints on the background fields. The important steps of this calculation will be carefully motivated in the following.

The idea is to start from the most general renormalizable action with the given field content. It is convenient to throw away immediately the tachyon term which is allowed by renormalizability, but which is not even BRST invariant for the undeformed BRST transformations, at least for a non-constant tachyon field. The starting point then reduces to the most general classically conformally invariant action. In order to write down a classically conformally invariant action (ghost number zero in each sector), we have to combine elementary fields to terms with conformal weight (1,1). There are no fields with negative conformal weight. The a priory possible elementary building blocks of ghost number (0,0) are thus

$$\begin{array}{ll}
\text{weight (0,0)} & x^M \\
\text{weight (1,0)} & \partial x^M, d_{z\boldsymbol{\alpha}}, \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \\
\text{weight (0,1)} & \bar{\partial} x^M, \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}, \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} \\
\text{weight (1,1)} & \partial\bar{\partial} x^M, \bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}, \partial\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}, \bar{\partial} d_{z\boldsymbol{\alpha}}, \partial\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}
\end{array}$$

We now can combine an arbitrary function of x^M (background field) with either a (1,1)-building block or with one (1,0) combined with one (0,1) building block. Via partial integration, a $\partial\bar{\partial}x^M$-term with an arbitrary x-dependent coefficient can always be rewritten as a $\partial x^M \bar{\partial} x^N$-term[1]. Before writing down the resulting action, let us note that we will immediately absorb the x-dependent coefficient coming with $\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ in a reparametrization of $\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ so that we simply get the free ghost kinetic term $\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}$. Likewise for the hatted variables.

The most general classically conformally invariant (or renormalizable, adding Tachyon term) action with the same field content (including the pure spinor constraint on the ghosts) with independently conserved left and right ghost number now reads

$$
\boxed{\begin{aligned}
S \;=\; \int d^2z \;\; & \tfrac{1}{2}\partial x^M \underbrace{(G_{MN}(\vec{x}) + B_{MN}(\vec{x}))}_{\equiv O_{MN}(\vec{x})}\bar{\partial}x^N + \bar{\partial}x^M E_M{}^{\boldsymbol{\alpha}}(\vec{x})\, d_{z\boldsymbol{\alpha}} + \partial x^M E_M{}^{\hat{\boldsymbol{\alpha}}}(\vec{x})\, \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}} + \\
& + d_{z\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}(\vec{x})\, \hat{d}_{\bar{z}\hat{\boldsymbol{\beta}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}} C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}(\vec{x})\, \boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}(\vec{x})\, \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} d_{z\boldsymbol{\gamma}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}(\vec{x})\, \boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} + \\
& + \underbrace{\left(\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\bar{\partial}x^M \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\vec{x})\right)}_{\equiv \nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}} \boldsymbol{\omega}_{z\boldsymbol{\beta}} + \underbrace{\left(\partial\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\partial x^M \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}(\vec{x})\right)}_{\equiv \hat{\nabla}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}} \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} + \\
& + \tfrac{1}{2}L_{z\bar{z}a}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) + \tfrac{1}{2}\hat{L}_{\bar{z}z\hat{a}}(\hat{\boldsymbol{\lambda}}\gamma^{\hat{a}}\hat{\boldsymbol{\lambda}})
\end{aligned}} \tag{5.1}
$$

Note that we denote with $\vec{x}$ the complete set x^M of superspace coordinates, while $\vec{x}$ will only denote the bosonic subset x^m. As stated already above, the kinetic ghost term $\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ can always be brought to this simple form by a redefinition of $\boldsymbol{\omega}$. We will discuss this and other worldsheet reparametrizations below in detail. The motivation for the definition of the covariant derivative $\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}$ will also be given at a later point. For the moment, $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(x)$ is just an arbitrary coefficient function or background field. Like in the flat case, we implement the pure spinor constraints via two Lagrange multipliers.

In order to complete the theory, we need two BRST operators which reduce to the well known ones in the flat case. Their nilpotency and (anti)holomorphicity will be checked later and lead to the supergravity constraints. For the moment, let us just write down the most general ansatz of their currents, which have to be of conformal weight (1,0) and (0,1) and ghost number (1,0) and (0,1) respectively

$$
\boldsymbol{j}_z \;=\; \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\left(d_{z\boldsymbol{\alpha}} + \Upsilon^{(2)}{}_{\boldsymbol{\alpha}M}(\vec{x})\,\partial_z x^M + \boldsymbol{\lambda}^{\boldsymbol{\gamma}}\Upsilon^{(3)}{}_{\boldsymbol{\alpha\gamma}}{}^{\boldsymbol{\beta}}(\vec{x})\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right), \quad \boldsymbol{j}_{\bar{z}} = 0 \tag{5.2}
$$

$$
\hat{\boldsymbol{\jmath}}_{\bar{z}} \;=\; \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}} + \hat{\Upsilon}^{(2)}_{\hat{\boldsymbol{\alpha}}M}(\vec{x})\,\partial_{\bar{z}} x^M + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\gamma}}}\hat{\Upsilon}^{(3)}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\beta}}}(\vec{x})\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\right), \quad \hat{\boldsymbol{\jmath}}_z = 0 \tag{5.3}
$$

Like for the ghost kinetic term, we have immediately absorbed any $\vec{x}$-dependent coefficient $\Upsilon^{(1)}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\vec{x})$ coming with $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\beta}}$ and its hatted version in a redefinition of $d_{z\boldsymbol{\beta}}$ and $\hat{d}_{\bar{z}\hat{\boldsymbol{\beta}}}$.[2] Of course one can further redefine $d_{z\boldsymbol{\alpha}}$ and $\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}$, such that we arrive at the standard form $\boldsymbol{j}_z = \boldsymbol{\lambda}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\jmath}}_{\bar{z}} = \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}d_{\bar{z}\hat{\boldsymbol{\alpha}}}$. This does not change the general form of the action. We will discuss the reparametrizations more carefully in the next section.

The following observation is important to reduce the computations one has to do. Let us first define

$$
\hat{O}_{MN} \;\equiv\; O_{NM}, \quad \left(\hat{G} = G, \hat{B} = -B, \hat{H} = -H\right) \tag{5.4}
$$

$$
\hat{\mathcal{P}}^{\hat{\boldsymbol{\gamma}}\boldsymbol{\gamma}} \;\equiv\; \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \tag{5.5}
$$

$$
\hat{S}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\alpha}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\beta}} \;\equiv\; S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} \tag{5.6}
$$

Then – rather obviously – the following statement holds

Proposition 3 (left-right symmetry) *The complete theory (action +BRST operators) is invariant under the exchange of hatted and unhatted objects if at the same time their indices are flipped from hatted to unhatted and from z to $\bar{z}$ and vice verse, and ∂ is exchanged with $\bar{\partial}$:*

$$
\begin{gathered}
d \leftrightarrow \hat{d}, \boldsymbol{\lambda} \leftrightarrow \hat{\boldsymbol{\lambda}}, \boldsymbol{\omega} \leftrightarrow \hat{\boldsymbol{\omega}}, L \leftrightarrow \hat{L}, O \leftrightarrow \hat{O}, \mathcal{P} \leftrightarrow \hat{\mathcal{P}}, S \leftrightarrow \hat{S}, C \leftrightarrow \hat{C}, \Omega \leftrightarrow \hat{\Omega}, \nabla \leftrightarrow \hat{\nabla}, \Upsilon^{(i)} \leftrightarrow \hat{\Upsilon}^{(i)}, \boldsymbol{j} \leftrightarrow \hat{\boldsymbol{\jmath}} \\
\partial \leftrightarrow \bar{\partial}, \text{ indices: } \boldsymbol{\alpha} \leftrightarrow \hat{\boldsymbol{\alpha}}, z \leftrightarrow \bar{z}
\end{gathered} \tag{5.7}
$$

In particular the replacement $O \leftrightarrow \hat{O}$ implies due to (5.4) that

$$
B \leftrightarrow -B, \qquad G \leftrightarrow G \tag{5.8}
$$

[1]This, however, contributes to the surface term. In the case of open strings, adding a $\partial\bar{\partial}x^M$-term is therefore equivalent to the modification of the boundary part of the action. ◇

[2]If one wants to study degenerate limits of the theory, one should remember and reintroduce the coefficients $\Upsilon^{(1)}$, $\hat{\Upsilon}^{(1)}$ and the one coming with the ghost kinetic terms. ◇

Simple eom's Before we close this section, let us quickly give the equations of motion of those worldsheet variables (all but x^K) which can be seen from the target superspace point of view as tangent or cotangent vectors. This refers to the form of their reparametrizations that will be discussed on page 49. Their equations of motion are comparatively simple:

$$\frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} = \bar\partial x^M E_M{}^{\boldsymbol{\gamma}} + \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.9}$$

$$\frac{\delta S}{\delta \hat d_{\bar z\hat{\boldsymbol{\gamma}}}} = \partial x^M E_M{}^{\hat{\boldsymbol{\gamma}}} + d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}} C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \tag{5.10}$$

$$\frac{\delta S}{\delta \boldsymbol{\omega}_{z\boldsymbol{\beta}}} = -\left(\nabla_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\left(C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right)\right) \equiv -\mathcal{D}_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \tag{5.11}$$

$$\frac{\delta S}{\delta \hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}} = -\left(\hat\nabla_{z}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right)\right) \equiv -\hat{\mathcal{D}}_{z}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} \tag{5.12}$$

$$\frac{\delta S}{\delta \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} = -\left(\nabla_{\bar z}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} - \left(C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right)\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right) + L_{z\bar z a}(\gamma^a\boldsymbol{\lambda})_{\boldsymbol{\alpha}} \equiv -\mathcal{D}_{\bar z}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} + L_{z\bar z a}(\gamma^a\boldsymbol{\lambda})_{\boldsymbol{\alpha}} \tag{5.13}$$

$$\frac{\delta S}{\delta \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} = -\left(\hat\nabla_{z}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}} - \left(\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right)\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right) + \hat L_{z\bar z a}(\gamma^a\hat{\boldsymbol{\lambda}})_{\hat{\boldsymbol{\alpha}}} \equiv -\hat{\mathcal{D}}_{z}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}} + \hat L_{z\bar z a}(\gamma^a\hat{\boldsymbol{\lambda}})_{\hat{\boldsymbol{\alpha}}} \tag{5.14}$$

$$\frac{\delta S}{\delta L_{z\bar z a}} = \frac{1}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}), \qquad \frac{\delta S}{\delta \hat L_{z\bar z a}} = \frac{1}{2}(\hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}}) \tag{5.15}$$

In (5.11)-(5.14) we have introduced yet two other "covariant derivatives" $\mathcal{D}_{\bar z}$ and $\hat{\mathcal{D}}_z$:

$$\mathcal{D}_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \equiv \bar\partial\boldsymbol{\lambda}^{\boldsymbol{\beta}} + A_{\bar z\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \qquad A_{\bar z\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \equiv \bar\partial x^M \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.16}$$

$$\hat{\mathcal{D}}_{z}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} \equiv \partial\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + \hat A_{z\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}, \qquad \hat A_{z\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \equiv \partial x^M \hat\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \tag{5.17}$$

These covariant derivatives are introduced simply for calculational convenience and we do not give a geometric interpretation – although this might be interesting. For the covariant derivatives $\nabla_{\bar z}$ and $\hat\nabla_z$ defined in (5.1) instead, there exists a simple geometric interpretation. They are pullbacks of the covariant target super tangent space derivatives with connection coefficients $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\hat\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ to the worldsheet. The reason why these two background fields can be seen as connections will be given in the following.

Note that the derivation of the still missing variational derivative with respect to x^K is quite involved and will only be given in section 5.5 on page 54 using a covariant variational principle.

5.2 Vielbeins, worldsheet reparametrizations and target space symmetries

There are several ways to reparametrize the worldsheet fields in the above action and the BRST currents. One can use such reparametrizations to simplify the form of the action (as we did already implicitly in order to get a simple ghost kinetic term) or of the BRST currents.

Before we come to the first convenient reparametrization, let us observe the following: The two background fields $E_M{}^{\boldsymbol{\alpha}}$ and $E_M{}^{\hat{\boldsymbol{\alpha}}}$, combined to a 42×32 matrix $E_M{}^{\boldsymbol{\mathcal{A}}}$,$\boldsymbol{\mathcal{A}} \in \{\boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}}\}$ have maximal rank 32 in a small perturbation around the string in flat background. Or in other words, the quadratic block $E_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}}$ is invertible[3]. It can thus be completed by some $E_M{}^a$ to an invertible 42×42 matrix which we can interpret as (super)vielbein. The only requirement for $E_M{}^a$ to be a valid completion is that its bosonic sub-matrix $E_m{}^a$ is invertible[4]. The "background field" $E_M{}^a$ does not appear in the action and nothing should depend on it. Let us from now on use the completed vielbein $E_M{}^A$ and its inverse $E_A{}^M$ to switch from curved to flat indices and vice verse. In particular we define

$$G_{AB} \equiv E_A{}^M G_{MN} E_B{}^N \tag{5.18}$$

For later usage we denote the components of the pullback of the vielbein E^A to the worldsheet as

$$\Pi_z^A \equiv \partial x^M E_M{}^A \tag{5.19}$$

$$\Pi_{\bar z}^A \equiv \bar\partial x^M E_M{}^A \tag{5.20}$$

In flat space, $\Pi^a_{z/\bar z}$ will just be the supersymmetric momentum and the fermionic component will reduce to the worldsheet derivative of the fermionic coordinates: $\Pi^{\boldsymbol{\mathcal{A}}}_{z/\bar z} \overset{\text{flat}}{\to} \partial_{z/\bar z}\boldsymbol{\theta}^{\mathcal{A}}$.

Let us now study the possible reparametrizations of the worldsheet variables systematically.

[3]Again it might be interesting to study also degenerate limits. ◇

[4]The bosonic supermatrix $\begin{pmatrix} E_m{}^a & E_m{}^{\boldsymbol{\mathcal{A}}} \\ E_{\boldsymbol{\mathcal{M}}}{}^a & E_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}} \end{pmatrix}$ is invertible, iff its bosonic blocks $(E_m{}^a)$ and $(E_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}})$ are invertible. ◇

Possible reparametrizations We denote by $\phi^{\mathcal{I}}_{\text{all}}$ the collection of all worldsheet fields. If we make some reparametrization $\tilde{\phi}^{\mathcal{I}}_{\text{all}} = f[\phi^{\mathcal{I}}_{\text{all}}]$, the Jacobi matrix has to be invertible in order to lead to equivalent equations of motion:

$$\frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}(\sigma)} = \int d^2\tilde{\sigma} \;\; \frac{\delta \tilde{\phi}^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})}{\delta \phi^{\mathcal{I}}_{\text{all}}(\sigma)} \frac{\delta S}{\delta \tilde{\phi}^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})} \tag{5.21}$$

The following reparametrizations are the most general ones which respect the conformal weight as well as the left and right-moving ghost numbers (note that the Lagrange multipliers have ghost number $(-2,0)$ and $(0,-2)$ respectively):

$$\begin{aligned}
\tilde{x}^M &= f^M(\vec{x}) && (5.22)\\
\tilde{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} &= \Lambda_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}(\vec{x})\boldsymbol{\lambda}^{\boldsymbol{\beta}}, \qquad \tilde{\hat{\boldsymbol{\lambda}}}^{\hat{\boldsymbol{\alpha}}} = \hat{\Lambda}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}(\vec{x})\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} && (5.23)\\
\tilde{d}_{z\boldsymbol{\alpha}} &= \Xi^{(1)}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\vec{x})d_{z\boldsymbol{\beta}} + \Xi^{(2)}{}_{\boldsymbol{\alpha}M}(\vec{x})\partial x^M + \Xi^{(3)}{}_{\boldsymbol{\alpha}\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}(\vec{x})\boldsymbol{\lambda}^{\boldsymbol{\gamma}}\boldsymbol{\omega}_{z\boldsymbol{\delta}} && (5.24)\\
\tilde{\hat{d}}_{\bar{z}\hat{\boldsymbol{\alpha}}} &= \hat{\Xi}^{(1)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\alpha}}}(\vec{x})\hat{d}_{\bar{z}\hat{\boldsymbol{\beta}}} + \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}N}(\vec{x})\bar{\partial}x^N + \hat{\Xi}^{(3)\hat{\boldsymbol{\delta}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}(\vec{x})\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\gamma}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\delta}}} && (5.25)\\
\tilde{\boldsymbol{\omega}}_{z\boldsymbol{\alpha}} &= \Xi^{(4)}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\vec{x})\boldsymbol{\omega}_{z\boldsymbol{\beta}}, \qquad \tilde{\hat{\boldsymbol{\omega}}}_{\bar{z}\hat{\boldsymbol{\alpha}}} = \hat{\Xi}^{(4)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\alpha}}}(\vec{x})\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} && (5.26)\\
\tilde{L}_{z\bar{z}a} &= \Xi^{(5)}{}_a{}^b(\vec{x})L_{z\bar{z}b}, \qquad \tilde{\hat{L}}_{\bar{z}za} = \hat{\Xi}^{(5)b}_{a}(\vec{x})\hat{L}_{\bar{z}zb} && (5.27)
\end{aligned}$$

f^M has to be an invertible function and Λ, $\Xi^{(1)}, \Xi^{(4)}, \Xi^{(5)}$ and their hatted equivalents have to be invertible matrices. For a general reparametrization, $\Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ can be a general invertible matrix, but if we want to leave the form of the action invariant, it has to be an element of the spin group or a simple scaling. We will discuss that below. Note also, that we have already used $\Xi^{(4)}$ and $\Xi^{(1)}$ and their hatted versions to get a simple ghost-kinetic term in the action and a simple first term of the BRST operator.

Shift reparametrization Let us first study the effect of the shift-reparametrizations

$$\begin{aligned}
d_{z\boldsymbol{\alpha}} &= \tilde{d}_{z\boldsymbol{\alpha}} - \Xi^{(2)}{}_{\boldsymbol{\alpha}M}(\vec{x})\partial x^M - \Xi^{(3)}{}_{\boldsymbol{\alpha}\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}(\vec{x})\boldsymbol{\lambda}^{\boldsymbol{\gamma}}\boldsymbol{\omega}_{z\boldsymbol{\delta}}, \qquad \Xi^{(1)}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} && (5.28)\\
\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}} &= \tilde{\hat{d}}_{\bar{z}\hat{\boldsymbol{\alpha}}} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}N}(\vec{x})\bar{\partial}x^N - \hat{\Xi}^{(3)\hat{\boldsymbol{\delta}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}(\vec{x})\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\gamma}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\delta}}}, \qquad \hat{\Xi}^{(1)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\alpha}}} = \delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && (5.29)
\end{aligned}$$

on the form of the action. Plugging the above reparametrization into (5.1)-(5.3), the form of the action and the BRST currents does not change if the background fields are redefined accordingly. The shift-reparametrization thus induces an effective transformation of the background fields:

$$\begin{aligned}
\tilde{E}_N{}^{\boldsymbol{\gamma}} &= E_N{}^{\boldsymbol{\gamma}} - \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}B}E_N{}^B, \qquad \tilde{E}_M{}^{\hat{\boldsymbol{\gamma}}} = E_M{}^{\hat{\boldsymbol{\gamma}}} - \Xi^{(2)}{}_{\boldsymbol{\alpha}A}E_M{}^A\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}} && (5.30)\\
\tilde{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} - C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}A}E_M{}^A - E_M{}^{\boldsymbol{\gamma}}\Xi^{(3)}{}_{\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \Xi^{(3)}{}_{\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}A}E_M{}^A && (5.31)\\
\tilde{\hat{\Omega}}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} &= \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} - \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}}\Xi^{(2)}{}_{\boldsymbol{\alpha}A}E_M{}^A - E_M{}^{\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}} + \Xi^{(2)}{}_{\boldsymbol{\alpha}A}E_M{}^A\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}} && (5.32)\\
\tilde{C}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} &= C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} - \Xi^{(3)}{}_{\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}, \qquad \tilde{\hat{C}}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}} = \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}} - \mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}} && (5.33)\\
\tilde{S}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} &= S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\Xi^{(3)}{}_{\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}} - \Xi^{(3)}{}_{\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}} && (5.34)\\
\tilde{\Upsilon}^{(2)}_{\boldsymbol{\alpha}M} &= \Upsilon^{(2)}{}_{\boldsymbol{\alpha}M} - \Xi^{(2)}{}_{\boldsymbol{\alpha}M}, \qquad \tilde{\hat{\Upsilon}}^{(2)}{}_{\hat{\boldsymbol{\alpha}}N} = \hat{\Upsilon}^{(2)}_{\hat{\boldsymbol{\alpha}}N} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}N} && (5.35)\\
\tilde{\Upsilon}^{(3)\boldsymbol{\beta}}_{\boldsymbol{\alpha}\boldsymbol{\gamma}} &= \Upsilon^{(3)}{}_{\boldsymbol{\alpha}\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}} - \Xi^{(3)}{}_{\boldsymbol{\alpha}\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}}, \qquad \tilde{\hat{\Upsilon}}^{(3)}{}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\beta}}} = \hat{\Upsilon}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}} - \hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}} && (5.36)
\end{aligned}$$

Finally we have the transformation of $O_{MN} = G_{MN} + B_{MN}$ which we split after the transformation again into its symmetric and antisymmetric part:

$$\tilde{G}_{MN} = E_M{}^A E_N{}^B \times \tag{5.37}$$

$$\begin{pmatrix}
G_{ab} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(a|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|b)} & G_{a\boldsymbol{\beta}} - \Xi^{(2)}{}_{\boldsymbol{\beta}a} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(a|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\boldsymbol{\beta})} & G_{a\hat{\boldsymbol{\beta}}} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\beta}}a} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(a|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\hat{\boldsymbol{\beta}})}\\
G_{\boldsymbol{\alpha}b} - \Xi^{(2)}{}_{\boldsymbol{\alpha}b} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(\boldsymbol{\alpha}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|b)} & G_{\boldsymbol{\alpha}\boldsymbol{\beta}} - 2\Xi^{(2)}{}_{(\boldsymbol{\alpha}\boldsymbol{\beta})} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(\boldsymbol{\alpha}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\boldsymbol{\beta})} & G_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} - \Xi^{(2)}{}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(\boldsymbol{\alpha}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\hat{\boldsymbol{\beta}})}\\
G_{\hat{\boldsymbol{\alpha}}b} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}b} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|b)} & G_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} - \Xi^{(2)}{}_{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\boldsymbol{\beta})} & G_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} - 2\hat{\Xi}^{(2)}_{(\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}})} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}(\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\hat{\boldsymbol{\beta}})}
\end{pmatrix}_{AB}$$

$$\tilde{B}_{MN} = E_M{}^A E_N{}^B \times \tag{5.38}$$

$$\begin{pmatrix}
B_{ab} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[a|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|b]} & B_{a\boldsymbol{\beta}} - \Xi^{(2)}{}_{\boldsymbol{\beta}a} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[a|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\boldsymbol{\beta}]} & B_{a\hat{\boldsymbol{\beta}}} + \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\beta}}a} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[a|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\hat{\boldsymbol{\beta}}]}\\
B_{\boldsymbol{\alpha}b} + \Xi^{(2)}{}_{\boldsymbol{\alpha}b} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[\boldsymbol{\alpha}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|b]} & B_{\boldsymbol{\alpha}\boldsymbol{\beta}} + 2\Xi^{(2)}{}_{[\boldsymbol{\alpha}\boldsymbol{\beta}]} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[\boldsymbol{\alpha}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\boldsymbol{\beta}]} & B_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} + \Xi^{(2)}{}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} + \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[\boldsymbol{\alpha}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\hat{\boldsymbol{\beta}}]}\\
B_{\hat{\boldsymbol{\alpha}}b} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}b} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|b]} & B_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} - \Xi^{(2)}{}_{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}} - \hat{\Xi}^{(2)}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\boldsymbol{\beta}]} & B_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} - 2\hat{\Xi}^{(2)}_{[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}]} + 2\Xi^{(2)}{}_{\boldsymbol{\gamma}[\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\gamma}}|\hat{\boldsymbol{\beta}}]}
\end{pmatrix}_{AB}$$

Interestingly, looking at (5.37), one can bring G_{AB} to the block diagonal form $G_{AB} = \text{diag}\,(G_{ab}, 0, 0)$ at least for vanishing $\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}$. For general $\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}$, this is less clear because the equations become at first sight quadratic[5]

[5]Note that the matrices in (5.37) and (5.38) do not yet correspond to $\tilde{G}_{AB}$ and $\tilde{B}_{AB}$ given by $\tilde{G}_{MN} = \tilde{E}_M{}^A\tilde{E}_N{}^B\tilde{G}_{AB}$ and the equivalent equation for $\tilde{B}_{MN}$, as we have expressed $\tilde{G}_{MN}$ and $\tilde{B}_{MN}$ in terms of the untransformed vielbeins. Due to (5.30), the

in the transformation parameters. It is thus more convenient to use the shift reparametrization to bring the BRST-currents to their standard form, i.e. simply shift $\Upsilon^{(2)}$, $\Upsilon^{(3)}$, and their hatted counterparts to zero. From now on we will thus use the simple BRST-currents:

$$\boxed{\begin{aligned} \boldsymbol{j}_z &= \boldsymbol{\lambda}^{\boldsymbol{\alpha}} d_{z\boldsymbol{\alpha}}, \quad \boldsymbol{j}_{\bar{z}} = 0 && (5.39)\\ \hat{\boldsymbol{j}}_{\bar{z}} &= \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}, \quad \hat{\boldsymbol{j}}_z = 0 && (5.40)\end{aligned}}$$

In [13] the authors start with both, the simple form of the BRST currents as well as the above mentioned special form of G_{AB} and thus a reduced rank of G_{MN}. As we cannot reach both at the same time with the shift reparametrizations, the simplified form of the symmetric two-tensor has to be a result of BRST invariance or likewise on-shell holomorphicity of the BRST-current. We will discover this result soon. Only then we will use the freedom of the choice of the auxiliary vielbein components $E_M{}^a$ (which do not appear in the action), in order to fix G_{ab} to η_{ab}, or at least proportional to it. For the moment, however, we do not assume any restrictions on G_{MN}, $E_M{}^a$ and G_{AB} apart from the invertability of $E_m{}^a$.

Local target space symmetries There are still many reparametrizations left and we could try to further simplify the form of the action. It is, however, convenient not to fix all freedom. As we do not want to destroy the form of action and BRST currents that we have already obtained, the freedom consists of 'stabilizing' reparametrizations. I.e. we have to restrict to those reparametrizations out of (5.22)-(5.27) which leave the form of the action (5.1) and the simple BRST currents (5.39) and (5.40) invariant if one transforms the background fields accordingly. These reparametrizations are in general not symmetries from the worldsheet point of view as the compensating transformation of the background fields corresponds to a change of the coupling constants. However, as the action remains formally invariant, all the constraints on the background fields which will be derived later will also remain formally invariant. From the target space point of view the transformations of the background fields (going along with the $\vec{x}$-dependent reparametrizations) thus correspond to local symmetries of the target space effective theory. What we have done so far by e.g. eliminating the coefficient fields $\Upsilon^{(i)}$ in the BRST operator, corresponds to a target space gauge fixing of auxiliary background fields.

Residual shift symmetry Any further shift reparametrization of $d_{z\boldsymbol{\alpha}}$ and $\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}$ changes off-shell the form of the BRST currents (5.39) and (5.40). But we may still allow changes of the current up to the pure spinor constraint. The pure spinor constraint generates a gauge transformation as we will see in the next section. Any change of the BRST currents proportional to the pure spinor constraint thus can be compensated by a gauge transformation. Under the reparametrizations

$$\begin{aligned} d_{z\boldsymbol{\alpha}} &= \tilde{d}_{z\boldsymbol{\alpha}} - \Xi^{(3)}{}_b{}^{\boldsymbol{\delta}}(\vec{x})(\gamma^b\boldsymbol{\lambda})_{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\delta}}, \qquad \Rightarrow \Xi^{(3)}{}_{\boldsymbol{\alpha\gamma}}{}^{\boldsymbol{\delta}} \equiv \gamma^b_{\boldsymbol{\alpha\gamma}}\Xi^{(3)}{}_b{}^{\boldsymbol{\delta}} && (5.41)\\ \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}} &= \tilde{\hat{d}}_{\bar{z}\hat{\boldsymbol{\alpha}}} - \hat{\Xi}^{(3)\hat{\boldsymbol{\delta}}}_b(\vec{x})(\gamma^b\hat{\boldsymbol{\lambda}})_{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\delta}}}, \qquad \Rightarrow \hat{\Xi}^{(3)\hat{\boldsymbol{\delta}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}} \equiv \gamma^b_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\delta}}}_b && (5.42)\end{aligned}$$

the BRST currents change to

$$\begin{aligned} \boldsymbol{j}_z &= \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\tilde{d}_{z\boldsymbol{\alpha}} - \Xi^{(3)}{}_b{}^{\boldsymbol{\delta}}(\vec{x})(\boldsymbol{\lambda}\gamma^b\boldsymbol{\lambda})\boldsymbol{\omega}_{z\boldsymbol{\delta}}, \quad \boldsymbol{j}_{\bar{z}} = 0 && (5.43)\\ \hat{\boldsymbol{j}}_{\bar{z}} &= \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\tilde{\hat{d}}_{\bar{z}\hat{\boldsymbol{\alpha}}} - \hat{\Xi}^{(3)\hat{\boldsymbol{\delta}}}_b(\vec{x})(\hat{\boldsymbol{\lambda}}\gamma^b\hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\delta}}}, \quad \hat{\boldsymbol{j}}_z = 0 && (5.44)\end{aligned}$$

Global symmetries like the BRST transformation can always be redefined by a gauge transformation without changing their physical meaning. Doing this brings us back to the simple form of the BRST currents. The transformation of the background fields under this reparametrization is

$$\begin{aligned} \tilde{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} - E_M{}^{\boldsymbol{\gamma}}\gamma^b_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}} && (5.45)\\ \tilde{\hat{\Omega}}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} &= \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} - E_M{}^{\hat{\boldsymbol{\gamma}}}\gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_b && (5.46)\\ \tilde{C}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} &= C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} - \gamma^b_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}, \qquad \tilde{\hat{C}}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}} = \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\alpha}} - \mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}}\gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_b && (5.47)\\ \tilde{S}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} &= S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\gamma^b_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}} + C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_b - \gamma^a_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_a{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}\hat{\Xi}^{(3)\hat{\boldsymbol{\beta}}}_b && (5.48)\end{aligned}$$

This target space gauge symmetry will be fixed at a later point in section 5.11 on page 71.

vielbeins transformation has the form

$$\tilde{E}_M{}^A = \left(E_M{}^c, E_M{}^{\boldsymbol{\gamma}}, E_M{}^{\hat{\boldsymbol{\gamma}}}\right)\begin{pmatrix} \delta_c{}^a & -\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\delta}}c} & -\Xi^{(2)}{}_{\boldsymbol{\delta}c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\\ 0 & \delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}} - \mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\delta}}\boldsymbol{\gamma}} & -\Xi^{(2)}{}_{\boldsymbol{\delta\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\\ 0 & -\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}\hat{\Xi}^{(2)}_{\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\gamma}}} & \delta_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}} - \Xi^{(2)}{}_{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\end{pmatrix}$$

For non-vanishing $\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}$, the inverse of this matrix would enter the final form of $\tilde{G}_{AB}$ and make the problem of finding a reparametrization with $\tilde{G}_{AB} = \operatorname{diag}(\tilde{G}_{ab}, 0, 0)$ more complicated. ⋄

Superdiffeomorphisms Let us now consider the general reparametrizations (5.22) of the superspace-embedding functions x^M which correspond to target space super-diffeomorphisms.

$$\tilde{x}^M = f^M(\vec{x}) \tag{5.49}$$

The worldsheet derivatives of the embedding functions transform like target space vectors

$$\bar{\partial}\tilde{x}^M = \partial\tilde{x}^M/\partial x^N \cdot \bar{\partial}x^N \tag{5.50}$$

For the action and the BRST-operators to remain form-invariant, the background fields have to transform tensorial according to the appearance of the curved index M, e.g. $\tilde{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\tilde{\vec{x}}) = \Omega_{N\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}(\vec{x})\,\partial x^N/\partial\tilde{x}^M$. All objects with only flat indices or no indices have to transform like scalars. In this way we observe that the resulting effective equations for the background fields will be superdiffeomorphism invariant.

Gauge transformation of the B-field One of the gauge transformations of the background fields is a bit special, as it is not related to a worldsheet reparametrization. It is the shift $B \mapsto B + \mathbf{d}\Lambda$ with some one-form Λ. This does not change the action at all, as the total derivative term simply drops out (for closed strings). It is, however, again not a worldsheet symmetry, as we do not transform the worldsheet fields but the coupling constants. The background field-constraints will in the end be the same for the transformed B and we thus have again a gauge symmetry from the target space point of view.

Local Lorentz transformations and local scale transformations Next we consider reparametrizations of the ghost $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$. An admissible reparametrizations (5.23) of $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$ turns the pure spinor term $L_{z\bar{z}a}(\boldsymbol{\lambda}^T\gamma^a\boldsymbol{\lambda})$ into $L_{z\bar{z}a}(\tilde{\boldsymbol{\lambda}}^T\Lambda^{-1}\gamma^a\Lambda^{T\,-1}\tilde{\boldsymbol{\lambda}})$. In order to obtain the old pure spinor term also in the new variables, the reparametrization of the ghosts has to be accompanied by an appropriate reparametrization $L_{z\bar{z}b} = \Lambda_b{}^a(\vec{x})\cdot\tilde{L}_{z\bar{z}a}$ of the Lagrange multiplier $L_{z\bar{z}a}$. The condition for the invariance of the pure spinor term under the reparametrization then reads[6]

$$\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} \stackrel{!}{=} \Lambda_b{}^a(\Lambda^{-1})_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\gamma^b_{\boldsymbol{\gamma}\boldsymbol{\delta}}(\Lambda^{-1})_{\boldsymbol{\beta}}{}^{\boldsymbol{\delta}} \tag{5.51}$$

For infinitesimal reparametrizations we can rewrite it as

$$2L_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]} \stackrel{!}{=} L_b{}^a\gamma^b_{\boldsymbol{\alpha}\boldsymbol{\beta}} \qquad \text{(infini)} \tag{5.52}$$

$$\text{with } \Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \equiv \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad \Lambda_a{}^b \equiv \delta_a^b + L_a{}^b \tag{5.53}$$

[6]The fact that we use the index structure $\Lambda_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}$ instead of $\Lambda^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$ is only for later notational convenience. It is not necessarily related to using NW-conventions, although $\tilde{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} = \boldsymbol{\lambda}^{\boldsymbol{\beta}}\Lambda_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}$ contains a nice NW-contraction. For us the reason is simply that the alternative index position would be very inconvenient for the associated connection. The symbol $\Omega_{M\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}$ is just much simpler to type (and looks better) than $\Omega_M{}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$. $\diamond$

To obey this, both reparametrizations are restricted to local Lorentz transformations and local scale transformations[7]. The infinitesimal generators thus have the following explicit form:

$$L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = L^{(D)\boldsymbol{\beta}}_{\boldsymbol{\alpha}} + L^{(L)\boldsymbol{\beta}}_{\boldsymbol{\alpha}}, \qquad L_a{}^b = L^{(D)b}_a + L^{(L)b}_a \tag{5.54}$$

$$L^{(D)\boldsymbol{\beta}}_{\boldsymbol{\alpha}} \equiv \frac{1}{2}L^{(D)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad L^{(L)\boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \frac{1}{4}L^{(L)}_{ab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad L^{(L)}_{ab} = -L^{(L)}_{ba} \tag{5.55}$$

$$L^{(D)b}_a \equiv L^{(D)}\delta^b_a, \qquad L^{(L)b}_a = L^{(L)}_{cd}\delta^{[c}_a\eta^{d]b}, \qquad L^{(L)}_{cd} = -L^{(L)}_{dc} \tag{5.56}$$

The reparametrization so far reads

$$\tilde{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} = \Lambda_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \tag{5.57}$$

$$\tilde{L}_{z\bar{z}a} = \Lambda^{-1b}_a L_{z\bar{z}b} \tag{5.58}$$

Note that in our notation Λ contains both, Lorentz transformations and scale transformations (dilatations).

In order to maintain the special form of the ghost kinetic term and of the BRST-operator, we likewise have to transform

$$\tilde{d}_{z\boldsymbol{\alpha}} = (\Lambda^{-1})_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}d_{z\boldsymbol{\beta}} \tag{5.59}$$

$$\tilde{\boldsymbol{\omega}}_{z\boldsymbol{\alpha}} = (\Lambda^{-1})_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \tag{5.60}$$

with infinitesimally $(\Lambda^{-1})_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} - L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$. The background fields can again be reparametrized in a way that the complete action plus the BRST operators remain form-invariant: Just transform every background field with unhatted spinorial indices accordingly. E.g.

$$\tilde{C}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = (\Lambda^{-1})_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\Lambda_{\boldsymbol{\delta}}{}^{\boldsymbol{\beta}}C_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}, \quad \ldots \tag{5.61}$$

Only the field $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ must not transform like a tensor, but like a connection, in order to keep the form-invariance of the action

$$\tilde{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = -\partial_M\Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + (\Lambda^{-1})_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\Lambda_{\boldsymbol{\delta}}{}^{\boldsymbol{\beta}}\Omega_{M\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} \tag{5.62}$$

This is exactly the reason why we have combined it to a covariant derivative in the ghost kinetic term right from the beginning. For the effective field equations all this means that they will be invariant under a local Lorentz transformation and dilatation acting on all the indices of the background fields which are coupled to the ghosts, the ghost-momenta and the variables $d_{z\boldsymbol{\alpha}}$, or in other words, acting on all unhatted flat spinorial indices.

[7]The 32×32 unity and the antisymmetrized Γ-matrices $\Gamma^{a_1\cdots a_p}$ (see appendix D on page 167ff) form a basis of the vector space of all 32×32 matrices. The 16×16 sub-matrices $\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$, $\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$, …, $\gamma^{a_1\cdots a_{10}}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$ in the block-diagonal (they vanish for an odd number p of bosonic antisymmetrized indices, see (D.110) on page 177) therefore span all the 16×16 matrices. And due to the relations (D.128)-(D.131) on page 178, i.e. $\gamma^{[p]}\propto\gamma^{[n-p]}$, already the matrices $\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$, $\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$ and $\gamma^{a_1\cdots a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$ form a complete basis of all 16×16-matrices. We thus can expand the infinitesimal generator $L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$ of the reparametrization matrix (i.e. $\Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} = \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} + L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$) as follows:

$$L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} = \frac{1}{2}L^{(D)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} + \frac{1}{4}L^{(L)}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} + L_{a_1\ldots a_4}\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$$

Plugging this expansion into the condition (5.52) yields

$$L_b{}^a\gamma^b_{\boldsymbol{\alpha\beta}} \stackrel{!}{=} 2L_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]} = L^{(D)}\gamma^a_{\boldsymbol{\alpha\beta}} + \frac{1}{2}L^{(L)}_{a_1a_2}\underbrace{\gamma^{a_1a_2}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]}}_{\propto\gamma^{[1]}_{\boldsymbol{\alpha\beta}}+\underbrace{\gamma^{[3]}_{[\boldsymbol{\alpha\beta}]}}_{0}} + 2L_{a_1\ldots a_4}\underbrace{\gamma^{a_1\ldots a_4}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]}}_{\propto\underbrace{\gamma^{[3]}_{[\boldsymbol{\alpha\beta}]}}_{0}+\gamma^{[5]}_{\boldsymbol{\alpha\beta}}} \qquad (*)$$

Below the curly bracket, we have indicated the schematic expansion (D.112) of page 177. Due to (D.111), all the $\gamma^{[3]}$'s vanish because of the graded antisymmetrization. We can thus concentrate on the $\gamma^{[1]}$ and $\gamma^{[5]}$-part:

$$\gamma^{a_1a_2}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]} \stackrel{(D.114)}{=} 2\gamma^{[a_1}{}_{\boldsymbol{\alpha\beta}}\eta^{a_2]a}$$

$$\gamma^{a_1\ldots a_4}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]} \stackrel{(D.114)}{=} \gamma^{a_1\ldots a_4a}{}_{\boldsymbol{\alpha\beta}}$$

The righthand side of (*) has to be a linear combination of γ^a's which is not true with a remaining $\gamma^{[5]}$-term $L_{a_1\ldots a_4}\gamma^{a_1\ldots a_4a}{}_{\boldsymbol{\alpha\beta}}$. We thus have to demand

$$L_{a_1\ldots a_4} \stackrel{!}{=} 0$$

With this condition, (*) and therefore (5.52) are fulfilled and the relation between the reparametrization of the ghosts and of the Lagrange multipliers is given by

$$L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} = \frac{1}{2}L^{(D)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} + \frac{1}{4}L^{(L)}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}$$

$$L_b{}^a = L^{(D)}\delta^a_b + L^{(L)}_{bc}\eta^{ca} \qquad \diamond$$

.

We get an equivalent but in the beginning completely independent local Lorentz transformation and scaling $\hat{\Lambda}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ acting on the hatted indices. In addition we may redefine the bosonic vielbein $E^a = \mathbf{d}x^M E_M{}^a$, which we introduced by hand. Remember, it is related to G_{AB} via $G_{MN} = E_M{}^A G_{AB} E_N{}^B$ and we did not yet restrict G_{AB}. The matrices $E_M{}^a$ (of maximal rank 10) can thus be redefined by an arbitrary GL(10) transformation on the index a, accompanied by a compensating transformation of G_{AB}. At a later point, we will obtain a restriction on G_{AB} which then allows only Lorentz and scale transformations $\check{\Lambda}_a{}^b$ acting on the index a of $E_M{}^a$. This transformation, acting on bosonic flat indices only, is again independent of the other two local structure group transformations (acting on the spinorial indices). The relation of the three transformations will in the end be fixed (see page 92) by a convenient gauge fixing of some torsion components. In contrast to the fermionic transformations, the bosonic local Lorentz transformation is not coupled to a reparametrization of an elementary field (from the worldsheet point of view), but only to the transformation of G_{ab}:

$$\tilde{E}_M{}^a = \check{\Lambda}_c{}^a E_M{}^c \tag{5.63}$$

$$\tilde{G}_{ab} = (\check{\Lambda}^{-1})_a{}^c G_{cd} (\check{\Lambda}^{-1})_b{}^d \tag{5.64}$$

The transformation of the background fields is determined by their flat indices. Combining the bosonic and fermionic flat indices to $A \equiv (a, \boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}})$, we have a block diagonal **structure group transformation** acting on the target super tangent space:

$$\underline{\Lambda}_A{}^B \equiv \begin{pmatrix} \check{\Lambda}_a{}^b & 0 & 0 \\ 0 & \Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat{\Lambda}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.65}$$

All three blocks are independent. $\Lambda_a{}^b$ instead, which is acting on the Lagrange multiplier (but on no background field!), was induced by $\Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ via the invariance of $\gamma^a_{\boldsymbol{\alpha\beta}}$. Also keep in mind that $\check{\Lambda}_a{}^b$ is so far not restricted to Lorentz transformations or scalings. It will be so at a later point.

5.3 Connection

We have seen in equation (5.62) on the preceding page that $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ transform like connections under structure group transformations. Let us introduce some auxiliary target space field $\check{\Omega}_{Ma}{}^b$ which transforms like a connection under the transformation $\check{\Lambda}_a{}^b$ of the bosonic tangent space. As the field $\check{\Omega}_{Ma}{}^b$ does not appear in the worldsheet action, nothing should depend on it in the end. We can now combine the three objects to a structure group connection on the target super tangent space (let's call it the **mixed connection**)

$$\underline{\Omega}_{MA}{}^B \equiv \begin{pmatrix} \check{\Omega}_{Ma}{}^b & 0 & 0 \\ 0 & \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.66}$$

The underline will help us later to distinguish this connection from alternative choices. This underline will decorate all objects referring to this connection. The corresponding superspace connection coefficients $\underline{\Gamma}_{MN}{}^K$ are now given via

$$0 \stackrel{!}{=} \underline{\nabla}_M E_N{}^A \equiv \partial_M E_N{}^A - \underline{\Gamma}_{MN}{}^K E_K{}^A + \underline{\Omega}_{MB}{}^A E_N{}^B \tag{5.67}$$

Due to the block-diagonal form of the connection, the curvature $\underline{R}_A{}^B \equiv \mathbf{d}\underline{\Omega}_A{}^B - \underline{\Omega}_A{}^C \wedge \underline{\Omega}_C{}^B$ is block diagonal as well

$$\underline{R}_A{}^B = \begin{pmatrix} \check{R}_a{}^b & 0 & 0 \\ 0 & R_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat{R}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.68}$$

and the upper index of the torsion $\underline{T}^A \equiv \mathbf{d}E^A - E^C \wedge \underline{\Omega}_C{}^A$ tells us by which block of the connection it is determined:

$$\underline{T}^A = (\check{T}^a, T^{\boldsymbol{\alpha}}, \hat{T}^{\hat{\boldsymbol{\alpha}}}) \tag{5.69}$$

Remark Although the connection coefficients which act on the spinorial indices have the correct transformation properties, we did not yet check that they are Lie algebra valued, i.e. that the matrices $\Omega_M{}_{\cdot}{}^{\cdot}$ and $\hat{\Omega}_M{}_{\cdot}{}^{\cdot}$ are not general matrices, but are restricted to the structure group algebra of Lorentz and scale transformations. We will show this partwise below in section 5.4 when we discuss the antighost gauge symmetry and will complete the argument when we study the holomorphicity of the BRST current in section 5.7. Let us already here give the result for completeness:

$$\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}\Omega^{(D)}_M \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{2}\hat{\Omega}^{(D)}_M \delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{\Omega}^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{5.70}$$

The labels (D) and (L) distinguish the dilatation (or scaling) part from the Lorentz part.

This special form of the connection of course induces a special form of the curvature (see (5.68) and (F.88),(F.90) and (F.92) on page F.90). The curvature is blockdiagonal in the last two indices (5.68) and each block decays into a scale (or dilatation) part and a Lorentz part:

$$\underline{R}_{MNC}{}^{D} = \operatorname{diag}\left(\check{R}_{MNc}{}^{d}, R_{MN\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}, \hat{R}_{MN\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\right) \tag{5.71}$$

$$\check{R}_{MNc}{}^{d} = \check{F}^{(D)}_{MN}\delta^d_c + \check{R}^{(L)}_{MN\,c}{}^{d}, \qquad \check{F}^{(D)}_{MN} = \frac{1}{10}\check{R}_{MNc}{}^{c} \tag{5.72}$$

$$R_{MN\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \frac{1}{2}F^{(D)}_{MN}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \frac{1}{4}R^{(L)}_{MNa_1}{}^{b}\eta_{ba_2}\gamma^{a_1a_2}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}, \qquad F^{(D)}_{MN} = -\frac{1}{8}R_{MN\boldsymbol{\gamma}}{}^{\boldsymbol{\gamma}} \tag{5.73}$$

$$\hat{R}_{MN\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}} = \frac{1}{2}\hat{F}^{(D)}_{MN}\delta_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}} + \frac{1}{4}\hat{R}^{(L)}_{MNa_1}{}^{b}\eta_{ba_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}, \qquad \hat{F}^{(D)}_{MN} = -\frac{1}{8}\hat{R}_{MN\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.74}$$

with the scale field strength

$$\check{F}^{(D)} \equiv \mathbf{d}\check{\Omega}^{(D)}, \qquad F^{(D)} \equiv \mathbf{d}\Omega^{(D)}, \qquad \hat{F}^{(D)} \equiv \mathbf{d}\hat{\Omega}^{(D)} \tag{5.75}$$

The major part of the covariant derivation of the last equation of motion in section 5.5, where we have not yet completed the argument that the mixed connection is structure group valued, does not refer to this fact. Only the variation of the pure spinor term will be affected and this will be discussed carefully.

5.4 Antighost gauge symmetry

The pure spinor constraints $\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}} = 0$ are first class constraints at least in the flat case and thus generate gauge symmetries. The same should be true in the curved case. We can see this fact, however, without referring to the Hamiltonian language, simply as a consistency condition on the equations of motion.

For the ghost field we have two equations of motion which have to be consistent in order to allow any solutions:

$$\frac{\delta S}{\delta \boldsymbol{\omega}_{z\boldsymbol{\beta}}} = -\left(\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\left(\bar{\partial}x^M\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\right)\right) \equiv -\mathcal{D}_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \tag{5.76}$$

$$\frac{\delta S}{\delta L_{z\bar{z}a}} = \frac{1}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) \tag{5.77}$$

Every linear combination of the second line, $\frac{\mu_a}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda})$, obviously is still on-shell zero for any set of local parameters μ_a. When we act with $\bar{\partial}$ on this expression, the result still has to vanish on-shell. I.e. for any μ_a, we need to have:

$$\begin{aligned} 0 \underset{\text{on-shell}}{\overset{!}{=}}\; & \bar{\partial}\left(\frac{\mu_a}{2}\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}\right) = \qquad\qquad \forall\mu_a(z,\bar{z}) \\ \overset{(5.16)}{=}\; & \bar{\partial}\mu_a\cdot\underbrace{\frac{1}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda})}_{\frac{\delta S}{\delta L_{z\bar{z}a}}} + \mu_a(\boldsymbol{\lambda}\gamma^a)_{\boldsymbol{\beta}}\underbrace{\mathcal{D}_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}}_{-\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}}} - \mu_a\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underbrace{\left(\Pi^C_{\bar{z}}\Omega_{C[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}} + C_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}S_{[\boldsymbol{\alpha}|\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\right)}_{A_{\bar{z}[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \end{aligned} \tag{5.78}$$

The first two terms in the last line vanish on-shell, so we may concentrate on the rest. Following footnote 7 on page 49 (with $A_{\bar{z}[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}$ taking the role of $L_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}$) we can expand $A_{\bar{z}[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}$ in antisymmetrized γ-matrices and obtain for the last term in (5.78)

$$\begin{aligned} -\mu_a\boldsymbol{\lambda}^{\boldsymbol{\alpha}}A_{\bar{z}[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]}\boldsymbol{\lambda}^{\boldsymbol{\beta}} &= -\mu_a\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\left(\frac{1}{2}A^{(D)}_{\bar{z}}\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} + \frac{1}{2}A^{(L)}_{\bar{z}\,a_1a_2}\gamma^{[a_1}{}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\eta^{a_2]a} + A_{\bar{z}\,a_1\ldots a_4}\gamma^{a_1\ldots a_4a}{}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\right)\boldsymbol{\lambda}^{\boldsymbol{\beta}} = \\ &= -\underbrace{\left(A^{(D)}_{\bar{z}}\delta^b_a + A^{(L)b}_{\bar{z}\,a}\right)}_{\equiv A_{\bar{z}\,a}{}^{b}}\mu_b\cdot\underbrace{\frac{1}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda})}_{\frac{\delta S}{\delta L_{z\bar{z}a}}} - \mu_a A_{\bar{z}\,a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4a}\boldsymbol{\lambda}) \end{aligned} \tag{5.79}$$

It is natural to view $A_{\bar{z}\,a}{}^{b}$ as the connection coefficients corresponding to $\mathcal{D}_{\bar{z}}$ when acting on bosonic indices. It is built from the expansion coefficients of $A_{\bar{z}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ which are in turn built from the expansion coefficients of

$\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$, $C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}$ and $S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}$ (all seen as matrices in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ – compare again to footnote 7 on page 49)[8]

$$\mathcal{D}_{\bar z}\mu_a \equiv \bar\partial\mu_a - A_{\bar z a}{}^b\mu_b, \qquad A_{\bar z a}{}^b \equiv \underbrace{\bar\partial x^M\Omega_{Ma}{}^b}_{\Pi^C_{\bar z}\Omega_{Ca}{}^b} + C_a{}^{b\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{a\hat{\boldsymbol{\alpha}}}{}^{b\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.80}$$

$$\text{with}\quad \Omega_{Ma}{}^b \equiv \Omega^{(D)}_M\delta^b_a + \Omega^{(L)}_{Ma}{}^b \qquad \Leftarrow \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}\Omega^{(D)}_M\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \underbrace{\Omega_{Ma_1\ldots a_4}}_{=0\text{ (later)}}\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{5.81}$$

$$C_a{}^{b\hat{\boldsymbol{\gamma}}} \equiv C^{\hat{\boldsymbol{\gamma}}}\delta^b_a + C^{\hat{\boldsymbol{\gamma}}}{}_{ac}\eta^{cb} \qquad \Leftarrow C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = \frac{1}{2}C^{\hat{\boldsymbol{\gamma}}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}C^{\hat{\boldsymbol{\gamma}}}{}_{ab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \underbrace{C^{\hat{\boldsymbol{\gamma}}}{}_{a_1\ldots a_4}}_{=0\text{ (later)}}\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{5.82}$$

$$S_{a\hat{\boldsymbol{\alpha}}}{}^{b\hat{\boldsymbol{\beta}}} \equiv S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\delta^b_a + S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{ac}\eta^{cb} \qquad \Leftarrow S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} = \frac{1}{2}S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{ac}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \underbrace{S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{a_1\ldots a_4}}_{=0\text{ (later)}}\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{5.83}$$

The coefficient $\Omega_{Ma_1\ldots a_4}$ and the other $\gamma^{[4]}$-coefficients do not enter the definitions of $\Omega_{Ma}{}^b$, $C_a{}^{b\hat{\boldsymbol{\gamma}}}$ and $S_{a\hat{\boldsymbol{\alpha}}}{}^{b\hat{\boldsymbol{\beta}}}$. At a later point we will find that the $\gamma^{[4]}$-coefficients actually have to vanish, which then implies $\mathcal{D}_{\bar z}\gamma^a_{\boldsymbol{\alpha\beta}} = 0$. This is the actual motivation for this choice of bosonic connection. It is tempting to argue that

$$A_{\bar z\, a_1\ldots a_4} \equiv \Pi^C_{\bar z}\Omega_{Ca_1\ldots a_4} + \hat d_{\bar z\hat{\boldsymbol{\gamma}}}C^{\hat{\boldsymbol{\gamma}}}{}_{a_1\ldots a_4} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{a_1\ldots a_4}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.84}$$

has to vanish already at this point, in order for all the terms in (5.78) to vanish on-shell. But the condition will be a bit weaker, as there is yet another equation of motion applicable[9]. We can replace $\Pi^{\boldsymbol{\gamma}}_{\bar z}$ (appearing in ((5.84)) and (5.80), and defined in (5.20)) with the equation of motion (5.9):$\Pi^{\boldsymbol{\gamma}}_{\bar z} = \frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} - \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}$. Putting now all the last equations together, we arrive at

$$\begin{aligned}\bar\partial\left(\frac{\mu_a}{2}\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}\right) &= \mathcal{D}_{\bar z}\mu_a\cdot\frac{\delta S}{\delta L_{z\bar z a}} - \mu_a(\boldsymbol{\lambda}\gamma^a)_{\boldsymbol{\beta}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}} - \mu_a\Omega_{\boldsymbol{\gamma}a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} + \\ &\quad -\mu_a\Big[\Pi^{\{c,\hat{\boldsymbol{\gamma}}\}}_{\bar z}\Omega_{\{c,\hat{\boldsymbol{\gamma}}\}a_1\ldots a_4} + \hat d_{\bar z\hat{\boldsymbol{\gamma}}}\left(C^{\hat{\boldsymbol{\gamma}}}{}_{a_1\ldots a_4} - \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\Omega_{\boldsymbol{\gamma}a_1\ldots a_4}\right) + \\ &\quad +\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{a_1\ldots a_4} - \hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}a_1\ldots a_4}\right)\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\Big](\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\end{aligned} \tag{5.85}$$

The dummy indices in curly brackets $\{c,\hat{\boldsymbol{\gamma}}\}$ in the second line simply should indicate a sum over c and $\hat{\boldsymbol{\gamma}}$ only, and not over $\boldsymbol{\gamma}$. The first line on the righthand side vanishes on-shell. The next two lines also have to vanish for every μ_a, because the left-hand side vanishes on-shell. At this point we cannot make use of further equations of motion. In particular the equation of motion for x^K, which we have not yet derived, would be of conformal weight (1,1) (containing terms like $\partial\bar\partial x^M$) and would therefore not be applicable. For consistency of the equations of motion, we thus get the following restrictions on the background fields

$$\boxed{\begin{aligned}\Omega_{c\,a_1\ldots a_4} &= \Omega_{\hat{\boldsymbol{\gamma}}\,a_1\ldots a_4} = 0 &(5.86)\\ C^{\hat{\boldsymbol{\gamma}}}{}_{a_1\ldots a_4} &= \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\Omega_{\boldsymbol{\gamma}a_1\ldots a_4} &(5.87)\\ S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{a_1\ldots a_4} &= \hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}a_1\ldots a_4} &(5.88)\end{aligned}}$$

This condition is weaker as the one given in [13] (see footnote (9)). It coincides exactly iff we impose in addition $\Omega_{\boldsymbol{\gamma}\,a_1\ldots a_4} = 0$ (see the remark at the end of this section). This additional restriction will, however, only be a result of BRST invariance.

According to Noether, every symmetry transformation corresponds to a divergence free current and vice verse. For a given current j^ζ, we can calculate the corresponding transformations by reading of the coefficients

[8] The coefficients $\Omega^{(D)}_M$ and $\Omega^{(L)}_{M\,a_1a_2}$ can be extracted from the given $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ using $\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}} = -16$ and $\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\gamma_{b_2b_1\,\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} = -32\delta^{a_1a_2}_{b_1b_2}$ (graded version of (D.140) on page 178)

$$\begin{aligned}\Omega_M &= -\frac{1}{8}\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}}\\ \Omega_{Ma_1a_2} &= -\frac{1}{8}\gamma_{a_1a_2\,\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \qquad \diamond\end{aligned}$$

[9] In the original derivation of the supergravity constraints from Berkovits' pure spinor string in [13] it is argued that the action has to be invariant under the gauge transformation $\delta\omega_{\boldsymbol{\alpha}} = \mu_a(\gamma^a\boldsymbol{\lambda})_{\boldsymbol{\alpha}}$ (the gauge symmetry generated by the pure spinor constraint in flat space). In our notation this implies exactly $A_{\bar z\,a_1\ldots a_4} = 0$. However, there is no reason a priory, why the form of the gauge symmetry should not be modified in curved space, as long as this modification vanishes for the flat case. We will indeed discover such a modification in the following, and with this modification the restriction on the background fields is weaker. Nevertheless we will obtain the same result in the end, as $A_{\bar z\,a_1\ldots a_4} = 0$ will be a consequence of BRST invariance later. $\diamond$

of the variational derivatives of S in the off-shell divergence of the current (see (E.7)):

$$\partial_\zeta j^\zeta_{(\rho)} = -\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \tag{5.89}$$

If we take $j_z \equiv \frac{\mu_{za}}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}), \quad j_{\bar{z}} \equiv 0$, the condition (5.78) tells that the current is on-shell divergence free. We have chosen a parameter of weight $(1,0)$, in order to get a current of correct weight. From (5.85) we can now read off the corresponding symmetry transformations:

$$\begin{aligned} \delta_{(\mu)}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} &= \mu_{za}(\boldsymbol{\lambda}\gamma^a)_{\boldsymbol{\alpha}} && (5.90)\\ \delta_{(\mu)}L_{z\bar{z}a} &= -\mathcal{D}_{\bar{z}}\mu_{za} && (5.91)\\ \delta_{(\mu)}d_{z\boldsymbol{\gamma}} &= \mu_{za}\Omega_{\boldsymbol{\gamma}a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda}) && (5.92)\end{aligned}$$

The current is divergence free for arbitrary (local) μ_{za} and we therefore have a gauge symmetry. This is the **antighost gauge symmetry** generated by the pure spinor constraint. For a flat background we have $\Omega_{\boldsymbol{\gamma}a_1\ldots a_4} = 0$ and the transformation reduces to the usual form. As stated several times already, we will obtain $\Omega_{\boldsymbol{\gamma}a_1\ldots a_4} = 0$ also in the curved background, but only later as a result of BRST invariance.

With the same reasoning we get a gauge transformation corresponding to the pure spinor constraint on the hatted ghost fields. This leads to equivalent restrictions on the hatted connection $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ and also on $\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\beta}}$ (seen as matrix in $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$). The background field $S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}$ is special, because the hatted version of the expansion (5.83) together with the condition (5.88) is again a condition on the expansion of S, now in its hatted indices. Once it is seen as matrix in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and once as matrix in $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$. This is better treatable in the special case considered in the following remark.

Remark on $\Omega_{\boldsymbol{\gamma}a_1\ldots a_4} = \hat{\Omega}_{\hat{\boldsymbol{\gamma}}a_1\ldots a_4} = 0$: Although we will discover these two additional constraints only later in (5.153) on page 60, it is nice to have everything at one place. So let us continue the discussion of $S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}$ in this case. As indicated above, we can expand it in two steps:

$$\begin{aligned} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} &= \frac{1}{2}S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}S_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}{}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \\ &= \frac{1}{2}\left(\frac{1}{2}S\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}S_{a_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\right)\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \\ &\quad + \frac{1}{4}\left(\frac{1}{2}\hat{S}_{a_1a_2}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}S_{a_1a_2b_1b_2}\gamma^{b_1b_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\right)\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \end{aligned} \tag{5.93}$$

Let us summarize the result for all the involved fields:

$$\boxed{\begin{aligned} \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \frac{1}{2}\Omega^{(D)}_M\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{2}\hat{\Omega}^{(D)}_M\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{\Omega}^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && (5.94)\\ C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} &= \frac{1}{2}C^{\hat{\boldsymbol{\gamma}}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}C^{\hat{\boldsymbol{\gamma}}}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} = \frac{1}{2}\hat{C}^{\boldsymbol{\gamma}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{C}^{\boldsymbol{\gamma}}_{a_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && (5.95)\\ S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} &= \frac{1}{4}S\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{8}S_{a_1a_2}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \\ &\quad + \frac{1}{8}\hat{S}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{16}S_{a_1a_2b_1b_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\gamma^{b_1b_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && (5.96)\end{aligned}}$$

Seen as a matrix in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (or $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ respectively), they are sums of generators of Lorentz and scale transformations. Remembering the definition of $\mathcal{D}_{\bar{z}}$ given in (5.16) and its extension to bosonic indices in (5.80), it leaves invariant the γ-matrices:[10]

$$\mathcal{D}_{\bar{z}}\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0, \qquad \hat{\mathcal{D}}_z\gamma^a_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0 \tag{5.97}$$

The expressions $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}$ and $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ are the only gauge invariant quantities (on the constraint surface $\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda} = 0$) which are linear in ghost and antighost. The reasoning is as follows: the most general combination is $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}X_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ with some general matrix $X_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ which can be expanded in $\gamma^{[0]}$, $\gamma^{[2]}$ and $\gamma^{[4]}$. Upon acting with a gauge transformation on this term, we get the products $\gamma^{[0]}\gamma^{[1]} = \gamma^{[1]}$, $\gamma^{[2]}\gamma^{[1]} \propto \gamma^{[1]} + \gamma^{[3]}$, and

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$$\mathcal{D}_{\bar{z}}\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \underbrace{\bar{\partial}\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}}}_{=0} + \left(\bar{\partial}x^M\Omega_{Mb}{}^a + C_b{}^{a\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}S_{b\hat{\boldsymbol{\alpha}}}{}^{a\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\right)\gamma^b_{\boldsymbol{\alpha}\boldsymbol{\beta}} - 2\left(\bar{\partial}x^M\Omega_{M[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}} + C_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}S_{[\boldsymbol{\alpha}|\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\right)\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]} \qquad \diamond$$

$\gamma^{[4]}\gamma^{[1]} \propto \gamma^{[3]} + \gamma^{[5]}$. As $\gamma^{[5]}$ does not vanish when contracted with two ghosts, the $\gamma^{[4]}$-part of the expansion has to vanish and we have shown the above statement. The gauge invariant expression $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}$ is nothing but the ghost current (5.143), while $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ is part of the Lorentz current, which is discussed in Berkovits' papers.

5.5 Covariant variational principle & EOM's

Remember the form of the action (5.1):

$$\begin{aligned} S &= \int d^2z \quad \frac{1}{2}\Pi_z^A \underbrace{(G_{AB}+B_{AB})}_{\equiv O_{AB}} \Pi_{\bar z}^B + \Pi_{\bar z}^{\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} + \Pi_z^{\hat{\boldsymbol{\gamma}}} \hat d_{\bar z\hat{\boldsymbol{\gamma}}} + d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \\ &\quad + \boldsymbol{\lambda}^{\boldsymbol{\alpha}} C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} d_{z\boldsymbol{\gamma}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \\ &\quad + \nabla_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \hat\nabla\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \frac{1}{2}L_{z\bar z a}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) + \frac{1}{2}\hat L_{z\bar z a}(\hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}}) \end{aligned} \tag{5.98}$$

In order to check if the BRST currents (5.39) and (5.40) are on-shell conserved (holomorphic and antiholomorphic respectively), it is first of all necessary to calculate the remaining classical equation of motion, the variation with respect to x^K. Remember, the other equations of motion were given already in (5.9)-(5.15) on page 45.

Covariant variation Deriving the variational derivative with respect to x^K is quite involved if we do not organize it properly. In the end we want to have equations which transform covariantly under superdiffeomorphisms and local structure group transformations. We therefore want to introduce a method where we stay covariant right from the beginning, e.g. a target space covariant variation of the action. In order to motivate the following definitions, let us consider only the variation of one simple term of the Lagrangian, e.g. the RR-term:

$$\begin{aligned} &\delta\left(d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}(\vec x)\hat d_{\bar z\hat{\boldsymbol{\gamma}}}\right) = \\ &= \delta d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + d_{z\boldsymbol{\gamma}}\delta x^M\partial_M\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\delta\hat d_{\bar z\hat{\boldsymbol{\gamma}}} = \qquad (5.99) \\ &= \underbrace{\left(\delta d_{z\boldsymbol{\gamma}} - \delta x^M\Omega_{M\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}}d_{z\boldsymbol{\beta}}\right)}_{\equiv\delta_{cov}d_{z\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + d_{z\boldsymbol{\gamma}}\underbrace{\delta x^M\underline{\nabla}_M\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}}_{\equiv\delta_{\underline{cov}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\underbrace{\left(\delta\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \delta x^M\hat\Omega_{M\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}\hat d_{\bar z\hat{\boldsymbol{\alpha}}}\right)}_{\equiv\delta_{c\hat ov}\hat d_{\bar z\hat{\boldsymbol{\gamma}}}} \qquad (5.100) \end{aligned}$$

In order to arrive at the target space covariant expression $\underline{\nabla}_M\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}$, it is thus convenient to group part of the x^K-variation to the variation of $d_{z\boldsymbol{\gamma}}$ or $\hat d_{\bar z\hat{\boldsymbol{\gamma}}}$ as done above. Of course we could have chosen any connection for the above rewriting, as long as we use for each contracted index pair the same connection. For the variation of the complete action, however, it is most convenient to choose the mixed connection, defined in (5.66),

$$\underline{\Omega}_{MA}{}^B \equiv \begin{pmatrix} \check\Omega_{Ma}{}^b & 0 & 0 \\ 0 & \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.101}$$

Like for the structure group transformation, the connection $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ acts on the unhatted fermionic indices and (!) on $L_{z\bar z a}$, while $\hat\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ acts on the hatted indices and (!) on $L_{\bar z z a}$. The third independent block $\check\Omega_{Ma}{}^b$ acts only on the bosonic indices that appear via the bosonic vielbein and not on elementary fields.

Similar considerations as for the RR-term hold for the other terms of the action. This motivates the definition of the **covariant variation** of the elementary fields in the above spirit:

$$\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \equiv \delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}} + \delta x^M\Omega_{M\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}, \qquad \delta_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} \equiv \delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}} - \delta x^M\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \tag{5.102}$$

$$\delta_{cov}d_{z\boldsymbol{\alpha}} \equiv \delta d_{z\boldsymbol{\alpha}} - \delta x^M\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}d_{z\boldsymbol{\beta}}, \qquad \delta_{cov}L_{z\bar z a} \equiv \delta L_{z\bar z a} - \delta x^M\Omega_{Ma}{}^bL_{z\bar z b} \tag{5.103}$$

$$\delta_{c\hat ov}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} \equiv \delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} + \delta x^M\hat\Omega_{M\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}, \qquad \delta_{c\hat ov}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}} \equiv \delta\boldsymbol{\omega}_{\bar z\hat{\boldsymbol{\alpha}}} - \delta x^M\hat\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.104}$$

$$\delta_{c\hat ov}\hat d_{\bar z\hat{\boldsymbol{\alpha}}} \equiv \delta\hat d_{\bar z\hat{\boldsymbol{\alpha}}} - \delta x^M\hat\Omega_{M\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat d_{\bar z\hat{\boldsymbol{\alpha}}}, \qquad \delta_{c\hat ov}\hat L_{\bar z z a} \equiv \delta\hat L_{\bar z z a} - \delta x^M\hat\Omega_{Ma}{}^b\hat L_{\bar z z b} \tag{5.105}$$

$$\delta_{\underline{cov}}x^K \equiv \delta x^K \tag{5.106}$$

Unfortunately this idea is not completely new. Similar versions of covariant variations have been presented in [63, 64] which in turn refer to [65, 66]. As already indicated in (5.100), we understand the covariant variation acting on arbitrary background tensor fields $T_{MA}^{NB}(\vec x)$ as

$$\begin{aligned} \delta_{\underline{cov}}T_{MA}^{NB}(\vec x) &\equiv \delta x^K\underline{\nabla}_K T_{MA}^{NB} = \qquad (5.107) \\ &= \delta T_{MA}^{NB} + \delta x^K\left(\underline{\Gamma}_{KL}{}^N T_{MA}^{LB} + \underline{\Omega}_{KC}{}^B T_{MA}^{NC} - \underline{\Gamma}_{KM}{}^L T_{LA}^{NB} - \underline{\Omega}_{KA}{}^C T_{MC}^{NB}\right) \qquad (5.108) \end{aligned}$$

In the last line we discover that the covariant variation acts on background fields in the same way as it acts on elementary fields if the index structure is the same. Note that the covariant variation cannot be understood as a variation (of e.g. x^K) in the ordinary sense. The covariant variation is simply a derivation which only reduces to an ordinary variation when acting on target space scalars, e.g. on the Lagrangian.

From the target space point of view, also objects like $\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}$ (target space covariant worldsheet derivatives of worldsheet variables) transform tensorial under structure group transformations and diffeomorphisms. The covariant variation is then simply defined according to their target space transformation properties:

$$\delta_{cov}\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \equiv \delta\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + \delta x^K \Omega_{K\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \tag{5.109}$$

$$\delta_{c\hat{o}v}\hat{\nabla}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} \equiv \delta\hat{\nabla}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + \delta x^K \hat{\Omega}_{K\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\nabla}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} \tag{5.110}$$

This is also the reason why the Lagrange multiplier is varied with help of the connection $\Omega_{Ma}{}^b$ (defined in (5.81) on page 52) which is induced by $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$, and not with the independent $\check{\Omega}_{Ma}{}^b$ that we have introduced to act on the bosonic vielbein indices: In the reparametrization corresponding to the structure group transformations, the transformation of the Lagrange multiplier is directly coupled to the transformation of the ghost.

Next we define the **covariant variational derivatives** $\frac{\delta_{cov}S}{\delta\phi^{\mathcal{I}}_{\text{all}}}$ via

$$\delta S \equiv \int_\Sigma d^2z \quad \delta_{cov}\phi^{\mathcal{I}}_{\text{all}}(z,\bar{z})\frac{\delta_{cov}S}{\delta\phi^{\mathcal{I}}_{\text{all}}(z,\bar{z})} \tag{5.111}$$

We will soon give a statement about the relation to the ordinary variational derivative. But let us first collect some tools to calculate it. In order to arrive at the righthand side of (5.111), we need to extract the covariant variations of the elementary fields. In expressions like $\delta_{cov}\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}$ in (5.109) this would require to commute e.g. the covariant variation δ_{cov} with the covariant derivative $\nabla_{\bar{z}}$ and then do some partial integration. It was probably already noticed by the reader that the covariant variation resembles very much the target space covariant worldsheet derivative $\nabla_{z/\bar{z}}$ anyway. In fact the latter can be seen as a special case of it, namely when we have $\delta\phi^{\mathcal{I}}_{\text{all}} = \partial_{z/\bar{z}}\phi^{\mathcal{I}}_{\text{all}}$. Let us therefore consider the commutators of two arbitrary covariant variations which will contain the desired commutator $[\delta_{cov},\nabla_{\bar{z}}]$ in the mentioned special case:

$$\left[\delta^{(1)}_{\underline{cov}},\delta^{(2)}_{\underline{cov}}\right]x^K = \left[\delta^{(1)},\delta^{(2)}\right]x^K + 2\delta^{(1)}x^M\underline{T}_{MN}{}^K\delta^{(2)}x^N \tag{5.112}$$

$$\begin{aligned}\left[\delta^{(1)}_{\underline{cov}},\delta^{(2)}_{\underline{cov}}\right]\varphi^{AM}{}_B &= \left[\delta^{(1)},\delta^{(2)}\right]_{\underline{cov}}\varphi^{AM}{}_B + \\ &\quad +2\delta^{(1)}x^K\delta^{(2)}x^L\left(\underline{R}_{KLC}{}^A\varphi^{CM}{}_B + \underline{R}_{KLN}{}^M\varphi^{AN}{}_B - \underline{R}_{KLB}{}^C\varphi^{AM}{}_C\right)\end{aligned} \tag{5.113}$$

Here $\varphi^{AM}{}_B$ is just a representative example for some elementary or composite field which transforms tensorial under the target space transformations (super-diffeomorphisms and local structure group transformations).

The covariant variation of the complete action coincides with the ordinary one as all indices are contracted. However, the covariant variational derivative defined in (5.111), differs from the ordinary variational derivatives. The important thing is, that nevertheless they define a set of equations of motion which is equivalent the usual one – and target space covariant. Let us see the equivalence explicitly and reformulate the ordinary variation into the covariant one:

$$\begin{aligned}\delta S &= \int d^2z \quad \delta d_{z\boldsymbol{\gamma}}\frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} + \delta\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}} + \delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + \delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}} + \delta\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}} + \\ &\quad +\delta L_{z\bar{z}a}\frac{\delta S}{\delta L_{z\bar{z}a}} + \delta\hat{L}_{\bar{z}za}\frac{\delta S}{\delta\hat{L}_{\bar{z}za}} + \delta x^K\frac{\delta S}{\delta x^K} = \end{aligned} \tag{5.114}$$

$$\begin{aligned}&= \int d^2z \quad \delta_{cov}d_{z\boldsymbol{\gamma}}\frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} + \delta_{c\hat{o}v}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}} + \delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + \delta_{c\hat{o}v}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \delta_{cov}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}} + \delta_{c\hat{o}v}\boldsymbol{\omega}_{\bar{z}\hat{\boldsymbol{\beta}}}\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}} + \\ &\quad +\delta_{cov}L_{z\bar{z}a}\frac{\delta S}{\delta L_{z\bar{z}a}} + \delta_{c\hat{o}v}\hat{L}_{z\bar{z}a}\frac{\delta S}{\delta\hat{L}_{z\bar{z}a}} + \delta x^K\Big(\frac{\delta S}{\delta x^K} + \Omega_{K\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}d_{z\boldsymbol{\delta}}\frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} + \hat{\Omega}_{K\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}}\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}} - \Omega_{K\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + \\ &\quad -\hat{\Omega}_{K\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \Omega_{K\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}} + \hat{\Omega}_{K\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\boldsymbol{\omega}_{\bar{z}\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}} + \Omega_{Ka}{}^bL_{z\bar{z}b}\frac{\delta S}{\delta L_{z\bar{z}a}} + \hat{\Omega}_{Ka}{}^b\hat{L}_{z\bar{z}b}\frac{\delta S}{\delta\hat{L}_{z\bar{z}a}}\Big)\end{aligned} \tag{5.115}$$

We can now read off the **covariant variational derivative** $\frac{\delta S_{cov}}{\delta x^K}$ w.r.t. x^K as the coefficient of δx^K:[11]

$$\begin{aligned}\frac{\delta_{\underline{cov}} S}{\delta x^K} &= \frac{\delta S}{\delta x^K} + \Omega_{K\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} d_{z\boldsymbol{\delta}} \frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} + \hat{\Omega}_{K\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}} \hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}} \frac{\delta S}{\delta \hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}} - \Omega_{K\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} \boldsymbol{\lambda}^{\boldsymbol{\beta}} \frac{\delta S}{\delta \boldsymbol{\lambda}^{\boldsymbol{\alpha}}} - \hat{\Omega}_{K\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}} \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} \frac{\delta S}{\delta \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \\ &\quad + \Omega_{K\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} \boldsymbol{\omega}_{z\boldsymbol{\alpha}} \frac{\delta S}{\delta \boldsymbol{\omega}_{z\boldsymbol{\beta}}} + \hat{\Omega}_{K\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}} \boldsymbol{\omega}_{\bar{z}\hat{\boldsymbol{\alpha}}} \frac{\delta S}{\delta \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}} + \Omega_{Ka}{}^{b} L_{z\bar{z}b} \frac{\delta S}{\delta L_{z\bar{z}a}} + \hat{\Omega}_{Ka}{}^{b} \hat{L}_{z\bar{z}b} \frac{\delta S}{\delta \hat{L}_{z\bar{z}a}}\end{aligned} \tag{5.116}$$

All the other variational derivatives (5.9)-(5.15) remain untouched:

$$\frac{\delta_{\underline{cov}} S}{\delta d_{z\boldsymbol{\alpha}}} = \frac{\delta S}{\delta d_{z\boldsymbol{\alpha}}}, \quad \ldots, \quad \frac{\delta_{\underline{cov}} S}{\delta \hat{L}_{\bar{z}za}} = \frac{\delta S}{\delta \hat{L}_{\bar{z}za}} \tag{5.117}$$

According to (5.116), $\delta_{\underline{cov}} S/\delta x^K$ coincides with $\delta S/\delta x^K$ when all the other equations of motion are fulfilled. This leads to the following obvious but important statement:

Proposition 4 *Setting the covariant variational derivatives defined via (5.116) and (5.117) to zero, leads to a set of equations which is equivalent to the equations of motion obtained by the ordinary variational derivatives:*

$$\frac{\delta_{\underline{cov}} S}{\delta\left(x^K, d_{z\boldsymbol{\alpha}}, \boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \boldsymbol{\omega}_{z\boldsymbol{\alpha}}, \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}, \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}, \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\alpha}}}, L_{z\bar{z}a}, \hat{L}_{\bar{z}za}\right)} = 0 \iff \frac{\delta S}{\delta\left(x^K, d_{z\boldsymbol{\alpha}}, \boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \boldsymbol{\omega}_{z\boldsymbol{\alpha}}, \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}, \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}, \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\alpha}}}, L_{z\bar{z}a}, \hat{L}_{\bar{z}za}\right)} = 0 \tag{5.118}$$

The covariant variational derivatives in turn are obtained by using the covariant variation defined in (5.102)-(5.109) and the commutators (5.112) and (5.113).

The last equation of motion We are now ready to calculate the last equation of motion, the variation with respect to x^K. Admittedly introducing a new tool like the covariant variation for just one equation seems a bit of overkill. However, in any case we would have been forced during the calculation to organize the result into covariant expressions and the covariant variation gives a general recipe how to do that. Although we described the covariant variation for the Berkovits string, it is a tool which is very useful in any other nonlinear sigma model. In addition it should be noted that the above concept works for an arbitrary connection and not only for the connection $\underline{\Omega}_{MA}{}^{B}$ or the corresponding $\underline{\Gamma}_{MN}{}^{K}$. The calculation just simplifies at some points, if one restricts to connections with special properties, or to connections which are already present in the action. E.g. only because we are choosing $\underline{\Omega}_{MA}{}^{B}$, we can make use of (5.112) and (5.113) in order to commute the covariant variation with the target space covariant worldsheet derivative. In addition we will make use of the fact that the covariant variation annihilates the vielbein:

$$\delta_{\underline{cov}} E_M{}^A(\vec{x}) = 0 \tag{5.119}$$

Note also how the antisymmetrized covariant derivative of the B-field can be written in terms of its exterior derivative H and the torsion:

$$\underline{\boldsymbol{\nabla}} B \equiv \underline{\nabla}_{\boldsymbol{M}} B_{\boldsymbol{MM}} = \mathbf{d}B - \imath_{\underline{T}} B = H_{\boldsymbol{MMM}} - 2\underline{T}_{\boldsymbol{MM}}{}^{K} B_{K\boldsymbol{M}} \tag{5.120}$$

The important contributions to the (covariant) variation of the action come from the covariant variation of the (spacetime covariant) worldsheet derivatives of the elementary fields, like $\delta_{cov}\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$ and $\delta_{\underline{cov}}\Pi^A_{z/\bar{z}}$. For the latter we have (compare to the equation before (2.12) in [59])

$$\delta_{\underline{cov}}\Pi^A_{z/\bar{z}} \stackrel{(5.119)}{=} \delta_{\underline{cov}}\partial_{z/\bar{z}}x^K \cdot E_K{}^A = \tag{5.121}$$

$$\stackrel{(5.112)}{=} \underline{\nabla}_{z/\bar{z}}\delta x^K \cdot E_K{}^A + 2\delta x^M \underline{T}_{MN}{}^A \partial_{z/\bar{z}} x^N \tag{5.122}$$

For the ghost terms we obtain curvature expressions instead of torsion expressions:

$$\delta_{cov}\nabla_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \stackrel{(5.113)}{=} \nabla_{\bar{z}}\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + 2\delta x^K \bar{\partial} x^L R_{KL\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \tag{5.123}$$

$$\delta_{c\hat{o}v}\hat{\nabla}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} \stackrel{(5.113)}{=} \hat{\nabla}_z\delta_{c\hat{o}v}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + 2\delta x^K \partial x^L \hat{R}_{KL\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} \tag{5.124}$$

[11] Note the analogy to the tangent space covariant derivative of some multivector valued form

$$K(x, \boldsymbol{e}, \tilde{\boldsymbol{e}}) \equiv K_{a_1\ldots a_k}{}^{b_1\ldots b_{k'}}(x) \cdot \boldsymbol{e}^{a_1} \cdots \boldsymbol{e}^{a_k} \tilde{\boldsymbol{e}}_{b_1} \cdots \tilde{\boldsymbol{e}}_{b_{k'}}$$

written in the following way

$$\nabla_m K = \partial_m K(x, \boldsymbol{e}, \tilde{\boldsymbol{e}}) - \boldsymbol{e}^a \Omega_{ma}{}^b \frac{\partial}{\partial \boldsymbol{e}^b} K + \tilde{\boldsymbol{e}}_a \Omega_{mb}{}^a \frac{\partial}{\partial \tilde{\boldsymbol{e}}_b} K \qquad \diamond$$

As a last ingredient, before we vary the action, we should note a specialty of the pure spinor term. The covariant variation on the Lagrange multiplier is chosen in such a way that the covariant variation of $\gamma^a_{\boldsymbol{\alpha\beta}}$ is almost zero. But as we discussed at length in section 5.4 on page 51 the structure group is not yet for all components of the connection reduced to Lorentz plus scale transformations and we have in general a non-vanishing $\gamma^{[4]}$-part $\Omega_{\boldsymbol{\gamma} a_1\ldots a_4}$. At least formally we therefore obtain a non-vanishing covariant derivative on $\gamma^a_{\boldsymbol{\alpha\beta}}$ (with $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ acting on the spinorial indices and $\Omega_{Ma}{}^b$ of (5.81) acting on the bosonic one):

$$\nabla_M\gamma^a_{\boldsymbol{\alpha\beta}} \quad = \quad -2E_M{}^{\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}\,a_1\ldots a_4}\gamma^{a_1\ldots a_4}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\delta}}\gamma^a_{\boldsymbol{\delta}|\boldsymbol{\beta}]} \stackrel{(D.114)}{=} -2E_M{}^{\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}\,a_1\ldots a_4}\gamma^{a_1\ldots a_4 a}{}_{\boldsymbol{\alpha\beta}} \tag{5.125}$$

Due to (5.116) and (5.117) we know already that only the variational derivative with respect to x^K gets modified while the others remain untouched. We therefore collect the terms which are proportional to the x^K-variation only. In particular we do not need to consider the first term respectively of the above two equations. For completeness, however, we keep the total derivatives coming from the corresponding partial integration. Apart from the variation of $\Pi^A_{z/\bar z}$, $\nabla_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}}$ and $\hat\nabla_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}$ we only have covariant variations of the background fields. The (covariant) variation of the action (5.98) thus takes the following form

$$\begin{aligned}
\delta S \quad = \quad & \int d^2z \quad \delta x^K\Big[\frac{1}{2}\Pi^A_z\underline{\nabla}_K O_{AB}\Pi^B_{\bar z} + d_{z\boldsymbol{\gamma}}\underline{\nabla}_K\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{\nabla}_K C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_K\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}d_{z\boldsymbol{\gamma}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_K S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\Big] + \\
& +\frac{1}{2}\underbrace{\left(\underline{\nabla}_z\delta x^K\cdot E_K{}^A + 2\delta x^M\underline{T}_{MN}{}^A\partial_z x^N\right)}_{\delta_{\underline{cov}}\Pi^A_z}O_{AB}\Pi^B_{\bar z} + \frac{1}{2}\Pi^A_z O_{AB}\underbrace{\left(\underline{\nabla}_{\bar z}\delta x^K\cdot E_K{}^B + 2\delta x^M\underline{T}_{MN}{}^B\partial_{\bar z}x^N\right)}_{\delta_{\underline{cov}}\Pi^B_{\bar z}} + \\
& +\underbrace{\left(\underline{\nabla}_{\bar z}\delta x^K\cdot E_K{}^{\boldsymbol{\gamma}} + 2\delta x^M\underline{T}_{MN}{}^{\boldsymbol{\gamma}}\partial_{\bar z}x^N\right)}_{\delta_{\underline{cov}}\Pi^{\boldsymbol{\gamma}}_{\bar z}}d_{z\boldsymbol{\gamma}} + \underbrace{\left(\underline{\nabla}_z\delta x^K\cdot E_K{}^{\hat{\boldsymbol{\gamma}}} + 2\delta x^M\underline{T}_{MN}{}^{\hat{\boldsymbol{\gamma}}}\partial_z x^N\right)}_{\delta_{\underline{cov}}\Pi^{\hat{\boldsymbol{\gamma}}}_z}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +\underbrace{2\delta x^K\bar\partial x^L R_{KL\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}}_{\delta_{cov}\nabla_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}}-\nabla_{\bar z}\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \underbrace{2\delta x^K\partial x^L\hat R_{KL\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}}_{\delta_{c\hat ov}\hat\nabla_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}-\hat\nabla_z\delta_{c\hat ov}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \\
& -\delta x^K E_K{}^{\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}\,a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\cdot L_{z\bar z a} - \delta x^K E_K{}^{\hat{\boldsymbol{\gamma}}}\hat\Omega_{\hat{\boldsymbol{\gamma}}\,a_1\ldots a_4}(\hat{\boldsymbol{\lambda}}\gamma^{a_1\ldots a_4 a}\hat{\boldsymbol{\lambda}})\cdot\hat L_{\bar z z a} + \\
& +\delta_{cov}d_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta d_{z\boldsymbol{\alpha}}} + \delta_{c\hat ov}\hat d_{\bar z\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat d_{\bar z\hat{\boldsymbol{\alpha}}}} + \delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + \delta_{c\hat ov}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \delta_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}}} + \delta_{c\hat ov}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}}} + \\
& +\delta_{cov}L_{z\bar z a}\frac{\delta S}{\delta L_{z\bar z a}} + \delta_{c\hat ov}\hat L_{\bar z z a}\frac{\delta S}{\delta\hat L_{\bar z z a}} + \partial_{\bar z}\left(\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right) + \partial_z\left(\delta_{c\hat ov}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right)
\end{aligned} \tag{5.126}$$

We finally make a partial integration for the terms in the third and fourth line (keeping again the total derivatives as a reference for future studies of the open string) and arrive at

$$\begin{aligned}
\delta S \quad = \quad & \int d^2z \quad \delta x^K E_K{}^C\Big[-\frac{1}{2}O_{CB}\underline{\nabla}_z\Pi^B_{\bar z} - \frac{1}{2}\underline{\nabla}_{\bar z}\Pi^A_z O_{AC} + \\
& +\frac{1}{2}\Pi^A_z\left(\underline{\nabla}_C O_{AB} - \underline{\nabla}_A O_{CB} - \underline{\nabla}_B O_{AC} + 2\underline{T}_{CA}{}^D O_{DB} + 2O_{AD}\underline{T}_{CB}{}^D\right)\Pi^B_{\bar z} + \\
& -\delta_C{}^{\boldsymbol{\gamma}}\underline{\nabla}_{\bar z}d_{z\boldsymbol{\gamma}} - \delta_C{}^{\hat{\boldsymbol{\gamma}}}\underline{\nabla}_z\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + 2\underline{T}_{CB}{}^{\boldsymbol{\gamma}}\Pi^B_{\bar z}d_{z\boldsymbol{\gamma}} + 2\underline{T}_{CA}{}^{\hat{\boldsymbol{\gamma}}}\Pi^A_z\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +2\Pi^B_{\bar z}R_{CB\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\Pi^A_z\hat R_{CA\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \\
& +d_{z\boldsymbol{\gamma}}\underline{\nabla}_C\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{\nabla}_C C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_C\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}d_{z\boldsymbol{\gamma}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_C S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} - \delta_C{}^{\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}\,a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\cdot L_{z\bar z a} - \delta_C{}^{\hat{\boldsymbol{\gamma}}}\hat\Omega_{\hat{\boldsymbol{\gamma}}\,a_1\ldots a_4}(\hat{\boldsymbol{\lambda}}\gamma^{a_1\ldots a_4 a}\hat{\boldsymbol{\lambda}})\cdot\hat L_{\bar z z a}\Big] + \\
& +\delta_{cov}d_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta d_{z\boldsymbol{\alpha}}} + \delta_{c\hat ov}\hat d_{\bar z\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat d_{\bar z\hat{\boldsymbol{\alpha}}}} + \delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + \delta_{c\hat ov}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \delta_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}}} + \delta_{c\hat ov}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}}} + \\
& +\delta_{cov}L_{z\bar z a}\frac{\delta S}{\delta L_{z\bar z a}} + \delta_{c\hat ov}\hat L_{\bar z z a}\frac{\delta S}{\delta\hat L_{\bar z z a}} + \\
& +\partial_{\bar z}\left(\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \frac{1}{2}\Pi^A_z O_{AK}\delta x^K + \delta x^K\cdot E_K{}^{\boldsymbol{\gamma}}d_{z\boldsymbol{\gamma}}\right) + \\
& +\partial_z\left(\delta_{c\hat ov}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \frac{1}{2}\delta x^K O_{KB}\Pi^B_{\bar z} + \delta x^K\cdot E_K{}^{\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}}\right)
\end{aligned} \tag{5.127}$$

Now we can read off the covariant variational derivative with respect to x^K. But let us note two further relations first:

$$\underline{\nabla}_C O_{AB} - \underline{\nabla}_A O_{CB} - \underline{\nabla}_B O_{AC} = \stackrel{(5.120)}{=} \quad 3H_{CAB} - 2\underline{T}_{AB}{}^D B_{DC} - 2\underline{T}_{CA}{}^D B_{DB} - 2\underline{T}_{BC}{}^D B_{DA} + \underline{\nabla}_C G_{AB} - \underline{\nabla}_A G_{CB} - \underline{\nabla}_B G_{AC} \tag{5.128}$$

and

$$\underline{\nabla}_z \Pi^D_{\bar z} \stackrel{(5.112)}{=} \underline{\nabla}_{\bar z}\Pi^D_z + 2\Pi^A_z \Pi^B_{\bar z}\underline{T}_{AB}{}^D \tag{5.129}$$

In addition we define

$$\underline{T}_{AB|C} \equiv \underline{T}_{AB}{}^D G_{DC} \tag{5.130}$$

Note that we use the symmetric rank two tensor G_{AB} only to pull indices down. Pulling them up again is in general not possible as G_{AB} might be degenerate. In fact we will learn soon that it has to be degenerate.

The **final result** of the variation now reads

$$\begin{aligned}
\delta S \;=\; & \int d^2z \quad \delta x^K \frac{\delta_{\underline{cov}} S}{\delta x^K} + \delta_{cov} d_{z\boldsymbol{\alpha}} \frac{\delta S}{\delta d_{z\boldsymbol{\alpha}}} + \delta_{c\hat{o}v}\hat d_{\bar z\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta \hat d_{\bar z\hat{\boldsymbol{\alpha}}}} + \\
& +\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + \delta_{c\hat{o}v}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} + \delta_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}}} + \delta_{c\hat{o}v}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}}\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}}} + \\
& +\delta_{cov}L_{z\bar z a}\frac{\delta S}{\delta L_{z\bar z a}} + \delta_{c\hat{o}v}\hat L_{\bar z z a}\frac{\delta S}{\delta \hat L_{\bar z z a}} + \\
& +\partial_{\bar z}\left(\delta_{cov}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \frac12 \Pi^A_z O_{AK}\delta x^K + \delta x^K\cdot E_K{}^{\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}}\right) + \\
& +\partial_z\left(\delta_{c\hat{o}v}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \frac12\delta x^K O_{KB}\Pi^B_{\bar z} + \delta x^K\cdot E_K{}^{\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}}\right)
\end{aligned} \tag{5.131}$$

with the following covariant variational derivatives or equations of motion (remember (5.9)-(5.15)):

$$\begin{aligned}
\frac{\delta_{\underline{cov}} S}{\delta x^K} \;=\; & E_K{}^C\Big[\underbrace{-\underline{\nabla}_{\bar z}\Pi^D_z}_{-\underline{\nabla}_z\Pi^D_{\bar z}+2\Pi^A_z\Pi^B_{\bar z}\underline{T}_{AB}{}^D} G_{DC} + \Pi^A_z\left(\frac32 H_{CAB} - \underline{T}_{AB|C} + 2\underline{T}_{C(A|B)} + \frac12\underline{\nabla}_C G_{AB} - \underline{\nabla}_{(A}G_{B)C}\right)\Pi^B_{\bar z} + \\
& -\delta_C{}^{\boldsymbol{\gamma}}\nabla_{\bar z} d_{z\boldsymbol{\gamma}} - \delta_C{}^{\hat{\boldsymbol{\gamma}}}\hat\nabla_z\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + 2T_{CB}{}^{\boldsymbol{\gamma}}\Pi^B_{\bar z} d_{z\boldsymbol{\gamma}} + 2\hat T_{CA}{}^{\hat{\boldsymbol{\gamma}}}\Pi^A_z\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +d_{z\boldsymbol{\gamma}}\underline{\nabla}_C\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{\nabla}_C C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_C\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} d_{z\boldsymbol{\gamma}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_C S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} - \delta_C{}^{\boldsymbol{\gamma}}\Omega_{\boldsymbol{\gamma}\,a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\cdot L_{z\bar z a} - \delta_C{}^{\hat{\boldsymbol{\gamma}}}\hat\Omega_{\hat{\boldsymbol{\gamma}}\,a_1\ldots a_4}(\hat{\boldsymbol{\lambda}}\gamma^{a_1\ldots a_4 a}\hat{\boldsymbol{\lambda}})\cdot\hat L_{\bar z z a} + \\
& +2\Pi^B_{\bar z}R_{CB\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\Pi^A_z\hat R_{CA\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\Big]
\end{aligned} \tag{5.132}$$

$$\frac{\delta S}{\delta d_{z\boldsymbol{\gamma}}} = \Pi^{\boldsymbol{\gamma}}_{\bar z} + \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} \tag{5.133}$$

$$\frac{\delta S}{\delta \hat d_{\bar z\hat{\boldsymbol{\gamma}}}} = \Pi^{\hat{\boldsymbol{\gamma}}}_z + d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}} C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \tag{5.134}$$

$$\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\beta}}} = -\left(\nabla_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\left(C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right)\right) \equiv -\mathcal{D}_{\bar z}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \tag{5.135}$$

$$\frac{\delta S}{\delta\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}} = -\left(\hat\nabla_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right)\right) \equiv -\hat{\mathcal{D}}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}} \tag{5.136}$$

$$\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} = -\left(\nabla_{\bar z}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} - \left(C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat d_{\bar z\hat{\boldsymbol{\gamma}}} - \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right)\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right) + L_{z\bar z a}(\gamma^a\boldsymbol{\lambda})_{\boldsymbol{\alpha}} \equiv -\mathcal{D}_{\bar z}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} + L_{z\bar z a}(\gamma^a\boldsymbol{\lambda})_{\boldsymbol{\alpha}} \tag{5.137}$$

$$\frac{\delta S}{\delta\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}} = -\left(\hat\nabla_z\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}} - \left(\hat C_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\right)\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}\right) + \hat L_{z\bar z a}(\gamma^a\hat{\boldsymbol{\lambda}})_{\hat{\boldsymbol{\alpha}}} \equiv -\hat{\mathcal{D}}_z\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\alpha}}} + \hat L_{z\bar z a}(\gamma^a\hat{\boldsymbol{\lambda}})_{\hat{\boldsymbol{\alpha}}} \tag{5.138}$$

$$\frac{\delta S}{\delta L_{z\bar z a}} = \frac12(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}), \qquad \frac{\delta S}{\delta\hat L_{z\bar z a}} = \frac12(\hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}}) \tag{5.139}$$

Note that we used for the covariant variation an independent connection $\check\Omega_{Ma}{}^b$ for the bosonic subspace. This connection is a priory not a background field of the string metric. We are free to choose it in a convenient way.

5.6 Ghost current

Let us assign ghost numbers $(1,0)$ and $(-1,0)$ to the fields $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$ and $\boldsymbol{\omega}_{z\boldsymbol{\alpha}}$. The corresponding transformation (with some global transformation parameter ρ) is

$$\delta\boldsymbol{\lambda}^{\alpha} = \rho\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \qquad \delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}} = -\rho\boldsymbol{\omega}_{z\boldsymbol{\alpha}} \tag{5.140}$$

For the action to remain unchanged, we also need to transform the Lagrange multiplier

$$\delta L_{z\bar{z}a} = -2\rho L_{z\bar{z}a} \tag{5.141}$$

which therefore has ghost number -2. Varying the action with a local parameter, we arrive at

$$\delta S = \int_\Sigma d^2z \quad \bar{\partial}\rho\cdot(\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}) + \text{bdry-terms} \tag{5.142}$$

According to (E.42) and footnote 4 on page 186, we can read off the ghost current as

$$j^{gh} = \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} \tag{5.143}$$

It has the same form as in flat space.

In section 5.7, we will derive the BRST transformations of the worldsheet fields from the given BRST current via "inverse Noether" (see (E.7)). The idea is to calculate the divergence of the current and try to express it in terms of the equations of motion. The transformations of the worldsheet fields can then be read off as coefficients. This avoids switching to the Hamiltonian formalism and using the Poisson bracket to generate the transformations. It might be instructive to see, how "inverse Noether" works for the simple example of the ghost current before we come to the BRST current later:

$$\begin{aligned}
-\delta\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} &\overset{!}{=} \bar{\partial}(\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}) = \\
&= \mathcal{D}_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\cdot\boldsymbol{\omega}_{z\boldsymbol{\alpha}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\mathcal{D}_{\bar{z}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} = \\
&= -\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\left(-\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + L_a(\gamma^a\boldsymbol{\lambda})_{\boldsymbol{\alpha}}\right) = \\
&= \boldsymbol{\omega}_{z\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\alpha}}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\frac{\delta S}{\delta\boldsymbol{\lambda}^{\boldsymbol{\alpha}}} + 2L_{z\bar{z}a}\frac{\delta S}{\delta L_{z\bar{z}a}}
\end{aligned} \tag{5.144}$$

From this one can read off the transformations with which we had begun.

The ghost current and the corresponding transformations for the hatted variables are obtained via proposition 3 on page 44.

5.7 Holomorphic BRST current

We now come to the main part of the derivation of the supergravity constraints from the pure spinor string. The pure spinor string in flat background had two (graded) commuting and nilpotent BRST differentials which defined the physical spectrum. Putting the string in a curved background is a matter of consistent deformation. I.e., gauge symmetries and BRST symmetries have to survive. They may be deformed, but the number of physical degrees of worldsheet variables cannot simply change as soon as there is a backreaction from the background that was produced by the strings themselves. This is a similar consistency like the demand for vanishing quantum anomalies. It is therefore legitimate to demand (apart from the two antighost gauge symmetries) also two (graded) commuting BRST symmetries. Remember, we already have simplified in (5.39) and (5.40) the general ansatz for the BRST currents by reparametrizations to the simple form

$$\boldsymbol{j}_z = \boldsymbol{\lambda}^{\boldsymbol{\gamma}}d_{z\boldsymbol{\gamma}}, \quad \boldsymbol{j}_{\bar{z}} = 0 \tag{5.145}$$

$$\hat{\boldsymbol{\jmath}}_{\bar{z}} = \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}, \quad \hat{\boldsymbol{\jmath}}_z = 0 \tag{5.146}$$

Instead of deriving the corresponding BRST transformations in the Hamiltonian formalism using the Poisson bracket, we stay in the Lagrangian formalism and apply Noether's theorem (see (E.15)) inversely in the sense that we try to express the divergence of the given currents as linear combinations of the equations of motion in order to derive the corresponding transformations:

$$\bar{\partial}\boldsymbol{j}_z \overset{!}{=} -\mathbf{s}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} = -\mathbf{s}_{\underline{cov}}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta_{\underline{cov}}S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{5.147}$$

$$\partial\hat{\boldsymbol{\jmath}}_{\bar{z}} \overset{!}{=} -\hat{\mathbf{s}}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} = -\hat{\mathbf{s}}_{\underline{cov}}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta_{\underline{cov}}S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{5.148}$$

Here $\phi^{\mathcal{I}}_{\rm all}$ is the collection of all the worldsheet fields. BRST invariance of the action is according to Noether equivalent to having this special form of the divergences of the currents. These two equations thus do three things at the same time: The possibility to write the divergence of the currents as linear combinations of the equations of motion fixes the precise form of the BRST current. At the same time it puts constraints on the background fields: all terms not proportional to equations of motion have to vanish. And finally it determines the form of the (covariant) BRST transformations.

After determining the BRST transformation, the nilpotency conditions $\mathbf{s}^2 = 0, [\mathbf{s}, \hat{\mathbf{s}}] = 0$ and $\hat{\mathbf{s}}^2 = 0$ put further constraints on the background fields including the torsion. Some torsion components can then be further simplified by using two of the three local Lorentz transformations and scale transformations which leads to only one remaining local Lorentz transformation and one local scale transformation. Putting these restrictions on some torsion components induces via the Bianchi identities further constraints on other components. All the constraints on the torsion and other functionals of the background fields combine finally to the target space supergravity equations of motion. Note that our approach differs from the one in [13] in two major points. First of all we stay in the Lagrangian formalism throughout. Second, we first check the holomorphicity and then the nilpotency. In fact, we need to do so, because only in the first step we can determine the BRST transformations of the worldsheet fields which we need in the Lagrangian formalism to check nilpotency. The BRST transformations have so far been given only for the heterotic string in [14], so that the transformations in the type II case are a new result.

Let us now perform in more detail the program sketched above:

$$\begin{aligned}
\bar{\partial}\boldsymbol{j}_z &= \mathcal{D}_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} + \boldsymbol{\lambda}^{\boldsymbol{\gamma}}\mathcal{D}_{\bar{z}} d_{z\boldsymbol{\gamma}} = && (5.149)\\
&= -d_{z\boldsymbol{\gamma}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\gamma}}\mathcal{D}_{\bar{z}} d_{z\boldsymbol{\gamma}} && (5.150)
\end{aligned}$$

In the following we will replace all occurrences of $\mathcal{D}_{\bar{z}} d_{z\boldsymbol{\gamma}}$, $\Pi_z^{\hat{\boldsymbol{\gamma}}}$, $\Pi_{\bar{z}}^{\boldsymbol{\gamma}}$, $\mathcal{D}_{\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$, $\hat{\mathcal{D}}_z\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}$, $\mathcal{D}_{\bar{z}}\boldsymbol{\omega}_{z\boldsymbol{\alpha}}$, $\hat{\mathcal{D}}_z\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\alpha}}}$, $\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}$ and $\hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}}$ by the equations of motion (5.132)-(5.139). In the end, all terms which are not proportional to the equations of motion have to vanish which leads to some of the supergravity constraints while the terms proportional to the equations of motion tell us the BRST transformation of the elementary fields. In order to extract $\mathcal{D}_{\bar{z}} d_{z\boldsymbol{\gamma}}$ from the x^K-equation of motion (5.132), let us project (5.132) to a flat spinorial index $\boldsymbol{\alpha}$ using some index relabeling:

$$\begin{aligned}
\mathcal{D}_{\bar{z}} d_{z\boldsymbol{\alpha}} &= -E_{\boldsymbol{\alpha}}{}^K\frac{\delta_{\underline{cov}}S}{\delta x^K} - \underline{\nabla}_{\bar{z}}\Pi_z^D G_{D\boldsymbol{\alpha}} + \\
&\quad +\Pi_z^C\left(\frac{3}{2}H_{\boldsymbol{\alpha}CD} - \underline{T}_{CD|\boldsymbol{\alpha}} + 2\underline{T}_{\boldsymbol{\alpha}(C|D)} + \frac{1}{2}\underline{\nabla}_{\boldsymbol{\alpha}}G_{CD} - \underline{\nabla}_{(C}G_{D)\boldsymbol{\alpha}}\right)\Pi_{\bar{z}}^D + \\
&\quad +2T_{\boldsymbol{\alpha}D}{}^{\boldsymbol{\gamma}}\Pi_{\bar{z}}^D d_{z\boldsymbol{\gamma}} + 2\hat{T}_{\boldsymbol{\alpha}C}{}^{\hat{\boldsymbol{\gamma}}}\Pi_z^C\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + \\
&\quad +d_{z\boldsymbol{\gamma}}\left(\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} - C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\right)\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\underline{\nabla}_{\boldsymbol{\alpha}}C_{\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(\underline{\nabla}_{\boldsymbol{\alpha}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} + S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}}\right)\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} d_{z\boldsymbol{\gamma}} + \\
&\quad +\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\boldsymbol{\alpha}}S_{\boldsymbol{\alpha}_2\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} - \Omega_{\boldsymbol{\alpha}\,a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\cdot L_{z\bar{z}a} + \\
&\quad +2\Pi_{\bar{z}}^D R_{\boldsymbol{\alpha}D\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\Pi_z^C\hat{R}_{\boldsymbol{\alpha}C\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} && (5.151)
\end{aligned}$$

Already at this point we can determine some constraints on the background fields. The divergence of the BRST current given in (5.150) has to become a linear combination of the equations of motion. The term $\underline{\nabla}_{\bar{z}}\Pi_z^D G_{D\boldsymbol{\alpha}}$ in (5.151) cannot be compensated by any other term and it also cannot be replaced by a further equation of motion. The same is true for our beloved $\Omega_{\boldsymbol{\alpha}\,a_1\ldots a_4}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_4 a}\boldsymbol{\lambda})\cdot L_{z\bar{z}a}$. Using in addition proposition 3 for the constraints from the antiholomorphicity of the right-mover BRST current, we can demand

$$\boxed{\begin{aligned}
G_{A\mathcal{B}} &\overset{!}{=} 0 \quad (\text{only } G_{ab}\neq 0) && (5.152)\\
\Omega_{\boldsymbol{\alpha}\,a_1\ldots a_4} &\overset{!}{=} 0, \qquad \hat{\Omega}_{\hat{\boldsymbol{\alpha}}\,a_1\ldots a_4}\overset{!}{=}0 && (5.153)
\end{aligned}}$$

With (5.153) we have finally obtained the missing ingredient for the reduction of the spinorial connection coefficients to Lorentz plus scale transformations as it was summarized already in the remark on page 53 at the end of the section 5.4 about the antighost gauge symmetry.

Equation (5.152) allows us to choose a frame where $G_{ab} = e^{2\Phi}\eta_{ab}$, such that we reduce also the bosonic structure group to Lorentz plus scale transformations. Let us discuss this in more detail in the following intermezzo.

Intermezzo about the reduced bosonic structure group

Due to (5.152) we know that G_{AB} is of the block-diagonal form $G_{AB} = \mathrm{diag}\,(G_{ab}, 0, 0)$. This means that the symmetric rank two tensor is of the form

$$G_{MN} = E_M{}^a G_{ab} E_N{}^b \tag{5.154}$$

In particular we have $G_{mn} = E_m{}^a G_{ab} E_n{}^b$. As the $E_M{}^a$ were introduced by hand, we may choose $E_m{}^a$ orthonormal as usual, i.e. such that G_{ab} becomes the Minkowski metric. This is at least for the leading component $G_{mn}(\vec{x})$ (i.e. $\vec{\boldsymbol{\theta}} = 0$) a familiar thing to do, but it holds also in the $\vec{\boldsymbol{\theta}}$-dependent case:

Proposition 5 *For all symmetric rank two tensor fields $G_{mn}(\overbrace{\vec{x}}^{\{\vec{x},\vec{\boldsymbol{\theta}}\}})$ whose real body ($\vec{\boldsymbol{\theta}} = 0$-part) has signature (1,9), there exists locally a frame $E_m{}^a(\vec{x})$, such that*

$$G_{mn}(\underbrace{\vec{x}}_{\{\vec{x},\vec{\boldsymbol{\theta}}\}}) = E_m{}^a(\vec{x})\eta_{ab} E_n{}^b(\vec{x}) \tag{5.155}$$

Note: In contrast to the ordinary bosonic version, the entries of the matrices are supernumbers.

Proof Due to usual linear algebra, there is an orthonormal basis with respect to the real symmetric matrix $G_{mn}(\vec{x})$, i.e. we can always find locally $E_m{}^a(\vec{x})$, s.t. (5.155) is fulfilled for $\vec{\boldsymbol{\theta}} = 0$. In order to prove the same for $\vec{\boldsymbol{\theta}} \neq 0$, we will make a $\vec{\boldsymbol{\theta}}$-expansion of (5.155) and show that we can always construct a solution $E_m{}^a(\vec{x}, \vec{\boldsymbol{\theta}})$ for arbitrary $\vec{\boldsymbol{\theta}}$ from the bosonic solution $E_m{}^a(\vec{x})$. Remember the notations $x^{\boldsymbol{\mathcal{M}}} \equiv \boldsymbol{\theta}^{\mathcal{M}}$ and $G_{mn}| = G_{mn}|_{\vec{\boldsymbol{\theta}}=0}$. The $\vec{\boldsymbol{\theta}}$-expansion of (5.155) then reads

$$\begin{aligned}
&\sum_{n\geq 0} \frac{1}{n!} x^{\boldsymbol{\mathcal{M}}_1} \cdots x^{\boldsymbol{\mathcal{M}}_n} \left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} G_{mn}\right)| \stackrel{!}{=} \\
&\stackrel{!}{=} \sum_{k,l\geq 0} \frac{1}{k!} x^{\boldsymbol{\mathcal{K}}_1} \cdots x^{\boldsymbol{\mathcal{K}}_k} \left(\partial_{\boldsymbol{\mathcal{K}}_1} \cdots \partial_{\boldsymbol{\mathcal{K}}_k} E_m{}^a\right)| \, \eta_{ab} \frac{1}{l!} x^{\boldsymbol{\mathcal{L}}_1} \cdots x^{\boldsymbol{\mathcal{L}}_l} \left(\partial_{\boldsymbol{\mathcal{L}}_1} \cdots \partial_{\boldsymbol{\mathcal{L}}_l} E_n{}^b\right)\Big| = \\
&= \sum_{n\geq 0} \frac{1}{n!} x^{\boldsymbol{\mathcal{M}}_1} \cdots x^{\boldsymbol{\mathcal{M}}_n} \sum_{m=0}^{n} \binom{n}{m} \left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_m} E_m{}^a\right)| \, \eta_{ab} \left(\partial_{\boldsymbol{\mathcal{M}}_{m+1}} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} E_n{}^b\right)\Big|
\end{aligned} \tag{5.156}$$

At $n = 0$ we have the solvable bosonic equation $G_{mn}(\vec{x}) \stackrel{!}{=} E_m{}^a(\vec{x})\eta_{ab} E_n{}^b(\vec{x})$ to start with. At higher orders n we have

$$\begin{aligned}
&\left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} G_{mn}\right) \stackrel{!}{=}\Big| \\
&\stackrel{!}{=} \sum_{m=0}^{n} \binom{n}{m} \left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_m} E_m{}^a\right)| \, \eta_{ab} \left(\partial_{\boldsymbol{\mathcal{M}}_{m+1}} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} E_n{}^b\right)\Big| = \\
&= 2\, E_m{}^a| \, \eta_{ab} \left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} E_n{}^b\right)\Big| + \sum_{m=1}^{n-1} \binom{n}{m} \left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_m} E_m{}^a\right)| \, \eta_{ab} \left(\partial_{\boldsymbol{\mathcal{M}}_{m+1}} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} E_n{}^b\right)\Big|
\end{aligned} \tag{5.157}$$

We thus have the iterative explicit expression for the n-th $\vec{\boldsymbol{\theta}}$-derivative of the vielbein in terms of the $(n-1)$-th and all lower derivatives.

$$\begin{aligned}
&\left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} E_n{}^d\right) =\Big| \\
&= \frac{1}{2}\eta^{cd} \, E_c{}^m| \left[\left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} G_{mn}\right)| - \sum_{m=1}^{n-1} \binom{n}{m} \left(\partial_{\boldsymbol{\mathcal{M}}_1} \cdots \partial_{\boldsymbol{\mathcal{M}}_m} E_m{}^a\right)| \, \eta_{ab} \left(\partial_{\boldsymbol{\mathcal{M}}_{m+1}} \cdots \partial_{\boldsymbol{\mathcal{M}}_n} E_n{}^b\right)\Big| \right]
\end{aligned} \tag{5.158}$$

This completes the proof of the proposition. □

In spite of the above proposition, we will not fix G_{ab} to η_{ab}, but only up to a conformal factor. This is of course possible by a redefinition of $E_M{}^a$ with the square root of this conformal factor. The reason for us to do this is the fact that we have for the spinorial indices not only Lorentz-, but also scale transformations. It seems natural to keep this scale invariance also for the bosonic indices, as long as we do not fix the fermionic one (in particular if we aim at structure group invariant γ-matrices $\gamma^a_{\boldsymbol{\alpha\beta}}$). We thus introduce an auxiliary **compensator field** $\Phi(\vec{x})$ and choose $E_m{}^a$ such that

$$\boxed{G_{ab} = e^{2\Phi}\eta_{ab}} \tag{5.159}$$

As soon as $E_m{}^a(\vec{x})$ is chosen appropriately, the remaining vielbein components $E_{\boldsymbol{\mathcal{M}}}{}^a$ are uniquely determined via:

$$G_{\boldsymbol{\mathcal{M}}n} \stackrel{!}{=} E_{\boldsymbol{\mathcal{M}}}{}^a e^{2\Phi}\eta_{ab} E_n{}^b \qquad \Rightarrow E_{\boldsymbol{\mathcal{M}}}{}^a = G_{\boldsymbol{\mathcal{M}}n} E_b{}^n e^{-2\Phi}\eta^{ba} \tag{5.160}$$

In summary this means that there is locally always a choice for the bosonic 1-form $E^a = \mathbf{d}x^M E_M{}^a$, such that $G_{MN} = E_M{}^a e^{2\Phi}\eta_{ab}E_N{}^b$ or $G_{MN} = E_M{}^a\eta_{ab}E_N{}^b$, if one does not introduce the compensator field. The latter form of G_{MN} was the starting point in [13], probably motivated by the integrated vertex operator of the flat space.

With the compensator field included, the bosonic structure group with infinitesimal generator $\check{L}_a{}^b$ (compare to page 50 with $\check{\Lambda}_a{}^b = \delta_a^b + \check{L}_a{}^b$) is – like the fermionic ones – restricted to Lorentz plus scale transformations. We should of course also restrict the auxiliary connection accordingly.

$$\check{L}_a{}^b = \check{L}^{(D)}\delta_a^b + \check{L}_a^{(L)b}, \qquad \check{L}_{ab} \equiv \check{L}_a{}^c\eta_{cb} = -\check{L}_{ba} \tag{5.161}$$

$$\check{\Omega}_{Ma}{}^b = \check{\Omega}_M^{(D)}\delta_a^b + \check{\Omega}_a^{(L)b}, \qquad \check{\Omega}_{Mab} \equiv \check{\Omega}_{Ma}{}^c\eta_{cb} = -\check{\Omega}_{Mba} \tag{5.162}$$

The compensator field is a scalar with respect to superdiffeomorphisms. With respect to the structure group, however, it has to transform in a special way, in order to make G_{ab} transforming covariantly. The infinitesimal transformation of G_{ab} under structure group transformations is $\delta G_{ab} = -2\check{L}_{(a|}{}^c G_{c|b)} = -2\check{L}^{(D)}G_{ab}$ (see (5.64) on page 50). This transformation results in a simple shift of the compensator field. For the same reason, also the covariant derivative contains a shift of Φ:

$$\delta\Phi = -\check{L}^{(D)} \tag{5.163}$$

$$\check{\nabla}_M\Phi \equiv \partial_M\Phi - \check{\Omega}_M^{(D)} \tag{5.164}$$

$$\underline{\nabla}_M G_{AB} = 2\check{\nabla}_M\Phi G_{AB} \qquad (= \partial_M G_{AB} - 2\underline{\Omega}_{M(A|}{}^C G_{C|B)}) \tag{5.165}$$

Let us return to the calculation of the divergence of the BRST current and let us finally replace $\mathcal{D}_{\bar z}d_{z\boldsymbol{\alpha}}$in (5.150) by the x^K equation of motion given in (5.151) (already using (5.152) and (5.153))[12]:

$$\begin{aligned}
\bar\partial \boldsymbol{j}_z = & -d_{z\boldsymbol{\gamma}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\gamma}}} - \boldsymbol{\lambda}^{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}{}^K\frac{\delta_{\underline{cov}}S}{\delta x^K} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^C\underbrace{\left(\frac{3}{2}H_{\boldsymbol{\alpha}CD} + 2\underline{T}_{\boldsymbol{\alpha}(C|D)} + \check{\nabla}_{\boldsymbol{\alpha}}\Phi G_{CD}\right)}_{\equiv Y_{\boldsymbol{\alpha}CD}}\Pi_{\bar z}^D + \\
& +2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}T_{\boldsymbol{\alpha}D}{}^{\boldsymbol{\gamma}}\Pi_{\bar z}^D d_{z\boldsymbol{\gamma}} + 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{T}_{\boldsymbol{\alpha}C}{}^{\hat{\boldsymbol{\gamma}}}\Pi_z^C\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\gamma}}\left(\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} - C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\right)\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\underline{\nabla}_{\boldsymbol{\alpha}}C_{\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(\underline{\nabla}_{\boldsymbol{\alpha}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} + S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}}\right)\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}d_{z\boldsymbol{\gamma}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\boldsymbol{\alpha}}S_{\boldsymbol{\alpha}_2\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + \\
& +2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_{\bar z}^D R_{\boldsymbol{\alpha}D\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^C\hat{R}_{\boldsymbol{\alpha}C\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}
\end{aligned} \tag{5.166}$$

Before we plug in further equations of motion (replacing $\Pi_{\bar z}^{\boldsymbol{\delta}}$ and $\Pi_z^{\hat{\boldsymbol{\gamma}}}$) we should notice that we can already read off some more constraints. Namely $Y_{\boldsymbol{\alpha}cd} = Y_{\boldsymbol{\alpha}c\hat{\boldsymbol{\delta}}} = Y_{\boldsymbol{\alpha}\boldsymbol{\gamma}d} = Y_{\boldsymbol{\alpha}\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} = 0$. The first constraint $Y_{\boldsymbol{\alpha}cd} = 0$ can be separated into symmetric and antisymmetric part of the indices c and d. In addition, we already add everywhere the constraints coming from the right-moving BRST current , using proposition 3 on page 44 ($H \to -H$, $\check{T} \to \check{T}$, $\check{\nabla} \to \check{\nabla}$)[13].

---

[12]The comparison of the rewritten bosonic x^K-equation

$$\begin{aligned}
& \frac{1}{2}\underline{\nabla}_{\bar z}(\Pi_z^e G_{ea}) + \frac{1}{2}\underline{\nabla}_z(\Pi_{\bar z}^e G_{ea}) = \\
= & -E_a{}^K\frac{\delta_{\underline{cov}}S}{\delta x^K} + \Pi_z^C\left(\frac{3}{2}H_{aCD} + 2\underline{T}_{a(C|D)} + \check{\nabla}_a\Phi G_{CD}\right)\Pi_{\bar z}^B + 2T_{aD}{}^{\boldsymbol{\gamma}}\Pi_{\bar z}^D d_{z\boldsymbol{\gamma}} + 2\hat{T}_{aC}{}^{\hat{\boldsymbol{\gamma}}}\Pi_z^C\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +d_{z\boldsymbol{\gamma}}\underline{\nabla}_a\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{\nabla}_a C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_a\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}d_{z\boldsymbol{\gamma}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_a S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + 2\Pi_{\bar z}^D R_{aD\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\Pi_z^C\hat{R}_{aC\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}
\end{aligned}$$

$$\begin{aligned}
\text{with} \quad \nabla_{\bar z}d_{z\boldsymbol{\alpha}} = & -E_{\boldsymbol{\alpha}}{}^K\frac{\delta_{\underline{cov}}S}{\delta x^K} + \Pi_z^C\left(\frac{3}{2}H_{\boldsymbol{\alpha}CD} + 2\underline{T}_{\boldsymbol{\alpha}(C|D)} + \check{\nabla}_{\boldsymbol{\alpha}}\Phi G_{CD}\right)\Pi_{\bar z}^D + 2T_{\boldsymbol{\alpha}D}{}^{\boldsymbol{\gamma}}\Pi_{\bar z}^D d_{z\boldsymbol{\gamma}} + 2\hat{T}_{\boldsymbol{\alpha}C}{}^{\hat{\boldsymbol{\gamma}}}\Pi_z^C\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \\
& +d_{z\boldsymbol{\gamma}}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\underline{\nabla}_{\boldsymbol{\alpha}}C_{\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{d}_{\bar z\hat{\boldsymbol{\gamma}}} + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\boldsymbol{\alpha}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}d_{z\boldsymbol{\gamma}} + \\
& +\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\boldsymbol{\alpha}}S_{\boldsymbol{\alpha}_2\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}} + 2\Pi_{\bar z}^D R_{\boldsymbol{\alpha}D\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\Pi_z^C\hat{R}_{\boldsymbol{\alpha}C\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\omega}}_{\bar z\hat{\boldsymbol{\beta}}}
\end{aligned}$$

and with $\hat{\nabla}_z\hat{d}_{\bar z\hat{\boldsymbol{\alpha}}}$ suggests the introduction of

$$d_{za} \equiv \frac{1}{2}\Pi_z^e G_{ea}, \qquad d_{\bar z a} \equiv \frac{1}{2}\Pi_{\bar z}^e G_{ea} \qquad \diamond$$

---

[13]At first we should remember that $\underline{T}_{AC}{}^B = \mathrm{diag}\,(\check{T}_{AC}{}^b, T_{AC}{}^{\boldsymbol{\beta}}, \hat{T}_{AC}{}^{\hat{\boldsymbol{\beta}}})$. As G_{bd} are the only non-vanishing components of G_{BD},

$$
\boxed{\begin{array}{rcll}
H_{\boldsymbol{\mathcal{A}}cd} &=& 0 & (5.167)\\
\check{T}_{\boldsymbol{\mathcal{A}}(c|d)} &=& -\frac{1}{2}\check{\nabla}_{\boldsymbol{\mathcal{A}}}\Phi G_{cd} & (5.168)\\
\left.\begin{array}{rcl}\frac{3}{2}H_{\boldsymbol{\alpha}c\hat{\boldsymbol{\delta}}}+\check{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}|c} &=& 0\\ -\frac{3}{2}H_{\hat{\boldsymbol{\alpha}}c\boldsymbol{\delta}}+\check{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\delta}|c} &=& 0\end{array}\right\} && \Rightarrow\quad H_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}c}=\check{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}|c}=0 & (5.169)\\
\frac{3}{2}H_{\boldsymbol{\alpha\gamma}d}+\check{T}_{\boldsymbol{\alpha\gamma}|d} &=& 0, \qquad -\frac{3}{2}H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}d}+\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}|d}=0 & (5.170)\\
H_{\boldsymbol{\alpha\gamma}\hat{\boldsymbol{\delta}}} &=& 0, \qquad H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}\boldsymbol{\delta}}=0 & (5.171)
\end{array}}
$$

So far we have used only the equations of motion obtained by the variational derivative with respect to the antighosts and with respect to x^K. There still remain the ones with respect to the ghosts, with respect to the Lagrange multipliers and with respect to $d_{z\boldsymbol{\alpha}}$ and $\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}$. The first ones simply will not enter the calculation and the pure spinor constraints (coming from the Lagrange multipliers) will be used at the very end. So let us remind ourselves the variational derivatives with respect to $d_{z\boldsymbol{\alpha}}$ and $\hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}$ ((5.134) and (5.133)):

$$
\Pi_{\bar{z}}^{\boldsymbol{\delta}} = \frac{\delta S}{\delta d_{z\boldsymbol{\delta}}}-\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}-\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}, \qquad \Pi_{z}^{\hat{\boldsymbol{\gamma}}}=\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}}-d_{z\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}-\boldsymbol{\lambda}^{\boldsymbol{\alpha}}C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \tag{5.172}
$$

Together with the new constraints (5.167)-(5.171) we plug them into the divergence (5.166) of the BRST current In a last effort we sort all the terms with respect to the appearance of the elementary fields and finally arrive at

$$
\begin{aligned}
\bar{\partial}\boldsymbol{j}_z = {} & -d_{z\boldsymbol{\gamma}}\frac{\delta S}{\delta\boldsymbol{\omega}_{z\boldsymbol{\gamma}}}-\boldsymbol{\lambda}^{\boldsymbol{\alpha}}E_{\boldsymbol{\alpha}}{}^{K}\frac{\delta_{\underline{cov}}}{\delta x^K}S+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Big(\frac{3}{2}\Pi_z^{\boldsymbol{\gamma}}H_{\boldsymbol{\alpha\gamma\delta}}+2T_{\boldsymbol{\alpha\delta}}{}^{\boldsymbol{\gamma}}d_{z\boldsymbol{\gamma}}-2\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}R_{\boldsymbol{\alpha}_2\boldsymbol{\delta\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}+\Pi_z^c\underbrace{3H_{\boldsymbol{\alpha}c\boldsymbol{\delta}}}_{2\check{T}_{\boldsymbol{\alpha\delta}|c}\,(5.170)}\Big)\frac{\delta S}{\delta d_{z\boldsymbol{\delta}}}+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Big(2\hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}}+2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}\Big)\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}}+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^c\Big(-\underbrace{3H_{\boldsymbol{\alpha}c\boldsymbol{\delta}}}_{2\check{T}_{\boldsymbol{\alpha\delta}|c}\,(5.170)}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}+2\hat{T}_{\boldsymbol{\alpha}c}{}^{\hat{\boldsymbol{\gamma}}}\Big)\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^{\boldsymbol{\gamma}}\left(2\hat{T}_{\boldsymbol{\alpha\gamma}}{}^{\hat{\boldsymbol{\gamma}}}-\frac{3}{2}H_{\boldsymbol{\alpha\gamma\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\gamma}}\left(2T_{\boldsymbol{\alpha}d}{}^{\boldsymbol{\gamma}}\right)\Pi_{\bar{z}}^d+2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\gamma}}\left(T_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}{}^{\boldsymbol{\gamma}}\right)\Pi_{\bar{z}}^{\hat{\boldsymbol{\delta}}}+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}d_{z\boldsymbol{\gamma}}\left(\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}-C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}-2T_{\boldsymbol{\alpha\delta}}{}^{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}-2\hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\Pi_z^c\Big(-\underbrace{3H_{\boldsymbol{\alpha}c\boldsymbol{\delta}}}_{2\check{T}_{\boldsymbol{\alpha\delta}|c}\,(5.170)}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}}+2\hat{R}_{\boldsymbol{\alpha}c\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\Big)\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\Pi_z^{\boldsymbol{\gamma}}\left(2\hat{R}_{\boldsymbol{\alpha\gamma}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}-\frac{3}{2}H_{\boldsymbol{\alpha\gamma\delta}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}}\right)\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}+\\
&+\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}d_{z\boldsymbol{\gamma}}\left(\underline{\nabla}_{\boldsymbol{\alpha}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}+S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}}-2T_{\boldsymbol{\alpha\delta}}{}^{\boldsymbol{\gamma}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}}-2\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}\right)\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}}+\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}
\end{aligned} \tag{5.173}
$$

---

the contraction of the upper torsion index with G_{BD} projects out the first block-diagonal and we can write

$$\underline{T}_{AC|D}=\check{T}_{AC|D}$$

The next important observation is that the constraints are independent of the choice of the auxiliary bosonic connection $\check{\Omega}_{Ma}{}^{b}$, as it should be . The only condition is that it obeys $\check{\Omega}_{M(a|b)}=\check{\Omega}_M^{(D)}G_{ab}$ which we used during the derivation by taking $\underline{\nabla}_M G_{AB}=2\check{\nabla}_M\Phi G_{AB}$ (see (5.165)). Remember also that $\check{\nabla}_{\boldsymbol{\alpha}}\Phi=E_{\boldsymbol{\alpha}}{}^{M}\partial_M\Phi-\check{\Omega}_{\boldsymbol{\alpha}}^{(D)}$ (5.164). $\check{\Omega}_{Ma}{}^{b}$ enters the terms $Y_{\boldsymbol{\alpha}CD}$ (defined in (5.166) and containing the constraints) only in the combination $2\check{T}_{\boldsymbol{\alpha}(C|D)}-\check{\Omega}_{\boldsymbol{\alpha}}^{(D)}G_{CD}$, where it completely cancels:

$$
\begin{aligned}
2\check{T}_{\boldsymbol{\alpha}(C|D)}-\check{\Omega}_{\boldsymbol{\alpha}}^{(D)}G_{CD} &= 2(\mathbf{d}E^b)_{\boldsymbol{\alpha}(C|}G_{b|D)}+\check{\Omega}_{\boldsymbol{\alpha}(C|D)}-\underbrace{\check{\Omega}_{(C|\boldsymbol{\alpha}|D)}}_{=0}-\check{\Omega}_{\boldsymbol{\alpha}}^{(D)}G_{CD}=\\
&= 2E_{\boldsymbol{\alpha}}{}^{M}E_{(C|}{}^{N}\partial_{[M}E_{N]}{}^{b}G_{b|D)}
\end{aligned}
$$

In particular the connection does not enter at all the following torsion component:

$$\check{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}|c} = (\mathbf{d}E^d)_{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}G_{dc}$$

The constraints (5.168)-(5.170) are therefore independent of the choice of $\check{\Omega}_{Ma}{}^{b}$. In particular, we can choose $\Omega_{Ma}{}^{b}$ (defined by $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ via $\nabla_M\gamma^c_{\boldsymbol{\alpha\beta}}=0$) or $\hat{\Omega}_{Ma}{}^{b}$ (defined by $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ via $\hat{\nabla}_M\gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}=0$). ◇

where we defined an extra symbol for the terms coming quadratic in the ghost $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$:

$$\begin{aligned} X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} \quad \equiv \quad & 2\left(R_{[\boldsymbol{\alpha}_1|d|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}}\right)\Pi_{\bar{z}}^{d}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + 2\Pi_{\bar{z}}^{\hat{\boldsymbol{\delta}}}\left(R_{[\boldsymbol{\alpha}_1|\hat{\boldsymbol{\delta}}|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}}\right)\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \\ & + \left(\underline{\nabla}_{[\boldsymbol{\alpha}_1}C_{\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} - 2\hat{T}_{[\boldsymbol{\alpha}_1|\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}C_{|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}} - 2R_{[\boldsymbol{\alpha}_1|\boldsymbol{\delta}|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \\ & + \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\left(\underline{\nabla}_{[\boldsymbol{\alpha}_1}S_{\boldsymbol{\alpha}_2]\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + 2\hat{R}_{[\boldsymbol{\alpha}_1|\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}C_{|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} + 2R_{[\boldsymbol{\alpha}_1|\boldsymbol{\delta}|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}}\right)\boldsymbol{\omega}_{z\boldsymbol{\beta}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} \end{aligned} \tag{5.174}$$

Summarizing, we observe that we managed – with the help of the equations of motion – to turn the simple equation (5.150) into a quite lengthy one ... We are not going to copy the whole long equation again for the next step. The only equation of motion that we may still apply, is the pure spinor constraint

$$\frac{\delta S}{\delta L_{z\bar{z}a}} = \frac{1}{2}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) \tag{5.175}$$

We therefore can concentrate on the term $\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}$, where the pure spinor combination $\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}$ might appear. As discussed in footnote 7 on page 49 (see also the appendix-subsection D.3.3 on page 178), all graded antisymmetric 16×16 matrices can be expanded in $\gamma^{[1]}$ and $\gamma^{[5]}$:

$$X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} \quad \equiv \quad X_a\gamma^a_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} + X_{a_1\ldots a_5}\gamma^{a_1\ldots a_5}_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} \tag{5.176}$$

$$X_a \quad \stackrel{(D.143)}{=} \quad \frac{1}{16}\gamma_a^{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} \qquad \left(= -\frac{1}{16}\gamma_a^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}\right) \tag{5.177}$$

$$X_{a_1\ldots a_5} \quad \stackrel{(D.143)}{=} \quad \frac{1}{32\cdot 5!}\gamma_{a_5\ldots a_1}^{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} \tag{5.178}$$

We can use this to rewrite the quadratic ghost term as follows:

$$\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2} \quad = \quad -\frac{1}{8}\gamma_a^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}\frac{\delta S}{\delta L_{z\bar{z}a}} + \frac{1}{32\cdot 5!}\gamma_{a_1\ldots a_5}^{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}(\boldsymbol{\lambda}\gamma^{a_1\ldots a_5}\boldsymbol{\lambda}) \tag{5.179}$$

This was the last ingredient to determine all remaining constraints on the background fields and also to be able to read off all BRST transformations (including the one for the Lagrange multiplier). Let us start with the constraints. In addition to (5.167)-(5.171), we get the following constraints on the background fields:

$$
\begin{aligned}
\hat{T}_{\alpha c}{}^{\hat{\gamma}} &= \underbrace{\check{T}_{\alpha\delta|c}}_{\frac{3}{2}H_{\alpha c\delta}\,(5.170)}\mathcal{P}^{\delta\hat{\gamma}}, \qquad T_{\hat{\alpha}c}{}^{\gamma} = \underbrace{\check{T}_{\hat{\alpha}\hat{\delta}|c}}_{-\frac{3}{2}H_{\hat{\alpha}c\hat{\delta}}\,(5.170)}\mathcal{P}^{\gamma\hat{\delta}} &(5.180)\\
\hat{T}_{\alpha\gamma}{}^{\hat{\gamma}} &= \frac{3}{4}H_{\alpha\gamma\delta}\mathcal{P}^{\delta\hat{\gamma}}, \qquad T_{\hat{\alpha}\hat{\gamma}}{}^{\gamma} = -\frac{3}{4}H_{\hat{\alpha}\hat{\gamma}\hat{\delta}}\mathcal{P}^{\gamma\hat{\delta}} &(5.181)\\
T_{\alpha d}{}^{\gamma} &= 0, \qquad \hat{T}_{\hat{\alpha}d}{}^{\hat{\gamma}} = 0 &(5.182)\\
T_{\alpha\hat{\delta}}{}^{\gamma} &= 0, \qquad \hat{T}_{\hat{\alpha}\delta}{}^{\hat{\gamma}} = 0, \quad \stackrel{(5.169)}{\Rightarrow} \underline{T}_{\alpha\hat{\alpha}}{}^{K} = 0 &(5.183)\\
C_{\alpha}{}^{\gamma\hat{\gamma}} &= \underline{\nabla}_{\alpha}\mathcal{P}^{\gamma\hat{\gamma}} - 2T_{\alpha\delta}{}^{\gamma}\mathcal{P}^{\delta\hat{\gamma}} - 2\underbrace{\hat{T}_{\alpha\hat{\delta}}{}^{\hat{\gamma}}}_{=0\,(5.183)}\mathcal{P}^{\gamma\hat{\delta}} &(5.184)\\
\hat{C}_{\hat{\alpha}}{}^{\hat{\gamma}\gamma} &= \underline{\nabla}_{\hat{\alpha}}\mathcal{P}^{\gamma\hat{\gamma}} - 2\hat{T}_{\hat{\alpha}\hat{\delta}}{}^{\hat{\gamma}}\mathcal{P}^{\gamma\hat{\delta}} &(5.185)\\
\hat{R}_{\alpha c\hat{\alpha}}{}^{\hat{\beta}} &= \underbrace{\frac{3}{2}H_{\alpha c\delta}}_{\check{T}_{\alpha\delta|c}\,(5.170)}\hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\delta}, \qquad R_{\hat{\alpha}c\alpha}{}^{\beta} = \underbrace{-\frac{3}{2}H_{\hat{\alpha}c\hat{\delta}}}_{\check{T}_{\hat{\alpha}\hat{\delta}|c}\,(5.170)}C_{\alpha}{}^{\beta\hat{\delta}} &(5.186)\\
\hat{R}_{\alpha\gamma\hat{\alpha}}{}^{\hat{\beta}} &= \frac{3}{4}H_{\alpha\gamma\delta}\hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\delta}, \qquad R_{\hat{\alpha}\hat{\gamma}\alpha}{}^{\beta} = -\frac{3}{4}H_{\hat{\alpha}\hat{\gamma}\hat{\delta}}C_{\alpha}{}^{\beta\hat{\delta}} &(5.187)\\
S_{\alpha\hat{\alpha}}{}^{\gamma\hat{\beta}} &= -\underline{\nabla}_{\alpha}\underbrace{\hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\gamma}}_{\underline{\nabla}_{\hat{\alpha}}\mathcal{P}^{\gamma\hat{\beta}}-2\hat{T}_{\hat{\alpha}\hat{\delta}}{}^{\hat{\beta}}\mathcal{P}^{\gamma\hat{\delta}}\,(5.185)} + 2T_{\alpha\delta}{}^{\gamma}\hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\delta} + 2\hat{R}_{\alpha\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}}\mathcal{P}^{\gamma\hat{\gamma}} &(5.188)\\
S_{\alpha\hat{\alpha}}{}^{\beta\hat{\gamma}} &= -\underline{\nabla}_{\hat{\alpha}}\underbrace{C_{\alpha}{}^{\beta\hat{\gamma}}}_{\underline{\nabla}_{\alpha}\mathcal{P}^{\beta\hat{\gamma}}-2T_{\alpha\delta}{}^{\beta}\mathcal{P}^{\delta\hat{\gamma}}\,(5.184)} + 2\hat{T}_{\hat{\alpha}\hat{\delta}}{}^{\hat{\gamma}}C_{\alpha}{}^{\beta\hat{\delta}} + 2R_{\hat{\alpha}\gamma\alpha}{}^{\beta}\mathcal{P}^{\gamma\hat{\gamma}} &(5.189)\\
\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}R_{d\alpha_1\alpha_2}{}^{\beta} &= 0, \qquad \gamma^{\hat{\alpha}_1\hat{\alpha}_2}_{a_1\ldots a_5}\hat{R}_{d\hat{\alpha}_1\hat{\alpha}_2}{}^{\hat{\beta}} = 0 &(5.190)\\
\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}R_{\hat{\delta}\alpha_1\alpha_2}{}^{\beta} &= 0, \qquad \gamma^{\hat{\alpha}_1\hat{\alpha}_2}_{a_1\ldots a_5}\hat{R}_{\delta\hat{\alpha}_1\hat{\alpha}_2}{}^{\hat{\beta}} = 0 &(5.191)\\
\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}\left(\underline{\nabla}_{\alpha_2}C_{\alpha_1}{}^{\beta\hat{\gamma}}\right) &= 2\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}\Big(R_{\alpha_2\delta\alpha_1}{}^{\beta}\mathcal{P}^{\delta\hat{\gamma}} - \underbrace{\hat{T}_{\alpha_1\hat{\delta}}{}^{\hat{\gamma}}}_{=0\,(5.183)}C_{\alpha_2}{}^{\beta\hat{\delta}}\Big), \quad \text{plus hatted version } \ldots &(5.192)\\
\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}\left(\underline{\nabla}_{\alpha_2}S_{\alpha_1\hat{\alpha}}{}^{\beta\hat{\beta}}\right) &= 2\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}\left(\hat{R}_{\alpha_1\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}}C_{\alpha_2}{}^{\beta\hat{\gamma}} - R_{\alpha_2\delta\alpha_1}{}^{\beta}\hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\delta}\right), \quad \text{plus hatted version} \ldots &(5.193)
\end{aligned}
$$

Note that on the constraint surface the condition $\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}X_{\alpha_1\alpha_2} = 0$ is equivalent to the vanishing of $X_{\alpha_1\alpha_2}$ when contracted with two ghost fields:

$$
\gamma^{\alpha_1\alpha_2}_{a_1\ldots a_5}X_{\alpha_1\alpha_2} = 0 \quad \stackrel{(5.176)-(5.178)}{\Longleftrightarrow} \quad X_{[\alpha_1\alpha_2]} = \frac{1}{16}(\gamma_a^{\alpha_4\alpha_3}X_{\alpha_3\alpha_4})\gamma^a_{\alpha_1\alpha_2} \quad \stackrel{(\lambda\gamma^a\lambda)=0}{\Longleftrightarrow} \quad \lambda^{\alpha_1}X_{\alpha_1\alpha_2}\lambda^{\alpha_2} = 0 \tag{5.194}
$$

The above equivalences hold for general bispinors, not only for the one defined in (5.174). It is not necessary to memorize the constraints (5.192) and (5.193) as they are a consequence of other constraints anyway. We will show this fact at the end of section 5.11 on page 71.

Let us now devote a new section to the BRST transformations that we can likewise read off from (5.173).

5.8 The covariant BRST transformations

Remember that we started on page 59 with the demand $\bar{\partial}\boldsymbol{j}_z \stackrel{!}{=} -\mathbf{s}_{\underline{cov}}\phi^{\mathcal{I}}_{\rm all}\frac{\delta_{\underline{cov}}S}{\delta\phi^{\mathcal{I}}_{\rm all}}$. The covariant BRST transformations $\mathbf{s}_{\underline{cov}}\phi^{\mathcal{I}}_{\rm all}$ have to be understood in the sense of the covariant variation defined in (5.102)-(5.106). We have for example $\mathbf{s}_{\underline{cov}}\hat{\lambda}^{\hat{\alpha}} = \mathbf{s}_{c\hat{o}v}\hat{\lambda}^{\hat{\alpha}} = \mathbf{s}\hat{\lambda}^{\hat{\alpha}} + \mathbf{s}x^M\hat{\Omega}_{M\hat{\beta}}{}^{\hat{\alpha}}\hat{\lambda}^{\hat{\beta}}$. When the constraints of the end of last section are fulfilled, we can read off the covariant BRST transformations $\mathbf{s}_{\underline{cov}}\phi^{\mathcal{I}}_{\rm all}$ from equation (5.173) together with (5.179). Again we give at the same time (using proposition 3 on page 44) the results for the right-mover BRST-symmetry $\hat{\mathbf{s}}$, defined via[14] $\partial\hat{\boldsymbol{j}}_{\bar{z}} \stackrel{!}{=} -\hat{\mathbf{s}}_{\underline{cov}}\phi^{\mathcal{I}}_{\rm all}\frac{\delta_{\underline{cov}}S}{\delta\phi^{\mathcal{I}}_{\rm all}}$:

[14] Another way to write down the BRST transformations for $d_{z\delta}$ and $\hat{d}_{\bar{z}\hat{\gamma}}$ is the following

$$
\begin{aligned}
\mathbf{s}_{cov}d_{z\delta} &= -\frac{3}{2}\lambda^{\alpha}\Pi_z^{\{c,\gamma\}}H_{\alpha\{c,\gamma\}\delta} - \lambda^{\alpha}\underline{T}_{\alpha\delta}{}^{\{c,\gamma\}}\{G_{cd}\Pi_z^d\,,\,2d_{z\gamma}\} + 2\lambda^{\alpha}\lambda^{\alpha_2}R_{\alpha_2\delta\alpha}{}^{\beta}\omega_{z\beta}\\
\mathbf{s}_{c\hat{o}v}\hat{d}_{\bar{z}\hat{\gamma}} &= -\frac{3}{2}\lambda^{\alpha}\Pi_{\bar{z}}^{\{d,\hat{\delta}\}}\underbrace{H_{\alpha\{d,\hat{\delta}\}\hat{\gamma}}}_{=0} - \lambda^{\alpha}\underbrace{\underline{T}_{\alpha\hat{\gamma}}{}^{\{d,\hat{\delta}\}}}_{=0}\{G_{dc}\Pi_{\bar{z}}^c\,,\,2\hat{d}_{\bar{z}\hat{\delta}}\} - 2\lambda^{\alpha}\hat{\lambda}^{\hat{\alpha}}\hat{R}_{\alpha\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}}\hat{\omega}_{\bar{z}\hat{\beta}}
\end{aligned}
$$

$$\begin{aligned}
\mathbf{s}x^M &= \boldsymbol{\lambda}^{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}}{}^M, \qquad \hat{\mathbf{s}}x^M = \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} E_{\hat{\boldsymbol{\alpha}}}{}^M && (5.195)\\
\mathbf{s}_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} &= 0 = \hat{\mathbf{s}}_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \qquad \hat{\mathbf{s}}_{c\hat{o}v}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} = 0 = \mathbf{s}_{c\hat{o}v}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} && (5.196)\\
\mathbf{s}_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} &= d_{z\boldsymbol{\alpha}}, \qquad \hat{\mathbf{s}}_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} = 0, \qquad \hat{\mathbf{s}}_{c\hat{o}v}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\alpha}}} = \hat{d}_{\bar{z}\hat{\boldsymbol{\alpha}}}, \qquad \mathbf{s}_{c\hat{o}v}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\alpha}}} = 0 && (5.197)\\
\mathbf{s}_{cov}d_{z\boldsymbol{\delta}} &= -\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^c \underbrace{3H_{\boldsymbol{\alpha}c\boldsymbol{\delta}}}_{2\check{T}_{\boldsymbol{\alpha\delta}|c}} - \frac{3}{2}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^{\boldsymbol{\gamma}} H_{\boldsymbol{\alpha\gamma\delta}} - 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}} T_{\boldsymbol{\alpha\delta}}{}^{\boldsymbol{\gamma}} d_{z\boldsymbol{\gamma}} + 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2} R_{\boldsymbol{\alpha}_2\boldsymbol{\delta\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} && (5.198)\\
\hat{\mathbf{s}}_{c\hat{o}v}\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}} &= \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\Pi_{\bar{z}}^c \underbrace{3H_{\hat{\boldsymbol{\alpha}}c\hat{\boldsymbol{\delta}}}}_{-2\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}|c}} + \frac{3}{2}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\Pi_{\bar{z}}^{\hat{\boldsymbol{\gamma}}} H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}} - 2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + 2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}_2}\hat{R}_{\hat{\boldsymbol{\alpha}}_2\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\boldsymbol{\omega}_{\bar{z}\hat{\boldsymbol{\beta}}} && (5.199)\\
\mathbf{s}_{c\hat{o}v}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} &= -2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underbrace{\hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}}_{=0}\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}} - 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} && (5.200)\\
\hat{\mathbf{s}}_{cov}d_{z\boldsymbol{\gamma}} &= -2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\underbrace{T_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}}_{=0} d_{z\boldsymbol{\delta}} - 2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} && (5.201)\\
\mathbf{s}_{cov}L_{z\bar{z}a} &= \frac{1}{8}\gamma_a^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}, \qquad \hat{\mathbf{s}}_{cov}L_{z\bar{z}a} = 0, \qquad \hat{\mathbf{s}}_{c\hat{o}v}\hat{L}_{\bar{z}z\,a} = \frac{1}{8}\gamma_a^{\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}\hat{X}_{\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}, \qquad \mathbf{s}_{c\hat{o}v}\hat{L}_{\bar{z}z} = 0 && (5.202)
\end{aligned}$$

The composite object $X_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}$ is given in (5.174). Let us for completeness also give the BRST transformation of the supersymmetric momentum

$$\mathbf{s}_{\underline{cov}}\Pi^A_{z/\bar{z}} \stackrel{(5.122)}{=} \nabla_{z/\bar{z}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\delta_{\boldsymbol{\alpha}}{}^A + 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi^B_{z/\bar{z}}\underline{T}_{\boldsymbol{\alpha}B}{}^A \tag{5.203}$$

$$\hat{\mathbf{s}}_{\underline{cov}}\Pi^A_{z/\bar{z}} \stackrel{(5.122)}{=} \hat{\nabla}_{z/\bar{z}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\delta_{\hat{\boldsymbol{\alpha}}}{}^A + 2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\Pi^B_{z/\bar{z}}\underline{T}_{\hat{\boldsymbol{\alpha}}B}{}^A \tag{5.204}$$

All these BRST transformations are similar to those for the heterotic string, given in [14]. There it was also noted that the BRST transformations always contain a Lorentz transformation (multiplication with the connection). We have absorbed this term into the definition of the covariant variation. The advantage is that we then have expressions all the time that are covariant with respect to the target space structure group. Although the ordinary BRST differential $\mathbf{s}$ is needed to calculate the cohomology (as it squares to zero), the calculations are simpler if they are performed with $\mathbf{s}_{\underline{cov}}$ and only in the end transferred to $\mathbf{s}$. When acting on a target space scalar, the two coincide anyway.

5.9 Graded commutation of left- and right-moving BRST differential

We have started in flat background with two independent BRST symmetries, the left-moving and the right-moving one, which both squared to zero and graded commuted. As they define the physical spectrum and identify physically equivalent states, these facts should not change in a consistent theory, at least on-shell. This is similar to the fact that gauge symmetries should not be broken. We have already derived the constraints coming from a vanishing divergence of the BRST currents. The ansatz for the currents was such that this corresponds to holomorphicity for $\boldsymbol{j}_z$ and antiholomorphicity for $\hat{\boldsymbol{\jmath}}_{\bar{z}}$. Having on-shell a holomorphic $\boldsymbol{j}_z$ and an antiholomorphic $\hat{\boldsymbol{\jmath}}_{\bar{z}}$ is in a conformal theory already enough to make the corresponding symmetries commute. For example on the level of operators, the operator product between a holomorphic and an antiholomorphic current always vanishes on-shell. The same is true for the charges which generate the symmetry. The on-shell

In the second line for the first two terms, we have just used a complicated way to write zero. The reason was to bring it to a form similar to the one in the first line. In any case, at least the first line suggests again the introduction of the variables

$$d_{zc} \equiv \frac{1}{2}G_{cd}\Pi_z^d, \qquad d_{\bar{z}c} \equiv \frac{1}{2}G_{cd}\Pi_{\bar{z}}^d$$

that we already proposed in footnote 12 on page 62. Indeed, their BRST transformation takes the form

$$\mathbf{s}_{c\bar{o}v}d_{z\,c} = -\frac{3}{2}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^{\boldsymbol{\beta}}H_{\boldsymbol{\alpha\beta}c} - 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\check{T}_{\boldsymbol{\alpha}c}{}^d d_{z\,d}$$

Using $H_{a\boldsymbol{\beta}c} = T_{\boldsymbol{\alpha}c}{}^{\boldsymbol{\delta}} = 0$ and at (least for $\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda} = 0$) $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}R_{\boldsymbol{\alpha}_2 d\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0$, the transformation of $d_{z\,c}$ takes the same form as the one of $d_{z\boldsymbol{\delta}}$ and we can write

$$\mathbf{s}_{\underline{cov}}d_{z\{d,\boldsymbol{\delta}\}} = -\frac{3}{2}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\Pi_z^{\{c,\boldsymbol{\gamma}\}}H_{\boldsymbol{\alpha}\{c,\boldsymbol{\gamma}\}\{d,\boldsymbol{\delta}\}} - 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{T}_{\boldsymbol{\alpha}\{d,\boldsymbol{\delta}\}}{}^{\{c,\boldsymbol{\gamma}\}}d_{z\{c,\boldsymbol{\gamma}\}} - 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}R_{\{d,\boldsymbol{\delta}\}\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \quad \text{for } (\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) = 0$$

We suggest to introduce d_{zd} as an independent variable into the action, with an on-shell value $d_{zc} \equiv \frac{1}{2}G_{cd}\Pi_z^d$. Doing this, one would arrive at a formalism where the G_{MN} term is replaced by a first order term, while the B_{MN} term remains. This would therefore be a mixed first-second order formalism which would be suitable to couple it to e.g. the components of a generalized complex structure. ◇

vanishing of the commutators is all that we can demand for consistency. Therefore we do not expect any additional information from the graded commutation of left- and right-moving BRST differential. Nevertheless it is instructive to calculate the graded commutators and consider it as a further check. In particular it is interesting to see the terms which prevent an off-shell commutation of the differentials. The starting point is the request that we have

$$[\hat{\mathbf{s}}, \mathbf{s}]\phi^{\mathcal{I}}_{\text{all}} \stackrel{!}{=} \delta_{(\mu)}\phi^{\mathcal{I}}_{\text{all}} + \delta_{(\hat{\mu})}\phi^{\mathcal{I}}_{\text{all}} + \delta_{\text{triv}}\phi^{\mathcal{I}}_{\text{all}} \tag{5.205}$$

where $\delta_{\text{triv}}\phi^{\mathcal{I}}_{\text{all}}$ is a trivial and thus on-shell vanishing gauge transformation (see page 186 in the appendix) while $\delta_{(\mu)}$ and $\delta_{(\hat{\mu})}$ are the antighost gauge transformations. Spelled out in words, (5.205) means that the graded commutator $[\hat{\mathbf{s}}, \mathbf{s}]$ has to vanish on shell up to antighost gauge transformations. There are at least two ways to check this. Either we calculate the commutator of the transformations on each worldsheet field or we calculate the transformations of the Noether currents. This is directly related to calculating the Poisson brackets of the generating charges in the Hamiltonian formalism.

Determining $[\mathbf{s}, \hat{\mathbf{s}}]$ via the transformation of the currents We start with the defining equations of the BRST currents:

$$\bar{\partial}\boldsymbol{j}_z = -\mathbf{s}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}, \quad \partial\hat{\boldsymbol{\jmath}}_{\bar{z}} = -\hat{\mathbf{s}}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{5.206}$$

The current for the graded commutator $[\hat{\mathbf{s}}, \mathbf{s}]$ is given only on-shell by $\hat{\mathbf{s}}\boldsymbol{j}_z$ or $\mathbf{s}\hat{\boldsymbol{\jmath}}_{\bar{z}}$ (one would expect this from the Hamiltonian formalism). A correct off-shell expression can be obtained by acting on (5.206) with $\hat{\mathbf{s}}$ or $\mathbf{s}$ respectively. The derivation of the current $j_{[\hat{\mathbf{s}},\mathbf{s}]}$ corresponding to $[\mathbf{s}, \hat{\mathbf{s}}]$ was too simple and indeed not correct in the original version of this thesis, so that by now I have moved a more careful and general derivation into the appendix. From there we can adopt the result from equation (E.55) on page 188:

$$j_{[\hat{\mathbf{s}},\mathbf{s}]z} = \hat{\mathbf{s}}\boldsymbol{j}_z + \Big(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\hat{\mathbf{s}}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_{\bar{z}}\phi^{\mathcal{K}}_{\text{all}})}\Big)\cdot\mathbf{s}\phi^{\mathcal{K}}_{\text{all}}, \quad j_{[\hat{\mathbf{s}},\mathbf{s}]\bar{z}} = \Big(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\hat{\mathbf{s}}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial\phi^{\mathcal{K}}_{\text{all}})}\Big)\cdot\mathbf{s}\phi^{\mathcal{K}}_{\text{all}} \tag{5.207}$$

or equivalently (interchanging the role of $\mathbf{s}$ and $\hat{\mathbf{s}}$)

$$j_{[\hat{\mathbf{s}},\mathbf{s}]z} = \Big(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\mathbf{s}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_{\bar{z}}\phi^{\mathcal{K}}_{\text{all}})}\Big)\cdot\hat{\mathbf{s}}\phi^{\mathcal{K}}_{\text{all}}, \quad j_{[\hat{\mathbf{s}},\mathbf{s}]\bar{z}} = \mathbf{s}\hat{\boldsymbol{\jmath}}_{\bar{z}} + \Big(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\mathbf{s}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial\phi^{\mathcal{K}}_{\text{all}})}\Big)\cdot\hat{\mathbf{s}}\phi^{\mathcal{K}}_{\text{all}} \tag{5.208}$$

For consistency we need only that $[\mathbf{s}, \hat{\mathbf{s}}]$ vanishes up to trivial and other gauge transformations. It is thus enough to demand that the corresponding current $j_{[\hat{\mathbf{s}},\mathbf{s}]}$ vanishes on-shell, because on-shell vanishing currents correspond to gauge transformations (see proposition 6 on 184 in the appendix). If we take the expression for $j_{[\hat{\mathbf{s}},\mathbf{s}]z}$ from (5.208) and the expression for $j_{[\hat{\mathbf{s}},\mathbf{s}]\bar{z}}$ from (5.207), we can observe that both components of the current vanish on-shell without any extra conditions on the background fields! As claimed at the beginning of this section this happens due to the fact that left- and right-mover BRST currents $\boldsymbol{j}_z$ and $\hat{\boldsymbol{\jmath}}_{\bar{z}}$ are on-shell holomorphic and antiholomorphic respectively.

In principle we are already done with the commutator $[\mathbf{s}, \hat{\mathbf{s}}]$, but it is a good check to see, whether we obtain the same result if we do it the other way round and take the expression for $j_{[\hat{\mathbf{s}},\mathbf{s}]z}$ from (5.207) and the expression for $j_{[\hat{\mathbf{s}},\mathbf{s}]\bar{z}}$ from (5.208). This corresponds to demanding $\hat{\mathbf{s}}\boldsymbol{j}_z \stackrel{\text{on shell}}{=} 0$, $\mathbf{s}\hat{\boldsymbol{\jmath}}_{\bar{z}} \stackrel{\text{on shell}}{=} 0$. In order to calculate $\hat{\mathbf{s}}\boldsymbol{j}_z$, remember the form of the BRST current $\boldsymbol{j}_z = \boldsymbol{\lambda}^{\boldsymbol{\alpha}} d_{z\boldsymbol{\alpha}}$ (5.39) and also note that it is a target space scalar. The BRST differential can thus be replaced by the covariant one:

$$\hat{\mathbf{s}}\boldsymbol{j}_z = -\boldsymbol{\lambda}^{\boldsymbol{\gamma}}\hat{\mathbf{s}}_{cov}d_{z\boldsymbol{\gamma}} = -2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\boldsymbol{\lambda}^{\boldsymbol{\gamma}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \underset{(5.194)}{\overset{(5.191)}{=}} -\frac{1}{8}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\gamma_a^{\boldsymbol{\alpha}\boldsymbol{\gamma}}R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\underbrace{(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda})}_{2\frac{\delta S}{\delta L_{z\bar{z}}}} \tag{5.209}$$

Using the left-right-symmetry of proposition 3 on page 44 we get the corresponding expression for $\mathbf{s}\hat{\boldsymbol{\jmath}}_{\bar{z}}$. Both vanish on the pure spinor constraint surface $(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) = (\hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}}) = 0$ so that indeed the Noether current belonging to $[\hat{\mathbf{s}}, \mathbf{s}]$ vanishes on-shell and thus $[\mathbf{s}, \hat{\mathbf{s}}]$ will vanish on-shell up to gauge transformations.

If we wanted to know also the non-trivial gauge transformations that appear in the commutator, we would have to calculate also the additional on-shell vanishing terms that are added to $\hat{\mathbf{s}}\boldsymbol{j}_z$ in the expression of $j_{[\mathbf{s},\hat{\mathbf{s}}]z}$ in (5.207). It turns out that only $\Big(\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}}}\frac{\partial(\hat{\mathbf{s}}\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}})}{\partial(\bar{\partial}x^K)}\Big)\cdot sx^K = \frac{3}{2}\frac{\delta S}{\delta\hat{d}_{\bar{z}\hat{\boldsymbol{\delta}}}}\hat{\lambda}^{\hat{\boldsymbol{\alpha}}}\hat{\lambda}^{\hat{\boldsymbol{\gamma}}}H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}$ is contributing a priori. However, we will see later that $H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}$ is required to vanish from the nilpotency demand of the BRST transformation as well as from the Bianchi identities.

The (non-trivial) gauge transformations that will appear in the commutator $[\mathbf{s}, \hat{\mathbf{s}}]$ are thus given precisely by the above off-shell non-vanishing term (5.209). Namely if we take $\mu_{za} \equiv -\frac{1}{4}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\gamma_a^{\boldsymbol{\alpha}\boldsymbol{\gamma}}R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ we obtain

$$\hat{\mathbf{s}}\boldsymbol{j}_z = \tfrac{1}{2}\mu_{za}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) \tag{5.210}$$

which is precisely the current of the antighost gauge transformation given on the lefthand side of (5.78) with corresponding antighost gauge transformations $\delta_{(\mu)}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} = \mu_{za}(\boldsymbol{\lambda}\gamma^a)_{\boldsymbol{\alpha}}$ (5.90) and $\delta_{(\mu)}L_{z\bar{z}a} = -\mathcal{D}_{\bar{z}}\mu_{za}$ (5.91). Having a current that coincides with the one of a gauge transformation, the form of $[\mathbf{s}, \hat{\mathbf{s}}]$ can only differ by a trivial gauge transformation. In any case we have obtained the result that the commutator vanishes up to gauge transformations. A safe way to figure out potentially appearing trivial gauge transformations in the commutator is to calculate it on each single worldsheet field separately.

Acting on each field separately Although this method would lead to the precise off-shell form of all the commutators, we are for now satisfied with the result we already obtained and give the explicit commutator only for the most simple cases. Starting with the covariant BRST transformations of the elementary fields (given in (5.195)-(5.202) on page 66), we will first calculate the commutator $[\hat{\mathbf{s}}_{\underline{cov}}, \mathbf{s}_{\underline{cov}}]$ and only after that determine the ordinary commutator via the relations (5.112) and (5.113). For the embedding functions x^K, the ghosts $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}$ and the antighosts $\boldsymbol{\omega}_{z\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\alpha}}}$ the calculation is very simple and we immediately obtain

$$\left[\hat{\mathbf{s}}_{\underline{cov}}, \mathbf{s}_{\underline{cov}}\right] x^K = 0 \tag{5.211}$$

$$\left[\hat{\mathbf{s}}_{\underline{cov}}, \mathbf{s}_{\underline{cov}}\right] \boldsymbol{\lambda}^{\boldsymbol{\gamma}} = 0, \qquad \left[\mathbf{s}_{\underline{cov}}, \hat{\mathbf{s}}_{\underline{cov}}\right] \hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\gamma}}} = 0 \tag{5.212}$$

$$\left[\hat{\mathbf{s}}_{\underline{cov}}, \mathbf{s}_{\underline{cov}}\right] \boldsymbol{\omega}_{z\boldsymbol{\gamma}} = \hat{\mathbf{s}}_{\underline{cov}} d_{z\boldsymbol{\gamma}} = -2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}, \qquad \left[\mathbf{s}_{\underline{cov}}, \hat{\mathbf{s}}_{\underline{cov}}\right] \hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\gamma}}} = -2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} \hat{R}_{\boldsymbol{\alpha}\boldsymbol{\gamma}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} \tag{5.213}$$

The transformations of the remaining fields are much more complicated and we prefer not to study them. Let us now derive the ordinary commutators:

$$[\hat{\mathbf{s}}, \mathbf{s}] x^K \stackrel{(5.112)}{=} \underbrace{\left[\hat{\mathbf{s}}_{\underline{cov}}, \mathbf{s}_{\underline{cov}}\right] x^K}_{=0} - 2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} \underbrace{\underline{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\alpha}}{}^K}_{=0\,(5.183)} \boldsymbol{\lambda}^{\boldsymbol{\alpha}} = 0 \tag{5.214}$$

$$[\mathbf{s}, \hat{\mathbf{s}}]_{cov} \boldsymbol{\lambda}^{\boldsymbol{\gamma}} \stackrel{(5.113)}{=} \underbrace{\left[\mathbf{s}_{\underline{cov}}, \hat{\mathbf{s}}_{\underline{cov}}\right] \boldsymbol{\lambda}^{\boldsymbol{\gamma}}}_{=0} - 2\underbrace{\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} R_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\boldsymbol{\lambda}^{\boldsymbol{\beta}}}_{=0\,(5.191)} = 0 \tag{5.215}$$

$$\begin{aligned} [\mathbf{s}, \hat{\mathbf{s}}]_{cov} \boldsymbol{\omega}_{z\boldsymbol{\gamma}} &\stackrel{(5.113)}{=} \underbrace{\left[\hat{\mathbf{s}}_{\underline{cov}}, \mathbf{s}_{\underline{cov}}\right] \boldsymbol{\omega}_{z\boldsymbol{\gamma}}}_{=-2\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}} + 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}} R_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} = \\ &= 4\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} R_{\hat{\boldsymbol{\alpha}}[\boldsymbol{\alpha}\boldsymbol{\gamma}]}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} \end{aligned} \tag{5.216}$$

Again we get the corresponding equations for $\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}$ and $\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\gamma}}}$. The last line corresponds excactly to the gauge transformation with gauge parameter $\mu_{za} = -\frac{1}{4}\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\alpha}}}\gamma_a^{\boldsymbol{\alpha}\boldsymbol{\gamma}} R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}$ that we found already above. This is strictly speaking true only if $H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}} = 0$ (remember the off-shell terms that were mentioned after (5.209)), a constraint that we will obtain only in the next section from nilpotency. The explanation is that the different ways of calculating the same quantity $[\mathbf{s}, \hat{\mathbf{s}}]$ certainly assume the validity of the Bianchi identities which already at this point would imply the above extra constraint. However, we will do a careful analysis of the Bianchi identities only in the end, after having obtained the additional constraints from nilpotency. It is further interesting to see in (5.214), that some holomorphicity constraints like $\underline{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\alpha}}{}^K = 0$ are needed for the commutation. In fact, in [59] this constraint was derived by demanding a vanishing Poisson bracket between the two generators of the BRST symmetries. The constraint $\underline{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\alpha}}{}^K = 0$ did not appear in our derivation via the currents above. The reason is that we already started the derivation in (5.206) from an equation which assumes on-shell holomorphicity.

5.10 Nilpotency of the BRST differentials

While the last section was rather a check than bringing much new information, the nilpotency of the BRST differentials will give us additional constraints on the background fields. The nilpotency is essential to define the physical spectrum as in the flat case via the cohomology. It would be inconsistent if this prescription breakes down, as soon as a nonvanishing background is generated by the strings. Demanding nilpotency at least on-shell and up to gauge transformations is thus legitimate.

Nilpotency constraints from the BRST transformation of the current In the same way as in the previous section, we can examine the BRST-transformation of the BRST-current instead of studying nilpotency on every single worldsheet field. Start from the defining equation of the BRST current

$$\bar{\partial}\boldsymbol{j}_z = -\mathbf{s}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{5.217}$$

Again the current for the graded commutator $[\mathbf{s},\mathbf{s}] = 2\mathbf{s}^2$ is given only on-shell by $\mathbf{s}\boldsymbol{j}_z$ (what one would expect from the Hamiltonian formalism). To obtain the off-shell expression one can act with $\mathbf{s}$ for a second time on the above equation. From the appendix we can adopt the result from equation (E.55) on page 188:

$$j_{[\mathbf{s},\mathbf{s}]z} = \mathbf{s}\boldsymbol{j}_z + \left(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\mathbf{s}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_{\bar{z}}\phi^{\mathcal{K}}_{\text{all}})}\right)\cdot\mathbf{s}\phi^{\mathcal{K}}_{\text{all}}, \quad j_{[\mathbf{s},\mathbf{s}]\bar{z}} = \left(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\mathbf{s}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial\phi^{\mathcal{K}}_{\text{all}})}\right)\cdot\mathbf{s}\phi^{\mathcal{K}}_{\text{all}} \tag{5.218}$$

The BRST transformation of the BRST current $\mathbf{s}\boldsymbol{j}_z$ is therefore at least on-shell the Noether current for the transformation $2\mathbf{s}^2$. For consistency we need only that $\mathbf{s}^2$ vanishes up to trivial and other gauge transformations. Due to proposition 6 on page 184 in the appendix, every gauge transformation has (up to trivially conserved terms) an on-shell vanishing Noether current. Demanding that $\mathbf{s}\boldsymbol{j}_z$ vanishes on-shell is therefore a necessary condition.[15] Also due to proposition 6 it is a sufficient condition, as we know already that $\mathbf{s}\boldsymbol{j}_z$ is a Noether current for a symmetry transformation and if this current vanishes on-shell, the transformation can be extended to a local one, i.e. it is a gauge transformation.

As the BRST current is a target space scalar, we can replace the BRST differential with the covariant one when calculating $\mathbf{s}\boldsymbol{j}_z$:

$$\begin{aligned}\mathbf{s}\boldsymbol{j}_z &= \mathbf{s}_{\underline{cov}}\left(\boldsymbol{\lambda}^{\boldsymbol{\delta}} d_{z\boldsymbol{\delta}}\right) = -\boldsymbol{\lambda}^{\boldsymbol{\delta}}\mathbf{s}_{\underline{cov}} d_{z\boldsymbol{\delta}} = \\ &\overset{(5.198)}{=} -\boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underbrace{3H_{\boldsymbol{\alpha}c\boldsymbol{\delta}}}_{2\check{T}_{\boldsymbol{\alpha\delta}|c}}\Pi^c_z - \frac{3}{2}\boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}H_{\boldsymbol{\alpha\gamma\delta}}\Pi^{\boldsymbol{\gamma}}_z - 2\boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}T_{\boldsymbol{\alpha\delta}}{}^{\boldsymbol{\gamma}}d_{z\boldsymbol{\gamma}} + 2\boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}R_{\boldsymbol{\alpha}_2\boldsymbol{\delta\alpha}}{}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}}\end{aligned} \tag{5.219}$$

The only equations of motion, which can make $\mathbf{s}\boldsymbol{j}_z$ vanish on-shell are the pure spinor constraints $\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda} = 0$. We therefore get the following conditions on the background fields

$$\Rightarrow \boldsymbol{\lambda}^{\boldsymbol{\delta}}H_{\boldsymbol{\alpha}C\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} = 0, \qquad \boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}T_{\boldsymbol{\alpha\delta}}{}^{\boldsymbol{\gamma}} = 0, \qquad \boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2}R_{\boldsymbol{\alpha}_2\boldsymbol{\delta\alpha}_1}{}^{\boldsymbol{\beta}} = 0, \qquad \text{(on shell)} \tag{5.220}$$

Remembering that we have the constraints $\check{T}_{\boldsymbol{\alpha\delta}|c} = \frac{3}{2}H_{\boldsymbol{\alpha}c\boldsymbol{\delta}}$ (5.170) and $\hat{T}_{\boldsymbol{\alpha\delta}}{}^{\hat{\boldsymbol{\gamma}}} = \frac{3}{4}H_{\boldsymbol{\alpha\delta\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}$, we can extend the above condition on the torsion on all indices

$$\boldsymbol{\lambda}^{\boldsymbol{\delta}}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{T}_{\boldsymbol{\alpha\delta}}{}^{C} = 0 \qquad \text{(on-shell)} \tag{5.221}$$

All these on-shell conditions can be formulated in an off-shell version with the help of γ-matrices by using (5.194) on page 65. Either we write that the terms are linear combinations of $\gamma^{[1]}$'s, or equivalently we can write that the $\gamma^{[5]}$-part vanishes. We thus can rewrite the constraints on torsion and H-field as

$$\underline{T}_{\boldsymbol{\alpha\beta}}{}^{C} = \gamma^d_{\boldsymbol{\alpha\beta}}f_d{}^C \quad \text{with } f_d{}^C \equiv \frac{1}{16}\gamma_d^{\boldsymbol{\varepsilon\delta}}\underline{T}_{\boldsymbol{\delta\varepsilon}}{}^C \tag{5.222}$$

$$H_{C\boldsymbol{\alpha\beta}} = H_{Ca}\gamma^a_{\boldsymbol{\alpha\beta}} \quad \text{with } H_{Ca} \equiv \frac{1}{16}H_{C\boldsymbol{\delta\varepsilon}}\gamma_a^{\boldsymbol{\varepsilon\delta}} \tag{5.223}$$

In particular for $C = \boldsymbol{\gamma}$, due to the (graded) total antisymmetry of $H_{\boldsymbol{\gamma\alpha\beta}}$, this should at the same time be proportional to $\gamma^a_{\boldsymbol{\gamma\alpha}}$ and $\gamma^a_{\boldsymbol{\beta\gamma}}$:

$$H_{\boldsymbol{\gamma\alpha\beta}} \overset{(5.223)}{=} H_{[\boldsymbol{\gamma}|a}\gamma^a_{|\boldsymbol{\alpha\beta}]} \overset{(5.223)}{=} \frac{1}{16}H_{[\boldsymbol{\gamma}|\boldsymbol{\delta\varepsilon}}\gamma_a^{\boldsymbol{\varepsilon\delta}}\gamma^a_{|\boldsymbol{\alpha\beta}]} \overset{(5.223)}{=} \frac{1}{16}H_{\boldsymbol{\varepsilon}b}\gamma^b_{[\boldsymbol{\gamma}|\boldsymbol{\delta}}\gamma_a^{\boldsymbol{\varepsilon\delta}}\gamma^a_{|\boldsymbol{\alpha\beta}]} \underset{(D.160)}{\overset{(D.108)}{=}} \frac{1}{8}H_{[\boldsymbol{\gamma}|a}\gamma^a_{|\boldsymbol{\alpha\beta}]} \tag{5.224}$$

In the last step we used the Clifford algebra (D.108) for the first two γ's and then the Fierz identity (D.160) to throw away one of the resulting terms. Remember that the appendix about Γ-matrices doesn't use the graded summation convention. For the Fierz identity we thus have a (graded) antisymmetrization, instead of the symmetrization and for the Clifford algebra we get an extra minus sign because of the NW-definition of the Kronecker-delta.

The second and the last term of the above equation (5.224) contradict each other if they do not vanish and thus $H_{\boldsymbol{\varepsilon\alpha\beta}}$ has to vanish. The components $H_{\hat{\boldsymbol{\varepsilon}}\boldsymbol{\alpha\beta}}$ were constraint to be zero already before. Of the components in (5.223), we thus have only $H_{c\boldsymbol{\alpha\beta}}$ nonvanishing. Because of $\check{T}_{\boldsymbol{\alpha\beta}|c} = -\frac{3}{2}H_{c\boldsymbol{\alpha\beta}}$ (5.170) and $\hat{T}_{\boldsymbol{\alpha\beta}}{}^{\hat{\boldsymbol{\gamma}}} \overset{(5.181)}{=} \frac{3}{4}H_{\boldsymbol{\alpha\beta\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} = 0$, we have in addition

$$f_{dc} = -\frac{3}{2}H_{cd}, \quad f_d{}^{\hat{\boldsymbol{\gamma}}} = 0 \tag{5.225}$$

[15] There are no trivially conserved parts in $\mathbf{s}\boldsymbol{j}_z$. A trivially conserved part is of the form $\partial_\zeta S^{[\zeta\xi]}$ for some rank two tensor $S^{\zeta\xi}$. In the conformal gauge this would take the form $\partial_z S_{[\bar{z}z]}$ which is of conformal weight (2,1). Such a term is certainly not present in our current. ◇

The new constraints on H and on the torsion thus read (the constraints in brackets follow from the other ones in combination with (5.170) and (5.181) and are thus redundant):

$$\boxed{\begin{aligned} H_{\boldsymbol{\varepsilon\alpha\beta}} &= 0, \qquad H_{c\boldsymbol{\alpha\beta}} = -\frac{2}{3}\gamma^a_{\boldsymbol{\alpha\beta}} f_{ac} && (5.226)\\ T_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} &= \gamma^d_{\boldsymbol{\alpha\beta}} f_d{}^{\boldsymbol{\gamma}}, \qquad (\check{T}_{\boldsymbol{\alpha\beta}}{}^c = \gamma^d_{\boldsymbol{\alpha\beta}} f_d{}^c, \qquad \hat{T}_{\boldsymbol{\alpha\beta}}{}^{\hat{\boldsymbol{\gamma}}} = 0) && (5.227)\end{aligned}}$$

As a remark let us note that the action in flat superspace with the ordinary WZ-term of the GS-string corresponds to $H_{c\boldsymbol{\alpha\beta}} = -\frac{2}{3}\gamma_{c\,\boldsymbol{\alpha\beta}}$ and thus to $f_{ac} = \eta_{ac}$. We can now analyze in a similar way the constraint on the curvature in (5.220). As the pure spinor constraint is quadratic it can be equivalently written as $\boldsymbol{\lambda}^{\boldsymbol{\alpha}_1}\boldsymbol{\lambda}^{\boldsymbol{\alpha}_2} R_{[\boldsymbol{\alpha}_2\boldsymbol{\delta\alpha}_1]}{}^{\boldsymbol{\beta}} = 0$ (on-shell). For this expression, one can do the same reasoning as above with $H_{\boldsymbol{\varepsilon\alpha\beta}}$ and arrives at

$$\boxed{R_{[\boldsymbol{\gamma\delta\alpha}]}{}^{\boldsymbol{\beta}} = 0} \qquad (5.228)$$

We will get the same constraint from the Bianchi identities later in (5.586) in case one feels uncomfortable with that line of arguments.

Of course we get all the correponding constraints also in the hatted version from the right-mover BRST current according to the left-right symmetry of page 44:

$$\boxed{\begin{aligned} H_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} &= 0, \qquad H_{c\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = \frac{2}{3}\gamma^a_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \hat{f}_{ac} && (5.229)\\ \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} &= \gamma^d_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \hat{f}_d{}^{\hat{\boldsymbol{\gamma}}}, \qquad (\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c = \gamma^d_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \hat{f}_d{}^c, \qquad T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} = 0) && (5.230)\\ \hat{R}_{[\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\alpha}}]}{}^{\hat{\boldsymbol{\beta}}} &= 0 && (5.231)\end{aligned}}$$

Remember that the curvature is structure group valued in the last two indices and decays into Lorentz and scale part (see (F.90) in the appendix on page 195): $R_{\boldsymbol{\gamma\delta\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}F^{(D)}_{\boldsymbol{\gamma\delta}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + R^{(L)}_{\boldsymbol{\gamma\delta\alpha}}{}^{\boldsymbol{\beta}}$ with $R^{(L)}_{\boldsymbol{\gamma\delta\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{4}R^{(L)}{}_{\boldsymbol{\gamma\delta}a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$. The constraint (5.228), i.e. $0 = R_{[\boldsymbol{\gamma\delta\alpha}]}{}^{\boldsymbol{\beta}} \propto R_{\boldsymbol{\gamma\delta\alpha}}{}^{\boldsymbol{\beta}} + 2R_{\boldsymbol{\alpha}[\boldsymbol{\gamma\delta}]}{}^{\boldsymbol{\beta}}$, therefore implies that $R_{\boldsymbol{\alpha}[\boldsymbol{\gamma\delta}]}{}^{\boldsymbol{\beta}}$ is Lie algebra valued in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as well. This means in particular that $R_{\boldsymbol{\alpha}[\boldsymbol{\gamma\delta}]}{}^{\boldsymbol{\beta}}\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} = 0$. Let us finally give the trace (in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$) of (5.228) and its hatted equivalent (5.231):

$$\begin{aligned} 0 &= R_{[\boldsymbol{\gamma\delta\alpha}]}{}^{\boldsymbol{\alpha}} && (5.232)\\ &= \frac{1}{2}F^{(D)}_{[\boldsymbol{\gamma\delta}}\delta_{\boldsymbol{\alpha}]}{}^{\boldsymbol{\alpha}} + R^{(L)}_{[\boldsymbol{\gamma\delta\alpha}]}{}^{\boldsymbol{\alpha}} = && (5.233)\\ &= -\frac{9}{3}F^{(D)}_{\boldsymbol{\gamma\delta}} + \frac{2}{3}R^{(L)}_{\boldsymbol{\alpha}[\boldsymbol{\gamma\delta}]}{}^{\boldsymbol{\alpha}} && (5.234)\end{aligned}$$

The scale curvature can be expressed in terms of the Lorentz curvature as

$$F^{(D)}_{\boldsymbol{\gamma\delta}} = \frac{2}{9}R^{(L)}_{\boldsymbol{\alpha}[\boldsymbol{\gamma\delta}]}{}^{\boldsymbol{\alpha}}, \quad \hat{F}^{(D)}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}} = \frac{2}{9}\hat{R}^{(L)}_{\hat{\boldsymbol{\alpha}}[\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}]}{}^{\hat{\boldsymbol{\alpha}}} \qquad (5.235)$$

Nilpotency on the single fields Just to get a flavour of how the calculation would work if we act on each field twice with the BRST differential, we perform this for the simplest cases. One discovers immediately that acting on x^K and $\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$ twice with the covariant BRST transformation yields zero. The reformulation of $\mathbf{s}^2_{\underline{cov}}$ in terms of the square of the ordinary differential $\mathbf{s}^2$ gives a torsion or a curvature term respectively. These terms have to vanish on-shell in order to have an on-shell vanishing $\mathbf{s}^2$:

$$0 = \mathbf{s}^2_{\underline{cov}} x^K = \underbrace{\mathbf{s}^2 x^K}_{\stackrel{!}{=}0\,(\mathrm{on-shell})} + 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{T}_{\boldsymbol{\alpha\beta}}{}^K\boldsymbol{\lambda}^{\boldsymbol{\beta}} \qquad \Rightarrow \boldsymbol{\lambda}^{\boldsymbol{\alpha}}\underline{T}_{\boldsymbol{\alpha\beta}}{}^K\boldsymbol{\lambda}^{\boldsymbol{\beta}} \stackrel{!}{=} 0 \quad (\mathrm{on-shell}) \qquad (5.236)$$

$$0 = \mathbf{s}^2_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \underbrace{(\mathbf{s}^2)_{cov}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}}_{\stackrel{!}{=}0\,(\mathrm{on-shell})} + 2\boldsymbol{\lambda}^{\boldsymbol{\gamma}}\boldsymbol{\lambda}^{\boldsymbol{\delta}}R_{\boldsymbol{\gamma\delta\beta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \qquad \Rightarrow \boldsymbol{\lambda}^{\boldsymbol{\gamma}}\boldsymbol{\lambda}^{\boldsymbol{\delta}}R_{\boldsymbol{\gamma\delta\beta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\lambda}^{\boldsymbol{\beta}} \stackrel{!}{=} 0 \quad (\mathrm{on-shell}) \qquad (5.237)$$

On the antighosts we have $\mathbf{s}^2_{cov}\boldsymbol{\omega}_{z\boldsymbol{\alpha}} = \mathbf{s}_{cov}d_{z\boldsymbol{\alpha}}$ which will not vanish, but which will correspond to a gauge transformation. The same should be true for $L_{z\bar{z}a}$. The calculation of $\mathbf{s}^2 d_{z\boldsymbol{\gamma}}$ is quite involved to calculate and will probably contain also constraints that follow from the earlier ones via Bianchi identities. We will calculate the identities anyway in sections 5.B on page 91 and 5.C on page 100.

5.11 Residual shift-reparametrization

Before we are going to collect all the constraints on the background fields which we have obtained so far, let us eventually make use of the residual shift-symmetry discussed in the paragraph on page 47 (which in turn refers to the paragraph about shift-reparametrization on page 46). It is a target space symmetry that is based on a residual shift reparametrization of the fermionic momenta:

$$d_{z\boldsymbol{\alpha}} = \tilde{d}_{z\boldsymbol{\alpha}} - \Xi^{(3)}{}_b{}^{\boldsymbol{\delta}}(\vec{x})(\gamma^b\boldsymbol{\lambda})_{\boldsymbol{\alpha}}\boldsymbol{\omega}_{z\boldsymbol{\delta}} \tag{5.238}$$

The BRST current gets changed under this reparametrization by a linear combination of the pure spinor constraints (5.43), but this change can be undone by a redefinition of the BRST transformations with the corresponding antighost gauge transformations. This does of course not change the on-shell holomorphicity of the BRST current, as the pure spinor term vanishes on-shell.

Apart from the change of the BRST current, we have the following induced transformations of the background fields coming along with this reparametrization:

$$\tilde{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} - E_M{}^{\boldsymbol{\gamma}}\gamma^b_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}} \tag{5.239}$$

$$\tilde{C}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} - \gamma^b_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \tag{5.240}$$

$$\tilde{S}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} = S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}\gamma^b_{\boldsymbol{\gamma\alpha}}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}} \tag{5.241}$$

Note that the transformations of $C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}$ and $S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}$ are in agreement with the holomorphicity constraints (5.184) and (5.189), relating them to $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$. It is thus enough to memorize the transformation of the connection $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$. Remember now the definition of the torsion as $\underline{T}^A = \mathbf{d}E^A - E^B\wedge\underline{\Omega}_B{}^A$. This implies the following transformation of the corresponding torsion component (see also (F.66) in the appendix on page 193):

$$\tilde{T}_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}} = T_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}} - \gamma^b_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}\Xi^{(3)}{}_b{}^{\boldsymbol{\beta}} \tag{5.242}$$

Due to the nilpotency constraints we have $T_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}} \propto \gamma^b_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}$. In addition, the left-right symmetry of proposition 3 on page 44 induces the same statements for $\hat{T}_{\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}{}^{\hat{\boldsymbol{\beta}}}$ and the second residual shift symmetry related to the reparametrization of $\hat{d}_{\hat{\boldsymbol{\gamma}}}$. We can therefore completely fix the two residual gauge symmetries by choosing the (obviously accessible) gauge

$$\boxed{T_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} = 0, \qquad \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = 0} \tag{5.243}$$

We can now immediately take advantage of this additional (conventional) constraint and check the validity of the constraints (5.192) and (5.193) on page 65.

5.12 Further discussion of some selected constraints

There are some constraints which deserve further examination, before we move on to study the Bianchi identities. First, the four constraints (5.192), (5.193) and their hatted versions on page 65 do not look very useful as they stand. We will show that they are actually consequences of other constraints. Second, with (5.188) and (5.189) we have two equations for $S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}}$ and it is interesting to know whether they are equivalent or not. Let us start with this last problem:

Consistency of (5.188) and (5.189) In the following we will (actually just for convenience) frequently use the new conventional constraint $T_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} = 0 = \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}$ (5.243). Starting with (5.188), the tensor of interest is given by

$$\begin{aligned} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} &\overset{(5.188)}{\underset{(5.184)}{=}} -\underline{\nabla}_{\boldsymbol{\alpha}}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + 2\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = \\ &\overset{(F.28)}{=} -\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + 2\underbrace{\underline{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^D}_{=0\,(5.183)}\underline{\nabla}_D\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}} - 2R_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}\boldsymbol{\delta}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} - 2\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}} + 2\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} \end{aligned} \tag{5.244}$$

In order for this to be compatible with (5.189), i.e. with

$$S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} \overset{(5.189)}{\underset{(5.185)}{=}} -\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + 2R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}} \tag{5.245}$$

the curvature has to obey

$$R_{\hat{\boldsymbol{\alpha}}[\boldsymbol{\alpha\delta}]}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} - \hat{R}_{\boldsymbol{\alpha}[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}]}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}} = 0 \tag{5.246}$$

In fact, this condition will be a simple consequence of the torsion Bianchi identities that we will obtain in (5.595) and (5.596).

Check of (5.192) The constraint (5.192) contains the covariant derivative of $C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}$ for which we can use in turn the constraint (5.184) together with our new constraint (5.243).

$$\begin{aligned} &\underline{\nabla}_{[\boldsymbol{\alpha}_2} C_{\boldsymbol{\alpha}_1]}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} - 2R_{[\boldsymbol{\alpha}_2|\boldsymbol{\delta}|\boldsymbol{\alpha}_1]}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} = \\ \overset{(5.184)}{=} \quad &\underline{\nabla}_{[\boldsymbol{\alpha}_2}\underline{\nabla}_{\boldsymbol{\alpha}_1]}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} - 2R_{[\boldsymbol{\alpha}_2|\boldsymbol{\delta}|\boldsymbol{\alpha}_1]}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} = \\ \overset{(F.28)}{=} \quad &-\underline{T}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}{}^{D}\underline{\nabla}_D\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} + 3\underbrace{R_{[\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1\boldsymbol{\delta}]}{}^{\boldsymbol{\beta}}}_{=0\,(5.228)}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + \underbrace{\hat{R}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}}_{=0\,(5.187),(5.226)}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}} \end{aligned} \tag{5.247}$$

Only the first term remains, but recalling the nilpotency constraint (5.221) in combination with (5.194), we observe that also this term vanishes, when contracted with $\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5}$. The constraint (5.192) therefore does not give new information and will be omitted in future listings. The same is true of course for its hatted version due to the left-right symmetry.

Relating (5.193) to a Bianchi identity For the constraint (5.193) we have to consider the following combination

$$\begin{aligned} &\underline{\nabla}_{[\boldsymbol{\alpha}_2} S_{\boldsymbol{\alpha}_1]\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} - 2\hat{R}_{[\boldsymbol{\alpha}_1|\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}C_{|\boldsymbol{\alpha}_2]}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} + 2R_{[\boldsymbol{\alpha}_2|\boldsymbol{\delta}|\boldsymbol{\alpha}_1]}{}^{\boldsymbol{\beta}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}} = \\ \underset{(5.184)}{\overset{(5.188)}{=}} \quad &-\underline{\nabla}_{[\boldsymbol{\alpha}_2|}\left(\underline{\nabla}_{|\boldsymbol{\alpha}_1]}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} - 2\hat{R}_{|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\right) - 2\hat{R}_{[\boldsymbol{\alpha}_1|\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\underline{\nabla}_{|\boldsymbol{\alpha}_2]}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} + 2R_{[\boldsymbol{\alpha}_2|\boldsymbol{\delta}|\boldsymbol{\alpha}_1]}{}^{\boldsymbol{\beta}}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}} = \\ \overset{(F.28)}{=} \quad &\underline{T}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}{}^{C}\underline{\nabla}_C\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + \underbrace{\hat{R}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}}_{=0\,(5.187),(5.226)}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} - \underbrace{\hat{R}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\beta}}}}_{=0\,(5.187),(5.226)}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} + \\ &+2\underline{\nabla}_{[\boldsymbol{\alpha}_2|}\hat{R}_{|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} + \underbrace{2R_{[\boldsymbol{\alpha}_2\boldsymbol{\delta}\boldsymbol{\alpha}_1]}{}^{\boldsymbol{\beta}}}_{=0\,(5.228)}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} = \\ = \quad &\underline{T}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}{}^{C}\underline{\nabla}_C\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} + 2\underline{\nabla}_{[\boldsymbol{\alpha}_2|}\hat{R}_{|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} \end{aligned} \tag{5.248}$$

The first term vanishes again when contracted with $\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5}$ ((5.221) and (5.194)) and the constraint (5.193) reduces to

$$\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5}\underline{\nabla}_{[\boldsymbol{\alpha}_2|}\hat{R}_{|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = 0 \tag{5.249}$$

We will see in a second that this equation is automatically fulfilled when the Bianchi identity for the curvature is fulfilled. We will study the Bianchi identities at a later point, but not all of those for the curvature, because we intend to make use of Dragon's theorem, relating second to first Bianchi identity. Let us therefore write down at this point the Bianchi identity that we have in mind (see (F.52) on page 192):

$$\begin{aligned} 0 \quad &\overset{!}{=} \quad \underline{\nabla}_{[\boldsymbol{\alpha}_2|}\hat{R}_{|\boldsymbol{\alpha}_1\hat{\boldsymbol{\gamma}}]\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + 2\underline{T}_{[\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1|}{}^{D}\hat{R}_{D|\hat{\boldsymbol{\gamma}}]\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \\ &= \quad \frac{2}{3}\underline{\nabla}_{[\boldsymbol{\alpha}_2|}\hat{R}_{|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{3}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\underbrace{\hat{R}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}}_{=0\,(5.187),(5.226)} + \frac{4}{3}\underbrace{\underline{T}_{\hat{\boldsymbol{\gamma}}[\boldsymbol{\alpha}_2|}{}^{D}}_{=0\,(5.183)}\hat{R}_{D|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{2}{3}\underline{T}_{\boldsymbol{\alpha}_2\boldsymbol{\alpha}_1}{}^{D}\hat{R}_{D\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{aligned} \tag{5.250}$$

Once again the last torsion term vanishes when contracted with $\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5}$, so that the above Bianchi identity implies

$$\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5}\underline{\nabla}_{[\boldsymbol{\alpha}_2|}\hat{R}_{|\boldsymbol{\alpha}_1]\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.251}$$

which is even stronger than (5.249). Of course we also get a hatted version of this constraint.

5.13 BI's & Collected constraints

The next step ist to study all the Bianchi identities. The logic is as follows: We have obtained certain constraints on the H-field, on the torsion and on the curvature. As these objects are defined in terms of B-field, vielbein and connection via $H = \mathbf{d}B$, $T^A = \mathbf{d}E^A - E^B \wedge \Omega_B{}^A$ and $R_A{}^B = \mathbf{d}\Omega_A{}^B - \Omega_A{}^C \wedge \Omega_C{}^A$, the constraints can be seen as differential equations for the elementary fields. If one solved these equations and calculated again H-field, torsion and curvature, one would observe additional constraints that one had not seen in the beginning. Solving the differential equations is a very hard problem, but the additional constraints on the derived objects (H-field, torsion and curvature) can be obtained by the Bianchi identites, without knowing the explicit solutions for the elementary fields. Indeed the Bianchi identities can help to derive the solutions. Depending on the point of view, the identities are a direct consequence of either the nilpotency of the de Rham differential $\mathbf{d}^2 = 0$ (see appendix F on page 189) or of the Jacobi identity for the commutator. Their explicit form, using the schematic index notation of 147, reads:

$$\underline{\nabla}_{\boldsymbol{A}} H_{\boldsymbol{AAA}} + 3\underline{T}_{\boldsymbol{AA}}{}^{C} H_{C\boldsymbol{AA}} \stackrel{!}{=} 0 \tag{5.252}$$

$$\underline{\nabla}_{\boldsymbol{A}} \underline{T}_{\boldsymbol{AA}}{}^{D} + 2\underline{T}_{\boldsymbol{AA}}{}^{C} \underline{T}_{C\boldsymbol{A}}{}^{D} \stackrel{!}{=} \underline{R}_{\boldsymbol{AAA}}{}^{D} \tag{5.253}$$

$$\underline{\nabla}_{\boldsymbol{A}} \underline{R}_{\boldsymbol{AA}B}{}^{C} + 2\underline{T}_{\boldsymbol{AA}}{}^{D} \underline{R}_{D\boldsymbol{A}B}{}^{C} \stackrel{!}{=} 0 \tag{5.254}$$

Repeated bold indices at the same altitude are simply antisymmetrized ones. Dragon's theorem (see page 197) tells us that – when the torsion Bianchi identity is fulfilled – we can replace the curvature Bianchi identity by the weaker condition

$$\begin{aligned} &\underline{R}_{\boldsymbol{CC}B}{}^{A} \underline{T}_{\boldsymbol{CC}}{}^{B} = \\ &= \underline{\nabla}_{\boldsymbol{C}} \underline{\nabla}_{\boldsymbol{C}} \underline{T}_{\boldsymbol{CC}}{}^{A} + \underline{T}_{\boldsymbol{CC}}{}^{D} \underline{\nabla}_{D} \underline{T}_{\boldsymbol{CC}}{}^{A} + 2\left(\underline{\nabla}_{\boldsymbol{C}} \underline{T}_{\boldsymbol{CC}}{}^{B} + 2\underline{T}_{\boldsymbol{CC}}{}^{D} \underline{T}_{D\boldsymbol{C}}{}^{B}\right) \underline{T}_{B\boldsymbol{C}}{}^{A} \end{aligned} \tag{5.255}$$

We will anyway concentrate on the Bianchi identities for H-field and torsion, because they provide most directly useful new algebraic constraints.

Note that all constraints so far were obtained for objects based on $\underline{\Omega}_{MA}{}^{B} = \mathrm{diag}\,(\check{\Omega}_{Ma}{}^{b}, \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}})$, the mixed connection defined in (5.66) on page 50. It contains three a priori independent blocks which all decay further in a Lorentz and a scale connection. One of the important results from the study of the Bianchi identities is that the torsion components $\check{T}_{\boldsymbol{\alpha\beta}}{}^{c}$ and $\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{c}$ are related to $\gamma^{c}_{\boldsymbol{\alpha\beta}}$ and $\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$ respectively by a Lorentz plus scale transformation. It is discussed in an intermezzo on page 92 (and was also used in Berkovit's and Howe's original work [13]) that this can be used to fix two of the three independent blocks. One is thus left with one independent copy of Lorentz plus scale which should leave invariant $\gamma^{c}_{\boldsymbol{\alpha\beta}}$ and $\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$. After this partial gauge fixing, the mixed connection is not an appropriate choice any longer, as it does not in general respect the gauge. We therefore introduce three alternative connections, namely the **left-mover connection** (defined by $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and invariance of the gamma-matrices), the **right-mover connection** (defined by $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ and invariance of the gamma-matrices) and the **average connection** (see beginning of appendix G on page 199 for more details)

$$\Omega_{MA}{}^{B} \equiv \mathrm{diag}\,(\Omega_{Ma}{}^{b}, \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}), \quad \nabla_{M}\gamma^{c}_{\boldsymbol{\alpha\beta}} = \nabla_{M}\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0 \tag{5.256}$$

$$\hat{\Omega}_{MA}{}^{B} \equiv \mathrm{diag}\,(\hat{\Omega}_{Ma}{}^{b}, \hat{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}), \quad \hat{\nabla}_{M}\gamma^{c}_{\boldsymbol{\alpha\beta}} = \hat{\nabla}_{M}\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0 \tag{5.257}$$

$$\underset{\leftrightarrow}{\Omega}_{MA}{}^{B} \equiv \frac{1}{2}\left(\Omega_{MA}{}^{B} + \hat{\Omega}_{MA}{}^{B}\right) \tag{5.258}$$

In addition we define the **difference tensor**

$$\Delta_{MA}{}^{B} \equiv \hat{\Omega}_{MA}{}^{B} - \Omega_{MA}{}^{B} = \mathrm{diag}\,(\Delta_{Ma}{}^{b}, \Delta_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \Delta_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}) \tag{5.259}$$

The Bianchi identities (5.252)-(5.254) should of course also hold when all objects are based on the above newly defined connections. This does not put restrictions on $\Delta_{MA}{}^{B}$. All different versions (based on different connections) of the Bianchi identities will lead to equivalent information (see proposition 7 on page 193). As they are most conveniently written down in terms of the mixed connection, we will follow this path. Only the bosonic block $\check{\Omega}_{Ma}{}^{b}$ will, depending on possible simplifications, be chosen to coincide with either the left-mover connection $\Omega_{Ma}{}^{b}$ or the right-mover connection $\hat{\Omega}_{Ma}{}^{b}$. The corresponding calculations are lengthy and mostly not very elluminating, so we put them into the local appendices at the end of this part of the thesis. There we first start with collecting all constraints that we have derived so far in appendix 5.A on page 88 and then discuss the Bianchi identities in detail starting from page 91. Some conceptually more interesting discussions within these appendicies are seperated in intermezzi. The first intermezzo on page 92 is, as already mentioned, about the fixing of two of the three copies of Lorentz plus scale transformations. The next on page 97 is about how to determine the complete difference tensor from the obtained constraints. There is finally a third intermezzo on page 104 which discusses the relation between constraints on the RR-bispinors and constraints (or equations of motion) for the corresponding p-forms.

After all this work in the local appendices, we will now collect all the constraints on the background fields that we have obtained, including the ones from the Bianchi identities. If we later, within the derivation of the supergravity transformations of some component fields, make use of some explicit form of components of torsion, curvature or other background fields without giving the explicit equation number, the corresponding equation should be among the following ones.

Not all equations that we are going to write are independent. It is sometimes convenient to have them in different versions and grouped in different ways. In particular we will give for later convenience the explicit form of the torsion components based on left-mover, right-mover and average connection, although this contains a lot of redundancy.

Restricted structure group constraints The first set of constraints is related to the restriction of the structure group (of the supermanifold) to a a block diagonal form with three copies of Lorentz and scale transformations. This was discussed in a paragraph on pages 50-48, in the remark on page 53 and in the intermezzo on page 61. The following equations are taken from (5.94)-(5.96), (5.152) or (5.154) and (5.159)

$$\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}\Omega^{(D)}_M\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{2}\hat{\Omega}^{(D)}_M\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{\Omega}^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{5.260}$$

$$C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = \frac{1}{2}C^{\hat{\boldsymbol{\gamma}}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}C^{\hat{\boldsymbol{\gamma}}}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \qquad \hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} = \frac{1}{2}\hat{C}^{\boldsymbol{\gamma}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{C}^{\boldsymbol{\gamma}}_{a_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{5.261}$$

$$\begin{aligned} S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} &= \frac{1}{4}S\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{8}S_{a_1a_2}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \\ &\quad + \frac{1}{8}\hat{S}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{16}S_{a_1a_2b_1b_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\gamma^{b_1b_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{aligned} \tag{5.262}$$

$$G_{MN} = E_M{}^aG_{ab}E_N{}^b, \qquad G_{ab} = e^{2\Phi}\eta_{ab} \tag{5.263}$$

The above equations (without the last one) are equivalent to

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.264}$$

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} = 0 \tag{5.265}$$

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} = 0 \tag{5.266}$$

Further constraints on C and S and indirectly on $\mathcal{P}$ The constraints (5.184) and (5.185) on C and (5.188) and (5.189) on S (all on page 65) can be regarded as defining equations. We have already shown in section 5.12 that the two equations for S are equivalent up to Bianchi identities.

$$C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} = \underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \tag{5.267}$$

$$\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}\boldsymbol{\gamma}} = \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \tag{5.268}$$

$$S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}} = -\underline{\nabla}_{\boldsymbol{\alpha}}\underbrace{\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}}_{\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}}} + 2\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \tag{5.269}$$

$$S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = -\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\underbrace{C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}}_{\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}} + 2R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \tag{5.270}$$

In addition we have from the Bianchi identities the equations (5.637) and (5.638):

$$\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = -\frac{1}{2}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\cdot\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \left(T_{cd}{}^{\boldsymbol{\alpha}} - \frac{1}{2}\hat{\nabla}_{\boldsymbol{\gamma}}\Phi\mathcal{P}^{\boldsymbol{\alpha}\boldsymbol{\delta}}\tilde{\gamma}_{cd\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}\right)\tilde{\gamma}^{cd}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{5.271}$$

$$\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}} = -\frac{1}{2}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\gamma}}\Phi\cdot\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \left(\hat{T}_{cd}{}^{\hat{\boldsymbol{\alpha}}} - \frac{1}{2}\nabla_{\boldsymbol{\gamma}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\tilde{\gamma}_{cd\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}\right)\tilde{\gamma}^{cd}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{5.272}$$

In the intermezzo on page 104 we give a qualitative discussion how these equations are related to field equations for the corresponding RR-p-form-field-strengths. The above expressions for the spinorial derivatives of the RR-bispinors (which coincide with C and $\hat{C}$ according to (5.267) and (5.268)) already take into account the restricted structure group according to (5.261). In addition they imply upon taking the trace that

$$\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}} = 8\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi \quad \text{or} \quad \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}(e^{-8\Phi}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}) = 0 \tag{5.273}$$

$$\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}} = 8\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\alpha}}\Phi \quad \text{or} \quad \underline{\nabla}_{\boldsymbol{\alpha}}(e^{-8\Phi}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}) = 0 \tag{5.274}$$

Constraints on H Due to (5.167)-(5.171), (5.226), (5.229), (5.476), (5.477) and the total antisymmetry of H, its only nonvanishing components are

$$H_{abc} \neq 0 \quad \text{(in general)} \tag{5.275}$$

$$H_{\boldsymbol{\alpha}\boldsymbol{\beta}c} = -\frac{2}{3}\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}} \equiv -\frac{2}{3}e^{2\Phi}\eta_{cd}\gamma^d_{\boldsymbol{\alpha}\boldsymbol{\beta}} \tag{5.276}$$

$$H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c} = \frac{2}{3}\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \equiv \frac{2}{3}e^{2\Phi}\eta_{cd}\gamma^d_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \tag{5.277}$$

The vanishing components are thus (written a bit redundantly)

$$H_{ab\boldsymbol{C}} = H_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}C} = H_{\boldsymbol{\mathcal{A}}\boldsymbol{B}\boldsymbol{C}} = 0 \tag{5.278}$$

Note that the constraints for $H_{\boldsymbol{\alpha\beta}c}$ and $H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c}$ (coming from (5.476) and (5.477)) are related to the torsion constraints for $\check{T}_{\boldsymbol{\alpha\beta}}{}^c$ and $\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c$ and thus (as mentioned in the beginning of this section) contain the gauge fixing of two of the three initially independent Lorentz and scale transformations (5.65). This is explained in detail at page 92.

Further conditions on H, coming from the Bianchi identities (5.580), (5.581) and (5.582), are

$$\nabla_{\hat{\boldsymbol{\delta}}}H_{abc} = -4\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}} \tag{5.279}$$

$$\hat{\nabla}_{\boldsymbol{\delta}}H_{abc} = 4T_{[ab|}{}^{\boldsymbol{\varepsilon}}\tilde{\gamma}_{|c]\boldsymbol{\varepsilon\delta}} \tag{5.280}$$

$$\nabla_{[a}H_{bcd]} = -\frac{9}{2}H_{[ab|}{}^eH_{e|cd]} \tag{5.281}$$

More information on the torsion components $T_{ab}{}^{\boldsymbol{\varepsilon}}$ and $\hat{T}_{ab}{}^{\hat{\boldsymbol{\varepsilon}}}$ will be given in the corresponding paragraph below.

Constraints on the torsion Let us now collect the information of the constraints (5.168)-(5.170), (5.180)-(5.183), (5.227), (5.230), (5.243) and the Bianchi identities (5.474), (5.475), (5.521), (5.522), (5.527), (5.528), (5.538), (5.539), (5.634) and (5.635). The only (a priori) nonvanishing components of the torsion $\underline{T}_{AB}{}^C$ are

$$\check{T}_{\boldsymbol{\alpha\beta}}{}^c = \gamma^c_{\boldsymbol{\alpha\beta}}, \qquad \check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c = \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \tag{5.282}$$

$$T_{\boldsymbol{\alpha}b}{}^c = -\frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\delta^c_b - \frac{1}{2}\gamma_b{}^c{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\nabla_{\boldsymbol{\beta}}\Phi, \qquad \hat{T}_{\hat{\boldsymbol{\alpha}}b}{}^c = -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta^c_b - \frac{1}{2}\gamma_b{}^c{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi \tag{5.283}$$

$$T_{ab}{}^c = \frac{3}{2}H_{ab}{}^c, \qquad \hat{T}_{ab}{}^c = -\frac{3}{2}H_{ab}{}^c \tag{5.284}$$

$$T_{\hat{\boldsymbol{\alpha}}c}{}^{\boldsymbol{\gamma}} = \tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}, \qquad \hat{T}_{\boldsymbol{\alpha}c}{}^{\hat{\boldsymbol{\gamma}}} = \tilde{\gamma}_{c\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} \tag{5.285}$$

$$T_{ab}{}^{\boldsymbol{\gamma}} = \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ab\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}, \qquad \hat{T}_{ab}{}^{\hat{\boldsymbol{\gamma}}} = \frac{1}{16}\left(\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + 8\nabla_{\boldsymbol{\gamma}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\tilde{\gamma}_{ab\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}} \tag{5.286}$$

The remaining components vanish, which can be written (again a bit redundantly) as

$$\underline{T}_{\boldsymbol{\mathcal{AB}}}{}^{\boldsymbol{\mathcal{C}}} = \underline{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^C = T_{\boldsymbol{\alpha}d}{}^{\boldsymbol{\gamma}} = \hat{T}_{\hat{\boldsymbol{\alpha}}d}{}^{\hat{\boldsymbol{\gamma}}} = T_{\hat{\boldsymbol{\alpha}}b}{}^c = \hat{T}_{\boldsymbol{\alpha}b}{}^c = 0 \tag{5.287}$$

We obtain some additional constraints from the Bianchi identities (5.701), (5.689), (5.705) and (5.706):

$$\nabla_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}} = -2\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - 3H_{bce}\gamma^e_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} \tag{5.288}$$

$$\hat{\nabla}_{\boldsymbol{\alpha}}\hat{T}_{bc}{}^{\hat{\boldsymbol{\delta}}} = -2\tilde{\gamma}_{[b|\,\boldsymbol{\alpha\delta}}\underline{\nabla}_{|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 3H_{bce}\gamma^e_{\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} \tag{5.289}$$

$$\nabla_{[a}T_{bc]}{}^{\boldsymbol{\delta}} = -3H_{[ab|}{}^eT_{e|c]}{}^{\boldsymbol{\delta}} - 2\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} \tag{5.290}$$

$$\hat{\nabla}_{[a}\hat{T}_{bc]}{}^{\hat{\boldsymbol{\delta}}} = 3H_{[ab|}{}^e\hat{T}_{e|c]}{}^{\hat{\boldsymbol{\delta}}} - 2T_{[ab|}{}^{\boldsymbol{\varepsilon}}\tilde{\gamma}_{|c]\,\boldsymbol{\varepsilon\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} \tag{5.291}$$

Difference tensor With the help of the constraints obtained from the Bianchi identities the explicit form (5.543)-(5.549) of the difference tensor is derived in the intermezzo on page 97. The components with bosonic structure group indices are given by

$$\Delta_{Ab}{}^c: \quad \Delta_{ab|c} = -3H_{abc} \tag{5.292}$$

$$\Delta_{\boldsymbol{\alpha}b|c} = \nabla_{\boldsymbol{\alpha}}\Phi G_{bc} + \tilde{\gamma}_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi \tag{5.293}$$

$$\Delta_{\hat{\boldsymbol{\alpha}}b|c} = -\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi G_{bc} - \tilde{\gamma}_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi \tag{5.294}$$

They determine the components with fermionic structure group indices to be of the form

$$\Delta_{A\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}: \quad \Delta_{a\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = -\frac{3}{4}H_{abc}\tilde{\gamma}^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} \quad , \qquad \Delta_{a\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = -\frac{3}{4}H_{abc}\tilde{\gamma}^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.295}$$

$$\Delta_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} = \frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} + \frac{1}{4}\gamma_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi\gamma^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\,, \quad \Delta_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} - \frac{1}{4}\gamma_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\gamma^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.296}$$

$$\Delta_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} - \frac{1}{4}\gamma_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\gamma^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\,, \quad \Delta_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = \frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} + \frac{1}{4}\gamma_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi\gamma^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.297}$$

The above equations imply in particular for the scale part (via taking the trace)

$$\Rightarrow \Delta^{(D)}_a = 0 \tag{5.298}$$

$$\Delta^{(D)}_{\boldsymbol{\alpha}} = \nabla_{\boldsymbol{\alpha}}\Phi \tag{5.299}$$

$$\Delta^{(D)}_{\hat{\boldsymbol{\alpha}}} = -\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi \tag{5.300}$$

As we meet here the covariant derivatives of the compensator field, it is useful to add at this place also the constraints (5.527),(5.528) and (5.529) on the covariant derivative of the compensator field coming from the Bianchi identities:

$$\nabla_{\hat{\boldsymbol{\alpha}}}\Phi = \hat{\nabla}_{\boldsymbol{\alpha}}\Phi = \nabla_a\Phi = \hat{\nabla}_a\Phi = 0 \tag{5.301}$$

Remember that the covariant derivative of the compensator field is given by $\nabla_A\Phi = E_A{}^M(\partial_M\Phi - \Omega_M^{(D)})$.

Torsion constraints rewritten in various ways Due to the explicit knowledge of the difference tensor, we can write down all components of $T_{AB}{}^C$, $\hat{T}_{AB}{}^C$ and $\underset{\longleftrightarrow}{T}{}_{AB}{}^C$ (using e.g. $T_{AB}{}^{\hat{\boldsymbol{\gamma}}} = \hat{T}_{AB}{}^{\hat{\boldsymbol{\gamma}}} - \Delta_{[AB]}{}^{\hat{\boldsymbol{\gamma}}}$). They will be needed to derive the supersymmetry transformations in the corresponding gauge. Before we start, let us stress once more that the scale transformations (or dilatations) are still part of our superspace structure group. If one prefers to fix the compensator field Φ to zero immediately (which would correspond to [13]), one needs to restrict to the Lorentz part $\Omega_{MA}^{(L)}{}^B$, $\hat{\Omega}_{MA}^{(L)}{}^B$ or $\underset{\longleftrightarrow}{\Omega}{}_{MA}^{(L)}{}^B$ of the corresponding connection. The Lorentz part of the torsion can be obtained via

$$T_{AB}^{(L)C} \quad = \quad T_{AB}{}^C - \Omega_{[AB]}^{(D)}{}^C \quad \text{with } \Omega_{Mb}^{(D)c} = \Omega_M^{(D)}\delta_b^c \text{ and } \Omega_{M\boldsymbol{\mathcal{B}}}^{(D)\boldsymbol{\mathcal{C}}} = \tfrac{1}{2}\Omega_M^{(D)}\delta_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} \tag{5.302}$$

This will be made more explicit below for each case.

Let us now start with the *left-mover torsion*, whose components $T_{AB}{}^C$ are

$$T_{AB}{}^c \equiv \begin{pmatrix} T_{ab}{}^c & T_{a\boldsymbol{\beta}}{}^c & T_{a\hat{\boldsymbol{\beta}}}{}^c \\ T_{\boldsymbol{\alpha}b}{}^c & T_{\boldsymbol{\alpha\beta}}{}^c & T_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^c \\ T_{\hat{\boldsymbol{\alpha}}b}{}^c & T_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^c & T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c \end{pmatrix} = \begin{pmatrix} \frac{3}{2}H_{ab}{}^c & \frac{1}{2}\nabla_{\boldsymbol{\beta}}\Phi\delta_a^c + \frac{1}{2}\gamma_a{}^c{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi & 0 \\ -\frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\delta_b^c - \frac{1}{2}\gamma_b{}^c{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi & \gamma^c_{\boldsymbol{\alpha\beta}} & 0 \\ 0 & 0 & \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.303}$$

$$T_{AB}{}^{\boldsymbol{\gamma}} \equiv \begin{pmatrix} T_{ab}{}^{\boldsymbol{\gamma}} & T_{a\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & T_{a\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \\ T_{\boldsymbol{\alpha}b}{}^{\boldsymbol{\gamma}} & T_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} & T_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \\ T_{\hat{\boldsymbol{\alpha}}b}{}^{\boldsymbol{\gamma}} & T_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \end{pmatrix} = \begin{pmatrix} \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ab\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\varepsilon}}} & 0 & -\tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} \\ 0 & 0 & 0 \\ \tilde{\gamma}_{b\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} & 0 & 0 \end{pmatrix} \tag{5.304}$$

$$T_{AB}{}^{\hat{\boldsymbol{\gamma}}} \equiv \begin{pmatrix} T_{ab}{}^{\hat{\boldsymbol{\gamma}}} & T_{a\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & T_{a\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ T_{\boldsymbol{\alpha}b}{}^{\hat{\boldsymbol{\gamma}}} & T_{\boldsymbol{\alpha\beta}}{}^{\hat{\boldsymbol{\gamma}}} & T_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ T_{\hat{\boldsymbol{\alpha}}b}{}^{\hat{\boldsymbol{\gamma}}} & T_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \end{pmatrix} = \tag{5.305}$$

$$\begin{pmatrix} \frac{1}{16}\left(\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + 8\nabla_{\boldsymbol{\gamma}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\tilde{\gamma}_{ab\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}} & -\tilde{\gamma}_{a\,\boldsymbol{\beta\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & \frac{3}{8}H_{ade}\tilde{\gamma}^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ \tilde{\gamma}_{b\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & 0 & (-\frac{1}{8}\gamma_{de\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}\nabla_{\boldsymbol{\delta}}\Phi - \frac{1}{4}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}) \\ -\frac{3}{8}H_{bde}\tilde{\gamma}^{de}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}} & (\frac{1}{8}\gamma_{de\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}\nabla_{\boldsymbol{\delta}}\Phi + \frac{1}{4}\nabla_{\boldsymbol{\beta}}\Phi\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}) & (\frac{1}{4}\gamma_{de\,[\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi + \frac{1}{2}\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}}\Phi\delta_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}}) \end{pmatrix}$$

If we want to extract the Lorentz part, only a few of the components change. Remember $\nabla_a\Phi = 0$ and $\nabla_{\hat{\boldsymbol{\alpha}}}\Phi = 0$ and assume only for this step that Φ was fixed to zero, which implies $\nabla_M\Phi \to -\Omega_M^{(D)}$ and thus $\Omega_a^{(D)} = 0$ and $\Omega_{\hat{\boldsymbol{\alpha}}}^{(D)} = 0$. According to (5.302) we then have

$$T_{\boldsymbol{\alpha}b}^{(L)c} \overset{\Phi=0}{=} T_{\boldsymbol{\alpha}b}{}^c - \tfrac{1}{2}\Omega_{\boldsymbol{\alpha}}^{(D)}\delta_b{}^c = \tfrac{1}{2}\gamma_b{}^c{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\Omega_{\boldsymbol{\delta}}^{(D)} \tag{5.306}$$

$$T_{\boldsymbol{\alpha\beta}}^{(L)\boldsymbol{\gamma}} = T_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} - \tfrac{1}{2}\Omega_{[\boldsymbol{\alpha}}^{(D)}\delta_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}} = -\tfrac{1}{2}\Omega_{[\boldsymbol{\alpha}}^{(D)}\delta_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}} \tag{5.307}$$

$$T_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}^{(L)\hat{\boldsymbol{\gamma}}} \overset{\Phi=0}{=} T_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} - \tfrac{1}{4}\Omega_{\boldsymbol{\alpha}}^{(D)}\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = \tfrac{1}{8}\gamma_{de\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}\Omega_{\boldsymbol{\delta}}^{(D)} \tag{5.308}$$

All other components of $T^{(L)}$ coincide with T for $\Phi = 0$ (and $\nabla_M\Phi \to -\Omega_M^{(D)}$).

The *right-mover torsion* components $\hat{T}_{AB}{}^C$ are

$$\hat{T}_{AB}{}^c \equiv \begin{pmatrix} \hat{T}_{ab}{}^c & \hat{T}_{a\boldsymbol{\beta}}{}^c & \hat{T}_{a\hat{\boldsymbol{\beta}}}{}^c \\ \hat{T}_{\boldsymbol{\alpha}b}{}^c & \hat{T}_{\boldsymbol{\alpha\beta}}{}^c & \hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^c \\ \hat{T}_{\hat{\boldsymbol{\alpha}}b}{}^c & \hat{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^c & \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}H_{ab}{}^c & 0 & \frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi\delta_a^c + \frac{1}{2}\gamma_a{}^c{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi \\ 0 & \gamma^c_{\boldsymbol{\alpha\beta}} & 0 \\ -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_b^c - \frac{1}{2}\gamma_b{}^c{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi & 0 & \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.309}$$

$$\hat{T}_{AB}{}^{\boldsymbol{\gamma}} \equiv \begin{pmatrix} \hat{T}_{ab}{}^{\boldsymbol{\gamma}} & \hat{T}_{a\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & \hat{T}_{a\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \\ \hat{T}_{\boldsymbol{\alpha}b}{}^{\boldsymbol{\gamma}} & \hat{T}_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} & \hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \\ \hat{T}_{\hat{\boldsymbol{\alpha}}b}{}^{\boldsymbol{\gamma}} & \hat{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \end{pmatrix} = \tag{5.310}$$

$$\begin{pmatrix} \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ab\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}} & -\frac{3}{8}H_{ade}\tilde{\gamma}^{de}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & -\tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} \\ \frac{3}{8}H_{bde}\tilde{\gamma}^{de}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}} & (\frac{1}{4}\gamma_{de[\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\delta}}\Phi + \frac{1}{2}\nabla_{[\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}}) & (\frac{1}{8}\gamma_{de\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi + \frac{1}{4}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}) \\ \tilde{\gamma}_{b\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} & (-\frac{1}{8}\gamma_{de\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi - \frac{1}{4}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}) & 0 \end{pmatrix}$$

$$\hat{T}_{AB}{}^{\hat{\boldsymbol{\gamma}}} \equiv \begin{pmatrix} \hat{T}_{ab}{}^{\hat{\boldsymbol{\gamma}}} & \hat{T}_{a\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & \hat{T}_{a\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ \hat{T}_{\boldsymbol{\alpha}b}{}^{\hat{\boldsymbol{\gamma}}} & \hat{T}_{\boldsymbol{\alpha\beta}}{}^{\hat{\boldsymbol{\gamma}}} & \hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ \hat{T}_{\hat{\boldsymbol{\alpha}}b}{}^{\hat{\boldsymbol{\gamma}}} & \hat{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \end{pmatrix} = \begin{pmatrix} \frac{1}{16}\left(\underline{\nabla}_{\boldsymbol{\varepsilon}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + 8\nabla_{\boldsymbol{\varepsilon}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\tilde{\gamma}_{ab\,\boldsymbol{\delta}}{}^{\boldsymbol{\varepsilon}} & -\tilde{\gamma}_{a\,\boldsymbol{\beta\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & 0 \\ \tilde{\gamma}_{b\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.311}$$

In order to extract the Lorentz part, remember $\hat{\nabla}_a\Phi = 0$ and $\hat{\nabla}_{\boldsymbol{\alpha}}\Phi = 0$. For $\Phi = 0$ ($\nabla_M\Phi \to -\Omega_M^{(D)}$) this implies $\hat{\Omega}_a^{(D)} = 0$ and $\hat{\Omega}_{\boldsymbol{\alpha}}^{(D)} = 0$. According to (5.302) we then have

$$\hat{T}^{(L)c}_{\hat{\boldsymbol{\alpha}}b} \overset{\Phi=0}{=} \hat{T}_{\hat{\boldsymbol{\alpha}}b}{}^c - \tfrac{1}{2}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\alpha}}}\delta_b{}^c = \tfrac{1}{2}\gamma_b{}^c{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\delta}}} \tag{5.312}$$

$$\hat{T}^{(L)\hat{\boldsymbol{\gamma}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} - \tfrac{1}{2}\hat{\Omega}^{(D)}_{[\hat{\boldsymbol{\alpha}}}\delta_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}} = -\tfrac{1}{2}\hat{\Omega}^{(D)}_{[\hat{\boldsymbol{\alpha}}}\delta_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.313}$$

$$\hat{T}^{(L)\boldsymbol{\gamma}}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} \overset{\Phi=0}{=} \hat{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} - \tfrac{1}{4}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\alpha}}}\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = \tfrac{1}{8}\gamma_{de\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\delta}}} \tag{5.314}$$

All other components of $\hat{T}^{(L)}$ coincide with $\hat{T}$ for $\Phi = 0$ (and $\nabla_M\Phi \to -\Omega_M^{(D)}$).

Finally we give the components of the *average torsion* $\underset{\leftrightarrow}{T}_{AB}{}^C \equiv \frac{1}{2}\left(T_{AB}{}^C + \hat{T}_{AB}{}^C\right)$:

$$\underset{\leftrightarrow}{T}_{AB}{}^c \equiv \begin{pmatrix} \underset{\leftrightarrow}{T}_{ab}{}^c & \underset{\leftrightarrow}{T}_{a\boldsymbol{\beta}}{}^c & \underset{\leftrightarrow}{T}_{a\hat{\boldsymbol{\beta}}}{}^c \\ \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}b}{}^c & \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^c & \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^c \\ \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}b}{}^c & \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^c & \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \frac{1}{4}\nabla_{\boldsymbol{\beta}}\Phi\delta_a^c + \frac{1}{4}\gamma_a{}^c{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi & \frac{1}{4}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi\delta_a^c + \frac{1}{4}\gamma_a{}^c{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi \\ -\frac{1}{4}\nabla_{\boldsymbol{\alpha}}\Phi\delta_b^c - \frac{1}{4}\gamma_b{}^c{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi & \gamma^c_{\boldsymbol{\alpha}\boldsymbol{\beta}} & 0 \\ -\frac{1}{4}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_b^c - \frac{1}{4}\gamma_b{}^c{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi & 0 & \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.315}$$

$$\underset{\leftrightarrow}{T}_{AB}{}^{\boldsymbol{\gamma}} \equiv \begin{pmatrix} \underset{\leftrightarrow}{T}_{ab}{}^{\boldsymbol{\gamma}} & \underset{\leftrightarrow}{T}_{a\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & \underset{\leftrightarrow}{T}_{a\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \\ \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}b}{}^{\boldsymbol{\gamma}} & \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \\ \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}b}{}^{\boldsymbol{\gamma}} & \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}} \end{pmatrix} = \tag{5.316}$$

$$\begin{pmatrix} \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ab\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\varepsilon}}} & -\frac{3}{16}H_{ade}\tilde{\gamma}^{de}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} & -\tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} \\ \frac{3}{16}H_{bde}\tilde{\gamma}^{de}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}} & (\frac{1}{8}\gamma_{de[\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\delta}}\Phi + \frac{1}{4}\nabla_{[\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}}) & (\frac{1}{16}\gamma_{de\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi + \frac{1}{8}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}) \\ \tilde{\gamma}_{b\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} & (-\frac{1}{16}\gamma_{de\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi - \frac{1}{8}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}) & 0 \end{pmatrix}$$

$$\underset{\leftrightarrow}{T}_{AB}{}^{\hat{\boldsymbol{\gamma}}} \equiv \begin{pmatrix} \underset{\leftrightarrow}{T}_{ab}{}^{\hat{\boldsymbol{\gamma}}} & \underset{\leftrightarrow}{T}_{a\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & \underset{\leftrightarrow}{T}_{a\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}b}{}^{\hat{\boldsymbol{\gamma}}} & \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}b}{}^{\hat{\boldsymbol{\gamma}}} & \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}} & \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \end{pmatrix} = \tag{5.317}$$

$$\begin{pmatrix} \frac{1}{16}\left(\underline{\nabla}_{\boldsymbol{\varepsilon}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + 8\nabla_{\boldsymbol{\varepsilon}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\tilde{\gamma}_{ab\,\boldsymbol{\delta}}{}^{\boldsymbol{\varepsilon}} & -\tilde{\gamma}_{a\,\boldsymbol{\beta}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & \frac{3}{16}H_{ade}\tilde{\gamma}^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \\ \tilde{\gamma}_{b\,\boldsymbol{\alpha}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & 0 & (-\frac{1}{16}\gamma_{de\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}\nabla_{\boldsymbol{\delta}}\Phi - \frac{1}{8}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}) \\ -\frac{3}{16}H_{bde}\tilde{\gamma}^{de}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}} & (\frac{1}{16}\gamma_{de\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}\nabla_{\boldsymbol{\delta}}\Phi + \frac{1}{8}\nabla_{\boldsymbol{\beta}}\Phi\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}) & (\frac{1}{8}\gamma_{de\,[\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi + \frac{1}{4}\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}}\Phi\delta_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}}) \end{pmatrix}$$

The unfortunate situation that neither $\underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}$ nor $\underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}$ vanish raises the question whether the conventional constraints $T_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = \hat{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = 0$ were a clever choice or better should be replaced by a constraint on the average torsion.

Once more, in order to extract the Lorentz part, we need (for $\Phi = 0$) the constraints $\underset{\leftrightarrow}{\Omega}{}_a^{(D)} = 0$, $\underset{\leftrightarrow}{\Omega}{}_{\boldsymbol{\alpha}}^{(D)} = \frac{1}{2}\Omega_{\boldsymbol{\alpha}}^{(D)}$ and $\underset{\leftrightarrow}{\Omega}{}_{\hat{\boldsymbol{\alpha}}}^{(D)} = \frac{1}{2}\hat{\Omega}_{\hat{\boldsymbol{\alpha}}}^{(D)}$. According to (5.302) we then have

$$\underset{\leftrightarrow}{T}{}^{(L)c}_{\boldsymbol{\alpha}b} \overset{\Phi=0}{=} \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}b}{}^c - \tfrac{1}{4}\Omega^{(D)}_{\boldsymbol{\alpha}}\delta_b{}^c = \tfrac{1}{4}\gamma_b{}^c{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\Omega^{(D)}_{\boldsymbol{\delta}} \tag{5.318}$$

$$\underset{\leftrightarrow}{T}{}^{(L)\boldsymbol{\gamma}}_{\boldsymbol{\alpha}\boldsymbol{\beta}} \overset{\Phi=0}{=} \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} - \tfrac{1}{4}\Omega^{(D)}_{[\boldsymbol{\alpha}}\delta_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}} = -\tfrac{1}{8}\gamma_{de[\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}}\Omega^{(D)}_{\boldsymbol{\delta}} - \tfrac{1}{2}\Omega^{(D)}_{[\boldsymbol{\alpha}}\delta_{\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}} \tag{5.319}$$

$$\underset{\leftrightarrow}{T}{}^{(L)\hat{\boldsymbol{\gamma}}}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} \overset{\Phi=0}{=} \underset{\leftrightarrow}{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} - \tfrac{1}{8}\Omega^{(D)}_{\boldsymbol{\alpha}}\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = \tfrac{1}{16}\gamma_{de\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}\Omega^{(D)}_{\boldsymbol{\delta}} \tag{5.320}$$

$$\underset{\leftrightarrow}{T}{}^{(L)c}_{\hat{\boldsymbol{\alpha}}b} \overset{\Phi=0}{=} \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}b}{}^c - \tfrac{1}{4}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\alpha}}}\delta_b{}^c = \tfrac{1}{4}\gamma_b{}^c{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\delta}}} \tag{5.321}$$

$$\underset{\leftrightarrow}{T}{}^{(L)\hat{\boldsymbol{\gamma}}}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \overset{\Phi=0}{=} \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} - \tfrac{1}{4}\hat{\Omega}^{(D)}_{[\hat{\boldsymbol{\alpha}}}\delta_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}} = -\tfrac{1}{8}\gamma_{de\,[\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\delta}}} - \tfrac{1}{2}\hat{\Omega}^{(D)}_{[\hat{\boldsymbol{\alpha}}}\delta_{\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.322}$$

$$\underset{\leftrightarrow}{T}{}^{(L)\boldsymbol{\gamma}}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} \overset{\Phi=0}{=} \underset{\leftrightarrow}{T}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} - \tfrac{1}{8}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\alpha}}}\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = \tfrac{1}{16}\gamma_{de\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{de}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\delta}}} \tag{5.323}$$

The remaining components of $\underset{\leftrightarrow}{T}{}^{(L)}$ coincide with $\underset{\leftrightarrow}{T}$ for $\Phi = 0$ (and $\nabla_M\Phi \to -\Omega_M^{(D)}$).

Constraints on the curvature Induced by the restricted structure group constraints on the connection, we have such constraints likewise for the curvature (see (5.68) on page 50 and (F.88),(F.90) and (F.92) on page

F.90. The curvature is blockdiagonal and each part decays into a scale part and a Lorentz part:

$$\underline{R}_{ABC}{}^{D} = \mathrm{diag}\,(\check{R}_{ABc}{}^{d}, R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}, \hat{R}_{AB\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}) \tag{5.324}$$

$$\check{R}_{ABc}{}^{d} = \check{F}^{(D)}_{AB}\delta^{d}_{c} + \check{R}^{(L)}_{AB\,c}{}^{d}, \qquad \check{F}^{(D)}_{AB} = \frac{1}{10}\check{R}_{ABc}{}^{c} \tag{5.325}$$

$$R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \frac{1}{2}F^{(D)}_{AB}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \frac{1}{4}R^{(L)}_{ABa_1}{}^{b}\eta_{ba_2}\gamma^{a_1a_2}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}, \qquad F^{(D)}_{AB} = -\frac{1}{8}R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\gamma}} \tag{5.326}$$

$$\hat{R}_{AB\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{2}\hat{F}^{(D)}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{R}^{(L)}_{ABa_1}{}^{b}\eta_{ba_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}, \qquad \hat{F}^{(D)}_{AB} = -\frac{1}{8}\hat{R}_{AB\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.327}$$

with the scale field strength

$$\check{F}^{(D)} \equiv \mathbf{d}\check{\Omega}^{(D)}, \qquad F^{(D)} \equiv \mathbf{d}\Omega^{(D)}, \qquad \hat{F}^{(D)} \equiv \mathbf{d}\hat{\Omega}^{(D)} \tag{5.328}$$

The bosonic field strength is also obtained via the commutator of covariant derivatives acting on the compensator field Φ. Only the bosonic block $\check{\Omega}_{Ma}{}^{b}$ of the mixed connection $\underline{\Omega}_{MA}{}^{B}$ acts on Φ, because Φ is a compensator for the transformation of $G_{ab} = e^{2\Phi}\eta_{ab}$ (with bosonic indices only). But as the different blocks of the structure group got related by partial gauge fixing, we may as well act with the left- or right-mover connection on it:

$$\check{F}^{(D)}_{MN} = -\underline{\nabla}_{[M}\check{\nabla}_{N]}\Phi - \underline{T}_{MN}{}^{K}\check{\nabla}_{K}\Phi \tag{5.329}$$

$$F^{(D)}_{MN} = -\nabla_{[M}\nabla_{N]}\Phi - T_{MN}{}^{K}\nabla_{K}\Phi \tag{5.330}$$

$$\hat{F}^{(D)}_{MN} = -\hat{\nabla}_{[M}\hat{\nabla}_{N]}\Phi - \hat{T}_{MN}{}^{K}\hat{\nabla}_{K}\Phi \tag{5.331}$$

Finallly we collect the holomorphicity (5.186),(5.187),(5.190),(5.191) and nilpotency constraints (5.228),(5.231) on the curvature, together with the Bianchi identities (5.586), (5.587), (5.595), (5.596), (5.609), (5.610), (5.689) and (5.690):

$$R_{\hat{\boldsymbol{\alpha}}c\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}}, \qquad \hat{R}_{\boldsymbol{\alpha}c\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\delta}}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} \tag{5.332}$$

$$R_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0, \qquad \hat{R}_{\boldsymbol{\alpha}\boldsymbol{\gamma}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.333}$$

$$R_{c[\boldsymbol{\alpha}\boldsymbol{\beta}]}{}^{\boldsymbol{\gamma}} = \gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}T_{dc}{}^{\boldsymbol{\gamma}}, \qquad \hat{R}_{c[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\gamma}}} = \gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{T}_{dc}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.334}$$

$$R_{\hat{\boldsymbol{\gamma}}[\boldsymbol{\alpha}\boldsymbol{\beta}]}{}^{\boldsymbol{\delta}} = -\gamma^{e}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\tilde{\gamma}_{e\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}, \qquad \hat{R}_{\boldsymbol{\gamma}[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\delta}}} = -\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\tilde{\gamma}_{e\,\boldsymbol{\gamma}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} \tag{5.335}$$

$$R_{[\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\boldsymbol{\alpha}_3]}{}^{\boldsymbol{\beta}} = 0, \qquad \hat{R}_{[\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2\hat{\boldsymbol{\alpha}}_3]}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.336}$$

$$R_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} = \underline{\nabla}_{\boldsymbol{\alpha}}T_{bc}{}^{\boldsymbol{\delta}}\big|_{\check{\Omega}=\hat{\Omega}} + 4\tilde{\gamma}_{[b|\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}, \quad \hat{R}_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}} = \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\hat{T}_{bc}{}^{\hat{\boldsymbol{\delta}}}\Big|_{\check{\Omega}=\Omega} + 4\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\gamma}}}\tilde{\gamma}_{|c]\,\boldsymbol{\varepsilon}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} \tag{5.337}$$

Taking the trace of the first two curvature constraints (using (5.274) and (5.273)) gives further informations on the Dilatation-Field-strength (and thus indirectly also on the Lorentz curvature)

$$\hat{F}^{(D)}_{c\boldsymbol{\alpha}} = \tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi, \qquad F^{(D)}_{c\hat{\boldsymbol{\alpha}}} = \tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\alpha}}\Phi \tag{5.338}$$

$$\hat{F}^{(D)}_{\boldsymbol{\alpha}\boldsymbol{\gamma}} = 0, \qquad F^{(D)}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}} = 0 \tag{5.339}$$

Remaining BI's Finally we get a couple of constraints on curvature components where the structure group indices are bosonic. They are related to the above ones as we shall discuss after presenting them:

$$R_{\boldsymbol{\alpha}\boldsymbol{\beta}c}{}^{d} \stackrel{(5.719)}{=} -\nabla_{[\boldsymbol{\alpha}}\nabla_{\boldsymbol{\beta}]}\Phi\delta_{c}{}^{d} + \gamma_{c}{}^{d}{}_{[\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\beta}]}\nabla_{\boldsymbol{\delta}}\Phi + 3\gamma^{e}_{\boldsymbol{\alpha}\boldsymbol{\beta}}H_{ec}{}^{d} + \gamma_{c}{}^{e}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\gamma}}\Phi\gamma_{e}{}^{d}{}_{|\boldsymbol{\beta}]}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi \tag{5.340}$$

$$\hat{R}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c}{}^{d} \stackrel{(5.720)}{=} -\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}}\hat{\nabla}_{\hat{\boldsymbol{\beta}}]}\Phi\delta_{c}{}^{d} + \gamma_{c}{}^{d}{}_{[\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\beta}}]}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi - 3\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}H_{ec}{}^{d} + \gamma_{c}{}^{e}{}_{[\hat{\boldsymbol{\alpha}}|}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\gamma_{e}{}^{d}{}_{|\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi \tag{5.341}$$

$$R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{d} \stackrel{(5.727)}{=} \frac{1}{2}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi\delta^{d}_{c} + \frac{1}{2}\gamma_{c}{}^{d}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\gamma}}\Phi - 2\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} + 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\gamma^{d}_{\boldsymbol{\varepsilon}\boldsymbol{\alpha}} \tag{5.342}$$

$$\hat{R}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}c}{}^{d} \stackrel{(5.728)}{=} \frac{1}{2}\hat{\nabla}_{\boldsymbol{\beta}}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta^{d}_{c} + \frac{1}{2}\gamma_{c}{}^{d}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\boldsymbol{\beta}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\beta}}}\gamma^{d}_{\boldsymbol{\varepsilon}\boldsymbol{\beta}} + 2\tilde{\gamma}_{c\,\boldsymbol{\beta}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} \tag{5.343}$$

$$R_{\hat{\boldsymbol{\alpha}}[bc]d} \stackrel{(5.741)}{=} -\frac{1}{8}\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{d[b|\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}\gamma_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} + G_{d[b|}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi \tag{5.344}$$

$$\hat{R}_{\boldsymbol{\alpha}[bc]d} \stackrel{(5.740)}{=} -\frac{1}{8}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{d[b|\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\gamma_{|c]\boldsymbol{\varepsilon}\boldsymbol{\alpha}} + G_{d[b|}\tilde{\gamma}_{|c]\,\boldsymbol{\alpha}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi \tag{5.345}$$

$$R_{[abc]}{}^{d} \overset{(5.748)}{=} \frac{3}{2}\nabla_{[a}H_{bc]}{}^{d} + \frac{9}{2}H_{[ab|}{}^{e}H_{e|c]}{}^{d} + 2T_{[ab|}{}^{\varepsilon}T_{\varepsilon|c]}{}^{d} \qquad (5.346)$$

$$\hat{R}_{[abc]}{}^{d} \overset{(5.749)}{=} -\frac{3}{2}\hat{\nabla}_{[a}H_{bc]}{}^{d} + \frac{9}{2}H_{[ab|}{}^{e}H_{e|c]}{}^{d} + 2\hat{T}_{[ab|}{}^{\hat{\varepsilon}}\hat{T}_{\hat{\varepsilon}|c]}{}^{d} \qquad (5.347)$$

From the structure group constraints on the curvature, we know that the components split into Lorentz and scale part $R_{ABc}{}^{d} = F^{(D)}_{AB}\delta^{d}_{c} + R^{(L)}_{ABc}{}^{d}$. The same is true for the componets with fermionic structure group indices, where we had the split $R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \frac{1}{2}F^{(D)}_{AB}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \frac{1}{4}R^{(L)}_{ABc}{}^{d}\gamma^{c}{}_{d\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}$. The coefficients $F^{(D)}_{AB}$ and $R^{(L)}_{ABc}{}^{d}$ are the same, when the bosonic block of $\check{\Omega}_{Ma}{}^{b}$ was chosen to coincide with the left-mover connection. They can be extracted from $R_{ABc}{}^{d}$ just as $\frac{1}{10}$ of the trace part and as the antisymmetric part respectively. To extract the coefficients instead from $R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}$, we need the fermionic trace $\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\gamma}} = -16$ which yields $F^{(D)}_{AB} = -\frac{1}{8}R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\gamma}}$ and the identity $\gamma_{ab\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}\gamma^{cd}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = 32\delta^{cd}_{ab}$ that allows to extract the Lorentz part as $R^{(L)}_{ABc}{}^{d} = \frac{1}{8}\gamma_{c}{}^{d}{}_{\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}$. Then we can relate both curvature blocks directly in the following way:

$$R_{ABc}{}^{d} = -\frac{1}{8}R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\gamma}}\delta^{d}_{c} + \frac{1}{8}\gamma_{c}{}^{d}{}_{\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} \qquad (5.348)$$

$$R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \frac{1}{20}R_{ABc}{}^{c}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \frac{1}{4}R_{ABc}{}^{d}\gamma^{c}{}_{d\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} \qquad (5.349)$$

In the same way we can relate $\hat{R}_{ABc}{}^{d}$ and $\hat{R}_{AB\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}$ and compare their constraints which should reveal additional information. This was used for example in footnote 28 on page 111 to derive the constraint

$$\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi = -\tilde{\gamma}_{d\,\boldsymbol{\alpha\rho}}\mathcal{P}^{\boldsymbol{\rho}\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} \qquad (5.350)$$

on the compensator superfield.

5.14 The dilaton superfield

While we have found the covariant derivatives $\nabla_{a}\Phi = \hat{\nabla}_{a}\Phi = \nabla_{\hat{\boldsymbol{\alpha}}}\Phi = \hat{\nabla}_{\boldsymbol{\alpha}}\Phi$ of the compensator field Φ to be forced to vanish, the remaining components $\nabla_{\boldsymbol{\alpha}}\Phi = E_{\boldsymbol{\alpha}}{}^{M}(\partial_{M}\Phi - \Omega^{(D)}_{M})$ and $\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi = E_{\hat{\boldsymbol{\alpha}}}{}^{M}(\partial_{M}\Phi - \hat{\Omega}^{(D)}_{M})$ seem to contain physical fermionic degrees of freedom. Indeed, the leading components of the scale connections $\Omega^{(D)}_{\boldsymbol{\alpha}}$ and $\hat{\Omega}^{(D)}_{\hat{\boldsymbol{\alpha}}}$ were identified in [13] up to a constant factor with the dilatinos. As we have not yet fixed the local scale invariance (guaranteed by the compensator field Φ), those connections are not covariant and we take instead the just mentioned covariant derivatives of the compensator field. That is, we define the **dilatinos** as

$$\lambda_{\boldsymbol{\alpha}} \equiv \nabla_{\boldsymbol{\alpha}}\Phi|_{\vec{\boldsymbol{\theta}}=0}\,, \quad \hat{\lambda}_{\hat{\boldsymbol{\alpha}}} \equiv \hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\Big|_{\vec{\boldsymbol{\theta}}=0} \qquad (5.351)$$

We are still completely missing the dilaton itself, whose appearance is a bit hidden. It does not show up explicitely in the action. Although we did not manually include it via the Fradkin Tseytlin term, its physical degrees of freedom should already be present in this setting.[16] Usually one would suspect the dilatinos to be components at first order in $\vec{\boldsymbol{\theta}}$ of a scalar dilaton superfield instead of being the component of a (non-covariantly transforming) compensator field. The idea to recover such a scalar superfield is to equate its spinorial derivative with the covariant spinorial derivatives of the compensator field and let the algebra fix the missing bosonic derivative. So let us simply "define" the scalar **dilaton superfield** $\Phi_{(ph)}$ via

$$\nabla_{\boldsymbol{\alpha}}\Phi_{(ph)} \equiv \nabla_{\boldsymbol{\alpha}}\Phi, \quad \hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi_{(ph)} \equiv \hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi \qquad (5.352)$$

The different behaviour of the fields under scale transformations is reflected in the different action of the covariant derivative. While for the dilaton it acts like a partial derivative $\nabla_{\boldsymbol{\alpha}}\Phi_{(ph)} = E_{\boldsymbol{\alpha}}{}^{M}\partial_{M}\Phi_{(ph)}$, the action on the compensator field – as mentioned already above – includes a shift $\nabla_{\boldsymbol{\alpha}}\Phi = E_{\boldsymbol{\alpha}}{}^{M}(\partial_{M}\Phi_{(ph)} - \Omega^{(D)}_{M})$. Of course we have to make sure that this definition does not put additional restrictions on the already present field content, in particular on the scale field strength. As $\Phi_{(ph)}$ is supposed to be a scalar field (where the commutator of covariant derivatives does not contain any curvature terms), while Φ is a compensator field

[16] Thanks to N. Berkovits for clarifying this issue. In [13, 59] the dilaton was added as an extra field via the Fradkin-Tseytlin term $S_{FT} = \int \alpha' r\Phi_{(ph)}$ (with r being the worldsheet curvature) and then related to the already present field content via a quantum consistency argument. Their result was $E_{\boldsymbol{\alpha}}{}^{M}\partial_{M}\Phi_{(ph)} = 4\Omega_{\boldsymbol{\alpha}}$ and $E_{\hat{\boldsymbol{\alpha}}}{}^{M}\partial_{M}\Phi_{(ph)} = 4\hat{\Omega}_{\hat{\boldsymbol{\alpha}}}$. Because of the introduction of our compensator field Φ, their relations would modify in our case to

$$E_{\boldsymbol{\alpha}}{}^{M}\partial_{M}(\Phi_{(ph)} + 4\Phi) = 4\Omega_{\boldsymbol{\alpha}} \iff -4\nabla_{\boldsymbol{\alpha}}\Phi = \nabla_{\boldsymbol{\alpha}}\Phi_{(ph)}$$

$$E_{\hat{\boldsymbol{\alpha}}}{}^{M}\partial_{M}(\Phi_{(ph)} + 4\Phi) = 4\hat{\Omega}_{\hat{\boldsymbol{\alpha}}} \iff -4\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi = \hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi_{(ph)}$$

Our definition (5.352) of the dilaton is thus consistent with this result, although the definitions differ by a factor -4. ◇

(where the commutator of covariant derivatives contains the scale field strength), it is instructive to compare the derivative commutators acting on them:

$$\begin{aligned}\underline{\nabla}_{[\boldsymbol{\alpha}}\underline{\nabla}_{\boldsymbol{\beta}]}\Phi_{(ph)}\Big|_{\check{\Omega}=\Omega} &= -\underline{T}_{\boldsymbol{\alpha\beta}}{}^{C}\big|_{\check{\Omega}=\Omega}\nabla_C\Phi_{(ph)} = -\gamma^c_{\boldsymbol{\alpha\beta}}\nabla_c\Phi_{(ph)}\\ \underline{\nabla}_{[\boldsymbol{\alpha}}\underline{\nabla}_{\boldsymbol{\beta}]}\Phi\Big|_{\check{\Omega}=\Omega} &= -\underline{T}_{\boldsymbol{\alpha\beta}}{}^{C}\big|_{\check{\Omega}=\Omega}\nabla_C\Phi - F^{(D)}_{\boldsymbol{\alpha\beta}} = -F^{(D)}_{\boldsymbol{\alpha\beta}}\end{aligned} \tag{5.353}$$

Similar equations hold for the hatted indices. Consistency then requires

$$\gamma^c_{\boldsymbol{\alpha\beta}}\nabla_c\Phi_{(ph)} = F^{(D)}_{\boldsymbol{\alpha\beta}},\quad \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\nabla_c\Phi_{(ph)} = \hat{F}^{(D)}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \tag{5.354}$$

In contrast to $\nabla_c\Phi$ and $\hat{\nabla}_c\Phi$, the bosonic derivative $\nabla_c\Phi_{(ph)}$ of the dilaton superfield is in general nonzero. For the validity of the above 'definition' it is important to observe that because of the constraints (5.571) the equations (5.354) do not put an additional artificial restriction on $F^{(D)}_{\boldsymbol{\alpha\beta}}$ and $\hat{F}^{(D)}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$. Instead (5.354) consistently completes (5.352) to a complete superspace derivative of the superfield and we can use the supervielbein to switch to curved coordinates where the covariant derivative $\nabla_M\Phi_{(ph)}$ on the scalar field coincides with the partial derivative $\partial_M\Phi_{(ph)}$. Integrating it, we are just missing a constant, the dilaton zero mode (responsible for the string-coupling in the loop-expansion). The dilaton superfield is thus well-defined by (5.352) up to an integration constant.

5.15 Local SUSY-transformation of the fermionic fields

In order to make contact to generalized complex geometry, we are interested in the local supersymmetry transformations of the fermionic fields, i.e. the gravitino and the dilatino. Note that the superdiffeomorphisms and the local structure group transformations contain a huge number of auxiliary gauge degrees of freedom in the $\boldsymbol{\theta}$-expansion of the transformation parameters. The physical fields are recovered by choosing a gauge, in particular the so-called WZ-gauge. Remaining bosonic diffeomorphisms, local structure group transformations of the bosonic manifold and local supersymmetry are then part of the stabilizer transformations of the chosen gauge. In the appendix H on page 206, this procedure is carefully explained and the supergravity transformations are derived for a general setting, following roughly [17].

5.15.1 Connection to choose

As mentioned above, in the appendix H on page 206 we describe the ususal procedure of choosing the Wess Zumino gauge $E_{\boldsymbol{\mathcal{M}}}{}^{A}\big| = \delta_{\boldsymbol{\mathcal{M}}}{}^{A}$ and $\Omega_{\boldsymbol{\mathcal{M}}A}{}^{B}\big| = 0$ (see (H.76) and (H.92)). This gauge fixing is possible with any connection as long as it takes the same values (in the Lie algebra) as the gauge transformations (Remember, a connection is a Lie algebra valued one form). However, the present case is a bit special in the following sense: We have derived the supergravity constraints using the connection

$$\underline{\Omega}_{MA}{}^{B} \equiv \begin{pmatrix} \check{\Omega}_{Ma}{}^{b} & 0 & 0\\ 0 & \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0\\ 0 & 0 & \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\end{pmatrix} \tag{5.355}$$

After that we have coupled the independent structure group transformations of the three blocks by a gauge fixing s.t. $T_{\boldsymbol{\alpha\beta}}{}^{c} = \gamma^c_{\boldsymbol{\alpha\beta}}$ and $T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{c} = \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$. The remaining gauge symmetry has to leave this gauge fixing invariant which reduces the structure group to only one copy of the Lorentz group plus one scale group. The above connection however does not leave the gauge fixing invariant (the covariant derivatives do not vanish in general). In order to be consistent, we thus have to reformulate the equations in terms of a connection which leaves $\gamma^c_{\boldsymbol{\alpha\beta}}$ and $\gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$ invariant. Possible choices are either the left mover connection $\Omega_{MA}{}^{B}$ (defined by $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\nabla_M\gamma^c_{\boldsymbol{\alpha\beta}} = \nabla_M\gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0$) or the right-mover connection $\hat{\Omega}_{MA}{}^{B}$ (defined by $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$) or the **average connection**

$$\underset{\leftrightarrow}{\Omega}_{MA}{}^{B} \equiv \frac{1}{2}\left(\Omega_{MA}{}^{B} + \hat{\Omega}_{MA}{}^{B}\right) = \Omega_{MA}{}^{B} + \frac{1}{2}\Delta_{MA}{}^{B} \tag{5.356}$$

We will study the choices $\Omega_{MA}{}^{B}$ and $\underset{\leftrightarrow}{\Omega}_{MA}{}^{B}$. The first has the advantage that at least the left mover equations stay simple while the second has the advantage that the symmetry between left and right movers is preserved. Corresponding to the the first choice the connection part of the WZ gauge simply reads

$$\boxed{\Omega_{\boldsymbol{\mathcal{M}}A}{}^{B}\big| = 0}\quad \text{(gauge I)} \tag{5.357}$$

In this gauge all the equations derived in appendix H on page 206 hold literally. The average connection becomes in this gauge $\underset{\leftrightarrow}{\Omega}_{\boldsymbol{\mathcal{M}}A}{}^{B}\Big| = \frac{1}{2}\,\Delta_{\boldsymbol{\mathcal{M}}A}{}^{B}\big|$, while the mixed connection can be written as $\underline{\Omega}_{\boldsymbol{\mathcal{M}}A}{}^{B}\big|_{\check{\Omega}=\Omega,\boldsymbol{\theta}=0} =$

$\mathrm{diag}\,(0,0,\Delta_{\boldsymbol{\mathcal{M}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\big|)$. Alternatively to gauge-I we could put $\hat{\Omega}_{\boldsymbol{\mathcal{M}}\,A}{}^B\big| = 0$ or equivalently $\underleftrightarrow{\Omega}_{\boldsymbol{\mathcal{M}}\,A}{}^B\big| = -\frac{1}{2}\,\Delta_{\boldsymbol{\mathcal{M}}A}{}^B|$ which would be the same type of gauge with simply the role of hatted and unhatted variables interchanged.

However, a qualitatively different but likewise natural gauge fixing (preserving the symmetry in hatted and unhatted variables) is

$$\boxed{\underleftrightarrow{\Omega}_{\boldsymbol{\mathcal{M}}\,A}{}^B\Big| = 0} \quad \text{(gauge II)} \tag{5.358}$$

In this gauge we have to replace in all equations of appendix H on page 206 the objects $\Omega_{MA}{}^B$, ∇_M, $T_{MN}{}^A$ and $R_{MNA}{}^B$ with $\underleftrightarrow{\Omega}_{MA}{}^B$, $\underleftrightarrow{\nabla}_M$, $\underleftrightarrow{T}_{MN}{}^A$ and $\underleftrightarrow{R}_{MNA}{}^B$ respectively. The mixed connection in this gauge becomes $\underline{\Omega}_{\boldsymbol{\mathcal{M}}A}{}^B|_{\breve{\Omega}=\Omega,\boldsymbol{\theta}=0} = \mathrm{diag}\,(-\frac{1}{2}\Delta_{\boldsymbol{\mathcal{M}}a}{}^b, -\frac{1}{2}\Delta_{\boldsymbol{\mathcal{M}}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \frac{1}{2}\Delta_{\boldsymbol{\mathcal{M}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}})\Big|$.

5.15.2 Denoting the physical component fields

We will try (where possible) to use a small letter to denote the leading component of a superfield. One should keep in mind that the notation for the component fields is a bit subtle, because the bosonic vielbein offers a second useful possibility to change from flat to curved indices. We will also make use of this possibility for the component fields, but one has to be careful. Defining for example $h_{mnk} \equiv H_{mnk}|$ and then changing to flat indices with the bosonic vielbein, is different from first changing to flat indices with the supervielbein and then taking the leading component: $h_{abc} \neq H_{abc}|$. In the following we will provide the definitions of the component fields. If the same component field is given later with changed indices (flat to curved or vice verse), then this is done via the bosonic vielbein.

$$E_M{}^A| \equiv \begin{pmatrix} e_m{}^a & \psi_m{}^{\boldsymbol{\mathcal{A}}} \\ 0 & \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}} \end{pmatrix} \tag{5.359}$$

$$\Omega_{mA}{}^B| \equiv \omega_{mA}{}^B, \quad (\Omega_{\boldsymbol{\mathcal{M}}A}{}^B| = 0) \tag{5.360}$$

$$\Phi| \equiv \phi, \qquad \Phi_{(ph)}(\vec{x})\Big| \equiv \phi_{(ph)}(\vec{x}) \tag{5.361}$$

$$G_{mn}| \equiv e^{2\phi} g_{mn} = e_m{}^a e_n{}^b e^{2\phi}\eta_{ab} \tag{5.362}$$

$$B_{mn}| \equiv b_{mn}, \qquad H_{mnk}| \equiv h_{mnk} \quad \Rightarrow \quad h_{mnk} = \partial_{[m} b_{nk]} \tag{5.363}$$

The second line which defines the bosonic connection certainly has to be adjusted according to the superconnection on which the WZ-gauge is based. For gauge II the definition of the bosonic connection would thus change to $\underleftrightarrow{\Omega}_{mA}{}^B\Big| \equiv \underleftrightarrow{\omega}_{mA}{}^B$, $(\underleftrightarrow{\Omega}_{\boldsymbol{\mathcal{M}}A}{}^B\Big| = 0)$. In the fourth line we see that we can use the bosonic compensator field ϕ to switch from string frame (vanishing ϕ) to the Einstein frame where ϕ should be gauge fixed to be proportional to the dilaton. In the third line we have defined the bosonic **dilaton** $\phi_{(ph)}$ as the leading component of the dilaton superfield. In contrast to the compensator field, it contains a physical degree of freedom which cannot be gauged away.[17]

For the definition of the leading component of the RR-bispinor $\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$ we first need a motivating observation. Because of the definition of the dilaton superfield in (5.352) via the spinorial covariant derivative of the compensator field, the latter can be replaced in (5.274),(5.273) by the spinorial derivative of the dilaton superfield

[17]There are some more words to say about the remaining scale invariance. The fact that the definition of the bosonic metric includes the compensator field leads to a loss of the correspondance between scaling behaviour and flat index. Define alternatively

$$G_{mn}| \equiv \tilde{g}_{mn} = e_m{}^a e_n{}^b \tilde{g}_{ab} \quad (= e_m{}^a e_n{}^b e^{2\phi}\eta_{ab}), \qquad \tilde{g}^{mn} = e_a{}^m e_b{}^n g^{ab} \quad (\equiv e_a{}^m e_b{}^n e^{-2\phi}\eta^{ab})$$

For a scale transformation $\delta\phi = -\varphi$, we have the following transformations of the other fields:

$$\begin{aligned}
\delta e_m{}^a &= \varphi e_m{}^a, \quad \delta e_a{}^m = -\varphi e_a{}^m \\
\delta\tilde{g}_{ab} &= -2\varphi\,\tilde{g}_{ab}, \quad \delta\tilde{g}^{ab} = 2\varphi\tilde{g}^{ab} \qquad \leftrightarrow \qquad \delta\eta_{ab} = \delta\eta^{ab} = 0 \\
\delta\tilde{g}_{mn} &= \delta\tilde{g}^{mn} = 0 \qquad \leftrightarrow \qquad \delta g_{mn} = 2\varphi g_{mn}, \quad \delta g^{mn} \\
\delta b_{mn} &= \delta h_{mnk} = \delta\phi_{(ph)} = 0 \\
\delta \mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} &= \varphi \mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} \\
\delta\psi_m{}^{\boldsymbol{\mathcal{A}}} &= \tfrac{1}{2}\varphi\psi_m{}^{\boldsymbol{\mathcal{A}}} \\
\delta\lambda_{\boldsymbol{\mathcal{A}}} &= -\tfrac{1}{2}\varphi\lambda_{\boldsymbol{\mathcal{A}}}
\end{aligned}$$

While for the use of $\tilde{g}_{mn}$ and $\tilde{g}_{ab}$ the scaling behaviour is coupled to the flat indices, this is not the case for g_{mn} and η_{ab}. Before the scale invariance is not fixed, we thus should not use g_{mn} or η_{ab} to lower or raise indices.

Similar considerations hold for the covariant derivative. Denote for the moment the bosonic spacetime-connection with $\gamma_{mk}{}^l$. We will use it only in this footnote and should not mix it up with an antisymmetrized product of three γ-matrices. This spacetime connection will not be defined as the leading component of $\Gamma_{mk}{}^l$, but via

$$\nabla_m e_k{}^a = 0 \text{ with } \nabla_m \equiv \partial_m \pm \gamma_{ml}{}^k \pm \omega_{mA}{}^B$$

which implies $\gamma_{mk}{}^n e_n{}^a = \Gamma_{mk}{}^N\big| E_N{}^a|$. The scaling part of the so defined bosonic covariant derivative acts on $\tilde{g}_{mn}$ and $\tilde{g}_{ab}$ according to their indices but not on g_{mn} and η_{ab}. $\diamond$

and those equations can be rewritten as

$$\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}(e^{-8\Phi_{(ph)}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}) = 0 \quad = \quad , \quad \underline{\nabla}_{\boldsymbol{\alpha}}(e^{-8\Phi_{(ph)}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}) = 0 \tag{5.364}$$

This is the motivation to define the RR-fields as

$$\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} \equiv e^{-8\phi_{(ph)}}\ \mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}\Big| \tag{5.365}$$

We had defined the dilatino already in the previous section in (5.351). Having now the scalar dilaton superfield at hand, it is convenient to use (5.352) in order to write them as components of this superfield:

$$\lambda_{\boldsymbol{\mathcal{A}}} \equiv \nabla_{\boldsymbol{\mathcal{A}}}\Phi_{(ph)}\big| \tag{5.366}$$

The subtleties of having bosonic and superspace vielbein at the same time were mentioned already in the beginning of this subsection. An example for the issues is provided by the inverse vielbein whose leading components are given by

$$E_A{}^M\big| = \begin{pmatrix} e_a{}^m & -\psi_a{}^{\boldsymbol{\mathcal{M}}} \\ 0 & \delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}} \end{pmatrix} \tag{5.367}$$

where $e_a{}^m$ is the inverse of $e_m{}^a$ and the indices of the gravitino were converted via bosonic vielbein and fermionic Kronecker delta respectively:

$$e_a{}^m e_m{}^b = \delta_a^b \tag{5.368}$$
$$\psi_a{}^{\boldsymbol{\mathcal{M}}} \equiv e_a{}^m\psi_m{}^{\boldsymbol{\mathcal{A}}}\delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}} \tag{5.369}$$

In the same way we define

$$b_{ab} \equiv e_a{}^m e_b{}^n b_{mn} \tag{5.370}$$
$$h_{abc} \equiv e_a{}^m e_b{}^n e_c{}^n h_{mnk} \tag{5.371}$$
$$g_{ab} \equiv e_a{}^m e_b{}^n g_{mn} = \eta_{ab} \tag{5.372}$$

As mentioned above, these expressions do in general not coincide with the leading components of the corresponding superfields

$$G_{ab}\big| = e^{2\phi}\eta_{ab} - 2e_{[a}{}^m\psi_{b]}{}^{\boldsymbol{\mathcal{N}}}\ G_{m\boldsymbol{\mathcal{N}}}\big| + \psi_a{}^{\boldsymbol{\mathcal{M}}}\psi_b{}^{\boldsymbol{\mathcal{N}}}\ G_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}}\big| = \tag{5.373}$$
$$= e^{2\phi}\eta_{ab} - 2\psi_{[b}{}^{\boldsymbol{\mathcal{B}}}\ G_{a]\boldsymbol{\mathcal{B}}}\big| - \psi_a{}^{\boldsymbol{\mathcal{A}}}\psi_b{}^{\boldsymbol{\mathcal{B}}}\ G_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}\big| \tag{5.374}$$
$$B_{ab}\big| = b_{ab} - 2e_{[a}{}^m\psi_{b]}{}^{\boldsymbol{\mathcal{N}}}\ B_{m\boldsymbol{\mathcal{N}}}\big| + \psi_a{}^{\boldsymbol{\mathcal{M}}}\psi_b{}^{\boldsymbol{\mathcal{N}}}\ B_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}}\big| = \tag{5.375}$$
$$= b_{ab} - 2\psi_{[b}{}^{\boldsymbol{\mathcal{B}}}\ B_{a]\boldsymbol{\mathcal{B}}}\big| - \psi_a{}^{\boldsymbol{\mathcal{A}}}\psi_b{}^{\boldsymbol{\mathcal{B}}}\ B_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}\big| \tag{5.376}$$
$$H_{abc}\big| = h_{abc} - 3e_{[a}{}^m e_b{}^n\psi_{c]}{}^{\boldsymbol{\mathcal{K}}}\ H_{mn\boldsymbol{\mathcal{K}}}\big| + 3\psi_a{}^{\boldsymbol{\mathcal{M}}}\psi_b{}^{\boldsymbol{\mathcal{N}}}e_c{}^k\ H_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}k}\big| - \psi_a{}^{\boldsymbol{\mathcal{M}}}\psi_b{}^{\boldsymbol{\mathcal{N}}}\psi_c{}^{\boldsymbol{\mathcal{K}}}\ H_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{K}}}\big| = \tag{5.377}$$
$$= h_{abc} - 3\psi_{[c}{}^{\boldsymbol{\mathcal{C}}}\ H_{ab]\boldsymbol{\mathcal{C}}}\big| - 3\psi_{[a}{}^{\boldsymbol{\mathcal{A}}}\psi_{b|}{}^{\boldsymbol{\mathcal{B}}}\ H_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}|c]}\big| - \psi_a{}^{\boldsymbol{\mathcal{A}}}\psi_b{}^{\boldsymbol{\mathcal{B}}}\psi_c{}^{\boldsymbol{\mathcal{C}}}\ H_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}}}\big| \tag{5.378}$$

Note that for vanishing gravitino $\psi_m{}^{\boldsymbol{\mathcal{A}}}$ there is no difference between the usage of bosonic vielbein or supervielbein to change from flat to curved indices. For non-vanishing gravitino the expressions already simplify significantly, if we take into account the WZ-like gauge $B_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}}\big| = B_{m\boldsymbol{\mathcal{N}}}\big| = 0$ for the B-field and the supergravity constraints of H-field and rank-two tensor G_{AB}. The latter has G_{ab} as only nonvanishing component.

$$G\big|_{ab} = e^{2\phi}\eta_{ab} \tag{5.379}$$
$$B_{ab}\big| = b_{ab} \tag{5.380}$$
$$H_{abc}\big| = h_{abc} + 2e^{2\phi}e_{[a}{}^m e_b{}^n\gamma_{c]\,\boldsymbol{\alpha}\boldsymbol{\beta}}\psi_m{}^{\boldsymbol{\alpha}}\psi_n{}^{\boldsymbol{\beta}} - 2e^{2\phi}e_{[a}{}^m e_b{}^n\gamma_{c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}}\hat{\psi}_n{}^{\hat{\boldsymbol{\beta}}} \tag{5.381}$$

Let us eventually see how the **bosonic torsion**

$$t^a \equiv \mathbf{d}e^a - e^c \wedge \omega_c{}^a \tag{5.382}$$

is related to the leading component of the superspace torsion:

$$T_{mn}{}^a\big| = \partial_{[m}\ E_{n]}{}^a\big| + E_{[n}{}^c\big|\ \Omega_{m]c}{}^a\big| + E_{[n}{}^{\boldsymbol{\mathcal{C}}}\big|\ \Omega_{m]\boldsymbol{\mathcal{C}}}{}^a\big| = \tag{5.383}$$
$$= \partial_{[m}e_{n]}{}^a + e_{[n}{}^c\omega_{m]c}{}^a = t_{mn}{}^a \tag{5.384}$$

Rewriting the superspace connection in terms of components with flat indices yields

$$t_{mn}{}^a = e_m{}^c e_n{}^d\ T_{cd}{}^a\big| + 2e_{[m}{}^c\psi_{n]}{}^{\boldsymbol{\mathcal{D}}}\ T_{c\boldsymbol{\mathcal{D}}}{}^a\big| + \psi_m{}^{\boldsymbol{\mathcal{C}}}\psi_n{}^{\boldsymbol{\mathcal{D}}}\ T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^a\big| \tag{5.385}$$

which implies

$$t_{cd}{}^{a} = T_{cd}{}^{a}| + 2e_{[c|}{}^{m}\psi_{m}{}^{\boldsymbol{\mathcal{C}}}\, T_{\boldsymbol{\mathcal{C}}|d]}{}^{a}| + e_{c}{}^{m}e_{d}{}^{n}\psi_{m}{}^{\boldsymbol{\mathcal{C}}}\psi_{n}{}^{\boldsymbol{\mathcal{D}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^{a}| \tag{5.386}$$

Similarly we have for the **bosonic curvature**

$$r_{a}{}^{b} \equiv \mathbf{d}\omega_{a}{}^{b} - \omega_{a}{}^{c}\wedge\omega_{c}{}^{b} \tag{5.387}$$

the following relations to the superspace curvature:

$$R_{mna}{}^{b}\big| = r_{mna}{}^{b} \tag{5.388}$$

$$r_{cda}{}^{b} = R_{cda}{}^{b}\big| + 2e_{[c|}{}^{m}\psi_{m}{}^{\boldsymbol{\mathcal{C}}}\, R_{\boldsymbol{\mathcal{C}}|d]a}{}^{b}\big| + e_{c}{}^{m}e_{d}{}^{n}\psi_{m}{}^{\boldsymbol{\mathcal{C}}}\psi_{n}{}^{\boldsymbol{\mathcal{D}}}\, R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}a}{}^{b}\big| \tag{5.389}$$

For gauge II the above expressions again have to be understood in terms of the average connection. As we have not yet plugged any torsion or curvature constraints into the equations, they are still valid for both gauges.

5.15.3 The gravitino transformation

5.15.3.1 General form

In the appendix, the general form of the gravitino transformation is given in equation (H.209), which we repeat here for convenience:

$$\delta_{\varepsilon}\psi_{m}{}^{\boldsymbol{\mathcal{A}}} = \underbrace{\partial_{m}\varepsilon^{\boldsymbol{\mathcal{A}}} + \omega_{m\boldsymbol{\mathcal{C}}}{}^{\boldsymbol{\mathcal{A}}}\varepsilon^{\boldsymbol{\mathcal{C}}}}_{\nabla_{m}\varepsilon^{\boldsymbol{\mathcal{A}}}} + 2\varepsilon^{\boldsymbol{\mathcal{C}}}e_{m}{}^{b}\, T_{\boldsymbol{\mathcal{C}}b}{}^{\boldsymbol{\mathcal{A}}}\big| + 2\varepsilon^{\boldsymbol{\mathcal{C}}}\psi_{m}{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}\big| \tag{5.390}$$

where $\omega_{m\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{B}}} \equiv \Omega_{m\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{B}}}\big|$. The connection appearing explicitely and implicitely (in the torsion) in this transformation has to be the same connection as the one on which the WZ gauge fixing condition was put. The above equation can thus be understood literally if we choose gauge I (based on the left-mover connection $\Omega_{MA}{}^{B}$) while for gauge II (based on the average connection $\underset{\longleftrightarrow}{\Omega}_{MA}{}^{B}$) every implicit or explicit appearance of $\Omega_{MA}{}^{B}$ has to be replaced by $\underset{\longleftrightarrow}{\Omega}_{MA}{}^{B}$. We can continue the considerations for a while without deciding, whether we are in gauge I or gauge II, although the notation will suggest that we are in gauge I (with connection $\Omega_{MA}{}^{B}$).

For the transformation of the gravitino(s) given above, we still need additional information about the connection $\omega_{m\boldsymbol{\mathcal{C}}}{}^{\boldsymbol{\mathcal{A}}}$, which does not necessarily coincide with the Levi Civita connection. In bosonic manifolds, the connection is completely determined by torsion and (non)metricity, if a metric is given. If no metric is given, one can likewise demand the preservation of other structures or structure constants. In particular in 10-dimensional superspace we do not have a non-degenerate superspace-metric. Only the bosonic block G_{ab} of the symmetric rank two tensor G_{AB} has full rank. In order to determine the full superspace connection, one thus needs more than the information about the covariant derivative of the symmetric rank two tensor. A natural candidate is the covariant derivative of the gamma-matrices, the structure constants of the supersymmetry algebra. This logic is carefully described in appendix G.

The derivation of (5.390) in the appendix did not assume any restrictions on the structure group, apart from being blockdiagonal w.r.t. bosonic and fermionic indices. Right now, we make use of the fact that we have (for gauge I as well as for gauge II) a connection with

$$\nabla_{M}\gamma^{c}_{\boldsymbol{\alpha\beta}} \overset{!}{=} \nabla_{M}\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \overset{!}{=} 0 \tag{5.391}$$

which relates the three blocks of $\Omega_{MA}{}^{B}$ and restricts the structure group to local Lorentz and local scale transformations. It is convenient to write

$$\gamma^{c}_{\boldsymbol{\mathcal{A}\mathcal{B}}} \equiv \begin{pmatrix} \gamma^{c}_{\boldsymbol{\alpha\beta}} & 0 \\ 0 & \gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{5.392}$$

Only in type IIA this matrix coincides with $A\Gamma^{c}$ (where A is the intertwiner responsible for the Dirac-conjugate: $\bar{\Psi} = \Psi^{\dagger}A$).

We can then make use of equation (G.57) of appendix G, which relates the leading components of the superspace connection, in particular the ones with fermionic structure group indices

$$\omega_{m\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{B}}} \equiv \Omega_{m\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{B}}}\big| \;, \tag{5.393}$$

to the Levi Civita connection and a somewhat lengthy rest:

$$\begin{aligned}\omega_{m\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}} &= \omega^{(LC)\boldsymbol{\mathcal{A}}}_{m\boldsymbol{\mathcal{B}}}+\\ &+\frac{1}{4}e_m{}^a\Big\{2e^{-2\phi}\,T_{a[b|c]}\big|-e^{-2\phi}\,T_{bc|a}\big|-2\big(\nabla_{[b|}\Phi\big|-e_{[b|}{}^k\partial_k\phi\big)\eta_{|c]a}-2e_{[b}{}^n\eta_{c]a}\psi_n{}^{\boldsymbol{\mathcal{C}}}\,(\nabla_{\boldsymbol{\mathcal{C}}}\Phi)\big|+\\ &+\big(2e_a{}^k e_{[b}{}^n\eta_{c]d}-e_b{}^k e_c{}^n\eta_{ad}\big)\,\psi_k{}^{\boldsymbol{\mathcal{C}}}\psi_n{}^{\boldsymbol{\mathcal{D}}}\,T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^d\big|+\\ &+e^{-2\phi}\Big(2e_a{}^n\psi_n{}^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}[b|c]}\big|-2e_b{}^n\psi_n{}^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}(a|c)}\big|+2e_c{}^n\psi_n{}^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}(a|b)}\big|\Big)\Big\}\gamma^{bc}{}_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}\\ &-\frac{1}{2}\left(\psi_m{}^{\boldsymbol{\mathcal{C}}}\,\nabla_{\boldsymbol{\mathcal{C}}}\Phi\big|+e_m{}^a\,\nabla_a\Phi\big|-\partial_m\phi\right)\delta_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}\end{aligned} \tag{5.394}$$

where the Levi Civita connection $\omega^{(LC)\boldsymbol{\mathcal{A}}}_{m\boldsymbol{\mathcal{B}}}$ is the one with respect to the metric $g_{mn}=e_m{}^a\eta_{ab}e_n{}^b$. We should note that the Levi Civita connection is not a suitable connection for scale transformations, because it is only Lorentz group valued. The terms $\partial_k\phi$ with the partial derivative of the compensator field do not transform covariantly under scale transformations and are the minimal extension of the Levi Civita connection to make it a structure group valued connection. On the other hand, if one decides to simply fix ϕ to zero and thus ending up only with Lorentz transformations, these terms disappear. The last line which is dilatation-valued can then not any longer be seen as part of the connection.

Together with (5.390) the above expression for the connection determines the supergravity transformation of the gravitino. In order to plug in the explicit constraints for the torsion, we have to decide in which gauge we work.

5.15.3.2 In gauge I

In gauge I, we can take the above equations literally and plug in the corresponding torsion constraints (5.303)-(5.305). We will need in addition that according to (5.381) the leading component of the H-field with flat coordinates is related to the bosonic h-field via $H_{abc}|=h_{abc}+2e^{2\phi}e_{[a}{}^m e_b{}^n\gamma_{c]\,\boldsymbol{\alpha\beta}}\psi_m{}^{\boldsymbol{\alpha}}\psi_n{}^{\boldsymbol{\beta}}-2e^{2\phi}e_{[a}{}^m e_b{}^n\gamma_{c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}}\hat{\psi}_n{}^{\hat{\boldsymbol{\beta}}}$. The connection becomes

$$\begin{aligned}\omega_{m\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}} &= \omega^{(LC)\boldsymbol{\mathcal{A}}}_{m\boldsymbol{\mathcal{B}}}+\frac{1}{4}e_m{}^a\Big\{\tfrac{3}{2}h_{abc}e^{-2\phi}+2e_{[b|}{}^k\partial_k\phi\eta_{|c]a}+4e_a{}^k e_{[b}{}^n\eta_{c]d}\psi_k{}^{\boldsymbol{\gamma}}\psi_n{}^{\boldsymbol{\delta}}\gamma^d_{\boldsymbol{\gamma\delta}}+\\ &\quad -2e_b{}^k e_c{}^n\eta_{ad}\hat{\psi}_k{}^{\hat{\boldsymbol{\gamma}}}\hat{\psi}_n{}^{\hat{\boldsymbol{\delta}}}\gamma^d_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}-e_a{}^n\psi_n{}^{\boldsymbol{\gamma}}\gamma_{bc\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}}\Big\}\gamma^{bc}{}_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}+\\ &\quad +\tfrac{1}{2}\left(\partial_m\phi-\psi_m{}^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}}\right)\delta_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}\end{aligned} \tag{5.395}$$

The constraints needed for the left-mover version of the transformation (5.390) are rather simple. In particular all the components $T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\alpha}}$ vanish. The local supersymmetry transformation of the left-mover gravitino turns into

$$\boxed{\delta_\varepsilon\psi_m{}^{\boldsymbol{\alpha}}=\underbrace{\partial_m\varepsilon^{\boldsymbol{\alpha}}+\omega_{m\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}\varepsilon^{\boldsymbol{\gamma}}}_{\nabla_m\varepsilon^{\boldsymbol{\alpha}}}+2\varepsilon^{\hat{\boldsymbol{\gamma}}}e_m{}^b e^{2\phi+8\phi_{(ph)}}\gamma_{b\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}} \tag{5.396}$$

If we want to fix the local scale invariance by setting the compensator field to zero, this gauge has to be respected by the supersymmetry transformation which then has to be redefined according to (H.193) with a dilatation with parameter $\varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}}$, which would add a term $\frac{1}{2}(\varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}})\psi_m{}^{\boldsymbol{\alpha}}$ to the above transformation.

For the right-mover transformation, the torsion constraints are more involved and we arrive at

$$\begin{aligned}\delta_\varepsilon\psi_m{}^{\hat{\boldsymbol{\alpha}}} &= \underbrace{\partial_m\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}}+\omega_{m\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}}_{\nabla_m\varepsilon^{\hat{\boldsymbol{\alpha}}}}+\\ &\quad +\tfrac{1}{4}e_m{}^a\Big(-3e^{-2\phi}h_{abc}-6e_{[a}{}^m e_b{}^n\gamma_{c]\,\boldsymbol{\alpha\beta}}\psi_m{}^{\boldsymbol{\alpha}}\psi_n{}^{\boldsymbol{\beta}}+6e_{[a}{}^m e_b{}^n\gamma_{c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}}\hat{\psi}_n{}^{\hat{\boldsymbol{\beta}}}+\\ &\quad +e_a{}^n\psi_n{}^{\boldsymbol{\beta}}\gamma_{bc\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}}-e_a{}^n\hat{\psi}_n{}^{\hat{\boldsymbol{\beta}}}\gamma_{bc\,\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}}\Big)\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\gamma^{bc}{}_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}+\\ &\quad +\tfrac{1}{2}(\psi_m{}^{\boldsymbol{\beta}}\lambda_{\boldsymbol{\beta}}-\psi_m{}^{\hat{\boldsymbol{\beta}}}\hat{\lambda}_{\hat{\boldsymbol{\beta}}})\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}}+\\ &\quad +2\varepsilon^{\boldsymbol{\gamma}}e_m{}^b e^{2\phi+8\phi_{(ph)}}\gamma_{b\,\boldsymbol{\gamma\delta}}\mathfrak{p}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}+\\ &\quad \underbrace{+\tfrac{1}{4}(\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\gamma_{de\,\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}}-\varepsilon^{\boldsymbol{\gamma}}\gamma_{de\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}})\hat{\psi}_m{}^{\hat{\boldsymbol{\beta}}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}}_{\text{Lorentz trafo}}\underbrace{+\tfrac{1}{2}(\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\hat{\lambda}_{\hat{\boldsymbol{\gamma}}}-\varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}})\psi_m{}^{\hat{\boldsymbol{\alpha}}}}_{\text{dilatation}}\end{aligned} \tag{5.397}$$

It is obvious that "gauge I" prefers the left-movers and destroys the left-right symmetry. The last two terms correspond to a Lorentz and a scale transformation of the gravitino with gauge parameters $(\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\gamma_{de\,\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}}-$

$\varepsilon^{\boldsymbol{\gamma}}\gamma_{de\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}})$ and $(\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\hat{\lambda}_{\hat{\boldsymbol{\gamma}}} - \varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}})$ respectively. They could be removed by redefining the local supersymmetry transformation, but then they would show up for the left-mover. For the right-mover one can combine some terms, if we plug the explicit expression of the connection into the above equation:

$$\boxed{\begin{aligned}
\delta_\varepsilon \psi_m{}^{\hat{\boldsymbol{\alpha}}} \;=\;& \nabla_m^{(LC)}\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}} + \tfrac{1}{4}e_m{}^a\Big(-\tfrac{3}{2}h_{abc}e^{-2\phi} + 2e_{[b|}{}^k\partial_k\phi\eta_{|c]a} + \\
& -2e_b{}^m e_c{}^n\gamma_{a\,\boldsymbol{\alpha}\boldsymbol{\beta}}\psi_m{}^{\boldsymbol{\alpha}}\psi_n{}^{\boldsymbol{\beta}} + 4e_a{}^m e_{[b}{}^n\gamma_{c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}}\hat{\psi}_n{}^{\hat{\boldsymbol{\beta}}} - e_a{}^n\hat{\psi}_n{}^{\hat{\boldsymbol{\beta}}}\gamma_{bc\,\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}}\Big)\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\gamma^{bc}{}_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}} + \\
& +\tfrac{1}{2}(\partial_m\phi - \psi_m{}^{\hat{\boldsymbol{\beta}}}\hat{\lambda}_{\hat{\boldsymbol{\beta}}})\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}} + \\
& +2\varepsilon^{\boldsymbol{\gamma}}e_m{}^b e^{2\phi+8\phi_{(ph)}}\gamma_{b\,\boldsymbol{\gamma}\boldsymbol{\delta}}\mathfrak{p}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}} + \\
& +\underbrace{\tfrac{1}{4}(\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\gamma_{de\,\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}} - \varepsilon^{\boldsymbol{\gamma}}\gamma_{de\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}})\hat{\psi}_m{}^{\hat{\boldsymbol{\beta}}}\gamma^{de}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}}_{\text{Lorentz trafo}} + \underbrace{\tfrac{1}{2}(\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\hat{\lambda}_{\hat{\boldsymbol{\gamma}}} - \varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}})\psi_m{}^{\hat{\boldsymbol{\alpha}}}}_{\text{dilatation}} \qquad (5.398)
\end{aligned}}$$

Comparing the first three lines with the left-mover connection (5.395), we recognize its hatted version, i.e. the right-mover connection. The first three lines thus combine to $\hat{\nabla}_m\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}}$. We would have obtained the same result without the last line if we had started with the right-mover super-connection instead of the left-mover one. Using a different gauge thus corresponds to redefining the supersymmetry transformation by a local Lorentz and scale transformation. Also this transformation needs to be modified in the case that ϕ is fixed to zero. The stabilizing dilatation with parameter $\varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}}$ would add the term $\frac{1}{2}(\varepsilon^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}})\psi_m{}^{\hat{\boldsymbol{\alpha}}}$ and thus cancel the last term.

5.15.3.3 In gauge II

For gauge II, we need to replace the connection $\Omega_{MA}{}^B$ everywhere in the gravitino transformation (5.394) and (5.390) by the average connection $\underleftrightarrow{\Omega}_{MA}{}^B$ (with $\underleftrightarrow{\omega}_{m\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{B}}} \equiv \underleftrightarrow{\Omega}_{m\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{B}}}\Big|$). This implies that we also have to replace the torsion components $T_{AB}{}^C$ by $\underleftrightarrow{T}_{AB}{}^C$. The constraints on the corresponding torsion $\underleftrightarrow{T}_{AB}{}^C$ are collected in (5.315)-(5.317). The explicit form of the transformation becomes quite lengthy if we split the fermionic index $\boldsymbol{\mathcal{A}}$ into the left and right-mover spinorial indices $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$. For that reason it is advantageous to try to rewrite the constraints (5.315)-(5.317) with the combined fermionic indices. To this end we define

$$\epsilon_{(\boldsymbol{\mathcal{A}})} \;=\; \left\{\begin{array}{cc} 1 & \boldsymbol{\mathcal{A}} = \boldsymbol{\alpha} \\ -1 & \text{for } \boldsymbol{\mathcal{A}} = \hat{\boldsymbol{\alpha}} \end{array}\right. \qquad (5.399)$$

$$\mathcal{P}^{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}} \;\equiv\; \begin{pmatrix} 0 & \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} \\ \mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} & 0 \end{pmatrix} \qquad (5.400)$$

In order to keep the left-right symmetry we should think of $\hat{\epsilon}_{(\boldsymbol{\mathcal{A}})} \equiv -\epsilon_{(\boldsymbol{\mathcal{A}})}$. Remembering also the definition of $\gamma^c_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}$ in (5.392) and the relation of the spinorial derivative $\nabla_{\boldsymbol{\mathcal{A}}}\Phi_{(ph)}$ of the dilaton superfield to the one of the compensator (5.352), the torsion constraints (5.315)-(5.317) can be written as

$$\begin{aligned}
\underleftrightarrow{T}_{AB}{}^c \;&\equiv\; \begin{pmatrix} \underleftrightarrow{T}_{ab}{}^c & \underleftrightarrow{T}_{a\boldsymbol{\mathcal{B}}}{}^c \\ \underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}b}{}^c & \underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^c \end{pmatrix} = \\
&= \begin{pmatrix} 0 & \frac{1}{4}\nabla_{\boldsymbol{\mathcal{B}}}\Phi_{(ph)}\delta^c_a + \frac{1}{4}\gamma_a{}^c{}_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{D}}}\nabla_{\boldsymbol{\mathcal{D}}}\Phi_{(ph)} \\ -\frac{1}{4}\nabla_{\boldsymbol{\mathcal{A}}}\Phi_{(ph)}\delta^c_b - \frac{1}{4}\gamma_b{}^c{}_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{D}}}\nabla_{\boldsymbol{\mathcal{D}}}\Phi_{(ph)} & \gamma^c_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}} \end{pmatrix} \qquad (5.401) \\
\underleftrightarrow{T}_{AB}{}^{\boldsymbol{\mathcal{C}}} \;&\equiv\; \begin{pmatrix} \underleftrightarrow{T}_{ab}{}^{\boldsymbol{\mathcal{C}}} & \underleftrightarrow{T}_{a\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} \\ \underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}b}{}^{\boldsymbol{\mathcal{C}}} & \underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} \end{pmatrix} = \\
&= \begin{pmatrix} \left(\frac{1}{16}\underline{\nabla}_{\boldsymbol{\mathcal{E}}}\mathcal{P}^{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}} + \frac{1}{2}\nabla_{\boldsymbol{\mathcal{E}}}\Phi_{(ph)}\mathcal{P}^{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}\right)\tilde{\gamma}_{ab\,\boldsymbol{\mathcal{D}}}{}^{\boldsymbol{\mathcal{E}}} & -\epsilon_{(\boldsymbol{\mathcal{C}})}\frac{3}{16}H_{ade}\tilde{\gamma}^{de}{}_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} - \tilde{\gamma}_{a\,\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{D}}}\mathcal{P}^{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}} \\ \epsilon_{(\boldsymbol{\mathcal{C}})}\frac{3}{16}H_{bde}\tilde{\gamma}^{de}{}_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{C}}} + \tilde{\gamma}_{b\,\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{D}}}\mathcal{P}^{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}} & \frac{\epsilon_{(\boldsymbol{\mathcal{A}})}\epsilon_{(\boldsymbol{\mathcal{B}})}}{8}\left(\gamma_{de[\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{D}}}\gamma^{de}{}_{\boldsymbol{\mathcal{B}}]}{}^{\boldsymbol{\mathcal{C}}}\nabla_{\boldsymbol{\mathcal{D}}}\Phi_{(ph)} + 2\nabla_{[\boldsymbol{\mathcal{A}}}\Phi_{(ph)}\delta_{\boldsymbol{\mathcal{B}}]}{}^{\boldsymbol{\mathcal{C}}}\right) \end{pmatrix} \qquad (5.402)
\end{aligned}$$

In case that one has fixed the compensator superfield Φ already to zero, the Lorentz part of the above torsion differs according to (5.318)-(5.323) only in the following components:

$$\begin{aligned}
\underleftrightarrow{T}^{(L)c}_{\boldsymbol{\mathcal{A}}b} \;&\overset{\Phi=0}{=}\; \underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}b}{}^c + \tfrac{1}{4}\nabla_{\boldsymbol{\mathcal{A}}}\Phi_{(ph)}\delta_b{}^c = -\tfrac{1}{4}\gamma_b{}^c{}_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{D}}}\nabla_{\boldsymbol{\mathcal{D}}}\Phi_{(ph)} \qquad (5.403) \\
\underleftrightarrow{T}^{(L)\,\boldsymbol{\mathcal{C}}}_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}} \;&\overset{\Phi=0}{=}\; \underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} + \tfrac{1}{4}\nabla_{[\boldsymbol{\mathcal{A}}|}\Phi_{(ph)}\delta_{|\boldsymbol{\mathcal{B}}]}{}^{\boldsymbol{\mathcal{C}}} = \\
&=\; \tfrac{1}{8}\epsilon_{(\boldsymbol{\mathcal{A}})}\epsilon_{(\boldsymbol{\mathcal{B}})}\gamma_{de[\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{D}}}\gamma^{de}{}_{\boldsymbol{\mathcal{B}}]}{}^{\boldsymbol{\mathcal{C}}}\nabla_{\boldsymbol{\mathcal{D}}}\Phi_{(ph)} + \tfrac{\epsilon_{(\boldsymbol{\mathcal{A}})}\epsilon_{(\boldsymbol{\mathcal{B}})}+1}{4}\nabla_{[\boldsymbol{\mathcal{A}}|}\Phi_{(ph)}\delta_{|\boldsymbol{\mathcal{B}}]}{}^{\boldsymbol{\mathcal{C}}} \qquad (5.404)
\end{aligned}$$

For the components $\underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}}$ at $\vec{\boldsymbol{\theta}} = 0$ (appearing in (5.390)) we need to remember the dilatino-definition $\nabla_{\boldsymbol{\mathcal{A}}}\Phi_{(ph)}\big| = \lambda_{\boldsymbol{\mathcal{A}}}$ (5.366) and for $\underleftrightarrow{T}_{\boldsymbol{\mathcal{A}}b}{}^{\boldsymbol{\mathcal{C}}}$ at $\vec{\boldsymbol{\theta}} = 0$ we need $H_{abc}\big| = h_{abc} + 2e^{2\phi}e_{[a}{}^m e_b{}^n\gamma_{c]\,\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}\epsilon_{(\boldsymbol{\mathcal{A}})}\psi_m{}^{\boldsymbol{\mathcal{A}}}\psi_n{}^{\boldsymbol{\mathcal{B}}}$

(5.381), $\mathcal{P}^{\mathcal{A}\mathcal{B}}\big| = e^{8\phi_{(ph)}}\mathfrak{p}^{\mathcal{A}\mathcal{B}}$ (5.365), $\tilde{\gamma}^{de}{}_{\mathcal{A}}{}^{\mathcal{C}} = e^{-2\Phi}\gamma^{de}{}_{\mathcal{A}}{}^{\mathcal{C}}$, $\tilde{\gamma}_{b\,\mathcal{A}\mathcal{D}} = e^{2\Phi}\tilde{\gamma}_{b\,\mathcal{A}\mathcal{D}}$ and $\Phi| = \phi$. Now we can plug the constraints into (5.390) to arrive at:

$$
\begin{aligned}
\delta_\varepsilon \psi_m{}^{\mathcal{A}} \;=\; & \underbrace{\partial_m \varepsilon^{\mathcal{A}} + \underleftrightarrow{\omega}_{m\mathcal{C}}{}^{\mathcal{A}}\varepsilon^{\mathcal{C}}}_{\underleftrightarrow{\nabla}_m \varepsilon^{\mathcal{A}}} + \tfrac{1}{4}\varepsilon^{\mathcal{C}}\epsilon_{(\mathcal{C})}\gamma_{de[\mathcal{C}|}{}^{\mathcal{D}}\lambda_{\mathcal{D}}\psi_m{}^{\mathcal{B}}\epsilon_{(\mathcal{B})}\gamma^{de}{}_{|\mathcal{B}]}{}^{\mathcal{A}} + \tfrac{1}{2}\varepsilon^{\mathcal{C}}\psi_m{}^{\mathcal{B}}\epsilon_{(\mathcal{C})}\epsilon_{(\mathcal{B})}\lambda_{[\mathcal{C}}\delta_{\mathcal{B}]}{}^{\mathcal{A}} + \\
& + \varepsilon^{\mathcal{C}} e_m{}^b \left(\epsilon_{(\mathcal{C})}\frac{3}{8}\left(h_{bde}e^{-2\phi} + 2e_{[b}{}^k e_d{}^l \gamma_{e]\,\mathcal{D}\mathcal{B}}\epsilon_{(\mathcal{D})}\psi_k{}^{\mathcal{D}}\psi_l{}^{\mathcal{B}}\right)\gamma^{de}{}_{\mathcal{C}}{}^{\mathcal{A}} + 2e^{2\phi+8\phi_{(ph)}}\gamma_{b\,\mathcal{C}\mathcal{D}}\mathfrak{p}^{\mathcal{A}\mathcal{D}} \right)
\end{aligned} \tag{5.405}
$$

If we instead have $\Phi = 0$ and restrict to the Lorentz-part of the torsion, the last term in the first line has to be replaced by $\frac{1}{2}\varepsilon^{\mathcal{C}}\psi_m{}^{\mathcal{B}}(\epsilon_{(\mathcal{C})}\epsilon_{(\mathcal{B})} + 1)\lambda_{[\mathcal{C}}\delta_{\mathcal{B}]}{}^{\mathcal{A}}$ and the bosonic connection $\underleftrightarrow{\omega}_{m\mathcal{C}}{}^{\mathcal{A}}$ by its Lorentz part $\underleftrightarrow{\omega}^{(L)}_{m\mathcal{C}}{}^{\mathcal{A}}$. In order to determine the connection from (5.394) we make use of further torsion constraints from (5.401) and (5.402) and the constraint $\underleftrightarrow{\nabla}_a\Phi = 0$. The result is

$$
\begin{aligned}
\omega_{m\mathcal{B}}{}^{\mathcal{A}} = & \\
= \;& \omega^{(LC)}_{m\mathcal{B}}{}^{\mathcal{A}} + \tfrac{1}{4}e_m{}^a\Big\{ 2e_{[b|}{}^k\partial_k\phi\,\eta_{|c]a} + \left(2e_a{}^k e_{[b}{}^n\eta_{c]d} - e_b{}^k e_c{}^n\eta_{ad}\right)\psi_k{}^{\mathcal{C}}\psi_n{}^{\mathcal{D}}\gamma^d_{\mathcal{C}\mathcal{D}} - \tfrac{1}{2}e_a{}^n\psi_n{}^{\mathcal{C}}\gamma_{bc\,\mathcal{C}}{}^{\mathcal{D}}\lambda_{\mathcal{D}}\Big\}\gamma^{bc}{}_{\mathcal{B}}{}^{\mathcal{A}} \\
& -\tfrac{1}{4}\left(\psi_m{}^{\mathcal{C}}\lambda_{\mathcal{C}} - 2\partial_m\phi\right)\delta_{\mathcal{B}}{}^{\mathcal{A}}
\end{aligned} \tag{5.406}
$$

where the second line is the Lorentz part $\omega^{(L)}_{m\mathcal{B}}{}^{\mathcal{A}}$ of the connection. Some terms in the gravitino transformation can be further combined if we plug back this explicit expression for the connection into (5.405):

$$
\boxed{
\begin{aligned}
\delta_\varepsilon\psi_m{}^{\mathcal{A}} \;=\; & \nabla^{(LC)}_m\varepsilon^{\mathcal{A}} + 2e^{2\phi+8\phi_{(ph)}}\varepsilon^{\mathcal{B}}e_m{}^b\gamma_{b\,\mathcal{B}\mathcal{D}}\mathfrak{p}^{\mathcal{A}\mathcal{D}} + \\
& + \tfrac{1}{4}e_m{}^a\Big\{ 2e_{[b|}{}^k\partial_k\phi\,\eta_{|c]a} + \psi_k{}^{\mathcal{C}}\gamma^d_{\mathcal{C}\mathcal{D}}\left(2e_a{}^k e_{[b}{}^n\eta_{c]d} - e_b{}^k e_c{}^n\eta_{ad} + 3\epsilon_{(\mathcal{A})}\epsilon_{(\mathcal{D})}e_{[a}{}^k e_b{}^n\eta_{c]d}\right)\psi_n{}^{\mathcal{D}} + \\
& - e_a{}^n\psi_n{}^{\mathcal{C}}\tfrac{1}{2}\left(1 + \epsilon_{(\mathcal{A})}\epsilon_{(\mathcal{C})}\right)\gamma_{bc\,\mathcal{C}}{}^{\mathcal{D}}\lambda_{\mathcal{D}} + \tfrac{3}{2}\epsilon_{(\mathcal{A})}h_{abc}e^{-2\phi}\Big\}\gamma^{bc}{}_{\mathcal{B}}{}^{\mathcal{A}}\varepsilon^{\mathcal{B}} + \\
& - \tfrac{1}{2}\Big\{\tfrac{1}{2}\left(1 + \epsilon_{(\mathcal{A})}\epsilon_{(\mathcal{C})}\right)\psi_m{}^{\mathcal{C}}\lambda_{\mathcal{C}} - \partial_m\phi\Big\}\varepsilon^{\mathcal{A}} \\
& + \underbrace{\tfrac{1}{8}\epsilon_{(\mathcal{A})}(\varepsilon^{\mathcal{B}}\epsilon_{(\mathcal{B})}\gamma_{de\,\mathcal{B}}{}^{\mathcal{D}}\lambda_{\mathcal{D}})\psi_m{}^{\mathcal{C}}\gamma^{de}{}_{\mathcal{C}}{}^{\mathcal{A}}}_{\text{Lorentz trafo}} + \underbrace{\tfrac{1}{4}\epsilon_{(\mathcal{A})}(\varepsilon^{\mathcal{B}}\epsilon_{(\mathcal{B})}\lambda_{\mathcal{B}})\psi_m{}^{\mathcal{A}}}_{\text{dilatation}}
\end{aligned} \qquad (5.407)
}
$$

Note that we still have local structure group invariance, so that we can change the last terms by simply redefining the supersymmetry transformation with a Lorentz transformation and a dilatation. However, we cannot remove the terms for left- and rightmovers at the same time, because the corresponding gauge parameter differs in sign due to the factor $\epsilon(\mathcal{A})$ which is $+1$ for $\boldsymbol{\alpha}$ and -1 for $\hat{\boldsymbol{\alpha}}$. Note also that if the compensator superfield Φ was fixed to zero already in the beginning, the dilatation part changes to $\frac{1+\epsilon_{(\mathcal{A})}\epsilon_{(\mathcal{B})}}{4}\varepsilon^{\mathcal{B}}\lambda_{\mathcal{B}}\psi_m{}^{\mathcal{A}}$ and thus corresponds to a redefinition of the supersymmetry transformation by a dilatation with parameter $\frac{1}{2}\varepsilon^{\mathcal{B}}\lambda_{\mathcal{B}}$. This is the same minimal modification which is necessary when we only fix the leading component ϕ to zero in the end and need to stabilize it with a compensating dilatation according to according to (H.193). The above transformation can be seen as the final result, but it is at this point instructive to introduce eventually the split of the collective fermionic index into left and right-mover:

$$
\begin{aligned}
\delta_\varepsilon\psi_m{}^{\boldsymbol{\alpha}} \;=\; & \nabla^{(LC)}_m\varepsilon^{\boldsymbol{\alpha}} + 2e^{2\phi+8\phi_{(ph)}}\varepsilon^{\hat{\boldsymbol{\beta}}}e_m{}^b\gamma_{b\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}} + \\
& + \tfrac{1}{4}e_m{}^a\Big\{ 2e_{[b|}{}^k\partial_k\phi\,\eta_{|c]a} + 4e_a{}^k e_{[b}{}^n\eta_{c]d}\left(\psi_n{}^{\boldsymbol{\delta}}\psi_k{}^{\boldsymbol{\gamma}}\gamma^d_{\boldsymbol{\gamma}\boldsymbol{\delta}}\right) - 2e_b{}^k e_c{}^n\eta_{ad}\left(\hat{\psi}_n{}^{\hat{\boldsymbol{\delta}}}\hat{\psi}_k{}^{\hat{\boldsymbol{\gamma}}}\gamma^d_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\right) + \\
& - e_a{}^n\psi_n{}^{\boldsymbol{\gamma}}\gamma_{bc\,\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}} + \tfrac{3}{2}h_{abc}e^{-2\phi}\Big\}\gamma^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\varepsilon^{\boldsymbol{\beta}} - \tfrac{1}{2}\left(\psi_m{}^{\boldsymbol{\gamma}}\lambda_{\boldsymbol{\gamma}} - \partial_m\phi\right)\varepsilon^{\boldsymbol{\alpha}} + \\
& \underbrace{+\tfrac{1}{8}(\varepsilon^{\boldsymbol{\beta}}\gamma_{de\,\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}} - \hat{\varepsilon}^{\hat{\boldsymbol{\beta}}}\gamma_{de\,\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}})\psi_m{}^{\boldsymbol{\gamma}}\gamma^{de}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}}_{\text{Lorentz trafo}} + \underbrace{\tfrac{1}{4}(\varepsilon^{\boldsymbol{\beta}}\lambda_{\boldsymbol{\beta}} - \hat{\varepsilon}^{\hat{\boldsymbol{\beta}}}\hat{\lambda}_{\hat{\boldsymbol{\beta}}})\psi_m{}^{\boldsymbol{\alpha}}}_{\text{dilatation}}
\end{aligned} \tag{5.408}
$$

$$
\begin{aligned}
\delta_\varepsilon\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}} \;=\; & \nabla^{(LC)}_m\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}} + 2e^{2\phi+8\phi_{(ph)}}\varepsilon^{\boldsymbol{\beta}}e_m{}^b\gamma_{b\,\boldsymbol{\beta}\boldsymbol{\delta}}\mathfrak{p}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}} + \\
& + \tfrac{1}{4}e_m{}^a\Big\{ 2e_{[b|}{}^k\partial_k\phi\,\eta_{|c]a} + 4e_a{}^k e_{[b}{}^n\eta_{c]d}\left(\hat{\psi}_n{}^{\hat{\boldsymbol{\delta}}}\hat{\psi}_k{}^{\hat{\boldsymbol{\gamma}}}\gamma^d_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\right) - 2e_b{}^k e_c{}^n\eta_{ad}\left(\psi_n{}^{\boldsymbol{\delta}}\psi_k{}^{\boldsymbol{\gamma}}\gamma^d_{\boldsymbol{\gamma}\boldsymbol{\delta}}\right) + \\
& - e_a{}^n\hat{\psi}_n{}^{\hat{\boldsymbol{\gamma}}}\gamma_{bc\,\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}} - \tfrac{3}{2}h_{abc}e^{-2\phi}\Big\}\gamma^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{\varepsilon}^{\hat{\boldsymbol{\beta}}} - \tfrac{1}{2}\Big\{\hat{\psi}_m{}^{\hat{\boldsymbol{\gamma}}}\hat{\lambda}_{\hat{\boldsymbol{\gamma}}} - \partial_m\phi\Big\}\hat{\varepsilon}^{\hat{\boldsymbol{\alpha}}} + \\
& + \underbrace{\tfrac{1}{8}(\hat{\varepsilon}^{\hat{\boldsymbol{\beta}}}\gamma_{de\,\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\lambda}_{\hat{\boldsymbol{\delta}}} - \varepsilon^{\boldsymbol{\beta}}\gamma_{de\,\boldsymbol{\beta}}{}^{\boldsymbol{\delta}}\lambda_{\boldsymbol{\delta}})\hat{\psi}_m{}^{\hat{\boldsymbol{\gamma}}}\gamma^{de}{}_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}}_{\text{Lorentz trafo}} + \underbrace{\tfrac{1}{4}(\hat{\varepsilon}^{\hat{\boldsymbol{\beta}}}\hat{\lambda}_{\hat{\boldsymbol{\beta}}} - \varepsilon^{\boldsymbol{\beta}}\lambda_{\boldsymbol{\beta}})\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}}}_{\text{dilatation}}
\end{aligned} \tag{5.409}
$$

Comparing these results with the ones obtained in "gauge I", i.e. with (5.398) for $\delta_\varepsilon\hat{\psi}_m{}^{\hat{\boldsymbol{\alpha}}}$ and with (5.396) together with the left-mover connection (5.395) for $\delta_\varepsilon\psi_m{}^{\boldsymbol{\alpha}}$, we recognize that they again differ just in the last lines and are related by a local Lorentz and scale transformation.

One can rewrite the result a bit using (D.166) whose graded version reads

$$\gamma_{ab\beta}{}^{\delta}\gamma^{ab}{}_{\gamma}{}^{\alpha} = \gamma_{ab\gamma}{}^{\delta}\gamma^{ab}{}_{\beta}{}^{\alpha} + 8\gamma^{a}_{\beta\gamma}\gamma_a^{\alpha\delta} + 20\delta_{[\beta}{}^{\alpha}\delta_{\gamma]}{}^{\delta} \tag{5.410}$$

This leads to

$$\begin{aligned}\tfrac{1}{8}(\varepsilon^{\beta}\gamma_{ab\beta}{}^{\delta}\lambda_{\delta})\psi_m{}^{\gamma}\gamma^{ab}{}_{\gamma}{}^{\alpha} &= \tfrac{1}{8}(\psi_m{}^{\gamma}\gamma_{ab\gamma}{}^{\delta}\lambda_{\delta})\gamma^{ab}{}_{\beta}{}^{\alpha}\varepsilon^{\beta} + (\varepsilon^{\beta}\gamma^{a}_{\beta\gamma}\psi_m{}^{\gamma})\gamma_a^{\alpha\delta}\lambda_{\delta} + \\ &\quad + \tfrac{5}{4}(\psi_m{}^{\gamma}\lambda_{\gamma})\varepsilon^{\alpha} - \tfrac{5}{4}(\varepsilon^{\beta}\lambda_{\beta})\psi_m{}^{\alpha}\end{aligned} \tag{5.411}$$

but is of no real advantage. However, the above gravitino transformation simplifies significantly, if we consider it at $\psi_m{}^{\mathcal{A}} = \lambda_{\mathcal{A}} = 0$ which is of special interest when we want to consider a string vacuum with vanishing vacuum expectation value of the fermionic fields. In addition we finally fix the bosonic compensator field ϕ to zero and arrive at

$$\boxed{\delta_{\varepsilon}\psi_m{}^{\mathcal{A}}\big|_{\psi=\lambda=0} = \nabla_m^{(LC)}\varepsilon^{\mathcal{A}} + \tfrac{3}{8}\epsilon_{(\mathcal{A})}e_m{}^{a}h_{abc}\varepsilon^{\mathcal{B}}\gamma^{bc}{}_{\mathcal{B}}{}^{\mathcal{A}} + 2e^{8\phi_{(ph)}}\varepsilon^{\mathcal{B}}e_m{}^{b}\gamma_{b\,\mathcal{B}\mathcal{D}}\mathfrak{p}^{\mathcal{A}\mathcal{D}}} \tag{5.412}$$

For convenience of the reader we present the result again with the split of the fermionic index:

$$\begin{aligned}\delta_{\varepsilon}\psi_m{}^{\alpha}\big|_{\psi=\lambda=0} &= \nabla_m^{(LC)}\varepsilon^{\alpha} + \tfrac{3}{8}e_m{}^{a}h_{abc}\varepsilon^{\beta}\gamma^{bc}{}_{\beta}{}^{\alpha} + 2e^{8\phi_{(ph)}}\hat{\varepsilon}^{\hat{\beta}}e_m{}^{b}\gamma_{b\,\hat{\beta}\hat{\delta}}\mathfrak{p}^{\alpha\hat{\delta}} &(5.413)\\ \delta_{\varepsilon}\hat{\psi}_m{}^{\hat{\alpha}}\Big|_{\psi=\lambda=0} &= \nabla_m^{(LC)}\hat{\varepsilon}^{\hat{\alpha}} - \tfrac{3}{8}e_m{}^{a}h_{abc}\hat{\varepsilon}^{\hat{\beta}}\gamma^{bc}{}_{\hat{\beta}}{}^{\hat{\alpha}} + 2e^{8\phi_{(ph)}}\varepsilon^{\beta}e_m{}^{b}\gamma_{b\,\beta\delta}\mathfrak{p}^{\delta\hat{\alpha}} &(5.414)\end{aligned}$$

This differs from the form that one can find in the literature (e.g. [67]) by a redefinition $8\phi_{(ph)} \to \phi_{(ph)}$, $\mathfrak{p}^{\alpha\hat{\delta}} \to \frac{1}{32}\mathfrak{p}^{\alpha\hat{\delta}}$ and by a redefinition $3H_{mnk} \to H_{mnk}$ where the latter discrepancy was simply due to our different definition of the wedge product.

5.15.4 The dilatino transformation

According to (5.366), the dilatinos are related to the dilaton superfield via

$$\lambda_{\mathcal{A}} = \nabla_{\mathcal{A}}\Phi_{(ph)}\big| = \underset{\longleftrightarrow}{\nabla}_{\mathcal{A}}\Phi_{(ph)} \tag{5.415}$$

Note that for the dilaton $\Phi_{(ph)}$ (in contrast to the compensator field Φ) it does not make a difference with which connection we act, because it is a scalar field. As described in the appendix, the covariant derivative of the scalar field transforms like a vector under supergauge transformations which leads to the following simple local supersymmetry transformation of the dilatino (see in the appendix on page 227):

$$\delta_{\varepsilon}\lambda_{\mathcal{A}} = \varepsilon^{\mathcal{C}}\,\nabla_{\mathcal{C}}\nabla_{\mathcal{A}}\Phi_{(ph)}\big| \tag{5.416}$$

For the second action of the covariant derivative the connection of course plays a role and $\nabla_{\mathcal{C}}$ has to be replaced by $\underset{\longleftrightarrow}{\nabla}_{\mathcal{C}}$ in gauge II. The transformation can be rewritten in terms of the $\vec{\boldsymbol{\theta}}^2$ component of the dilaton superfeld according to H.239 on page 227 as

$$\begin{aligned}\delta_{\varepsilon}\lambda_{\mathcal{A}} &= -\varepsilon^{\mathcal{C}}\,T_{\mathcal{C}\mathcal{A}}{}^{b}\big|\,e_b{}^{k}\partial_k\phi_{(ph)} + \varepsilon^{\mathcal{C}}\,T_{\mathcal{C}\mathcal{A}}{}^{b}\big|\,\psi_b{}^{\mathcal{K}}\lambda_{\mathcal{K}} - \varepsilon^{\mathcal{C}}\,T_{\mathcal{C}\mathcal{A}}{}^{\mathcal{B}}\big|\,\lambda_{\mathcal{B}} + \\ &\quad + \varepsilon^{\mathcal{C}}\delta_{\mathcal{C}}{}^{\mathcal{M}}\delta_{\mathcal{A}}{}^{\mathcal{K}}\,\partial_{\mathcal{M}}\partial_{\mathcal{K}}\Phi_{(ph)}\big|\end{aligned} \tag{5.417}$$

In any case we need more information about constraints on the dilaton superfield, in order to write down the explicit transformation. In footnote 28 on page 111 we have derived a constraint on $\nabla_{\hat{\beta}}\nabla_{\alpha}\Phi = \nabla_{\hat{\beta}}\nabla_{\alpha}\Phi_{(ph)}$, and in a similar way it should be possible to extract more information on $\nabla_{\beta}\nabla_{\alpha}\Phi_{(ph)}$. Without such constraints it is therefore not yet very useful to write down the transformation in both gauges. An interesting difference of the two gauges, however, is the location of the dilatinos in the compensator superfield, which we will quickly discuss:

Gauge I In gauge I we have in particular $\Omega^{(D)}_{\mathcal{A}}\Big| = 0$. The constraint $\nabla_{\hat{\alpha}}\Phi = 0$ and the relation $\nabla_{\alpha}\Phi = \nabla_{\alpha}\Phi_{(ph)}$ thus imply

$$\begin{aligned}\partial_{\hat{\mu}}\Phi| &= 0 &(5.418)\\ \partial_{\mu}\Phi| &= \partial_{\mu}\Phi_{(ph)}\big| = \lambda_{\mu} &(5.419)\end{aligned}$$

The relation $\hat{\nabla}_{\hat{\alpha}}\Phi = \nabla_{\alpha}\Phi_{(ph)}$ (together with the above $\partial_{\hat{\mu}}\Phi| = 0$) and the constraint $\hat{\nabla}_{\alpha}\Phi = 0$ (together with the above $\partial_{\mu}\Phi| = \partial_{\mu}\Phi_{(ph)}|$) on the other hand imply

$$\begin{aligned}\hat{\Omega}^{(D)}_{\hat{\mu}}\Big| &= -\,\partial_{\hat{\mu}}\Phi_{(ph)}\big| = -\hat{\lambda}_{\hat{\mu}} &(5.420)\\ \hat{\Omega}^{(D)}_{\mu}\Big| &= \partial_{\mu}\Phi_{(ph)}\big| = \lambda_{\mu} &(5.421)\end{aligned}$$

Only one of the dilatinos is thus part of the compensator field, while both are contained in $\hat{\Omega}^{(D)}_{\mathcal{M}}$ which should in this gauge not be seen as scale part of the connection but as scale part of the difference tensor

$$\Delta^{(D)}_{\mathcal{M}}\Big| = \hat{\Omega}^{(D)}_{\mathcal{M}}\Big| - \Omega^{(D)}_{\mathcal{M}}\Big| = \hat{\Omega}^{(D)}_{\mathcal{M}}\Big| \tag{5.422}$$

Let me add one more step in this new version of the document. With the information that we already had in the first arXiv version (namely the constraint $\nabla_{\hat{\beta}}\nabla_{\alpha}\Phi = -\tilde{\gamma}_{d\,\alpha\rho}\mathcal{P}^{\rho\hat{\varepsilon}}\gamma^{d}_{\hat{\varepsilon}\hat{\beta}}$ of footnote 28), we can actually write down explicitely at least half of the supersymmetry transformation of the dilatino. Simply start with (5.416) and plug in everything we know

$$\begin{aligned}
\delta_{\varepsilon}\lambda_{\alpha} &= \varepsilon^{\gamma}\,\nabla_{\gamma}\nabla_{\alpha}\Phi_{(ph)}\big| + \hat{\varepsilon}^{\hat{\gamma}}\,\nabla_{\hat{\gamma}}\nabla_{\alpha}\Phi_{(ph)}\big| = \\
&= \varepsilon^{\gamma}\,\nabla_{\gamma}\nabla_{\alpha}\Phi_{(ph)}\big| - \hat{\varepsilon}^{\hat{\gamma}}e^{2\phi}\gamma_{d\,\alpha\rho}e^{8\phi_{(ph)}}\mathfrak{p}^{\rho\hat{\varepsilon}}\gamma^{d}_{\hat{\varepsilon}\hat{\gamma}} \\
\delta_{\varepsilon}\hat{\lambda}_{\hat{\alpha}} &= \varepsilon^{\gamma}\,\nabla_{\gamma}\nabla_{\hat{\alpha}}\Phi_{(ph)}\big| + \hat{\varepsilon}^{\hat{\gamma}}\,\nabla_{\hat{\gamma}}\nabla_{\hat{\alpha}}\Phi_{(ph)}\big| = \\
&= \varepsilon^{\gamma}\,\nabla_{\hat{\alpha}}\nabla_{\gamma}\Phi_{(ph)}\big| - 2\varepsilon^{\gamma}\,T_{\gamma\hat{\alpha}}{}^{C}\nabla_{C}\Phi_{(ph)}\big| + \hat{\varepsilon}^{\hat{\gamma}}\,\nabla_{\hat{\gamma}}\nabla_{\hat{\alpha}}\Phi_{(ph)}\big| = \\
&= -\varepsilon^{\gamma}e^{2\phi}\gamma_{d\,\gamma\rho}e^{8\phi_{(ph)}}\mathfrak{p}^{\rho\hat{\varepsilon}}\gamma^{d}_{\hat{\varepsilon}\hat{\alpha}} + \varepsilon^{\gamma}\left(\tfrac{1}{4}\gamma_{de\gamma}{}^{\delta}\gamma^{de}{}_{\hat{\alpha}}{}^{\hat{\gamma}}\lambda_{\delta} + \tfrac{1}{2}\lambda_{\gamma}\delta_{\hat{\alpha}}{}^{\hat{\gamma}}\right)\hat{\lambda}_{\hat{\delta}} + \hat{\varepsilon}^{\hat{\gamma}}\,\nabla_{\hat{\gamma}}\nabla_{\hat{\alpha}}\Phi_{(ph)}\big|
\end{aligned}$$

The second term of the last line would vanish for $\lambda = \psi = 0$. As mentioned before, we need some additional constraints on $\nabla_{\gamma}\nabla_{\alpha}\Phi_{(ph)}$ and $\nabla_{\hat{\gamma}}\nabla_{\hat{\alpha}}\Phi_{(ph)}$ to determine the missing second half of the transformations respectively.

Gauge II In gauge II the situation is fortunately more symmetric and we have $\underset{\leftrightarrow}{\Omega}{}^{(D)}_{\mathcal{A}}\Big| = 0$ and $\underset{\leftrightarrow}{\nabla}_{\mathcal{A}}\Phi = \frac{1}{2}\left(\nabla_{\mathcal{A}}\Phi + \hat{\nabla}_{\mathcal{A}}\Phi\right) = \frac{1}{2}\nabla_{\mathcal{A}}\Phi_{(ph)}$. This (together with $\hat{\nabla}_{\alpha}\Phi = \nabla_{\hat{\alpha}}\Phi = 0$) implies

$$\partial_{\mathcal{M}}\Phi| = \tfrac{1}{2}\lambda_{\mathcal{M}} \tag{5.423}$$

$$\Omega^{(D)}_{\hat{\mu}}\Big| = \tfrac{1}{2}\hat{\lambda}_{\hat{\mu}} = -\,\hat{\Omega}^{(D)}_{\hat{\mu}}\Big| \quad \Rightarrow \Delta^{(D)}_{\hat{\mu}}\Big| = -\hat{\lambda}_{\hat{\mu}} \tag{5.424}$$

$$\hat{\Omega}^{(D)}_{\mu}\Big| = \tfrac{1}{2}\lambda_{\mu} = -\,\Omega^{(D)}_{\mu}\Big| \quad \Rightarrow \Delta^{(D)}_{\mu}\Big| = \lambda_{\mu} \tag{5.425}$$

According to the first line both dilatinos are contained in the compensator superfield in this gauge. Their local supersymmetry transformation could thus also be determined by the transformation of the compensator superfield which is, however, of no advantage and gives the same result.

Again we add one more step with respect to the 1st arXiv version of this document, in order to obtain at least half of the SUSY transformation in an explicit form. In the gauge II, (5.416) becomes for $\mathcal{A} = \alpha$

$$\begin{aligned}
\delta_{\varepsilon}\lambda_{\alpha} &= \varepsilon^{\gamma}\,\underset{\leftrightarrow}{\nabla}_{\gamma}\nabla_{\alpha}\Phi_{(ph)}\Big| + \hat{\varepsilon}^{\hat{\gamma}}\,\underset{\leftrightarrow}{\nabla}_{\hat{\gamma}}\nabla_{\alpha}\Phi_{(ph)}\Big| = \\
&= \varepsilon^{\gamma}\,\nabla_{\gamma}\nabla_{\alpha}\Phi_{(ph)}\big| - \tfrac{1}{2}\varepsilon^{\gamma}\,\Delta_{\gamma\alpha}{}^{\delta}\nabla_{\delta}\Phi_{(ph)}\big| + \hat{\varepsilon}^{\hat{\gamma}}\,\nabla_{\hat{\gamma}}\nabla_{\alpha}\Phi_{(ph)}\big| - \tfrac{1}{2}\hat{\varepsilon}^{\hat{\gamma}}\,\Delta_{\hat{\gamma}\alpha}{}^{\delta}\nabla_{\delta}\Phi_{(ph)}\big| = \\
&= \varepsilon^{\gamma}\,\nabla_{\gamma}\nabla_{\alpha}\Phi_{(ph)}\big| - \left(\tfrac{1}{4}(\varepsilon^{\gamma}\lambda_{\gamma})\lambda_{\alpha} + \tfrac{1}{8}(\varepsilon^{\gamma}\gamma_{bc\,\gamma}{}^{\varepsilon}\lambda_{\varepsilon})\gamma^{bc}{}_{\alpha}{}^{\delta}\lambda_{\delta}\right) + \\
&\quad -\hat{\varepsilon}^{\hat{\gamma}}e^{2\phi}\gamma_{d\,\alpha\rho}e^{8\phi_{(ph)}}\mathfrak{p}^{\rho\hat{\varepsilon}}\gamma^{d}_{\hat{\varepsilon}\hat{\gamma}} + \left(\tfrac{1}{4}(\hat{\varepsilon}^{\hat{\gamma}}\hat{\lambda}_{\hat{\gamma}})\lambda_{\alpha} + \tfrac{1}{8}(\hat{\varepsilon}^{\hat{\gamma}}\gamma_{bc\,\hat{\gamma}}{}^{\hat{\varepsilon}}\hat{\lambda}_{\hat{\varepsilon}})\gamma^{bc}{}_{\alpha}{}^{\delta}\lambda_{\delta}\right)
\end{aligned}$$

For $\lambda = \psi = 0$ the terms in the brackets disappear and we end up with the same expression as in gauge I. In gauge II the transformation of $\hat{\lambda}_{\hat{\alpha}}$ can be simply obtained by the unbroken left-right symmetry.

The transformation of the remaining fields in a general form (constraints not yet plugged into the equations) can be found in the appendix after page 225.

5.A Constraints before the BI's

Reduced structure group constraints The following equations are taken from (5.94)-(5.96), (5.152) or (5.154) and (5.159)

$$\Omega_{M\alpha}{}^{\beta} = \frac{1}{2}\Omega^{(D)}_{M}\delta_{\alpha}{}^{\beta} + \frac{1}{4}\Omega^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\alpha}{}^{\beta}, \qquad \hat{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} = \frac{1}{2}\hat{\Omega}^{(D)}_{M}\delta_{\hat{\alpha}}{}^{\hat{\beta}} + \frac{1}{4}\hat{\Omega}^{(L)}_{Ma_1a_2}\gamma^{a_1a_2}{}_{\hat{\alpha}}{}^{\hat{\beta}} \tag{5.426}$$

$$C_{\alpha}{}^{\beta\hat{\gamma}} = \frac{1}{2}C^{\hat{\gamma}}\delta_{\alpha}{}^{\beta} + \frac{1}{4}C^{\hat{\gamma}}_{a_1a_2}\gamma^{a_1a_2}{}_{\alpha}{}^{\beta}, \qquad \hat{C}_{\hat{\alpha}}{}^{\hat{\beta}\gamma} = \frac{1}{2}\hat{C}^{\gamma}\delta_{\hat{\alpha}}{}^{\hat{\beta}} + \frac{1}{4}\hat{C}^{\gamma}_{a_1a_2}\gamma^{a_1a_2}{}_{\hat{\alpha}}{}^{\hat{\beta}} \tag{5.427}$$

$$\begin{aligned}
S_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} &= \frac{1}{4}S\delta_{\alpha}{}^{\beta}\delta_{\hat{\alpha}}{}^{\hat{\beta}} + \frac{1}{8}S_{a_1a_2}\delta_{\alpha}{}^{\beta}\gamma^{a_1a_2}{}_{\hat{\alpha}}{}^{\hat{\beta}} + \\
&\quad + \frac{1}{8}\hat{S}_{a_1a_2}\gamma^{a_1a_2}{}_{\alpha}{}^{\beta}\delta_{\hat{\alpha}}{}^{\hat{\beta}} + \frac{1}{16}S_{a_1a_2b_1b_2}\gamma^{a_1a_2}{}_{\alpha}{}^{\beta}\gamma^{b_1b_2}{}_{\hat{\alpha}}{}^{\hat{\beta}}
\end{aligned} \tag{5.428}$$

$$G_{MN} = E_M{}^{a}G_{ab}E_N{}^{b}, \qquad G_{ab} = e^{2\Phi}\eta_{ab} \tag{5.429}$$

The above equations (without the last one) are equivalent to

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0 \quad (5.430)$$

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}} = 0 \quad (5.431)$$

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\beta}}} = 0 \quad (5.432)$$

As discussed in the appendix G on page 199, the spinorial left-mover connection $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ induces via invariance of the small gamma-matrices a whole superspace left-mover connection $\Omega_{MA}{}^{B}$. Likewise the spinorial rightmover-connection $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ induces a superspace right-mover connection $\hat{\Omega}_{MA}{}^{B}$. The constraints (5.430) then apply in the same way for $\hat{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$:

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\check{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\check{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0 \qquad \text{for any } \check{\Omega} \text{ which is Lorentz plus scale} \quad (5.433)$$

Let us denote the difference one-form between the left-mover and the rightmover connection by

$$\Delta_{MA}{}^{B} \equiv \hat{\Omega}_{MA}{}^{B} - \Omega_{MA}{}^{B} = \begin{pmatrix} \Delta_{Ma}{}^{b} & 0 & 0 \\ 0 & \Delta_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \Delta_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \quad (5.434)$$

The above restrictions on the spinorial connections induces the same restrictions on the difference tensor

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\Delta_{C\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\Delta_{C\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0 \quad (5.435)$$

Further constraints on C and S and indirectly on $\mathcal{P}$ The constraints (5.184) and (5.185) on C and (5.188) and (5.189) on S (all on page 65) can be regarded as defining equations. We have already shown in section 5.12 that the two equations for S are equivalent up to Bianchi identities.

$$C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} = \underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \quad (5.436)$$

$$\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}\boldsymbol{\gamma}} = \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \quad (5.437)$$

$$S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}} = -\underline{\nabla}_{\boldsymbol{\alpha}}\underbrace{\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}}_{\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}}} + 2\hat{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \quad (5.438)$$

$$S_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = -\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\underbrace{C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}}_{\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}} + 2R_{\hat{\boldsymbol{\alpha}}\boldsymbol{\gamma}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}} \quad (5.439)$$

Combining them with the reduced structure group constraints (5.430),(5.431) and (5.432), we obtain:

$$\gamma^{a_1\ldots a_4}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = 0, \qquad \gamma^{a_1\ldots a_4}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\alpha}}}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}} = 0 \quad (5.440)$$

The reduced structure group of S instead doesn't provide additional information. It is induced[18] by the reduced structure group property of C and of the curvature R.

Constraints on H Due to (5.167)-(5.171), (5.226), (5.229) and the total antisymmetry of H, its only nonvanishing components are

$$H_{abc} \neq 0 \qquad \text{(in general)} \quad (5.441)$$

$$H_{\boldsymbol{\alpha}\boldsymbol{\beta}c} = -\frac{2}{3}\check{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}|c} \equiv -\frac{2}{3}\gamma^{a}_{\boldsymbol{\alpha}\boldsymbol{\beta}}f_{ac} \quad (5.442)$$

$$H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c} = \frac{2}{3}\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}|c} \equiv \frac{2}{3}\gamma^{a}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{f}_{ac} \quad (5.443)$$

The vanishing components are thus (written a bit redundantly)

$$H_{ab\boldsymbol{\mathcal{C}}} = H_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}C} = H_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}}} = 0 \quad (5.444)$$

[18] We have $\nabla_M\gamma^{a_1a_2}{}_{a_3a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \hat{\nabla}_M\gamma^{a_1a_2}{}_{a_3a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \check{\nabla}_M\gamma^{a_1a_2}{}_{a_3a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0$ by definition, because ∇_M and the others are defined via $\nabla_M\gamma^{a}_{\boldsymbol{\alpha}\boldsymbol{\beta}} = 0$. Therefore we have for the mixed connection

$$\underline{\nabla}_M\tilde{\gamma}^{a_1a_2a_3a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = -4\check{\nabla}_M\Phi\cdot\tilde{\gamma}^{a_1a_2a_3a_4}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + 4(\check{\Omega}-\Omega)_{Mc}{}^{[a_1|}\tilde{\gamma}^{c|a_2a_3a_4]}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$$

All terms on the righthand side are proportional to $\gamma^{[4]}$, and therefore we have schematically

$$\gamma^{[4]}S \propto \gamma^{[4]}\left(-\underline{\nabla}C + 2R\mathcal{P}\right) \propto -\underline{\nabla}\underbrace{(\gamma^{[4]}C)}_{=0} + \underbrace{(\underline{\nabla}\gamma^{[4]})C}_{\propto\gamma^{[4]}C=0} + 2\underbrace{\gamma^{[4]}R}_{=0}\mathcal{P} = 0$$

The reduced structure group condition $\gamma^{[4]}S = 0$ is thus a consequence of $\gamma^{[4]}C = 0$ and $\gamma^{[4]}R = 0$. ⋄

Constraints on the torsion Let us now collect the information of the constraints (5.168)-(5.170), (5.180)-(5.183) and (5.227),(5.230),(5.243). The only (a priori) nonvanishing components of the torsion $\underline{T}_{AB}{}^{C}$ are

$$\check{T}_{\boldsymbol{\mathcal{A}}(c|d)} = -\frac{1}{2}\check{\nabla}_{\boldsymbol{\mathcal{A}}}\Phi G_{cd} \tag{5.445}$$

$$\check{T}_{\boldsymbol{\alpha\beta}|c} = -\frac{3}{2}H_{\boldsymbol{\alpha\beta}c} = \gamma^{d}_{\boldsymbol{\alpha\beta}}f_{dc}, \qquad \check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}|c} = \frac{3}{2}H_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c} = \gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{f}_{dc} \tag{5.446}$$

$$\hat{T}_{\boldsymbol{\alpha}c}{}^{\hat{\boldsymbol{\gamma}}} = \check{T}_{\boldsymbol{\alpha\delta}|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} = \gamma^{d}_{\boldsymbol{\alpha\delta}}f_{dc}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}, \qquad T_{\hat{\boldsymbol{\alpha}}c}{}^{\boldsymbol{\gamma}} = \check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}|c}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} = \gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\hat{f}_{dc}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} \tag{5.447}$$

$$\underline{T}_{ab}{}^{C} \neq 0 \qquad \text{(in general)} \tag{5.448}$$

The remaining components all vanish, which can be written (again a bit redundantly) as

$$\underline{T}_{\boldsymbol{\mathcal{AB}}}{}^{\boldsymbol{\mathcal{C}}} = \underline{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}{}^{C} = T_{\boldsymbol{\alpha}d}{}^{\boldsymbol{\gamma}} = \hat{T}_{\hat{\boldsymbol{\alpha}}d}{}^{\hat{\boldsymbol{\gamma}}} = 0 \tag{5.449}$$

The above constraints are constraints on the torsion $\underline{T}_{AB}{}^{C} = (\check{T}_{AB}{}^{c}, T_{AB}{}^{\boldsymbol{\gamma}}, \hat{T}_{AB}{}^{\hat{\boldsymbol{\gamma}}})$, which is based on the mixed connection $\underline{\Omega}_{AB}{}^{C}$ defined in (5.66) on page 50. When solving the Bianchi identities in the next local appendix, the bosonic block $\check{\Omega}_{Ma}{}^{b}$ of the connection will be chosen for convenience to sometimes coincide with the left-mover connection $\Omega_{Ma}{}^{b}$ (induced by $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$) or with the right mover connection (induced by $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\beta}}$; see appendix G on page 199). Not only for the bosonic block, but also for the fermionic blocks, information on torsion based on left-or right-mover connection, instead of the mixed connection will be important later. This information is in principle given by the difference-tensor $\Delta_{MA}{}^{B}$, introduced above in (5.434). Complete knowledge of the difference tensor, allows to calculate the corresponding torsion components via

$$\hat{T}_{AB}{}^{C} - T_{AB}{}^{C} = \Delta_{[AB]}{}^{C} \tag{5.450}$$

Due to the block diagonality of the connection and the difference tensor, some of these torsion components do not contain the connection at all. If we denote by $\check{\Omega}_{MA}{}^{B}$ the connection which is induced by the bosonic block of the mixed connection (i.e. it is block diagonal and Lorentz plus scale, but otherwise arbitrary), then we have

$$\hat{T}_{\boldsymbol{\mathcal{AB}}}{}^{c} = T_{\boldsymbol{\mathcal{AB}}}{}^{c} = \check{T}_{\boldsymbol{\mathcal{AB}}}{}^{c} = (\mathbf{d}E^{c})_{\boldsymbol{\mathcal{AB}}} \tag{5.451}$$

$$\hat{T}_{\{a,\hat{\boldsymbol{\alpha}}\}\{b,\hat{\boldsymbol{\beta}}\}}{}^{\boldsymbol{\gamma}} = T_{\{a,\hat{\boldsymbol{\alpha}}\}\{b,\hat{\boldsymbol{\beta}}\}}{}^{\boldsymbol{\gamma}} = \check{T}_{\{a,\hat{\boldsymbol{\alpha}}\}\{b,\hat{\boldsymbol{\beta}}\}}{}^{\boldsymbol{\gamma}} = (\mathbf{d}E^{\boldsymbol{\gamma}})_{\{a,\hat{\boldsymbol{\alpha}}\}\{b,\hat{\boldsymbol{\beta}}\}} \tag{5.452}$$

$$\hat{T}_{\{a,\boldsymbol{\alpha}\}\{b,\boldsymbol{\beta}\}}{}^{\hat{\boldsymbol{\gamma}}} = T_{\{a,\boldsymbol{\alpha}\}\{b,\boldsymbol{\beta}\}}{}^{\hat{\boldsymbol{\gamma}}} = \check{T}_{\{a,\boldsymbol{\alpha}\}\{b,\boldsymbol{\beta}\}}{}^{\hat{\boldsymbol{\gamma}}} = (\mathbf{d}E^{\hat{\boldsymbol{\gamma}}})_{\{a,\boldsymbol{\alpha}\}\{b,\boldsymbol{\beta}\}} \tag{5.453}$$

The brackets $\{a,\boldsymbol{\alpha}\}\{b,\boldsymbol{\beta}\}$ shall denote that the equation holds if the index A is either a or $\boldsymbol{\alpha}$ (but not $\hat{\boldsymbol{\alpha}}$), while the index B is either b or $\boldsymbol{\beta}$ (but not $\hat{\boldsymbol{\beta}}$).

Constraints on the curvature Induced by the restricted structure group constraints on the connection, we have such constraints likewise for the curvature (see (5.68) on page 50 and (F.88),(F.90) and (F.92) on page F.90. The curvature is blockdiagonal and each part decays into a scale part and a Lorentz part:

$$\underline{R}_{ABC}{}^{D} = \mathrm{diag}\,(\check{R}_{ABc}{}^{d}, R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}, \hat{R}_{AB\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}) \tag{5.454}$$

$$\check{R}_{ABc}{}^{d} = \check{F}^{(D)}_{AB}\delta^{d}_{c} + \check{R}^{(L)}_{AB\,c}{}^{d}, \qquad \check{F}^{(D)}_{AB} = \frac{1}{10}\check{R}_{ABc}{}^{c} \tag{5.455}$$

$$R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \frac{1}{2}F^{(D)}_{AB}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \frac{1}{4}R^{(L)}_{ABa_1}{}^{b}\eta_{ba_2}\gamma^{a_1a_2}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}, \qquad F^{(D)}_{AB} = -\frac{1}{8}R_{AB\boldsymbol{\gamma}}{}^{\boldsymbol{\gamma}} \tag{5.456}$$

$$\hat{R}_{AB\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\gamma}}} = \frac{1}{2}\hat{F}^{(D)}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\hat{R}^{(L)}_{ABa_1}{}^{b}\eta_{ba_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}, \qquad \hat{F}^{(D)}_{AB} = -\frac{1}{8}\hat{R}_{AB\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.457}$$

with the scale field strength

$$\check{F}^{(D)} \equiv \mathbf{d}\check{\Omega}^{(D)}, \qquad F^{(D)} \equiv \mathbf{d}\Omega^{(D)}, \qquad \hat{F}^{(D)} \equiv \mathbf{d}\hat{\Omega}^{(D)} \tag{5.458}$$

The bosonic field strength is also obtained via the commutator of covariant derivatives acting on the compensator field Φ. Only the bosonic block $\check{\Omega}_{Ma}{}^{b}$ of the mixed connection $\underline{\Omega}_{MA}{}^{B}$ acts on Φ, because Φ is a compensator for the transformation of G_{ab} (with bosonic indices):

$$\check{F}^{(D)}_{MN} = -\underline{\nabla}_{[M}\check{\nabla}_{N]}\Phi - \underline{T}_{MN}{}^{K}\check{\nabla}_{K}\Phi \tag{5.459}$$

Finallly we had a couple of holomorphicity (5.186),(5.187),(5.190),(5.191) and nilpotency constraints (5.228),(5.231) on the curvature:

$$\hat{R}_{\boldsymbol{\alpha}c\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \underbrace{\check{T}_{\boldsymbol{\alpha\delta}|c}}_{\gamma^d_{\boldsymbol{\alpha\delta}} f_{dc}} \underbrace{\hat{C}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}\boldsymbol{\delta}}}_{\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}}}, \qquad R_{\hat{\boldsymbol{\alpha}}c\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \underbrace{\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}|c}}_{\gamma^d_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\hat{f}_{dc}} \underbrace{C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}}}_{\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}}} \tag{5.460}$$

$$\hat{R}_{\boldsymbol{\alpha\gamma}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = 0, \qquad R_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0 \tag{5.461}$$

$$\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5} R_{d\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}} = 0, \qquad \gamma^{\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}_{a_1\ldots a_5} \hat{R}_{d\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.462}$$

$$\gamma^{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}_{a_1\ldots a_5} R_{\hat{\boldsymbol{\delta}}\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2}{}^{\boldsymbol{\beta}} = 0, \qquad \gamma^{\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}_{a_1\ldots a_5} \hat{R}_{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.463}$$

$$R_{[\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\boldsymbol{\alpha}_3]}{}^{\boldsymbol{\beta}} = 0, \qquad \hat{R}_{[\hat{\boldsymbol{\alpha}}_1\hat{\boldsymbol{\alpha}}_2\hat{\boldsymbol{\alpha}}_3]}{}^{\hat{\boldsymbol{\beta}}} = 0 \tag{5.464}$$

Taking the trace of the first two curvature constraints gives further informations on dilatation-Field-strength and Lorentz curvature

$$\hat{F}^{(D)}_{\boldsymbol{\alpha}c} = -\frac{1}{8}\check{T}_{\boldsymbol{\alpha\delta}|c}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}, \qquad F^{(D)}_{\hat{\boldsymbol{\alpha}}c} = -\frac{1}{8}\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}|c}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}} \tag{5.465}$$

$$\hat{F}^{(D)}_{\boldsymbol{\alpha\gamma}} = 0, \qquad F^{(D)}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}} = 0 \tag{5.466}$$

The trace of the last curvature constraint we had provided already in (5.235):

$$F^{(D)}_{\boldsymbol{\gamma\delta}} = \frac{2}{9}R^{(L)}_{\boldsymbol{\alpha}[\boldsymbol{\gamma\delta}]}{}^{\boldsymbol{\alpha}}, \quad \hat{F}^{(D)}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}} = \frac{2}{9}\hat{R}^{(L)}_{\hat{\boldsymbol{\alpha}}[\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}]}{}^{\hat{\boldsymbol{\alpha}}} \tag{5.467}$$

5.B Bianchi identities for H

In this local appendix we will study explicitly all the Bianchi identities for the H-field. They are of the form

$$0 \stackrel{!}{=} \underline{\nabla}_{\boldsymbol{A}} H_{\boldsymbol{AAA}} + 3\underline{T}_{\boldsymbol{AA}}{}^{C} H_{C\boldsymbol{AA}} \tag{5.468}$$

This is equivalent to $\mathbf{d}H = 0$ and is independent of the connection, in particular independent of the precise form of $\check{\Omega}$. Sometimes it is thus convenient to calculate with the left-mover connection $\check{\Omega}_a{}^b = \Omega_a{}^b$ (the latter defined via $\nabla_M\gamma^a_{\boldsymbol{\alpha\beta}} = 0$, see appendix G on page 199) and sometimes we set $\check{\Omega}_a{}^b = \hat{\Omega}_a{}^b$ (defined via $\hat{\nabla}_M\gamma^a_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0$).

...

Let us now go back to the Bianchi identity (5.468), where we make use of $\underline{T}_{AB}{}^C$ instead of $T_{AB}{}^C$ or $\hat{T}_{AB}{}^C$. What we have just discussed is thus for the moment only relevant for the the third index being bosonic $C = c$, as we might choose $\underline{T}_{AB}{}^c \equiv \check{T}_{AB}{}^c$ to be either $T_{AB}{}^c$ or $\hat{T}_{AB}{}^c$.

Every index A of the Bianchi identity (5.468) can be either a, $\boldsymbol{\alpha}$ or $\hat{\boldsymbol{\alpha}}$. As all indices are antisymmetrized, we can distinguish the cases by specifying how often each type of index appears. We denote in brackets first the number of bosonic indices, then the number of unhatted fermionic indices and finally the number of hatted fermionic indices:($\#a$,$\#\boldsymbol{\alpha}$,$\#\hat{\boldsymbol{\alpha}}$). The sum has to add up to four: $\#a+\#\boldsymbol{\alpha}+\#\hat{\boldsymbol{\alpha}} = 4$. Each number is in $\{0,\ldots,4\}$ which has five elements. If $\#a$ is 0 there are five possibilities left for $\#\boldsymbol{\alpha}$ which fixes $\#\hat{\boldsymbol{\alpha}}$ =4-$\#\hat{\boldsymbol{\alpha}}$. If $\#a$ is 1, there are four possibilities left for $\#\boldsymbol{\alpha}$, and so on. Altogether there are $5+4+3+2+1 = 15$ distinct cases. However, some of them are related by the symmetry between hatted and unhatted indices: ($\#a$,$\#\boldsymbol{\alpha}$,$\#\hat{\boldsymbol{\alpha}}$)$\leftrightarrow$($\#a$,$\#\hat{\boldsymbol{\alpha}}$,$\#\boldsymbol{\alpha}$). This map has "fixed points" only for ($\#\hat{\boldsymbol{\alpha}}$,$\#\boldsymbol{\alpha}$)$\in \{(0,0),(1,1),(2,2)\}$. The effective number of equations we have to calculate is thus $\frac{15-3}{2} + 3 = 9$. In the following we go through all these cases.

- <u>(0,4,0)$\boldsymbol{\alpha\beta\gamma\delta}$ $\leftrightarrow$((0,0,4)$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}$):</u>[19]

$$0 \stackrel{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}} \underbrace{H_{\boldsymbol{\beta\gamma\delta}]}}_{=0\,(5.226)} + 3\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{C} H_{C|\boldsymbol{\gamma\delta}]} = \tag{5.469}$$

$$\stackrel{(5.226)}{\underset{(5.227)}{=}} 3\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{c} H_{c|\boldsymbol{\gamma\delta}]} = \tag{5.470}$$

$$= -2\gamma^d_{[\boldsymbol{\alpha\beta}|} f_d{}^c \gamma^e_{|\boldsymbol{\gamma}]\boldsymbol{\delta}} f_{ec} \tag{5.471}$$

[19]It might be confusing that we obtain in (5.470) a constraint not only on some components of H_{ABC}, but on a bilinear combination of H_{ABC} and $\underline{T}_{AB}{}^C$. At first sight this seems to contradict the equivalence to $\mathbf{d}H = 0$ which is clearly only a constraint on H. However, H_{ABC} depends on H (with components H_{MNK}) AND the vielbein. And the torsion component $\underline{T}_{\boldsymbol{\alpha\beta}}{}^c = (\mathbf{d}E^c)_{\boldsymbol{\alpha\beta}} + \underline{\Omega}_{\boldsymbol{\alpha\beta}}{}^c = (\mathbf{d}E^c)_{\boldsymbol{\alpha\beta}}$ happens to depend only on the vielbein. The bilinear constraint thus boils down to $(\mathbf{d}H)_{\boldsymbol{\alpha\beta\gamma\delta}} = 0$, as it should be. $\diamond$

The last line can only reduce to the Fierz identity $\gamma^d_{[\boldsymbol{\alpha\beta}|}\gamma_{d\,|\boldsymbol{\gamma}]\boldsymbol{\delta}} = 0$ for[20]

$$f_d{}^c G_{cb} f_e{}^b = (f\cdot G\cdot f^T)_{de} \overset{!}{\propto} G_{de} \propto \eta_{de} \tag{5.472}$$

The same is true for $\hat f$:

$$(\hat f\cdot G\cdot \hat f^T)_{ab} \propto G_{ab} \tag{5.473}$$

That means, f and $\hat f$ are proportional to a Lorentz transformation. In other words, *If nonzero, f and $\hat f$ are a composition of a Lorentz transformation and a scaling.*

Intermezzo on the fixing of two blocks of the structure group

The above result provides a possibility to relate the three (a priori independent) blocks of the structure group on the tangent space of the supermanifold. We can thus use the local Lorentz transformation (acting only on the unhatted spinor indices) and the local scale transformation (likewise acting only on the unhatted spinor indices) to fix f to unity and likewise use the hatted transformations to fix $\hat f$ to unity as it was done in [13]. We will do the same, although – regarding the subtleties discussed below – one should keep in mind that other kinds of gauge fixing might also have their advantages. The gauge fixing leads to the following constraints:

$$\boxed{\begin{aligned}
\check T_{\boldsymbol{\alpha\beta}}{}^c &= \gamma^c_{\boldsymbol{\alpha\beta}}, \qquad (f_a{}^b = \delta_a^b) && (5.474)\\
\check T_{\boldsymbol{\hat\alpha\hat\beta}}{}^c &= \gamma^c_{\boldsymbol{\hat\alpha\hat\beta}}, \qquad (\hat f_a{}^b = \delta_a^b) && (5.475)\\
\Rightarrow H_{\boldsymbol{\alpha\beta}c} &= -\frac{2}{3}\gamma^d_{\boldsymbol{\alpha\beta}}G_{dc} = -\frac{2}{3}e^{2\Phi}\gamma^d_{\boldsymbol{\alpha\beta}}\eta_{dc} \equiv -\frac{2}{3}\tilde\gamma_{c\,\boldsymbol{\alpha\beta}} && (5.476)\\
H_{\boldsymbol{\hat\alpha\hat\beta}c} &= \frac{2}{3}\gamma^d_{\boldsymbol{\hat\alpha\hat\beta}}G_{dc} = \frac{2}{3}e^{2\Phi}\gamma^d_{\boldsymbol{\hat\alpha\hat\beta}}\eta_{dc} \equiv \frac{2}{3}\tilde\gamma_{c\,\boldsymbol{\hat\alpha\hat\beta}} && (5.477)
\end{aligned}}$$

The constraints (5.474) and (5.475) have to be valid for any bosonic connection-block $\check\Omega_{Ma}{}^b$, in particular for the left and right-mover connections: $T_{\boldsymbol{\alpha\beta}}{}^c = \hat T_{\boldsymbol{\alpha\beta}}{}^c = \gamma^c_{\boldsymbol{\alpha\beta}}$. Due to $\Delta_{[\boldsymbol{\alpha\beta}]}{}^c = \Delta_{[\boldsymbol{\hat\alpha\hat\beta}]}{}^c = 0$, the constraints for $\check T_{\boldsymbol{\alpha\beta}}{}^c$ and $\check T_{\boldsymbol{\hat\alpha\hat\beta}}{}^c$ are constraints on the vielbein only. Having fixed the torsion components to the chiral gamma matrices, the latter should remain invariant under the reduced structure group. If we act with an infinitesimal transformation

$$L_a{}^b = L^{(D)}\delta_a^b + L_a^{(L)\,b}, \qquad \text{with } L^{(L)}_{ab} = -L^{(L)}_{ba} \tag{5.478}$$

on the bosonic index, it has to be compensated by the appropriate actions on the fermionic indices (compare to footnote 7 on page 49 for a derivation):

$$L_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}L^{(D)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}L^{(L)}_{ab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{5.479}$$

$$L_{\boldsymbol{\hat\alpha}}{}^{\boldsymbol{\hat\beta}} = \frac{1}{2}L^{(D)}\delta_{\boldsymbol{\hat\alpha}}{}^{\boldsymbol{\hat\beta}} + \frac{1}{4}L^{(L)}_{ab}\gamma^{ab}{}_{\boldsymbol{\hat\alpha}}{}^{\boldsymbol{\hat\beta}} \tag{5.480}$$

This guarantuees

$$\delta_{(L)}\gamma^a_{\boldsymbol{\alpha\beta}} \equiv L_c{}^a\gamma^c_{\boldsymbol{\alpha\beta}} - 2L_{[\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\gamma^a_{\boldsymbol{\beta}]\boldsymbol{\gamma}} = 0 \tag{5.481}$$

$$\delta_{(L)}\gamma^a_{\boldsymbol{\hat\alpha\hat\beta}} \equiv L_c{}^a\gamma^c_{\boldsymbol{\hat\alpha\hat\beta}} - 2L_{[\boldsymbol{\hat\alpha}}{}^{\boldsymbol{\hat\gamma}}\gamma^a_{\boldsymbol{\hat\beta}]\boldsymbol{\hat\gamma}} = 0 \tag{5.482}$$

[20]Let us make this somewhat fishy argument more precise and contract (5.471) with two chiral gamma matrices. In order to be able to apply some equations of appendix D we will switch for a moment to ungraded summation conventions (or equivalently perform a grading shift of the fermionic index). We also multiply the whole equation by $-\frac{3}{2}$ for convenience:

$$\begin{aligned}
0 &\overset{!}{=} 3\gamma_a^{\beta\alpha}\gamma_b^{\delta\gamma}\gamma^d_{(\alpha\beta|}f_d{}^c\gamma^e_{|\gamma)\delta}f_{ec} = \\
&= \gamma_a^{\beta\alpha}\gamma^d_{\alpha\beta}\gamma_b^{\delta\gamma}\gamma^e_{\gamma\delta}f_d{}^c f_{ec} + \gamma_a^{\beta\alpha}\gamma^d_{\gamma\alpha}\gamma_b^{\delta\gamma}\gamma^e_{\beta\delta}f_d{}^c f_{ec} + \gamma_a^{\beta\alpha}\gamma^d_{\beta\gamma}\gamma_b^{\delta\gamma}\gamma^e_{\alpha\delta}f_d{}^c f_{ec} = \\
&\underset{(D.108)}{\overset{(D.137)}{=\!=}} (16)^2 f_a{}^c f_{bc} + 2\cdot\left(\delta_a^d\delta_\gamma^\beta + \gamma_a{}^{d\,\beta}{}_\gamma\right)\left(\delta_b^e\delta_\beta^\gamma + \gamma_b{}^{e\,\gamma}{}_\beta\right) f_d{}^c f_{ec} = \\
&\underset{(D.140)}{\overset{(D.135)}{=}} (16)^2 f_a{}^c f_{bc} + 32\delta_a^d\delta_b^e f_d{}^c f_{ec} + 2\cdot 32 G_{af}\delta_{eb}^{fd} f_d{}^c f^e_c = \\
&= 16\cdot 18 f_a{}^c f_{bc} - 32 G_{ab} f_e{}^c f^e_c + 32 f_b{}^c f_{ac} = \\
&= 16\cdot 20\cdot f_a{}^c f_{bc} - 32 G_{ab} f_e{}^c f^e_c
\end{aligned}$$

We can now read off $f_a{}^c f_{bc} = (\frac{1}{10}f_e{}^c f^e_c)G_{ab}$ or $f\eta f^T = \frac{1}{10}\mathrm{tr}\,(f\eta f^T)\cdot\eta$, which means simply that $f_a{}^b$ is proportional to a Lorentz transformation. ◇

It is important to realize that $\gamma^a_{\alpha\beta}$ and $\gamma^a_{\hat\alpha\hat\beta}$ are not covariantly constant with respect to the mixed connection $\underline{\Omega}_{MA}{}^B$ that we have used so far. For the choice $\check\Omega_{Ma}{}^b = \Omega_{Ma}{}^b$ we get $\underline{\nabla}_M\gamma^a_{\hat\alpha\hat\beta} \neq 0$, for $\check\Omega_{Ma}{}^b = \hat\Omega_{Ma}{}^b$ we get $\underline{\nabla}_M\gamma^a_{\alpha\beta} \neq 0$ and for any other choice of $\check\Omega_{Ma}{}^b$ none of the γ-matrices will be covariantly conserved in general. Although all the equations written in terms of $\underline{\Omega}_{MA}{}^B$ remain of course formally valid, it is geometrically not a suitable connection any longer. Parallel transport would destroy our gauge. As mentioned at the beginning of section G on page 199, there are at least three natural choices for connections which leave the gamma matrices invariant, for example $\Omega_{MA}{}^B$ (defined by the left-mover connection), $\hat\Omega_{MA}{}^B$ (defined by the rightmover connection) and the average $\underset{\longleftrightarrow}{\Omega}_{MA}{}^B \equiv \frac{1}{2}\left(\Omega_{MA}{}^B + \hat\Omega_{MA}{}^B\right)$. These will be in particular relevant for the discussion of the WZ-gauge. For the further discussion of the Bianchi identities after this intermezzo, however, we stick formally to $\underline{\Omega}_{MA}{}^B$.

Type IIA/IIB Let us also give an important remark about the differences of type IIA and type IIB which become important only at this point. In type IIB, the hatted index $^{\hat\alpha}$ should be of the same chirality, while in type IIA, $^{\hat\alpha}$ should be of opposite chirality as $^{\alpha}$. This statement makes only sense, when the Lorentz-transformations of hatted and unhatted indeces are coupled, which was done only in the last steps above. Before, the distinction between IIA and IIB was merely deciding whether $\gamma^c_{\hat\alpha\hat\beta}$ is numerically equal to $\gamma^c_{\alpha\beta}$ (IIB) or to $\gamma^{c\,\alpha\beta}$ (IIA).

The transcription from the general equations (with hatted indices) to the case of **type IIB** is quite simple and direct, as the index positions do not change. The conditions $\nabla_M\gamma^c_{\alpha\beta} = 0$ and $\nabla_M\gamma^c_{\hat\alpha\hat\beta} = 0$ become numerically the same and imply that $\Omega_{M\hat\alpha}{}^{\hat\beta} = \Omega_{M\alpha}{}^{\beta}$ (same for the average connection). The hatted indices thus indeed transform with the same chirality (w.r.t. Lorentz) and in addition with the same representation of the scale transformation and the hats of the indices can simply be dropped.

For **type IIA** the situation is a bit more involved and requires some familiarity with the graded summation convention discussed around page 7 in the first part of the thesis. A downstairs hatted index $_{\hat\alpha}$ should in IIA in the end correspond to an upstairs unhatted index and vice verse. In a first step, we will still distinguish it from the unhatted index and write it (just for this paragraph) as a **tilded index** $^{\tilde\alpha}$ at opposite vertical position. NW conventions for the hatted indices would then correspond to NE conventions for the tilded index. We could stick to such mixed conventions (NW for the unhatted indices and NE for the tilded indices), but in order to make a comparison of the tilded with the undecorated index, it is better to switch back to NW for the tilded index as well. In principle this works as follows: spell out the NW summation conventions for the hatted indices explicitely, replace the hatted by the tilded in opposite vertical position and write it again in terms of the graded summation convention based on NW. We call this an **index-position-shift**. For example for the action of the covariant derivative on a spinor with upper hatted index, this yields

$$\nabla_M\psi^{\hat\alpha} = \partial_M\psi^{\hat\alpha} + \Omega_{M\hat\gamma}{}^{\hat\alpha}\psi^{\hat\gamma} = \tag{5.483}$$

$$= \partial_M\psi^{\hat\alpha} + \sum_{\hat\gamma}\underbrace{(-)^{\hat\gamma\hat\alpha+\hat\gamma}}_{1}\Omega_{M\hat\gamma}{}^{\hat\alpha}\psi^{\hat\gamma} = \tag{5.484}$$

$$= \partial_M\psi_{\tilde\alpha} - \sum_{\tilde\gamma}(-)^{\tilde\gamma\tilde\alpha}\Omega_M{}^{\tilde\gamma}{}_{\tilde\alpha}\psi_{\tilde\gamma} = \tag{5.485}$$

$$= \partial_M\psi_{\tilde\alpha} - \Omega_M{}^{\tilde\gamma}{}_{\tilde\alpha}\psi_{\tilde\gamma} \tag{5.486}$$

In order to get back our usual index position for the connection (first fermionic index down, second up), we finally define

$$\Omega_{M\tilde\gamma}{}^{\tilde\alpha} \equiv \Omega_M{}^{\tilde\alpha}{}_{\tilde\gamma} \quad (= \Omega_{M\hat\gamma}{}^{\hat\alpha}) \tag{5.487}$$

where the equalities should be understood as graded equalities in the sense of (1.29) on page 9. Upon this identification, the action of the covariant derivative on a lower tilded index takes the usual form $\nabla_M\psi_{\tilde\alpha} = \partial_M\psi_{\tilde\alpha} - \Omega_{M\tilde\alpha}{}^{\tilde\gamma}\psi_{\tilde\gamma}$. Equation (5.487) also guarantees that the action of a covariant derivative on a lower hatted index becomes the correct action on the corresponding upper tilded index, i.e. $\nabla_M\psi_{\hat\alpha} = \partial_M\psi_{\hat\alpha} - \Omega_{M\hat\alpha}{}^{\hat\gamma}\psi_{\hat\gamma} = \partial_M\psi^{\tilde\alpha} + \Omega_{M\tilde\gamma}{}^{\tilde\alpha}\psi^{\tilde\gamma} = \nabla_M\psi^{\tilde\alpha}$. Now we are finally able to compare the connections $\Omega_{M\tilde\alpha}{}^{\tilde\beta}$ and $\Omega_{M\alpha}{}^{\beta}$ and see whether we can identify them like in type IIB. First note that like for the symmetry algebra generators (5.479) and (5.480) themselves, the invariance conditions $\nabla_M\gamma^c_{\alpha\beta} = 0$ and $\nabla_M\gamma^c_{\hat\alpha\hat\beta} = 0$ determine the spinorial connections to be of the form (see again footnote 7 on page 49 for a derivation)

$$\Omega_{M\alpha}{}^{\beta} = \tfrac{1}{2}\Omega^{(D)}_M\delta_\alpha{}^\beta + \tfrac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_\alpha{}^\beta \tag{5.488}$$

$$\Omega_{M\hat\alpha}{}^{\hat\beta} = \tfrac{1}{2}\Omega^{(D)}_M\delta_{\hat\alpha}{}^{\hat\beta} + \tfrac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_{\hat\alpha}{}^{\hat\beta} \tag{5.489}$$

The Kronecker delta in the second line will be rewritten upon the index-position shift as $\delta_{\hat\alpha}{}^{\hat\beta} = \delta^{\hat\beta}_{\hat\alpha} = \delta^{\tilde\alpha}_{\tilde\beta} = -\delta^{\tilde\alpha}{}_{\tilde\beta}$. Finally we make use of the facts that $\gamma^{ab\,\tilde\alpha}{}_{\tilde\beta}$ is graded equal to $\gamma^{ab}{}_{\tilde\beta}{}^{\tilde\alpha}$ (according to (D.110) in the appendix),

$\delta^{\tilde{\boldsymbol{\alpha}}}{}_{\tilde{\boldsymbol{\beta}}}$ is graded equal to $\delta_{\tilde{\boldsymbol{\beta}}}{}^{\tilde{\boldsymbol{\alpha}}}$ and of the identification (5.487) to arrive at

$$\Omega_{M\tilde{\boldsymbol{\alpha}}}{}^{\tilde{\boldsymbol{\beta}}} = -\tfrac{1}{2}\Omega^{(D)}_{M}\delta_{\tilde{\boldsymbol{\alpha}}}{}^{\tilde{\boldsymbol{\beta}}} + \tfrac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_{\tilde{\boldsymbol{\alpha}}}{}^{\tilde{\boldsymbol{\beta}}} \tag{5.490}$$

Therefore the tilded indices transform in the same way under Lorentz, but with opposite sign under scale transformations as the untilded indices. Only when the scale transformations are fixed, tilded and untilded indices can be identified. This can be seen differently, by simply doing the identification and imposing $\nabla_M\gamma^c_{\boldsymbol{\alpha\beta}} = \nabla_M\gamma^{c\,\boldsymbol{\alpha\beta}} = 0$ which implies via the Clifford algebra ($\gamma^{(a|\,\boldsymbol{\alpha\gamma}}\gamma^{|b)}_{\boldsymbol{\gamma\beta}} = -\eta^{ab}\delta^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$, the graded version of (D.108) of page 176) that $\nabla_M\eta^{ab} = 0$. But scale transformation do not leave invariant the Minkowski metric. In summary, keeping the (anyway auxiliary) scale transformations unfixed seems a bit artificial in type IIA and is more natural in type IIB.

Let us now proceed with the discussion of the Bianchi identities for the H-field.

- <u>(0,3,1)$\boldsymbol{\alpha\beta\gamma}\hat{\boldsymbol{\delta}}$ $\leftrightarrow$((0,1,3)$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}\boldsymbol{\delta}$):</u>

$$0 \overset{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}H_{\boldsymbol{\beta\gamma}\hat{\boldsymbol{\delta}}]} + 3\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{C}H_{C|\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}]} = 0 \quad \text{(due to (5.169), (5.171), and (5.226))} \tag{5.491}$$

No new constraints from this one.

Remark: As in the above equation we will make use of all the constraints that we have derived from the BRST invariance and nilpotency. As it is cumbersome to specify each time explicitely which constraint we have used, we will not do it everywhere. Every constraint that we use without referring to its equation number will be taken from (5.167)-(5.171) (page 63), (5.180)-(5.193) (page 65), (5.226)-(5.231) (page 70) and (5.243) on page 71. These are all the framed equations. However, to the newly gained constraints within this local appendix (which will be framed as well) we will refer explicitely.

- <u>(0,2,2)$\boldsymbol{\alpha\beta}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}$:</u>

$$0 \overset{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}H_{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}]} + 3\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{C}H_{C|\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}]} = \tag{5.492}$$
$$\propto \underline{T}_{\boldsymbol{\alpha\beta}}{}^{c}H_{c\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}} + \underline{T}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}{}^{c}H_{c\boldsymbol{\alpha\beta}} = \tag{5.493}$$
$$\propto \gamma^a_{\boldsymbol{\alpha\beta}}f_a{}^c\gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\hat{f}_{bc} - \gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\hat{f}_b{}^c\gamma^a_{\boldsymbol{\alpha\beta}}f_{ac} = \tag{5.494}$$
$$= \gamma^a_{\boldsymbol{\alpha\beta}}\gamma^b_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\left(f_a{}^c\hat{f}_{bc} - \hat{f}_b{}^c f_{ac}\right) = 0 \tag{5.495}$$

- <u>(1,3,0)$\boldsymbol{\alpha\beta\gamma}d$ $\leftrightarrow$((1,0,3)$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}d$):</u>[21]

$$0 \overset{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}H_{\boldsymbol{\beta\gamma}d]} + 3\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{C}H_{C|\boldsymbol{\gamma}d]} = \tag{5.496}$$
$$= \frac{3}{4}\underline{\nabla}_{[\boldsymbol{\alpha}}H_{\boldsymbol{\beta\gamma}]d} + \frac{3}{2}\check{T}_{[\boldsymbol{\beta}|d}{}^{c}H_{c|\boldsymbol{\gamma\alpha}]} = \tag{5.497}$$
$$= -\frac{1}{2}\underline{\nabla}_{[\boldsymbol{\alpha}}(\gamma^c_{\boldsymbol{\beta\gamma}]}G_{cd}) - \check{T}_{[\boldsymbol{\beta}|d|c}\gamma^c_{|\boldsymbol{\gamma\alpha}]} = \tag{5.498}$$
$$\overset{\check{\Omega}_{Ma}{}^{b}=\Omega_{Ma}{}^{b}}{=} -\gamma^c_{[\boldsymbol{\beta\gamma}}\Big(\nabla_{\boldsymbol{\alpha}]}\Phi\cdot G_{cd} + \underbrace{T_{\boldsymbol{\alpha}]d|c}}_{T_{\boldsymbol{\alpha}][d|c]}-\frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\cdot G_{dc}=-T_{\boldsymbol{\alpha}]c|d}-\nabla_{\boldsymbol{\alpha}}\Phi\cdot G_{dc}}\Big) = \tag{5.499}$$
$$= \gamma^c_{[\boldsymbol{\beta\gamma}}T_{\boldsymbol{\alpha}]c|d} \tag{5.500}$$

---

[21] Remember $\check{T}_{\boldsymbol{\alpha}(c|d)} = -\frac{1}{2}\check{\nabla}_{\boldsymbol{\alpha}}\Phi G_{cd} = \frac{1}{2}E_{\boldsymbol{\alpha}}{}^{M}(\check{\Omega}^{(D)}_{M} - \partial_M\Phi)G_{cd}$. This can be reformulated as a condition on the vielbein only:

$$\check{T}_{\boldsymbol{\alpha}c|d} = (\mathbf{d}E^e)_{\boldsymbol{\alpha}c}G_{ed} + \underbrace{\check{\Omega}_{[\boldsymbol{\alpha}c]}{}^{e}G_{ed}}_{\equiv\check{\Omega}_{[\boldsymbol{\alpha}c]|d}}$$
$$\check{T}_{\boldsymbol{\alpha}(c|d)} = (\mathbf{d}E^e)_{\boldsymbol{\alpha}(c}G_{d)e} + \frac{1}{2}\check{\Omega}_{\boldsymbol{\alpha}(c|d)} =$$
$$= (\mathbf{d}E^e)_{\boldsymbol{\alpha}(c}G_{d)e} + \frac{1}{2}\check{\Omega}^{(D)}_{\boldsymbol{\alpha}}G_{cd}$$
$$\Rightarrow (\mathbf{d}E)_{\boldsymbol{\alpha}(c|d)} \equiv (\mathbf{d}E^e)_{\boldsymbol{\alpha}(c}G_{d)e} = -\frac{1}{2}E_{\boldsymbol{\alpha}}{}^{M}\partial_M\Phi\, G_{cd}$$

Reparametrizing $\tilde{E}_M{}^A \equiv e^{\Phi}E_M{}^A$, this can be rewritten as

$$(\mathbf{d}\tilde{E})_{\boldsymbol{\alpha}(c|d)} = \left(E_{[\boldsymbol{\alpha}|}{}^{M}\partial_M\Phi\cdot e^{\Phi}G_{|c]d} - \frac{1}{2}e^{\Phi}E_{\boldsymbol{\alpha}}{}^{M}\partial_M\Phi\, G_{cd}\right) = 0 \quad \text{or } \tilde{\check{T}}_{\boldsymbol{\alpha}(c|d)} = 0$$

in accordance with [13]. $\diamond$

In the fourth line we made the choice of $\check{\Omega}_{Ma}{}^{b}$ in such a way that $\underline{\nabla}_{\boldsymbol{\alpha}}\gamma^{c}_{\boldsymbol{\beta\gamma}} = 0$. In the following calculations we will use a lot of gamma-matrix identities from appendix D where we did not use graded conventions. We will therefore temporarily switch to non-graded conventions (or equivalently perform a grading shift of the fermionic indices).

As a first step to solve the constraint (5.500), let us contract it with $\gamma_a^{\alpha\beta}$:

$$0 \overset{!}{=} \gamma_a^{\alpha\beta}\gamma^c_{\alpha\beta}T_{\gamma c|d} + \gamma_a^{\alpha\beta}\gamma^c_{\gamma\alpha}T_{\beta c|d} + \gamma_a^{\alpha\beta}\gamma^c_{\beta\gamma}T_{\alpha c|d} = \tag{5.501}$$

$$\overset{(D.108),(D.110)}{=} 16T_{\gamma a|d} + 2\left(\delta^c_a\delta^\beta_\gamma + \gamma^c{}_a{}_\gamma{}^\beta\right)T_{\beta c|d} = \tag{5.502}$$

$$\Rightarrow 9T_{\gamma a|d} = \gamma_a{}^c{}_\gamma{}^\beta T_{\beta c|d} \tag{5.503}$$

Although the contraction with $\gamma_a^{\alpha\beta}$ looks like a projection, the new equation (5.503) still contains all the information of (5.500) (in the nongraded version, the graded antisymmetrization becomes an ordinary symmetrization):

$$\gamma^c_{(\beta\gamma}T_{\alpha)c|d} \overset{(5.503)}{=} \frac{1}{9}\gamma^c_{(\beta\gamma|}\gamma_c{}^e{}_{|\alpha)}{}^\delta T_{\delta e|d} = \tag{5.504}$$

$$\overset{(D.108),(D.110)}{=} \frac{1}{9}\gamma^c_{(\beta\gamma|}\left(\gamma_{c|\alpha)}{}^\varepsilon\gamma^e{}_\varepsilon{}^\delta - \delta^e_c\delta^\delta_{|\alpha)}\right)T_{\delta e|d} = \tag{5.505}$$

$$\overset{(D.160)}{=} -\frac{1}{9}\gamma^c_{(\beta\gamma|}T_{|\alpha)c|d} \tag{5.506}$$

Comparing the first and the last line leads back to (5.500). This was just to argue that we can forget now about (5.500), and take (5.503) as new starting point. Remember that we have already a constraint for the symmetrized part (in c and d) of $T_{\alpha c|d}$ and let let us in addition introduce a temporary notation for the yet unknown antisymmetrized part:

$$T_{\alpha(c|d)} = -\frac{1}{2}\nabla_\alpha\Phi G_{cd}, \qquad T_{\alpha[c|d]} \equiv \dot{T}_{\alpha cd} \tag{5.507}$$

Now we split (5.503) into its symmetric and its antisymmetric part in a and d (the symmetric part is multiplied by (-2) for convenience):[22]

$$9\nabla_\gamma\Phi G_{ad} = 2\tilde{\gamma}_{c(a|\gamma}{}^\beta\dot{T}_\beta{}^c{}_{|d)} = \tilde{\gamma}_{ca\gamma}{}^\beta\dot{T}_\beta{}^c{}_d + \tilde{\gamma}_{cd\gamma}{}^\beta\dot{T}_\beta{}^c{}_a \tag{5.508}$$

$$9\dot{T}_{\gamma ad} = \tilde{\gamma}_{[a|c\gamma}{}^\beta\dot{T}_\beta{}^c{}_{|d]} - \frac{1}{2}\tilde{\gamma}_{ad\gamma}{}^\beta\nabla_\beta\Phi \tag{5.509}$$

In order to solve this kind of equations, it always helps to take traces (we will use the trace of (5.508) soon) and to contract with several combinations of γ-matrices. Here it turns out to be useful to contract (5.508) with $\tilde{\gamma}^{ab}{}_\alpha{}^\gamma$. The antisymmetrization in the bosonic indices of the result will produce a term similar to the one in (5.509), s.th. the equations can then be combined. But let us first perform the contraction. We will use the following gamma-matrix identities (see (D.117) on page 177):

$$\tilde{\gamma}^{ab}\tilde{\gamma}_{ca} = \delta^a_a\gamma^b{}_c - \delta^b_a\gamma^a{}_c - \delta^a_c\gamma^b{}_a + \delta^a_a\delta^b_c\mathbb{1} - \delta^a_c\delta^b_a\mathbb{1} = 8\gamma^b{}_c + 9\delta^b_c\mathbb{1} \tag{5.510}$$

$$\tilde{\gamma}^{ab}\tilde{\gamma}_{cd} = \gamma^{ab}{}_{cd} + \delta^b_c\gamma^a{}_d + \delta^a_d\gamma^b{}_c - \delta^b_d\gamma^a{}_c - \delta^a_c\gamma^b{}_d + \delta^a_d\delta^b_c\mathbb{1} - \delta^a_c\delta^b_d\mathbb{1} \tag{5.511}$$

The $\gamma^{[4]}$ part in the second equation could be removed by taking a symmetrization. This, however, would in the end only lead back to (5.508). Instead, note that the same $\gamma^{[4]}$ is produced in the product $\gamma_d{}^b\gamma_c{}^a$. And this combination is more useful, as we can then apply $\gamma_c{}^a{}_\alpha{}^\beta T_\beta{}^c{}_a \overset{(5.508)}{=} 45\nabla_\alpha\Phi$:

$$-\gamma_d{}^b\gamma_c{}^a = \gamma^{ab}{}_{cd} + \delta^b_c\gamma^a{}_d - \delta^a_d\gamma^b{}_c + G^{ba}\tilde{\gamma}_{dc} + G_{cd}\tilde{\gamma}^{ba} - \delta^a_d\delta^b_c\mathbb{1} + G_{cd}G^{ba}\mathbb{1} \tag{5.512}$$

$$\Rightarrow \tilde{\gamma}^{ab}\tilde{\gamma}_{cd} = -\gamma_d{}^b\gamma_c{}^a + 2\delta^a_d\gamma^b{}_c - \delta^b_d\gamma^a{}_c - \delta^a_c\gamma^b{}_d - G^{ba}\tilde{\gamma}_{dc} - G_{cd}\tilde{\gamma}^{ba} + \left(2\delta^a_d\delta^b_c - \delta^a_c\delta^b_d - G_{cd}G^{ba}\right)\mathbb{1} \tag{5.513}$$

[22]The tilde on gamma matrices or antisymmetriced products between them just takes into account the correct scaling weight: $\gamma^a_{\alpha\beta}$ is invariant under scale transformations, if the transformations acting on bosonic and fermionic indices are coupled as in (5.478)-(5.480), i.e. if the fermionic scale transformation has an extra factor $\frac{1}{2}$. The bosonic metric $G_{ab} \equiv e^{2\Phi}\eta_{ab}$ and its inverse $G^{ab} \equiv e^{-2\Phi}\eta^{ab}$, used to lower and raise bosonic flat indices, however, are not scale invariant. Lowering an index of the gamma-matrix yields $\tilde{\gamma}_{a\,\alpha\beta} \equiv G_{ab}\gamma^b_{\alpha\beta} = e^{2\Phi}\gamma_{a\,\alpha\beta}$. The reason for the tilde is thus only to indicate that the gamma matrix is not the numerical one but has a Weyl factor in it which corresponds to the weight indicated by the index structure. Similarly we have

$$\tilde{\gamma}^{ab}{}_\alpha{}^\beta \equiv e^{-2\Phi}\gamma^{ab}{}_\alpha{}^\beta \qquad \diamond$$

The contraction of (5.508) with $\tilde{\gamma}^{ab}{}_{\alpha}{}^{\gamma}$ then yields (using (5.510), (5.513) and $\gamma_c{}^a{}_\alpha{}^\beta T_\beta{}^c{}_a \overset{(5.508)}{=} 45\nabla_\alpha\Phi$):

$$
\begin{aligned}
&9\tilde{\gamma}_d{}^b{}_\alpha{}^\gamma\nabla_\gamma\Phi \overset{!}{=} \\
&\overset{!}{=} \left(8\gamma^b{}_{c\alpha}{}^\beta + 9\delta^b_c\delta^\beta_\alpha\right)\dot{T}_\beta{}^c{}_d + && (5.514)\\
&\quad + \left(-\gamma_d{}^b\gamma_c{}^a + 2\delta^a_d\gamma^b{}_c - \delta^b_d\gamma^a{}_c - \delta^a_c\gamma^b{}_d - G^{ba}\tilde{\gamma}_{dc} - G_{cd}\tilde{\gamma}^{ba} + \left(2\delta^a_d\delta^b_c - \delta^a_c\delta^b_d - G_{cd}G^{ba}\right)\mathbb{1}\right)_\alpha{}^\beta\dot{T}_\beta{}^c{}_a = && (5.515)\\
&= 8\gamma^b{}_{c\alpha}{}^\beta\dot{T}_\beta{}^c{}_d + 9\dot{T}_\alpha{}^b{}_d + \\
&\quad -45\gamma_d{}^b{}_\alpha{}^\beta\nabla_\beta\Phi + 2\gamma^b{}_{c\alpha}{}^\beta\dot{T}_\beta{}^c{}_d + 45\delta^b_d\nabla_\alpha\Phi - \gamma^b{}_d\underbrace{\dot{T}^a{}_a}_{=0} + \\
&\quad -\tilde{\gamma}_{dc\,\alpha}{}^\beta\dot{T}_\beta{}^{cb} - \tilde{\gamma}^{ba}{}_\alpha{}^\beta\dot{T}_{\beta da} + 2\dot{T}_\alpha{}^b{}_d - \delta^b_d\underbrace{\dot{T}_\alpha{}^c{}_c}_{=0} - \dot{T}_{\alpha d}{}^b = && (5.516)\\
&= 45\gamma^b{}_{d\,\alpha}{}^\gamma\nabla_\gamma\Phi + 45\delta^b_d\nabla_\alpha\Phi + 12\dot{T}_\alpha{}^b{}_d + \\
&\quad +10\gamma^b{}_{c\alpha}{}^\beta\dot{T}_\beta{}^c{}_d + \tilde{\gamma}^{bc}{}_\alpha{}^\beta\dot{T}_{\beta cd} - \tilde{\gamma}_{dc\,\alpha}{}^\beta\dot{T}_\beta{}^{cb} && (5.517)
\end{aligned}
$$

Putting everything on one side and taking the antisymmetric part (in b,d) of this equation leads to

$$
\begin{aligned}
0 &\overset{!}{=} 54\tilde{\gamma}_{bd\alpha}{}^\gamma\nabla_\gamma\Phi + 12\dot{T}_{\alpha bd} + 12\tilde{\gamma}_{[b|c\,\alpha}{}^\beta\dot{T}_\beta{}^c{}_{|d]} = && (5.518)\\
&\overset{(5.509)}{=} 54\tilde{\gamma}_{bd\alpha}{}^\gamma\nabla_\gamma\Phi + 12\dot{T}_{\alpha bd} + 12\left(9\dot{T}_{\alpha bd} + \frac{1}{2}\tilde{\gamma}_{bd\alpha}{}^\beta\nabla_\beta\Phi\right) && (5.519)\\
\Rightarrow \dot{T}_{\alpha bd} &= -\frac{1}{2}\tilde{\gamma}_{bd\alpha}{}^\gamma\nabla_\gamma\Phi && (5.520)
\end{aligned}
$$

Let us switch back to the graded conventions. After this somewhat tedious calculation, we only need to combine this antisymmetric part ($T_{\boldsymbol{\alpha}[b|d]} \equiv \dot{T}_{\boldsymbol{\alpha}bd}$) with the symmetric one $T_{\boldsymbol{\alpha}(b|d)} = -\frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi G_{bd}$, in order to end up with the final result for the Bianchi identity (5.500)

$$\boxed{T_{\boldsymbol{\beta}c}{}^a = -\frac{1}{2}\nabla_{\boldsymbol{\beta}}\Phi\delta_c{}^a - \frac{1}{2}\gamma_c{}^a{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\gamma}}\Phi} \qquad (5.521)$$

Via the left-right symmetry, we get correspondingly

$$\boxed{\hat{T}_{\hat{\boldsymbol{\beta}}c}{}^a = -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi\delta_c{}^a - \frac{1}{2}\gamma_c{}^a{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi} \qquad (5.522)$$

- <u>(1,2,1)$\boldsymbol{\alpha\beta}\hat{\boldsymbol{\gamma}}d \leftrightarrow$((1,1,2)$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}d$):</u>

$$
\begin{aligned}
0 &\overset{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}H_{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}d]} + 3\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^E H_{E|\hat{\boldsymbol{\gamma}}d]} = && (5.523)\\
&= \frac{1}{4}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}H_{\boldsymbol{\alpha\beta}d} + \frac{1}{2}\hat{T}_{\boldsymbol{\alpha\beta}}{}^{\hat{\boldsymbol{\varepsilon}}}H_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\gamma}}d} + \frac{1}{2}\check{T}_{\hat{\boldsymbol{\gamma}}d}{}^e H_{e\boldsymbol{\alpha\beta}} = && (5.524)\\
&= -\frac{1}{6}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}(\gamma^c_{\boldsymbol{\alpha\beta}}f_{cd}) - \frac{1}{3}\check{T}_{\hat{\boldsymbol{\gamma}}d}{}^e\gamma^c_{\boldsymbol{\alpha\beta}}f_{ce} = && (5.525)\\
&\underset{\check{\Omega}=\Omega}{\overset{f_{ce}=G_{ce}\ (5.474)}{=}} -\frac{1}{3}\gamma^c_{\boldsymbol{\alpha\beta}}\underbrace{\left(\nabla_{\hat{\boldsymbol{\gamma}}}\Phi G_{cd} + T_{\hat{\boldsymbol{\gamma}}d|c}\right)}_{-T_{\hat{\boldsymbol{\gamma}}c|d}\ (5.168)} && (5.526)
\end{aligned}
$$

$$\overset{(5.445)}{\Rightarrow} \boxed{T_{\hat{\boldsymbol{\gamma}}c}{}^d = 0, \quad \nabla_{\hat{\boldsymbol{\gamma}}}\Phi = 0} \qquad (5.527)$$

Likewise we have

$$\boxed{\hat{T}_{\boldsymbol{\alpha}b}{}^c = 0, \qquad \hat{\nabla}_{\boldsymbol{\gamma}}\Phi = 0} \qquad (5.528)$$

These results can also be used to determine $\nabla_a\Phi$:

$$
\begin{aligned}
\underbrace{\hat{\nabla}_{[\boldsymbol{\alpha}}\hat{\nabla}_{\boldsymbol{\beta}]}\Phi}_{=0} &= -\hat{T}_{\boldsymbol{\alpha\beta}}{}^C\hat{\nabla}_C\Phi - \hat{F}^{(D)}_{\boldsymbol{\alpha\beta}} = \\
&= -\underbrace{\hat{T}_{\boldsymbol{\alpha\beta}}{}^c}_{=\gamma^c_{\boldsymbol{\alpha\beta}}}\hat{\nabla}_c\Phi - \hat{T}_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}}\underbrace{\hat{\nabla}_{\boldsymbol{\gamma}}\Phi}_{=0} - \underbrace{\hat{T}_{\boldsymbol{\alpha\beta}}{}^{\hat{\boldsymbol{\gamma}}}}_{=0}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \underbrace{\hat{F}^{(D)}_{\boldsymbol{\alpha\beta}}}_{=0\ (5.466)}
\end{aligned}
$$

The above equation and its hatted counterpart imply

$$\boxed{\hat{\nabla}_c\Phi = \nabla_c\Phi = 0} \qquad (5.529)$$

We can play this game once more and consider the commutator

$$\underbrace{\hat{\nabla}_{[\boldsymbol{\alpha}}\hat{\nabla}_{b]}\Phi}_{=0} = -\hat{T}_{\boldsymbol{\alpha}b}{}^{C}\hat{\nabla}_{C}\Phi - \hat{F}^{(D)}_{\boldsymbol{\alpha}b} = \tag{5.530}$$

$$= -\hat{T}_{\boldsymbol{\alpha}b}{}^{c}\underbrace{\hat{\nabla}_{c}\Phi}_{=0} - \hat{T}_{\boldsymbol{\alpha}b}{}^{\boldsymbol{\gamma}}\underbrace{\hat{\nabla}_{\boldsymbol{\gamma}}\Phi}_{=0} - \underbrace{\hat{T}_{\boldsymbol{\alpha}b}{}^{\hat{\boldsymbol{\gamma}}}}_{=\tilde{\gamma}_{b\boldsymbol{\alpha}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \hat{F}^{(D)}_{\boldsymbol{\alpha}b} \tag{5.531}$$

Due to (5.465) we have $\hat{F}^{(D)}_{\boldsymbol{\alpha}b} = -\frac{1}{8}\tilde{\gamma}_{b\,\boldsymbol{\alpha}\boldsymbol{\delta}}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}$ and therefore

$$\boxed{\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}} = 8\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}}\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi} \tag{5.532}$$

The hatted version of this equation reads

$$\boxed{\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}} = 8\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\beta}}\Phi} \tag{5.533}$$

- <u>(2,2,0)$ab\boldsymbol{\alpha}\boldsymbol{\beta}$ $\leftrightarrow$((2,0,2)$ab\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}$):</u>[23]

$$0 \overset{!}{=} \underline{\nabla}_{[a}H_{b\boldsymbol{\alpha}\boldsymbol{\beta}]} + 3\underline{T}_{[ab|}{}^{C}H_{C|\boldsymbol{\alpha}\boldsymbol{\beta}]} = \tag{5.534}$$

$$= \frac{1}{2}\underline{\nabla}_{[a}H_{b]\boldsymbol{\alpha}\boldsymbol{\beta}} + \frac{1}{2}\check{T}_{ab}{}^{c}H_{c\boldsymbol{\alpha}\boldsymbol{\beta}} + \frac{1}{2}\check{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{c}H_{cab} = \tag{5.535}$$

$$= \frac{1}{2}\underline{\nabla}_{[a|}(-\frac{2}{3}\gamma^{c}_{\boldsymbol{\alpha}\boldsymbol{\beta}}f_{c|b]}) - \frac{1}{2}\gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\left(\frac{2}{3}\check{T}_{ab}{}^{c}f_{dc} - f_{d}{}^{c}H_{cab}\right) = \tag{5.536}$$

$$\overset{f_{cb}=G_{cb}}{\underset{\check{\Omega}=\Omega}{=}} -\frac{1}{3}\gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\big(T_{ab|d} - \frac{3}{2}H_{dab} + 2\underbrace{\nabla_{[a}\Phi}_{0\,(5.529)}\;G_{b]d}\big) \tag{5.537}$$

Using $\frac{1}{16}\gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\gamma_{c}^{\boldsymbol{\alpha}\boldsymbol{\beta}} = \delta^{d}_{c}$ we get

$$\boxed{T_{ab|d} = \frac{3}{2}H_{abd}} \tag{5.538}$$

Likewise we have[24]

$$\boxed{\hat{T}_{ab|d} = -\frac{3}{2}H_{abd}} \tag{5.539}$$

Intermezzo on the difference tensor

We have finally obtained the last ingredient to calculate the explicit form of the difference tensor (5.434) between the connections $\hat{\Omega}$ and Ω. The difference tensor is block-diagonal like the connections and we have in particular $\Delta_{[\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}]}{}^{c} = 0$. Using $\Delta_{[AB]}{}^{c} = \hat{T}_{AB}{}^{c} - T_{AB}{}^{c}$ with $\hat{T}_{ab|c} = \frac{3}{2}H_{abc}$, $T_{ab|c} = -\frac{3}{2}H_{abc}$ and $\hat{T}_{\boldsymbol{\alpha}b|c} = T_{\hat{\boldsymbol{\alpha}}b|c} = 0$, we can give a simple expression for $\Delta_{[AB]}{}^{c}$. At the same time we have information about the difference tensor

[23]Combinatorically $[ab][\boldsymbol{\alpha}\boldsymbol{\beta}]$ arises 4 times in all 24 possibilities$\Rightarrow \frac{4}{24} = \frac{1}{6}$ $\diamond$

[24]As a consitency check, we compute the BI's for the index-combination $ab\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}$ explicitely with T (not $\hat{T}$):

$$0 \overset{!}{=} \nabla_{[a}H_{b\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}]} + 3T_{[ab|}{}^{C}H_{C|\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}]} =$$

$$= \frac{1}{2}\nabla_{[a}H_{b]\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} + \frac{1}{2}T_{ab}{}^{c}H_{c\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} + \frac{1}{2}T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{c}H_{cab} =$$

$$= \frac{1}{2}\nabla_{[a|}(\frac{2}{3}\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{f}_{c|b]}) + \frac{1}{2}\gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\left(\frac{2}{3}T_{ab}{}^{c}\hat{f}_{dc} + \hat{f}_{d}{}^{c}H_{cab}\right) =$$

$$\overset{\hat{f}_{cb}=G_{cb}}{=} \frac{1}{3}\nabla_{[a|}(\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}G_{c|b]}) + \frac{1}{3}\gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\left(T_{ab|d} + \frac{3}{2}H_{dab}\right) =$$

$$= \frac{1}{3}\nabla_{[a|}(\gamma^{c}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}})G_{c|b]} + \frac{1}{3}\gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\left(T_{ab|d} + \frac{3}{2}H_{dab} + 2\nabla_{a}\Phi G_{b]d}\right) =$$

$$= \frac{1}{3}\gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\Big(T_{ab|d} + \frac{3}{2}H_{dab}\quad \underbrace{-\Delta_{[a|d||b]}}_{+\Delta_{[ab]|d}-2\Delta_{[a}G_{b]d}}\Big) =$$

$$= \frac{1}{3}\gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\Big(\hat{T}_{ab|d} + \frac{3}{2}H_{dab}\Big) \quad \diamond$$

when it is symmetrized in its last two (bosonic) indices: $\Delta_{A(b|c)} = \big(\hat{\Omega}_A^{(D)} - \Omega_A^{(D)}\big) G_{bc} = \big(\nabla_A\Phi - \hat{\nabla}_A\Phi\big) G_{bc}$ with $\nabla_{\hat{\boldsymbol{\alpha}}}\Phi = \hat{\nabla}_{\boldsymbol{\alpha}}\Phi = \nabla_a\Phi = \hat{\nabla}_a\Phi = 0$. We can thus write down explicitely the antisymmetrized (in the first two indices) and the symmetrized (in the last two indices) difference tensor between left and right-mover connection

$$\Delta_{[AB]}{}^c = \begin{pmatrix} -3H_{ab}{}^c & -T_{a\boldsymbol{\beta}}{}^c & \hat{T}_{a\hat{\boldsymbol{\beta}}}{}^c \\ -T_{\boldsymbol{\alpha}b}{}^c & 0 & 0 \\ \hat{T}_{\hat{\boldsymbol{\alpha}}b}{}^c & 0 & 0 \end{pmatrix} \tag{5.540}$$

$$\Delta_{a(b|c)} = 0, \quad \Delta_{\boldsymbol{\alpha}(b|c)} = \nabla_{\boldsymbol{\alpha}}\Phi G_{bc}, \quad \Delta_{\hat{\boldsymbol{\alpha}}(b|c)} = -\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi G_{bc} \tag{5.541}$$

As $\Delta_{AB}{}^C$ is block diagonal in the last two indices, we know that $\Delta_{\boldsymbol{\mathcal{A}}b}{}^c = 2\Delta_{[\boldsymbol{\mathcal{A}}b]}{}^c$.For $\Delta_{ab}{}^c$ we can use (see (G.31))

$$\Delta_{ab|c} = \Delta_{[ab]|c} + \Delta_{[ca]|b} - \Delta_{[bc]|a} + \Delta_{a(c|b)} + \Delta_{b(c|a)} - \Delta_{c(b|a)} \tag{5.542}$$

The difference tensor with bosonic structure group indices is thus completely determined to be

$$\Delta_{Ab}{}^c: \quad \Delta_{ab|c} = -3H_{abc} \tag{5.543}$$

$$\Delta_{\boldsymbol{\alpha}b|c} = -2T_{\boldsymbol{\alpha}b|c} \overset{(5.521)}{=} \nabla_{\boldsymbol{\alpha}}\Phi G_{bc} + \tilde{\gamma}_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi \tag{5.544}$$

$$\Delta_{\hat{\boldsymbol{\alpha}}b|c} = 2\hat{T}_{\hat{\boldsymbol{\alpha}}b|c} \overset{(5.522)}{=} -\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi G_{bc} - \tilde{\gamma}_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi \tag{5.545}$$

This is consistent with (5.541) as well as with the left-right symmetry, if one defines $\hat{\Delta} \equiv -\Delta$. The components of the difference tensor with fermionic group indices are induced by the ones with bosonic group indices via

$$\Delta_{A\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = \frac{1}{2}\Delta_A^{(D)}\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} + \frac{1}{4}\Delta_{A[b|c]}\tilde{\gamma}^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}, \quad \Delta_{A\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = \frac{1}{2}\Delta_A^{(D)}\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} + \frac{1}{4}\Delta_{A[b|c]}\tilde{\gamma}^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.546}$$

Remember that this is due to the fact that both connections $\Omega_{MA}{}^B$ and $\hat{\Omega}_{MA}{}^B$ are defined to leave the chiral γ-matrices invariant. The components with fermionic group indices are accordingly

$$\Delta_{A\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}: \quad \Delta_{a\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = -\frac{3}{4}H_{abc}\tilde{\gamma}^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}, \quad \Delta_{a\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = -\frac{3}{4}H_{abc}\tilde{\gamma}^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.547}$$

$$\Delta_{\boldsymbol{\alpha\beta}}{}^{\boldsymbol{\gamma}} = \frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} + \frac{1}{4}\gamma_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi\gamma^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}, \quad \Delta_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} - \frac{1}{4}\gamma_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\gamma^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.548}$$

$$\Delta_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} = -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}} - \frac{1}{4}\gamma_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\gamma^{bc}{}_{\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}, \quad \Delta_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} = \frac{1}{2}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} + \frac{1}{4}\gamma_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi\gamma^{bc}{}_{\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.549}$$

We will use this difference tensor from now on frequently to change from one connection to another. Let us take immediate advantage of the difference tensor to rewrite some constraints on the curvature with the help of equation (193) of the appendix.

$$\hat{R}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \underbrace{R_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}}_{=0} + \hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Delta_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \gamma_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}}{}^c\Delta_{c\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \Delta_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\Delta_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}} = \tag{5.550}$$

$$= -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\gamma_{ab\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} - \frac{3}{4}\gamma_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}}{}^cH_{cab}\tilde{\gamma}^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \left(-\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}} - \frac{1}{4}\gamma_{ab\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\varepsilon}}}\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\right)\left(-\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\delta_{\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}} - \frac{1}{4}\gamma_{cd\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\varphi}}}\hat{\nabla}_{\hat{\boldsymbol{\varphi}}}\Phi\gamma^{cd}{}_{\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}}\right) = \tag{5.551}$$

$$= -\frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\gamma_{ab\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} - \frac{3}{4}\gamma_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}}{}^cH_{cab}\tilde{\gamma}^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{16}(\gamma_{ab\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\varepsilon}}}\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi)(\gamma_{cd\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\varphi}}}\hat{\nabla}_{\hat{\boldsymbol{\varphi}}}\Phi)\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\gamma^{cd}{}_{\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}} \tag{5.552}$$

In order to simplify the last term, let us suppress the fermionic indices for a moment. The last line then reads $\frac{1}{32}\Big((\gamma_{ab}\hat{\nabla}\Phi)(\gamma_{cd}\hat{\nabla}\Phi) - (\gamma_{cd}\hat{\nabla}\Phi)(\gamma_{ab}\hat{\nabla}\Phi)\Big)\gamma^{ab}\gamma^{cd}$. Now we can use

$$\gamma^{ab}\gamma^{cd} = \gamma^{abcd} + \eta^{bc}\gamma^{ad} + \eta^{ad}\gamma^{bc} - \eta^{ac}\gamma^{bd} - \eta^{bd}\gamma^{ac} + \eta^{bc}\eta^{ad} - \eta^{ac}\eta^{bd} \tag{5.553}$$

Due to the contraction with $(\gamma_{cd}\hat{\nabla}\Phi)(\gamma_{ab}\hat{\nabla}\Phi) - (ab \leftrightarrow cd)$, the $\gamma^{[4]}$-term and the $\gamma^{[0]}$-term ($\eta^{b[c}\eta^{d]a}$) disappear. We are left with

$$\frac{1}{32}\Big((\gamma_{ab}\hat{\nabla}\Phi)(\gamma_{cd}\hat{\nabla}\Phi) - (ab \leftrightarrow cd)\Big)\gamma^{ab}\gamma^{cd} = \frac{1}{4}(\gamma_{ab}\hat{\nabla}\Phi)\eta^{bc}(\gamma_{cd}\hat{\nabla}\Phi)\gamma^{ad} \tag{5.554}$$

The curvature component in question and its hatted version thus become

$$\boxed{\begin{aligned} \hat{R}_{\hat{\gamma}\hat{\gamma}\hat{\alpha}}{}^{\hat{\beta}} &= -\frac{1}{2}\hat{\nabla}_{\hat{\gamma}}\hat{\nabla}_{\hat{\gamma}}\Phi\delta_{\hat{\alpha}}{}^{\hat{\beta}} + \\ &+\frac{1}{4}\left(\gamma_{ad\hat{\gamma}}{}^{\hat{\delta}}\hat{\nabla}_{\hat{\gamma}}\hat{\nabla}_{\hat{\delta}}\Phi + (\gamma_{ab\hat{\gamma}}{}^{\hat{\varepsilon}}\hat{\nabla}_{\hat{\varepsilon}}\Phi)\eta^{bc}(\gamma_{cd\hat{\gamma}}{}^{\hat{\varphi}}\hat{\nabla}_{\hat{\varphi}}\Phi) - 3\gamma_{\hat{\gamma}\hat{\gamma}}{}^{c}H_{cad}e^{-2\Phi}\right)\gamma^{ad}{}_{\hat{\alpha}}{}^{\hat{\beta}} \qquad (5.555) \\ R_{\gamma\gamma\alpha}{}^{\beta} &= -\frac{1}{2}\nabla_{\gamma}\nabla_{\gamma}\Phi\delta_{\alpha}{}^{\beta} + \\ &+\frac{1}{4}\left(\gamma_{ad\gamma}{}^{\delta}\nabla_{\gamma}\nabla_{\delta}\Phi + (\gamma_{ab\gamma}{}^{\varepsilon}\nabla_{\varepsilon}\Phi)\eta^{bc}(\gamma_{cd\gamma}{}^{\varphi}\nabla_{\varphi}\Phi) + 3\gamma_{\gamma\gamma}{}^{c}H_{cad}e^{-2\Phi}\right)\gamma^{ad}{}_{\alpha}{}^{\beta} \qquad (5.556) \end{aligned}}$$

We can compare this result to the nilpotency constraint $R_{[\gamma\delta\alpha]}{}^{\beta} = 0$ or at least to its trace $F^{(D)}_{\alpha\beta} = \frac{2}{9}R^{(L)}_{\gamma[\alpha\beta]}{}^{\gamma}$: Scaling and Lorentz component of (5.556) are

$$F^{(D)}_{\gamma\gamma} = -\nabla_{\gamma}\nabla_{\gamma}\Phi \tag{5.557}$$

$$R^{(L)}_{\gamma\gamma\alpha}{}^{\beta} = \frac{1}{4}\left(\gamma_{ad\gamma}{}^{\delta}\nabla_{\gamma}\nabla_{\delta}\Phi + (\gamma_{ab\gamma}{}^{\varepsilon}\nabla_{\varepsilon}\Phi)\eta^{bc}(\gamma_{cd\gamma}{}^{\varphi}\nabla_{\varphi}\Phi) + 3\gamma^{c}_{\gamma\gamma}H_{cad}e^{-2\Phi}\right)\gamma^{ad}{}_{\alpha}{}^{\beta} \tag{5.558}$$

with trace

$$\begin{aligned} R^{(L)}_{\beta\gamma\alpha}{}^{\beta} &= \frac{1}{8}\gamma^{ad}{}_{\alpha}{}^{\beta}\gamma_{ad\beta}{}^{\delta}\nabla_{\gamma}\nabla_{\delta}\Phi - \frac{1}{8}\gamma^{ad}{}_{\alpha}{}^{\beta}\gamma_{ad\gamma}{}^{\delta}\nabla_{\beta}\nabla_{\delta}\Phi + \\ &+\frac{1}{4}(\gamma_{ab\gamma}{}^{\varepsilon}\nabla_{\varepsilon}\Phi)\eta^{bc}\gamma^{ad}{}_{\alpha}{}^{\beta}(\gamma_{dc\beta}{}^{\varphi}\nabla_{\varphi}\Phi) + \frac{3}{4}\gamma^{ad}{}_{\alpha}{}^{\beta}\gamma^{c}_{\beta\gamma}H_{cad}e^{-2\Phi} \end{aligned} \tag{5.559}$$

Now we use $\gamma^{ad}{}_{\alpha}{}^{\gamma}\gamma_{dc\gamma}{}^{\beta} = 8\gamma^{a}{}_{c\alpha}{}^{\beta} + 9\delta^{a}_{c}\delta_{\alpha}{}^{\beta}$ (D.118) and $\gamma^{ad}{}_{\alpha}{}^{\gamma}\gamma_{ad\gamma}{}^{\beta} = -90\delta_{\alpha}{}^{\beta}$ (D.120) to arrive at

$$\begin{aligned} R^{(L)}_{\beta\gamma\alpha}{}^{\beta} &= \frac{-90}{8}\nabla_{\gamma}\nabla_{\alpha}\Phi - \frac{1}{8}\gamma^{ad}{}_{\alpha}{}^{\beta}\gamma_{ad\gamma}{}^{\delta}\nabla_{\beta}\nabla_{\delta}\Phi + \\ &+2\gamma^{ac}{}_{\gamma}{}^{\varepsilon}\nabla_{\varepsilon}\Phi\gamma_{ac\alpha}{}^{\varphi}\nabla_{\varphi}\Phi + \\ &+\frac{3}{4}\gamma^{cad}{}_{\alpha\gamma}H_{cad}e^{-2\Phi} \end{aligned} \tag{5.560}$$

The antisymmetric part (in α, γ) is

$$\begin{aligned} R^{(L)}_{\beta[\gamma\alpha]}{}^{\beta} &= \frac{-90}{8}\nabla_{[\gamma}\nabla_{\alpha]}\Phi - \frac{1}{8}\gamma^{ad}{}_{[\alpha|}{}^{\beta}\gamma_{ad|\gamma]}{}^{\delta}\nabla_{[\beta}\nabla_{\delta]}\Phi = \\ &= \frac{45}{4}F^{(D)}_{\gamma\alpha} - \frac{1}{8}\gamma^{ad}{}_{[\alpha|}{}^{\beta}\gamma_{ad|\gamma]}{}^{\delta}F^{(D)}_{\delta\beta} \end{aligned} \tag{5.561}$$

Now we expand the scaling curvature in γ-matrices. Because of the graded antisymmetry, only $\gamma^{[1]}$ and $\gamma^{[5]}$ appear: $F^{(D)}_{\delta\beta} = F^{(D)}_{c}\gamma^{c}_{\delta\beta} + F^{(D)}_{c_1\ldots c_5}\gamma^{c_1\ldots c_5}_{\delta\beta}$. In $\gamma^{ad}{}_{\gamma}{}^{\delta}F^{(D)}_{\delta\beta}$ we then need the following multiplications of γ-matrices (D.115):

$$\gamma^{ad}{}_{\gamma}{}^{\delta}\gamma^{c}_{\delta\beta} = \gamma^{adc}_{\gamma\beta} + \eta^{dc}\gamma^{a}_{\gamma\beta} - \eta^{ac}\gamma^{d}_{\gamma\beta} = \tag{5.562}$$

$$= -\gamma^{dac}_{\beta\gamma} + 2\eta^{c[a}\gamma^{d]}_{\beta\gamma} \tag{5.563}$$

$$\gamma^{ad}{}_{\gamma}{}^{\delta}\gamma^{c_1\ldots c_5}_{\delta\beta} = \gamma^{adc_1\ldots c_5}_{\gamma\beta} + 5\eta^{d[c_1|}\gamma^{a|c_2\ldots c_5]}_{\gamma\beta} - 5\eta^{a[c_1|}\gamma^{d|c_2\ldots c_5]}_{\gamma\beta} - 20\eta^{a[c_1|}\eta^{d|c_2}\gamma^{c_3\ldots c_5]}_{\gamma\beta} = \tag{5.564}$$

$$= \gamma^{dac_1\ldots c_5}_{\beta\gamma} - 5\eta^{d[c_1|}\gamma^{a|c_2\ldots c_5]}_{\beta\gamma} + 5\eta^{a[c_1|}\gamma^{d|c_2\ldots c_5]}_{\beta\gamma} - 20\eta^{a[c_1|}\eta^{d|c_2}\gamma^{c_3\ldots c_5]}_{\beta\gamma} \tag{5.565}$$

For the expression $\gamma_{ad[\alpha|}{}^{\beta}\gamma^{ad}{}_{|\gamma]}{}^{\delta}F^{(D)}_{\delta\beta}$ in (5.561), we can make use of (D.121)-(D.123) and of the fact that $\gamma^{[3]}_{[\alpha\gamma]} = \gamma^{[7]}_{[\alpha\gamma]} = 0$:

$$\gamma_{ad[\alpha|}{}^{\beta}\gamma^{dac}_{\beta|\gamma]} = 72\gamma^{c}_{\alpha\gamma}, \quad \gamma_{ad[\alpha|}{}^{\beta}\eta^{c[a}\gamma^{d]}_{\beta|\gamma]} = 9\gamma^{c}_{\alpha\gamma} \tag{5.566}$$

$$\gamma_{ad[\alpha|}{}^{\beta}\gamma^{dac_1\ldots c_5}_{\beta|\gamma]} = 20\gamma^{c_1\ldots c_5}_{\alpha\gamma}, \quad \gamma_{ad[\alpha|}{}^{\beta}\eta^{a[c_1|}\gamma^{d|c_2\ldots c_5]}_{\beta|\gamma]} = 5\gamma^{c_1\ldots c_5}_{\alpha\gamma} \tag{5.567}$$

$$\gamma_{ad[\alpha|}{}^{\beta}\eta^{a[c_1|}\eta^{d|c_2}\gamma^{c_3\ldots c_5]}_{\beta|\gamma]} = 0 \tag{5.568}$$

The equation (*) thus becomes

$$R^{(L)}_{\beta[\gamma\alpha]}{}^{\beta} = \frac{45}{4}F^{(D)}_{\gamma\alpha} + \frac{54}{8}F^{(D)}_{c}\gamma^{c}_{\alpha\gamma} - \frac{70}{8}F^{(D)}_{c_1\ldots c_5}\gamma^{c_1\ldots c_5}_{\alpha\gamma} = \tag{5.569}$$

$$= \frac{9}{2}F^{(D)}_{\gamma\alpha} + \frac{31}{2}F^{(D)}_{c_1\ldots c_5}\gamma^{c_1\ldots c_5}_{\gamma\alpha} \tag{5.570}$$

From our nilpotency constraint (5.235) we can now deduce that $F^{(D)}_{c_1\ldots c_5}=0$ or equivalently that

$$\boxed{\gamma^{\boldsymbol{\alpha\beta}}_{a_1\ldots a_5}F^{(D)}_{\boldsymbol{\alpha\beta}}=0,\quad \gamma^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}_{a_1\ldots a_5}\hat{F}^{(D)}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}=0} \tag{5.571}$$

- <u>(2,1,1)$ab\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}$:</u>

$$0 \overset{!}{=} \underline{\nabla}_{[a}H_{b\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}]}+3\underline{T}_{[ab|}{}^{C}H_{C|\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}]}= \tag{5.572}$$

$$= -\underline{T}_{[a|\boldsymbol{\alpha}}{}^{C}H_{C|b]\hat{\boldsymbol{\beta}}}-\underline{T}_{[b|\hat{\boldsymbol{\beta}}}{}^{C}H_{C|a]\boldsymbol{\alpha}}= \tag{5.573}$$

$$\overset{f_{ac}=G_{ac}}{=} -\frac{2}{3}\tilde{\gamma}_{[a|\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\tilde{\gamma}_{|b]\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}+\frac{2}{3}\tilde{\gamma}_{[b|\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{|a]\,\boldsymbol{\gamma\alpha}}= \tag{5.574}$$

$$= \frac{2}{3}\tilde{\gamma}_{[a|\boldsymbol{\alpha\delta}}\tilde{\gamma}_{|b]\,\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\beta}}}\left(-\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}+\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)=0 \tag{5.575}$$

- <u>(3,1,0)$abc\boldsymbol{\delta}$ ↔((3,0,1)$abc\hat{\boldsymbol{\delta}}$):</u>

$$0 \overset{!}{=} \underline{\nabla}_{[a}H_{bc\hat{\boldsymbol{\delta}}]}+3\underline{T}_{[ab|}{}^{E}H_{E|c\hat{\boldsymbol{\delta}}]}= \tag{5.576}$$

$$= -\frac{1}{4}\check{\nabla}_{\hat{\boldsymbol{\delta}}}H_{abc}+\frac{3}{2}\underline{T}_{[ab|}{}^{E}H_{E|c]\hat{\boldsymbol{\delta}}}-\frac{3}{2}\underline{T}_{\hat{\boldsymbol{\delta}}[a|}{}^{E}H_{E|bc]}= \tag{5.577}$$

$$\overset{\check{\Omega}=\Omega}{=} -\frac{1}{4}\nabla_{\hat{\boldsymbol{\delta}}}H_{abc}-\frac{3}{2}\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}H_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}-\frac{3}{2}T_{\hat{\boldsymbol{\delta}}[a|}{}^{e}H_{e|bc]}= \tag{5.578}$$

$$\overset{f_{ab}=G_{ab}}{=} -\frac{1}{4}\nabla_{\hat{\boldsymbol{\delta}}}H_{abc}-\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}-\frac{3}{2}\underbrace{T_{\hat{\boldsymbol{\delta}}[a|}{}^{e}}_{=0\,(5.527)}H_{e|bc]} \tag{5.579}$$

$$\boxed{\begin{aligned}\nabla_{\hat{\boldsymbol{\delta}}}H_{abc} &= -4\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}} && (5.580)\\ \text{likewise } \hat{\nabla}_{\boldsymbol{\delta}}H_{abc} &= 4T_{[ab|}{}^{\boldsymbol{\varepsilon}}\tilde{\gamma}_{|c]\boldsymbol{\varepsilon\delta}} && (5.581)\end{aligned}}$$

- <u>(4,0,0)$abcd$:</u>

$$\boxed{0\overset{!}{=}\check{\nabla}_{[a}H_{bcd]}+3\check{T}_{[ab|}{}^{e}H_{e|cd]}} \tag{5.582}$$

5.C The Bianchi identities for the torsion

The Bianchi identity for the torsion reads

$$0\overset{!}{=}\underline{\nabla}_{\boldsymbol{A}}\underline{T}_{\boldsymbol{AA}}{}^{D}+2\underline{T}_{\boldsymbol{AA}}{}^{C}\underline{T}_{C\boldsymbol{A}}{}^{D}-\underline{R}_{\boldsymbol{AAA}}{}^{D} \tag{5.583}$$

Again, depending on what is more convenient, the bosonic part of the connection $\check{\Omega}_a{}^b$ will be chosen to be either $\Omega_a{}^b$ or $\hat{\Omega}_a{}^b$. Due to proposition 7 on page 193, both are equivalent. The index A can again be either a, $\boldsymbol{\alpha}$ or $\hat{\boldsymbol{\alpha}}$. For fixed upper index the numbers of their appearance as lower index are $\#a$, $\#\boldsymbol{\alpha}$, $\#\hat{\boldsymbol{\alpha}}\in\{0,1,2,3\}$. In analogy to the Bianchi identities for H, we have for each fixed upper index $4+3+2+1=10$ possibilities and thus altogether 30 possibilities. The symmetry between hatted and unhatted indices relates the 10 with upper index $\hat{\boldsymbol{\delta}}$ to the ten with upper index $\boldsymbol{\delta}$. The remaining 10 have again an internal symmetry with fixed points $(\#\boldsymbol{\alpha},\#\hat{\boldsymbol{\alpha}})\in\{(0,0),(1,1)\}$, so that there remain effectively $\frac{10-2}{2}+2=6$ of those 10. Altogether we have thus effectively 16 equations to study.

- <u>(delta|0,3,0)$_{\boldsymbol{\alpha\beta\gamma}}{}^{\boldsymbol{\delta}}$ ↔((hdelta|0,0,3)$_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}$),dim1:</u>

$$0 \overset{!}{=} \nabla_{[\boldsymbol{\alpha}}T_{\boldsymbol{\beta\gamma}]}{}^{\boldsymbol{\delta}}+2\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{E}T_{E|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}-R_{[\boldsymbol{\alpha\beta\gamma}]}{}^{\boldsymbol{\delta}}= \tag{5.584}$$

$$= 2\check{T}_{[\boldsymbol{\alpha\beta}|}{}^{e}\underbrace{T_{e|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}}_{=0}-R_{[\boldsymbol{\alpha\beta\gamma}]}{}^{\boldsymbol{\delta}} \tag{5.585}$$

$$\boxed{\begin{aligned}R_{[\boldsymbol{\alpha\beta\gamma}]}{}^{\boldsymbol{\delta}} &= 0 && (5.586)\\ \hat{R}_{[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}]}{}^{\hat{\boldsymbol{\delta}}} &= 0 && (5.587)\end{aligned}}$$

This is a confirmation of the nilpotency constraint (5.228) that we had derived earlier. Taking the trace yields

$$0 \stackrel{!}{=} R_{\alpha\beta\gamma}{}^{\gamma} + 2R_{\gamma[\alpha\beta]}{}^{\gamma} = \tag{5.588}$$

$$= -9F^{(D)}_{\alpha\beta} + 2R^{(L)}_{\gamma[\alpha\beta]}{}^{\gamma} \tag{5.589}$$

$$\boxed{F^{(D)}_{\alpha\beta} \stackrel{!}{=} \frac{2}{9}R^{(L)}_{\gamma[\alpha\beta]}{}^{\gamma}} \tag{5.590}$$

and

$$\boxed{\hat{F}^{(D)}_{\hat{\alpha}\hat{\beta}} \stackrel{!}{=} \frac{2}{9}\hat{R}^{(L)}_{\hat{\gamma}[\hat{\alpha}\hat{\beta}]}{}^{\hat{\gamma}}} \tag{5.591}$$

- (delta|0,2,1)$_{\alpha\beta\hat{\gamma}}{}^{\delta}$ ↔((hdelta|0,1,2)$_{\hat{\alpha}\hat{\beta}\gamma}{}^{\hat{\delta}}$)dim1:

$$0 \stackrel{!}{=} \nabla_{[\alpha}T_{\beta\hat{\gamma}]}{}^{\delta} + 2T_{[\alpha\beta|}{}^{E}T_{E|\hat{\gamma}]}{}^{\delta} - R_{[\alpha\beta\hat{\gamma}]}{}^{\delta} = \tag{5.592}$$

$$= \frac{2}{3}T_{\alpha\beta}{}^{e}T_{e\hat{\gamma}}{}^{\delta} - \frac{2}{3}R_{\hat{\gamma}[\alpha\beta]}{}^{\delta} = \tag{5.593}$$

$$\stackrel{f_c{}^e=\delta_c^e}{=} -\frac{2}{3}\gamma_{\alpha\beta}{}^{e}\tilde{\gamma}_{e\,\hat{\gamma}\hat{\delta}}\mathcal{P}^{\delta\hat{\delta}} - \frac{2}{3}R_{\hat{\gamma}[\alpha\beta]}{}^{\delta} \tag{5.594}$$

$$\boxed{\begin{aligned} R_{\hat{\gamma}[\alpha\beta]}{}^{\delta} &= -\gamma_{\alpha\beta}{}^{e}\tilde{\gamma}_{e\,\hat{\gamma}\hat{\delta}}\mathcal{P}^{\delta\hat{\delta}} && (5.595)\\ \hat{R}_{\gamma[\hat{\alpha}\hat{\beta}]}{}^{\hat{\delta}} &= -\gamma_{\hat{\alpha}\hat{\beta}}{}^{e}\tilde{\gamma}_{e\,\gamma\delta}\mathcal{P}^{\delta\hat{\delta}} && (5.596)\end{aligned}}$$

Again taking the trace gives additional information on the Dilatation part

$$R_{\hat{\gamma}\alpha\delta}{}^{\delta} - R_{\hat{\gamma}\delta\alpha}{}^{\delta} = 2\gamma_{\alpha\delta}{}^{e}\mathcal{P}^{\delta\hat{\delta}}\tilde{\gamma}_{e\,\hat{\delta}\hat{\gamma}} \tag{5.597}$$

$$-8F^{(D)}_{\hat{\gamma}\alpha} - \frac{1}{2}F^{(D)}_{\hat{\gamma}\alpha} - R^{(L)}_{\hat{\gamma}\delta\alpha}{}^{\delta} = 2\gamma_{\alpha\delta}{}^{e}\mathcal{P}^{\delta\hat{\delta}}\tilde{\gamma}_{e\,\hat{\delta}\hat{\gamma}} \tag{5.598}$$

$$\boxed{F^{(D)}_{\hat{\gamma}\alpha} = -\frac{4}{17}\gamma^{e}_{\alpha\delta}\mathcal{P}^{\delta\hat{\delta}}\tilde{\gamma}_{e\,\hat{\delta}\hat{\gamma}} - \frac{2}{17}R^{(L)}_{\hat{\gamma}\delta\alpha}{}^{\delta}} \tag{5.599}$$

$$\boxed{\hat{F}^{(D)}_{\gamma\hat{\alpha}} = -\frac{4}{17}\gamma^{e}_{\hat{\alpha}\hat{\delta}}\mathcal{P}^{\delta\hat{\delta}}\tilde{\gamma}_{e\,\delta\gamma} - \frac{2}{17}\hat{R}^{(L)}_{\gamma\hat{\delta}\hat{\alpha}}{}^{\hat{\delta}}} \tag{5.600}$$

- (delta|0,1,2)$_{\alpha\hat{\beta}\hat{\gamma}}{}^{\delta}$ ↔((hdelta|0,2,1)$_{\hat{\alpha}\beta\gamma}{}^{\hat{\delta}}$)dim1:

$$0 \stackrel{!}{=} \nabla_{[\alpha}T_{\hat{\beta}\hat{\gamma}]}{}^{\delta} + 2T_{[\alpha\hat{\beta}|}{}^{E}T_{E|\hat{\gamma}]}{}^{\delta} - R_{[\alpha\hat{\beta}\hat{\gamma}]}{}^{\delta} = \tag{5.601}$$

$$= \frac{2}{3}T_{\hat{\beta}\hat{\gamma}}{}^{e}\underbrace{T_{e\alpha}{}^{\delta}}_{=0} - \frac{1}{2}\underbrace{R_{\hat{\beta}\hat{\gamma}\alpha}{}^{\delta}}_{=0} = 0 \tag{5.602}$$

- (delta|0,0,3)$_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{\delta}$ ↔((hdelta|0,3,0)$_{\alpha\beta\gamma}{}^{\hat{\delta}}$)dim1:

$$0 \stackrel{!}{=} \nabla_{[\hat{\alpha}}T_{\hat{\beta}\hat{\gamma}]}{}^{\delta} + 2T_{[\hat{\alpha}\hat{\beta}|}{}^{E}T_{E|\hat{\gamma}]}{}^{\delta} - \underbrace{R_{[\hat{\alpha}\hat{\beta}\hat{\gamma}]}{}^{\delta}}_{=0} = \tag{5.603}$$

$$= 2T_{[\hat{\alpha}\hat{\beta}|}{}^{e}T_{e|\hat{\gamma}]}{}^{\delta} = \tag{5.604}$$

$$= -2\gamma_{[\hat{\alpha}\hat{\beta}|}{}^{e}\tilde{\gamma}_{e\,|\hat{\gamma}]\hat{\delta}}\mathcal{P}^{\delta\hat{\delta}} \stackrel{\text{Fierz}}{=} 0 \tag{5.605}$$

- (delta|1,2,0)$_{\alpha\beta c}{}^{\delta}$ ↔((hdelta|1,0,2)$_{\hat{\alpha}\hat{\beta}c}{}^{\hat{\delta}}$)dim$\frac{3}{2}$:

$$0 \stackrel{!}{=} \nabla_{[\alpha}T_{\beta c]}{}^{\delta} + 2T_{[\alpha\beta|}{}^{E}T_{E|c]}{}^{\delta} - R_{[\alpha\beta c]}{}^{\delta} = \tag{5.606}$$

$$= \frac{2}{3}T_{\alpha\beta}{}^{E}T_{Ec}{}^{\delta} + \frac{4}{3}T_{c[\alpha|}{}^{E}\underbrace{T_{E|\beta]}{}^{\delta}}_{=0} - \frac{2}{3}R_{c[\alpha\beta]}{}^{\delta} = \tag{5.607}$$

$$= \frac{2}{3}\gamma_{\alpha\beta}{}^{e}T_{ec}{}^{\delta} - \frac{2}{3}R_{c[\alpha\beta]}{}^{\delta} \tag{5.608}$$

$$\boxed{\begin{aligned} R_{c[\boldsymbol{\alpha}\boldsymbol{\beta}]}{}^{\boldsymbol{\delta}} &= \gamma_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{e}T_{ec}{}^{\boldsymbol{\delta}} && (5.609)\\ \hat{R}_{c[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\delta}}} &= \gamma_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{e}\hat{T}_{ec}{}^{\hat{\boldsymbol{\delta}}} && (5.610)\end{aligned}}$$

Taking the trace yields

$$\begin{aligned} 0 &= R_{c\boldsymbol{\alpha}\boldsymbol{\delta}}{}^{\boldsymbol{\delta}} - R_{c\boldsymbol{\delta}\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} - 2\gamma_{\boldsymbol{\alpha}\boldsymbol{\delta}}{}^{e}T_{ec}{}^{\boldsymbol{\delta}} = && (5.611)\\ &= -\frac{17}{2}F^{(D)}_{c\boldsymbol{\alpha}} - R^{(L)}_{c\boldsymbol{\delta}\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} - 2\gamma_{\boldsymbol{\alpha}\boldsymbol{\delta}}{}^{e}T_{ec}{}^{\boldsymbol{\delta}} && (5.612)\end{aligned}$$

$$\boxed{F^{(D)}_{c\boldsymbol{\alpha}} = -\frac{2}{17}R^{(L)}_{c\boldsymbol{\delta}\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} - \frac{4}{17}\gamma^{e}_{\boldsymbol{\alpha}\boldsymbol{\delta}}T_{ec}{}^{\boldsymbol{\delta}}} \qquad (5.613)$$

$$\boxed{\hat{F}^{(D)}_{c\hat{\boldsymbol{\alpha}}} = -\frac{2}{17}\hat{R}^{(L)}_{c\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}} - \frac{4}{17}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\hat{T}_{ec}{}^{\hat{\boldsymbol{\delta}}}} \qquad (5.614)$$

- $\underline{(\text{delta}|1{,}1{,}1)_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{\boldsymbol{\delta}} \leftrightarrow ((\text{hdelta}|1{,}1{,}1)_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}c}{}^{\hat{\boldsymbol{\delta}}})\dim\tfrac{3}{2}:}$[25]

$$\begin{aligned} 0 &\overset{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}T_{\hat{\boldsymbol{\beta}}c]}{}^{\boldsymbol{\delta}} + 2\underline{T}_{[\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} - R_{[\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c]}{}^{\boldsymbol{\delta}} = && (5.615)\\ &\overset{\check{\Omega}=\hat{\Omega}}{=} \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}\underbrace{T_{\hat{\boldsymbol{\beta}}c}{}^{\boldsymbol{\delta}}}_{\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}} + \frac{2}{3}\hat{T}_{c\boldsymbol{\alpha}}{}^{\hat{\boldsymbol{\varepsilon}}}\underbrace{T_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\delta}}}_{=0} + \frac{2}{3}\underbrace{\hat{T}_{c\boldsymbol{\alpha}}{}^{e}}_{=0}T_{e\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\delta}} - \frac{1}{3}\underbrace{R_{\hat{\boldsymbol{\beta}}c\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}}_{\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}} = && (5.616)\\ &= \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}\left(\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right) - \frac{1}{3}\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{\boldsymbol{\alpha}}P^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = && (5.617)\\ &= \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}\left(\tilde{\gamma}_{c\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\right)\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = && (5.618)\\ &= \tfrac{2}{3}\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\boldsymbol{\alpha}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = 0 && (5.619)\end{aligned}$$

- $\underline{(\text{delta}|1{,}0{,}2)_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c}{}^{\boldsymbol{\delta}} \leftrightarrow ((\text{hdelta}|1{,}2{,}0)_{\boldsymbol{\alpha}\boldsymbol{\beta}c}{}^{\hat{\boldsymbol{\delta}}})\dim\tfrac{3}{2}:}$

$$\begin{aligned} 0 &\overset{!}{=} \underline{\nabla}_{[\hat{\boldsymbol{\alpha}}}T_{\hat{\boldsymbol{\beta}}c]}{}^{\boldsymbol{\delta}} + 2\underline{T}_{[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} - R_{[\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c]}{}^{\boldsymbol{\delta}} = && (5.620)\\ &= \frac{2}{3}\underline{\nabla}_{[\hat{\boldsymbol{\alpha}}}\underbrace{T_{\hat{\boldsymbol{\beta}}]c}{}^{\boldsymbol{\delta}}}_{\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}} + \frac{2}{3}\check{T}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{e}T_{ec}{}^{\boldsymbol{\delta}} + \frac{4}{3}\check{T}_{c[\hat{\boldsymbol{\alpha}}|}{}^{e}T_{e|\hat{\boldsymbol{\beta}}]}{}^{\boldsymbol{\delta}} = && (5.621)\\ &\overset{\check{\Omega}=\hat{\Omega}}{=} \frac{4}{3}\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}|}\Phi\tilde{\gamma}_{c\,|\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + \frac{2}{3}\underline{\nabla}_{[\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\gamma}}} + \frac{2}{3}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}T_{ec}{}^{\boldsymbol{\delta}} + \frac{4}{3}\hat{T}_{[\hat{\boldsymbol{\alpha}}|c}{}^{e}\tilde{\gamma}_{e\,|\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = && (5.622)\\ &= \left(\frac{4}{3}\left(\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}|}\Phi\delta^{e}_{c} + \hat{T}_{[\hat{\boldsymbol{\alpha}}|c}{}^{e}\right)\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + \frac{2}{3}\underline{\nabla}_{[\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\delta^{e}_{c}\right)\tilde{\gamma}_{e\,|\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\gamma}}} + \frac{2}{3}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}T_{ec}{}^{\boldsymbol{\delta}} = && (5.623)\\ &= \frac{2}{3}\big(-\underbrace{2\hat{T}_{[\hat{\boldsymbol{\alpha}}|e|c}}_{\Delta_{[\hat{\boldsymbol{\alpha}}|e|c}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + \underline{\nabla}_{[\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}G_{ec}\big)\gamma^{e}_{|\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\gamma}}} + \frac{2}{3}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}T_{ec}{}^{\boldsymbol{\delta}} && (5.624)\end{aligned}$$

Contracting the above with $\gamma_e^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$ (using $\gamma_e^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\gamma^{f}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = -\gamma_e^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\gamma^{f}_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\alpha}}} = -\gamma_e^{\hat{\alpha}\hat{\beta}}\gamma^{f}_{\hat{\beta}\hat{\alpha}} = -16\delta^{f}_{e}$), we get

$$\begin{aligned} T_{ec}{}^{\boldsymbol{\delta}} &= \frac{1}{16}\left(\underline{\nabla}_{[\hat{\boldsymbol{\alpha}}|}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}G_{cd} - 2\hat{T}_{[\hat{\boldsymbol{\alpha}}|d:c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\gamma^{d}_{|\hat{\boldsymbol{\beta}}]\hat{\boldsymbol{\delta}}}\gamma_e^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = && (5.625)\\ &= \frac{1}{16}\left(2\hat{T}_{\hat{\boldsymbol{\alpha}}d|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}G_{cd}\right)\gamma^{d}_{\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\beta}}}\gamma_e^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} && (5.626)\end{aligned}$$

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$$\begin{aligned} \underline{\nabla}_{M}\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\big|_{\check{\Omega}=\Omega} &= 2\gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\nabla_{M}\Phi G_{dc} = 2\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\nabla_{M}\Phi\\ \underline{\nabla}_{M}\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\big|_{\check{\Omega}=\hat{\Omega}} &= 2\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\nabla_{M}\Phi - \Delta_{Mc}{}^{d}\tilde{\gamma}_{d\,\boldsymbol{\alpha}\boldsymbol{\beta}} =\\ &= \gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\left[2\nabla_{M}\Phi G_{dc} - \Delta_{Mc|d}\right] =\\ &= \gamma^{d}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\left[\left(\nabla_{M}\Phi + \hat{\nabla}_{M}\Phi\right)G_{dc} - \Delta^{(L)}_{Mcd}\right]\end{aligned}$$

And equivalently

$$\begin{aligned} \underline{\nabla}_{M}\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\Big|_{\check{\Omega}=\hat{\Omega}} &= 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\hat{\nabla}_{M}\Phi\\ \underline{\nabla}_{M}\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\Big|_{\check{\Omega}=\Omega} &= \gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\left[\left(\nabla_{M}\Phi + \hat{\nabla}_{M}\Phi\right)G_{dc} + \Delta^{(L)}_{Mcd}\right] \qquad \diamond\end{aligned}$$

$$\boxed{\begin{aligned} T_{ec}{}^{\boldsymbol{\delta}} &= \frac{1}{16}\left(2\hat{T}_{\hat{\boldsymbol{\alpha}}d|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}G_{cd}\right)\gamma^{d}_{\hat{\boldsymbol{\delta}}\hat{\boldsymbol{\beta}}}\gamma_e^{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} && (5.627)\\ \hat{T}_{ec}{}^{\hat{\boldsymbol{\delta}}} &= \frac{1}{16}\left(2T_{\boldsymbol{\alpha}d|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}G_{cd}\right)\gamma^{d}_{\boldsymbol{\delta}\boldsymbol{\beta}}\gamma_e^{\boldsymbol{\alpha}\boldsymbol{\beta}} && (5.628)\end{aligned}}$$

The product of γ-matrices can be further expanded.

$$\begin{aligned} T_{ec}{}^{\boldsymbol{\delta}} &= \frac{1}{16}\left(2\hat{T}_{\hat{\boldsymbol{\alpha}}d|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}G_{cd}\right)\left(\delta^d_e\delta_{\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}} + \gamma^d{}_{e\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}}\right) = && (5.629)\\ &= \frac{1}{16}\big(2\hat{T}_{\hat{\boldsymbol{\alpha}}e|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}G_{ce} + \underbrace{2\hat{T}_{\hat{\boldsymbol{\alpha}}d|c}\gamma^d{}_{e\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}}}_{-18\hat{T}_{\hat{\boldsymbol{\delta}}e|c}\ (5.503)}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{ce\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}}\big) && (5.630)\end{aligned}$$

The result should be antisymmetric in e and c. Remember now

$$\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}G_{ce} = 8\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi G_{ce} = -16\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\hat{T}_{\hat{\boldsymbol{\delta}}(c|e)} \tag{5.631}$$

and we get

$$\begin{aligned} T_{ec}{}^{\boldsymbol{\delta}} &= \frac{1}{16}\left(-16\hat{T}_{\hat{\boldsymbol{\delta}}e|c}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 16\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\hat{T}_{\hat{\boldsymbol{\delta}}(c|e)} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{ce\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}}\right) = && (5.632)\\ &= \frac{1}{16}\left(-16\hat{T}_{\hat{\boldsymbol{\delta}}[e|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{ce\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}}\right) && (5.633)\end{aligned}$$

Using $\hat{T}_{\hat{\boldsymbol{\delta}}[e|c]} = -\frac{1}{2}\gamma_{ec\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi$ leads to

$$\boxed{T_{ec}{}^{\boldsymbol{\delta}} = \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ec\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}} \tag{5.634}$$

$$\boxed{\hat{T}_{ec}{}^{\hat{\boldsymbol{\delta}}} = \frac{1}{16}\left(\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 8\nabla_{\boldsymbol{\gamma}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ec\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}} \tag{5.635}$$

Instead of solving for the torsion component, we can also solve for the covariant derivative of the RR-field:

$$\frac{1}{16}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{ec\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}} = T_{ec}{}^{\boldsymbol{\delta}} - \frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{ec\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}} \tag{5.636}$$

Together with (5.532) and the fact that $C_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}} = \underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}$ is structure group valued in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (as well as $\hat{C}$), we get

$$\boxed{\begin{aligned} \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} &= -\frac{1}{2}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\varphi}}}\hat{\nabla}_{\hat{\boldsymbol{\varphi}}}\Phi\cdot\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \left(T_{fg}{}^{\boldsymbol{\alpha}} - \frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\varphi}}}\Phi\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\phi}}}\tilde{\gamma}_{fg\,\hat{\boldsymbol{\phi}}}{}^{\hat{\boldsymbol{\varphi}}}\right)\tilde{\gamma}^{fg}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && (5.637)\\ \underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}} &= -\frac{1}{2}\mathcal{P}^{\boldsymbol{\varphi}\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\varphi}}\Phi\cdot\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \left(\hat{T}_{fg}{}^{\hat{\boldsymbol{\alpha}}} - \frac{1}{2}\nabla_{\boldsymbol{\varphi}}\Phi\mathcal{P}^{\boldsymbol{\phi}\hat{\boldsymbol{\alpha}}}\tilde{\gamma}_{fg\,\boldsymbol{\phi}}{}^{\boldsymbol{\varphi}}\right)\tilde{\gamma}^{fg}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} && (5.638)\end{aligned}}$$

Due to the algebra of covariant derivatives, the above equations also contain informations on the spacetime derivative of $\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$. It is thus of interest to study the commutator $\underline{\nabla}_{[\boldsymbol{\gamma}}\underline{\nabla}_{\boldsymbol{\alpha}]}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}}$:

$$\begin{aligned} &-\gamma^d_{\boldsymbol{\gamma}\boldsymbol{\alpha}}\underline{\nabla}_d\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}} + \underline{R}_{\boldsymbol{\gamma}\boldsymbol{\alpha}\boldsymbol{\delta}}{}^{\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}} + \underbrace{\underline{R}_{\boldsymbol{\gamma}\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\alpha}}}}_{=0}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\delta}}} = \underline{\nabla}_{[\boldsymbol{\gamma}}\underline{\nabla}_{\boldsymbol{\alpha}]}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\alpha}}} =\\ &= -\frac{1}{2}\underline{\nabla}_{[\boldsymbol{\gamma}|}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\delta}}\Phi\cdot\delta_{|\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} - \frac{1}{2}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\nabla_{[\boldsymbol{\gamma}|}\nabla_{\boldsymbol{\delta}}\Phi\cdot\delta_{|\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} +\\ &\quad +\underline{\nabla}_{[\boldsymbol{\gamma}|}\left(\hat{T}_{bc}{}^{\hat{\boldsymbol{\alpha}}} - \frac{1}{2}\nabla_{\boldsymbol{\delta}}\Phi\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\alpha}}}\tilde{\gamma}_{bc\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\right)\tilde{\gamma}^{bc}{}_{|\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} + \left(\hat{T}_{bc}{}^{\hat{\boldsymbol{\alpha}}} - \frac{1}{2}\nabla_{\boldsymbol{\delta}}\Phi\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\alpha}}}\tilde{\gamma}_{bc\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\right)\underline{\nabla}_{[\boldsymbol{\gamma}|}\tilde{\gamma}^{bc}{}_{|\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}}\Big|_{\breve{\Omega}=\Omega} \qquad (5.639)\end{aligned}$$

In particular, we obtain a Dirac-like operator acting on the first index of $\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$ if we contract the indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\begin{aligned} &-\gamma^d_{\boldsymbol{\gamma}\boldsymbol{\alpha}}\underline{\nabla}_d\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}} + \underbrace{R_{\boldsymbol{\gamma}\boldsymbol{\alpha}\boldsymbol{\delta}}{}^{\boldsymbol{\alpha}}}_{-4F^{(D)}_{\boldsymbol{\gamma}\boldsymbol{\delta}}=4\nabla_{[\boldsymbol{\gamma}}\nabla_{\boldsymbol{\delta}]}\Phi+4\gamma^c_{\boldsymbol{\gamma}\boldsymbol{\delta}}\nabla_c\Phi}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}} =\\ &= \frac{17}{2}\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\delta}}\Phi + \frac{17}{2}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\delta}}\Phi - \frac{1}{2}\underline{\nabla}_{\boldsymbol{\alpha}}\left(\hat{T}_{bc}{}^{\hat{\boldsymbol{\alpha}}} - \frac{1}{2}\nabla_{\boldsymbol{\delta}}\Phi\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\alpha}}}\tilde{\gamma}_{bc\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\right)\tilde{\gamma}^{bc}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}} +\\ &\quad -\frac{1}{2}\left(\hat{T}_{bc}{}^{\hat{\boldsymbol{\alpha}}} - \frac{1}{2}\nabla_{\boldsymbol{\delta}}\Phi\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\alpha}}}\tilde{\gamma}_{bc\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\right)\underline{\nabla}_{\boldsymbol{\alpha}}\tilde{\gamma}^{bc}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}} \qquad (5.640)\end{aligned}$$

In the same way we can obtain an equation for Dirac-like operator acting on the second index of $\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$, if we consider the hatted version of the above equation.

Plugging further torsion constraints into these equations yields rather lengthy expressions and we thus restrict ourselves to a qualitative discussion of the further steps which would lead to field equations for the RR-p-forms, to be presented in the following intermezzo.

Intermezzo on the RR-field-equations

As just mentioned above, the equation (5.640) and its hatted equivalent together with some other torsion constraints of before determine the equations of motion of the RR-field strengths. We will make a qualitive discussion and assume that the fermionic fields vanish so that the equations in WZ gauge basically reduce to $\gamma^d_{\boldsymbol{\gamma\alpha}}\underline{\nabla}_d\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}=0$ and $\gamma^d_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}\underline{\nabla}_d\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}=0$ where $\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}$ is the leading component of $\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\alpha}}}$ in the $\hat{\boldsymbol{\theta}}$-expansion (see page 81).

In order to see that this corresponds to reasonable equations for the RR-p-forms, let us first recall the translation of field equations on the bispinor fields $\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$ into the equations on the level of differential forms in the flat case. On the form level one expects for the RR-field strength's $g^{(p)}$ s.th. like $\mathbf{d}g^{(p)}=0$ and $\star\mathbf{d}\star g^{(p)}=0$. As it is discussed in the appendix on page 172 and in the following, this corresponds on the bispinor level precisely to two Dirac equations, one acting on the first index and one on the second, i.e. $\partial\!\!\!/_{\boldsymbol{\gamma\alpha}}\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}=\partial\!\!\!/_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}=0$ with $\partial\!\!\!/_{\boldsymbol{\alpha\beta}}=\gamma^m_{\boldsymbol{\alpha\beta}}\partial_m$. Of course the equations are not yet the full truth, as they do not reflect the curved background. In order to show the above correspondence, we need to distinguish between IIA (where $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$ are of opposite chirality) and type IIB (where $\boldsymbol{\alpha}$ and $\hat{\boldsymbol{\alpha}}$ are of the same chirality). We will frequently use equations from the appendix D on page 167 where we did not use the graded conventions. We will therefore consider in this intermezzo the spinorial indices ungraded in the summations (this refers to the graded summation convention discussed in the first part of this thesis; if you have not read that part, you can safely ignore the comment).

Assume we are in **type IIA** where we can expand the RR-bispinor in even antisymmetrized products of γ-matrices:

$$\mathfrak{p}^{\alpha}{}_{\beta} = 2g^{(0)}\underbrace{\delta^{\alpha}_{\beta}}_{\gamma^{[0]}}+2g^{(2)}_{a_1a_2}\underbrace{\gamma^{a_1a_2\,\alpha}{}_{\beta}}_{\gamma^{[2]}}+2g^{(4)}_{a_1a_2a_3a_4}\underbrace{\gamma^{a_1a_2a_3a_4\alpha}{}_{\beta}}_{\gamma^{[4]}} \tag{5.641}$$

$$2g^{(0)} = \frac{1}{16}\mathfrak{p}^{\alpha}{}_{\beta}\delta^{\beta}_{\alpha} \tag{5.642}$$

$$2g^{(2)}_{a_1a_2} = \frac{1}{32}\mathfrak{p}^{\alpha}{}_{\beta}\gamma_{a_2a_1}{}^{\beta}{}_{\alpha} \tag{5.643}$$

$$2g^{(4)}_{a_1a_2a_3a_4} = \frac{1}{16\cdot 4!}\mathfrak{p}^{\alpha}{}_{\beta}\gamma_{a_4a_3a_2a_1}{}^{\beta}{}_{\alpha} \tag{5.644}$$

Usually the coeficients $g^{(p)}_{a_1\ldots a_p}$ which correspond to p-forms (or better p-form field strengths) are denoted with a capital G, but we want to keep the capital letters reserved for superfields. The matrices $\gamma^{[0]},\gamma^{[2]}$ and $\gamma^{[4]}$ are the chiral blocks of the antisymmetrized products of the Dirac gamma matrices $\Gamma^{[2k]}$ which is block diagonal. Similarly, $\Gamma^{[2k+1]}$ is block off-diagonal and defines the chiral blocks $\gamma^{[2k+1]}$:

$$\Gamma^{[2k]\underline{\alpha}}{}_{\underline{\beta}}=\begin{pmatrix}\gamma^{[2k]\alpha}{}_{\beta} & 0\\ 0 & \gamma^{[2k]}{}_{\alpha}{}^{\beta}\end{pmatrix},\quad \Gamma^{[2k+1]\underline{\alpha}}{}_{\underline{\beta}}=\begin{pmatrix}0 & \gamma^{[2k]\alpha\beta}\\ \gamma^{[2k+1]}_{\alpha\beta} & 0\end{pmatrix} \tag{5.645}$$

The chiral blocks can be extracted via the chirality matrix $\Gamma^{\#}{}^{\underline{\alpha}}{}_{\underline{\beta}}=\begin{pmatrix}\gamma^{\#\alpha}{}_{\beta} & 0\\ 0 & \gamma^{\#}_{\alpha}{}^{\beta}\end{pmatrix}=\begin{pmatrix}\delta^{\alpha}_{\beta} & 0\\ 0 & -\delta^{\beta}_{\alpha}\end{pmatrix}$ which acts (when multiplied from the right) on the first coloumn as the identity and on the second one as minus the identity:

$$\frac{1}{2}\left(\Gamma^{[2k]}(\mathbb{1}+\Gamma^{\#})\right)^{\underline{\alpha}}{}_{\underline{\beta}} = \begin{pmatrix}\gamma^{[2k]\alpha}{}_{\beta} & 0\\ 0 & 0\end{pmatrix},\quad \frac{1}{2}\left(\Gamma^{[2k]}(\mathbb{1}-\Gamma^{\#})\right)^{\underline{\alpha}}{}_{\underline{\beta}}=\begin{pmatrix}0 & 0\\ 0 & \gamma^{[2k]}{}_{\alpha}{}^{\beta}\end{pmatrix} \tag{5.646}$$

$$\frac{1}{2}\left(\Gamma^{[2k+1]}(\mathbb{1}+\Gamma^{\#})\right)^{\underline{\alpha}}{}_{\underline{\beta}} = \begin{pmatrix}0 & 0\\ \gamma^{[2k+1]}_{\alpha\beta} & 0\end{pmatrix},\quad \frac{1}{2}\left(\Gamma^{[2k+1]}(\mathbb{1}-\Gamma^{\#})\right)^{\underline{\alpha}}{}_{\underline{\beta}}=\begin{pmatrix}0 & \gamma^{[2k+1]\alpha\beta}\\ 0 & 0\end{pmatrix} \tag{5.647}$$

Via the clifford map, the $\Gamma^{[2k]}$ get mapped to even forms. In addition we define the Hodge star operator such that it corresponds via this mapping to the multiplication of the chirality matrix from the right (see page 169). The chiral blocks thus get mapped as follows

$$\gamma^{[2k]\alpha}{}_{\beta} \overset{/^{-1}}{\mapsto} \tfrac{1}{2}(1+\star)e^{a_1}\wedge\ldots\wedge e^{a_{2k}},\qquad \gamma^{[2k]}{}_{\alpha}{}^{\beta}\overset{/^{-1}}{\mapsto}\tfrac{1}{2}(1-\star)e^{a_1}\wedge\ldots\wedge e^{a_{2k}} \tag{5.648}$$

$$\gamma^{[2k+1]}_{\alpha\beta} \overset{/^{-1}}{\mapsto} \tfrac{1}{2}(1+\star)e^{a_1}\wedge\ldots\wedge e^{a_{2k+1}},\qquad \gamma^{[2k+1]\alpha\beta}\overset{/^{-1}}{\mapsto}\tfrac{1}{2}(1-\star)e^{a_1}\wedge\ldots\wedge e^{a_{2k+1}} \tag{5.649}$$

and the bispinor field $\mathfrak{p}^{\alpha}{}_{\beta}$ therefore corresponds to an even self-dual formal sum of differential forms:

$$\mathfrak{p}^{\alpha}{}_{\beta} = \not{g}^{\alpha}{}_{\beta} \overset{\not{}^{-1}}{\mapsto} g \equiv g^{(0)}\Big(1 + \underbrace{\frac{1}{10!}\epsilon_{b_1\ldots b_{10}} e^{b_1} \wedge \ldots \wedge e^{b_{10}}}_{\star 1}\Big) + g^{(2)}_{a_1a_2}\Big(e^{a_1} \wedge e^{a_2} \underbrace{- \frac{1}{8!}\epsilon^{a_1a_2}{}_{b_1\ldots b_8} e^{b_1} \cdots e^{b_8}}_{+\star(e^{a_1}\wedge e^{a_2})}\Big) +$$
$$+ g^{(4)}_{a_1a_2a_3a_4}\Big(e^{a_1} \cdots e^{a_4} + \underbrace{\frac{1}{6!}\epsilon^{a_1\ldots a_4}{}_{b_1\ldots b_6} e^{b_1} \cdots e^{b_6}}_{\star(e^{a_1}\wedge\ldots\wedge e^{a_4})}\Big) \tag{5.650}$$

According to (D.46) and (D.47) in the appendix, the action of the Dirac operator $\gamma^c_{\gamma\alpha}\nabla_c$ on the first or $\gamma^{c\,\gamma\beta}\nabla_c$ on the second index (with a covariant derivative that leaves the gamma-matrices invariant) yields

$$\gamma^c_{\gamma\alpha}\nabla_c \mathfrak{p}^{\alpha}{}_{\beta} \overset{\not{}^{-1}}{\mapsto} \boldsymbol{\nabla} g + \star\boldsymbol{\nabla} \underbrace{\star g}_{g} \tag{5.651}$$

$$\nabla_c \mathfrak{p}^{\alpha}{}_{\beta} \cdot \gamma^{c\,\beta\gamma} \overset{\not{}^{-1}}{\mapsto} \boldsymbol{\nabla} g - \star\boldsymbol{\nabla} \underbrace{\star g}_{g} \tag{5.652}$$

When $\omega_{ab}{}^c = \omega^{(LC)c}_{ab} + \frac{3}{2}h_{ab}{}^c$ one might expect to get something like the h-twisted differential on the righthand side, but this is not true for a connection that respects the gamma-matrices as we assumed in the two equations above. The expression in (5.651) does not coincide with the h-twisted differential for this choice of connection. It is important therefore that we act with our "mixed" connection which acts on the first fermionic index with $\omega_{a\beta}{}^{\gamma} = \frac{1}{4}\left(\omega^{(LC)c}_{ab} + \frac{3}{2}h_{ab}{}^c\right)\gamma^b{}_{c\,\beta}{}^{\gamma}$ and on the second with $\hat{\omega}_a{}^{\beta}{}_{\gamma} = \frac{1}{4}\left(\omega^{(LC)c}_{ab} - \frac{3}{2}h_{ab}{}^c\right)\gamma^b{}_c{}^{\beta}{}_{\gamma}$. This mixed connection does not leave both gamma-matrix blocks $\gamma^c_{\alpha\beta}$ and $\gamma^{c\,\alpha\beta}$ invariant at the same time. Depending on the sign we choose for the action on the bosonic index, it either leaves invariant only the first or only the second. The calculation of above therefore does not go through in the same way and gets modified as follows:

Let us act with the left-mover connection $\omega_{ab}{}^c = \omega^{(LC)c}_{ab} + \frac{3}{2}h_{ab}{}^c$ on the bosonic indices and rewrite $\hat{\omega}_a{}^{\delta}{}_{\beta} = \omega_a{}^{\delta}{}_{\beta} + \Delta_a{}^{\delta}{}_{\beta} = \omega_a{}^{\delta}{}_{\beta} - \frac{3}{4}h_{ab}{}^c\gamma^b{}_c{}^{\delta}{}_{\beta}$. We then have[26]

$$\begin{aligned} \gamma^c_{\gamma\alpha}\, \underline{\nabla}_c \mathfrak{p}^{\alpha}{}_{\beta}|_{\hat{\omega}=\omega} &= \gamma^c_{\gamma\alpha}\nabla_c\mathfrak{p}^{\alpha}{}_{\beta} - \tfrac{3}{4}\gamma^c_{\gamma\alpha}h_{ca}{}^b\gamma^a{}_b{}^{\delta}{}_{\beta}\mathfrak{p}^{\alpha}{}_{\delta} = &(5.653)\\ &= \gamma^c_{\gamma\alpha}\nabla_c\mathfrak{p}^{\alpha}{}_{\beta} - \tfrac{3}{4}h_{cab}\gamma^{[c|}_{\gamma\alpha}\mathfrak{p}^{\alpha}{}_{\delta}\gamma^{|ab]\delta}{}_{\beta} &(5.654)\end{aligned}$$

In the last term, we have two matrix multiplications between three matrices (in the spinorial indices), which corresponds on the form side to two Clifford-multiplications. According to (5.649), the chiral gamma matrix $\gamma^c_{\gamma\alpha}$ can be seen as the Clifford map of the self-dual projection of the vielbein $\frac{1}{2}(1+\star)e^c$. The even form g, corresponding to $\mathfrak{p}^{\alpha}{}_{\delta}$, is given in (5.650) and $\gamma^{ab\delta}{}_{\beta}$ corresponds according to (5.648) to $\frac{1}{2}(1+\star)e^a \wedge e^b$. Now we need the explicit expression for the Clifford multiplication on the form-side and the fact that the Clifford multiplication of two self-dually projected forms yields either zero or the self dual projection of their Clifford multiplication (see equation (D.51) and below in the appendix):

$$\not{\omega}\,\not{\rho} \overset{\not{}^{-1}}{\mapsto} \sum_{k\geq 0}\frac{1}{k!}\omega\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}\rho \tag{5.655}$$

$$\not{\omega}^{(p)}\tfrac{1}{2}(\mathbb{1}+\Gamma^{\#})\;\not{\rho}^{(r)}\tfrac{1}{2}(\mathbb{1}+\Gamma^{\#}) = \begin{cases} \not{\omega}^{(p)}\;\not{\rho}^{(r)}\frac{1}{2}(\mathbb{1}+\Gamma^{\#}) & \text{for } r \text{ even}\\ 0 \quad \text{for } r \text{ odd}\end{cases} \tag{5.656}$$

[26] In order to better understand the sign in (5.653), note that the action of the connection on the fermionic indices was defined via graded conventions according to the first part of the thesis and that the second (lower) index of the RR-bispinor used to be an upper hatted index $\boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = \boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}}{}_{\boldsymbol{\beta}}$. The action of the covariant derivative is thus

$$\underline{\nabla}_m \boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} =_g \partial_m \boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} + \omega_{m\boldsymbol{\delta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\mathfrak{p}}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} + \hat{\omega}_{m\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}}\boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}$$

In this second part of the thesis we ususally did not denote the graded equal sign explicitely. It had to be understood as such, whenever graded indices appeared. For this explicit comparison, however, it is useful to make a distinction. In terms of ordinary equal sign and explicitely written summation (NW-conventions), this becomes:

$$\underline{\nabla}_m \boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = \partial_m \boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} + \sum_{\boldsymbol{\delta}}\underbrace{(-)^{\boldsymbol{\delta}+\boldsymbol{\delta}\boldsymbol{\alpha}}}_{1}\omega_{m\boldsymbol{\delta}}{}^{\boldsymbol{\alpha}}\boldsymbol{\mathfrak{p}}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} + \underbrace{(-)^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}}_{-1}\sum_{\hat{\boldsymbol{\delta}}}\underbrace{(-)^{\hat{\boldsymbol{\delta}}+\hat{\boldsymbol{\delta}}(\hat{\boldsymbol{\beta}}+\boldsymbol{\alpha})}}_{-1}\hat{\omega}_{m\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\beta}}}\boldsymbol{\mathfrak{p}}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}$$

In other words, if we consider the indices to carry no grading, we have

$$\begin{aligned} \underline{\nabla}_m \mathfrak{p}^{\alpha\hat{\beta}} &= \partial_m\mathfrak{p}^{\alpha\hat{\beta}} + \omega_{m\delta}{}^{\alpha}\mathfrak{p}^{\delta\hat{\beta}} + \hat{\omega}_{m\hat{\delta}}{}^{\hat{\beta}}\mathfrak{p}^{\alpha\hat{\delta}}\\ \text{or } \underline{\nabla}_m \mathfrak{p}^{\alpha}{}_{\beta} &= \partial_m\mathfrak{p}^{\alpha}{}_{\beta} + \omega_{m\delta}{}^{\alpha}\mathfrak{p}^{\delta}{}_{\beta} + \hat{\omega}_m{}^{\delta}{}_{\beta}\mathfrak{p}^{\alpha}{}_{\delta} \quad \diamond\end{aligned}$$

The differential forms g and $e^a \wedge e^b$ both are even so that now we can write down (using also (5.651)) the inverse Clifford map of (5.654)

$$\begin{aligned}
\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c \mathfrak{p}^\alpha{}_\beta|_{\breve{\omega}=\omega} &\overset{/^{-1}}{\mapsto} \boldsymbol{\nabla} g + \star\boldsymbol{\nabla}\underbrace{\star g}_{g} + \\
&-\tfrac{3}{4}h_{cab}\tfrac{1}{2}(1+\star)\Big\{\sum_{l\geq 0}\frac{1}{l!}\left(\sum_{k\geq 0}\frac{1}{k!}e^{[c|}\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}g\right)\frac{\overleftarrow{\partial}}{\partial e^{c_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{c_l}}\times \\
&\times\eta^{c_1d_1}\cdots\eta^{c_ld_l}\frac{\partial}{\partial e^{d_l}}\cdots\frac{\partial}{\partial e^{d_1}}\left(e^{|a}\wedge e^{b]}\right)\Big\} \qquad (5.657)\\
&= (1+\star)\boldsymbol{\nabla} g + \\
&-\tfrac{3}{4}h_{cab}\tfrac{1}{2}(1+\star)\Big\{\sum_{l\geq 0}\frac{1}{l!}\left(e^{[c|}\wedge g + \eta^{[c|b_1}\frac{\partial}{\partial e^{b_1}}g\right)\frac{\overleftarrow{\partial}}{\partial e^{c_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{c_l}}\times \\
&\times\eta^{c_1d_1}\cdots\eta^{c_ld_l}\frac{\partial}{\partial e^{d_l}}\cdots\frac{\partial}{\partial e^{d_1}}\left(e^{|a}\wedge e^{b]}\right)\Big\} = \qquad (5.658)\\
&= (1+\star)\boldsymbol{\nabla} g + \\
&-\tfrac{3}{8}(1+\star)\Big(\underbrace{h\wedge g}_{\imath_h g} - \underbrace{e^a e^b h_{ab}{}^c\frac{\partial}{\partial e^c}g}_{\frac{2}{3}\imath_t g} - \underbrace{e^a h_a{}^{bc}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{\frac{2}{3}\imath_{\tilde t}g=\frac{2}{3}\imath_{\tilde t}\star g} + \underbrace{h^{abc}\frac{\partial}{\partial e^a}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{\imath_{\tilde h}g=\imath_{\tilde h}\star g}\Big) \qquad (5.659)
\end{aligned}$$

In the last line below the underbraces we have considered the h-field $h \equiv h_{abc}e^a e^b e^c$ as a 3-form, the corresponding torsion $t = \frac{3}{2}h_{ab}{}^c e^a e^b \otimes e_c$ as a vector-valued 2-form, $\tilde t \equiv \frac{3}{2}h_a{}^{bc}e^a \otimes e_b e_c$ as a two-vector valued 1-form and $\tilde h \equiv h^{abc}e_a e_b e_c$ as a three-vector and have used the generalized definition of an interior product with respect to a multivector valued form, given in (6.13). Now we can use the result given in the appendix in equation (D.36) on page 171 and in the discussion below, which implies that

$$\star\,\imath_{\tilde t}\star g \;=\; \imath_t g, \qquad \star\imath_{\tilde h}\star g = \imath_h g = h\wedge g \qquad (5.660)$$

Remembering that $\boldsymbol{\nabla} = \mathbf{d} - \imath_t$, we thus get the final result

$$\boxed{\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta|_{\breve\omega=\omega} \overset{/^{-1}}{\mapsto} (1+\star)\Big\{\left(\mathbf{d} - \tfrac{3}{4}h\wedge\right)g - \tfrac{1}{2}\underbrace{\imath_t g}_{\text{or }\star\imath_t\star g}\Big\}} \qquad (5.661)$$

with $\imath_T g = \frac{3}{2}e^a e^b h_{ab}{}^c\frac{\partial}{\partial e^c}g$ and $\star\imath_t\star g = \imath_{\tilde t}g = \frac{3}{2}e^a h_a{}^{bc}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g$.

Let's do the same analysis for the Dirac-operator acting on the second index, which turns out to be a bit simpler, with only one Clifford multiplication:

$$\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta|_{\breve\omega=\omega}\,\gamma^{c\,\beta\gamma} \;=\; \nabla_c\mathfrak{p}^\alpha{}_\beta\cdot\gamma^{c\,\beta\gamma} - \tfrac{3}{4}\mathfrak{p}^\alpha{}_\delta\underbrace{h_{abc}\gamma^{ab\delta}{}_\beta\gamma^{c\,\beta\gamma}}_{h_{abc}\gamma^{abc\,\delta\gamma}} \qquad (5.662)$$

According to (5.647), $h_{abc}\gamma^{abc\,\delta\gamma} = \left(\frac{1}{2}\,\not h(1\!\!1 - \Gamma^\#)\right)^{\delta\gamma}$. Using (5.652) and the explicit expression (5.655) for the Clifford multiplication on the form-side, the above derivative operator is mapped to the following:

$$\begin{aligned}
&\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta|_{\breve\omega=\omega}\,\gamma^{c\,\beta\gamma} \overset{/^{-1}}{\mapsto} \qquad (5.663)\\
&\overset{/^{-1}}{\mapsto} \left(\boldsymbol{\nabla} g - \star\boldsymbol{\nabla}\underbrace{\star g}_{g}\right) + \\
&\quad -\tfrac{3}{8}(1-\star)\sum_{k\geq 0}\frac{1}{k!}g\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}h \qquad (5.664)\\
&= (1-\star)\boldsymbol{\nabla} g + \\
&\quad -\tfrac{3}{8}(1-\star)\Big\{\underbrace{h\wedge g}_{\imath_h g} - \underbrace{3e^a e^b h_{ab}{}^c\frac{\partial}{\partial e^c}g}_{2\imath_t g} + \underbrace{3e^a h_a{}^{bc}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{2\imath_{\tilde t}\star g} - \underbrace{h^{abc}\frac{\partial}{\partial e^a}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{\imath_{\tilde h}\star g}\Big\} \qquad (5.665)
\end{aligned}$$

Using again that $\star\imath_{\tilde t}\star g = \imath_t g$, $\star\imath_{\tilde h}\star g = \imath_h g = h\wedge g$, and $\boldsymbol{\nabla} g = \mathbf{d}g - \imath_t g$ we end up with

$$\boxed{\gamma^{c\,\gamma\beta}\,\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta|_{\breve\omega=\omega} \overset{/^{-1}}{\mapsto} (1-\star)\Big\{\left(\mathbf{d} - \tfrac{3}{4}h\wedge\right)g + \tfrac{1}{2}\underbrace{\imath_t g}_{\text{or }-\star\imath_t\star g}\Big\}} \qquad (5.666)$$

with $\imath_t g = \frac{3}{2}e^a e^b h_{ab}{}^c \frac{\partial}{\partial e^c} g$ and $\star\imath_t \star g = \imath_{\tilde{t}} g = \frac{3}{2}e^a h_a{}^{bc}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c} g$. If both actions of the Dirac operator vanish, we thus get the following condition on the form side (adding and subtracting (5.661) and (5.666) lead to equivalent equations)[27]

$$\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c \mathfrak{p}^\alpha{}_\beta\big|_{\breve{\omega}=\omega} = \gamma^{c\,\gamma\beta}\,\underline{\nabla}_c \mathfrak{p}^\alpha{}_\beta\big|_{\breve{\omega}=\omega} = 0 \quad\Longleftrightarrow\quad \left(\mathbf{d} - \tfrac{3}{4}h\wedge\right) g - \tfrac{1}{2}\star\imath_t \star g = 0 \tag{5.667}$$

Next we consider the **type IIB** case where we can expand the RR-bispinor in odd antisymmetrized products of γ-matrices (see (D.142) on page 179):

$$\begin{aligned}
\mathfrak{p}^{\alpha\beta} &= 2g^{(1)}_a \underbrace{\gamma^{a\,\alpha\beta}}_{\gamma^{[1]}} + 2g^{(3)}_{a_1a_2a_3}\underbrace{\gamma^{a_1a_2a_3\,\alpha\beta}}_{\gamma^{[3]}} + g^{(5)}_{a_1a_2a_3a_4a_5}\underbrace{\gamma^{a_1a_2a_3a_4a_5\alpha\beta}}_{\gamma^{[5]}} && (5.668)\\
2g^{(1)}_a &= \frac{1}{16}\mathfrak{p}^{\alpha\beta}\gamma_{a\,\beta\alpha} && (5.669)\\
2g^{(3)}_{a_1a_2a_3} &= \frac{1}{16\cdot 3!}\mathfrak{p}^{\alpha\beta}\gamma_{a_1a_2a_3\,\beta\alpha} && (5.670)\\
g^{(5)}_{a_1a_2a_3a_4a_5} &= \frac{1}{32\cdot 5!}\mathfrak{p}^{\alpha\beta}\gamma_{a_5a_4a_3a_2a_1\,\beta\alpha} && (5.671)
\end{aligned}$$

This is mapped to an odd anti self-dual form on the form-side

$$\mathfrak{p}^{\alpha\beta} \overset{/^{-1}}{\mapsto} (1-\star)\left(g^{(1)}_a e^a + g^{(3)}_{a_1a_2a_3} e^{a_1}\wedge e^{a_2}\wedge e^{a_3} + \frac{1}{2}g^{(5)}_{a_1\ldots a_5} e^{a_1}\wedge\ldots\wedge e^{a_5}\right) \equiv g \tag{5.672}$$

on the form-side. According to (D.46) and (D.47) in the appendix, the action of the Dirac operator $\gamma^c_{\gamma\alpha}\nabla_c$ on the first or on the second index (with a covariant derivative that leaves the gamma-matrices invariant) yields for an antiselfdual and odd g

$$\begin{aligned}
\gamma^c_{\gamma\alpha}\nabla_c\mathfrak{p}^{\alpha\beta} &\overset{/^{-1}}{\mapsto} (1-\star)\boldsymbol{\nabla} g && (5.673)\\
\nabla_c\mathfrak{p}^{\alpha\beta}\cdot\gamma^c_{\beta\gamma} &\overset{/^{-1}}{\mapsto} -(1+\star)\boldsymbol{\nabla} g && (5.674)
\end{aligned}$$

Instead of a connection that leaves the gamma-matrices invariant, we have again the mixed connection acting differently on left- and right-movers. We thus act on the first fermionic index of $\mathfrak{p}^{\alpha\beta}$ with $\omega_{a\beta}{}^\gamma = \frac{1}{4}(\omega^{(LC)c}_{ab} + \frac{3}{2}h_{ab}{}^c)\gamma^b{}_{c\,\beta}{}^\gamma$ and on the second with $\hat{\omega}_{a\beta}{}^\gamma = \frac{1}{4}(\omega^{(LC)c}_{ab} - \frac{3}{2}h_{ab}{}^c)\gamma^b{}_{c\,\beta}{}^\gamma$. Again we decide to act on the bosonic indices with the left mover connection $\omega_{ab}{}^c = \omega^{(LC)c}_{ab} + \frac{3}{2}h_{ab}{}^c$ and rewrite $\hat{\omega}_{c\delta}{}^\beta = \omega_{c\delta}{}^\beta + \Delta_{c\delta}{}^\beta = \omega_{c\delta}{}^\beta - \frac{3}{4}h_{ca}{}^b\gamma^a{}_{b\,\delta}{}^\beta$. We then have for the action of the Dirac operator (based on the mixed connection) on the second index

$$\begin{aligned}
&\underline{\nabla}_c\mathfrak{p}^{\alpha\beta}\big|_{\breve{\omega}=\omega}\cdot\gamma^c_{\beta\gamma} = \\
&= \nabla_c\mathfrak{p}^{\alpha\beta}\cdot\gamma^c_{\beta\gamma} - \tfrac{3}{4}\mathfrak{p}^{\alpha\delta}h_{abc}\gamma^{abc}{}_{\delta\gamma} && (5.675)\\
&= \nabla_c\mathfrak{p}^{\alpha\beta}\cdot\gamma^c_{\beta\gamma} - \tfrac{3}{4}(\not{g}\ \not{h})^\alpha{}_\gamma \overset{/^{-1}}{\mapsto} && (5.676)\\
&\overset{/^{-1}}{\mapsto} -(1+\star)\boldsymbol{\nabla} g - \tfrac{3}{8}(1+\star)\sum_{k\geq 0}\frac{1}{k!}g\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}h = && (5.677)\\
&= -(1+\star)(\mathbf{d} - \imath_t)g + \tfrac{3}{8}(1+\star)\underbrace{h\wedge g}_{\imath_h g} - \tfrac{3}{8}(1+\star)\underbrace{3e^a e^b h_{ab}{}^c\frac{\partial}{\partial e^c}g}_{2\imath_t g} + \\
&\quad + \tfrac{3}{8}(1+\star)\underbrace{3e^a h_a{}^{bc}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{2\imath_{\tilde{t}}g = 2\star\imath_t\star g} - \tfrac{3}{8}(1+\star)\underbrace{h^{abc}\frac{\partial}{\partial e^a}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{\imath_{\tilde{h}}g = \star\imath_h\star g} && (5.678)
\end{aligned}$$

[27] We could try to absorb the somewhat disturbing contribution of $\star\imath_t\star g$ or $\imath_t g$ by reintroducing $\boldsymbol{\nabla} g$ via $\imath_t g = -\boldsymbol{\nabla} g + \mathbf{d}g$. The result, however, looks even less natural and the twisted differential gets modified at intermediate steps. The equations (5.661), (5.666) and (5.667) take the following form

$$\begin{aligned}
\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta\big|_{\breve{\omega}=\omega} &\overset{/^{-1}}{\mapsto} \tfrac{1}{2}(1+\star)\Big\{\boldsymbol{\nabla} g + \left(\mathbf{d} - \tfrac{3}{2}h\wedge\right) g\Big\}\\
\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta\big|_{\breve{\omega}=\omega}\gamma^{c\,\beta\gamma} &\overset{/^{-1}}{\mapsto} \tfrac{1}{2}(1-\star)\Big\{-\boldsymbol{\nabla} g + 3\left(\mathbf{d} - \tfrac{1}{2}h\wedge\right) g\Big\}\\
\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta\big|_{\breve{\omega}=\omega} = \underline{\nabla}_c\mathfrak{p}^\alpha{}_\beta\big|_{\breve{\omega}=\omega}\gamma^{c\,\beta\gamma} = 0 &\Longleftrightarrow 2\left(\mathbf{d} - \tfrac{3}{4}h\wedge\right) g + \star\boldsymbol{\nabla}\star g - \star\mathbf{d}\star g = 0 \quad \diamond
\end{aligned}$$

After collecting all the terms, we arrive at

$$\boxed{\underline{\nabla}_c \mathfrak{p}^{\alpha\beta}\big|_{\breve{\omega}=\omega} \cdot \gamma^c_{\beta\gamma} \overset{/^{-1}}{\mapsto} -(1+\star)\Big\{\big(\mathbf{d} - \tfrac{3}{4}h\wedge\big)\, g + \tfrac{1}{2}\underbrace{\imath_t g}_{\text{or } -\star\imath_t\star g}\Big\}} \tag{5.679}$$

For the action of the Dirac operator on the first index, finally, we have

$$\begin{aligned}
&\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c \mathfrak{p}^{\alpha\beta}\big|_{\breve{\omega}=\omega} = \\
&= \quad \gamma^c_{\gamma\alpha}\nabla_c \mathfrak{p}^{\alpha\beta} - \tfrac{3}{4}h_{abc}\gamma^c_{\gamma\alpha}\mathfrak{p}^{\alpha\delta}\gamma^{ab}{}_{\delta}{}^{\beta} \overset{/^{-1}}{\mapsto} && (5.680)\\
&\overset{/^{-1}}{\mapsto} \quad (1-\star)\boldsymbol{\nabla} g + \\
&\qquad -\tfrac{3}{8}h_{abc}(1-\star)\sum_{k\geq 0}\frac{1}{k!}\left(e^c\wedge g + \eta^{cd}\frac{\partial}{\partial e^d}g\right)\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1 b_1}\cdots\eta^{a_k b_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}\left(e^a\wedge e^b\right) && (5.681)\\
&= \quad (1-\star)\underbrace{\boldsymbol{\nabla} g}_{\mathbf{d}g-\imath_t g} + \\
&\qquad -\tfrac{3}{8}(1-\star)\Big\{\underbrace{h\wedge g}_{\imath_h g} - \underbrace{e^a e^b h_{ab}{}^c\frac{\partial}{\partial e^c}g}_{\frac{2}{3}\imath_t g} - \underbrace{e^a h_a{}^{bc}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{\frac{2}{3}\imath_{\tilde{t}} g = \frac{2}{3}\star\imath_t\star g} + \underbrace{h^{abc}\frac{\partial}{\partial e^a}\frac{\partial}{\partial e^b}\frac{\partial}{\partial e^c}g}_{\imath_{\tilde{h}} g = \star\imath_h\star g}\Big\} && (5.682)
\end{aligned}$$

The terms then combine to

$$\boxed{\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c \mathfrak{p}^{\alpha\beta}\big|_{\breve{\omega}=\omega} \overset{/^{-1}}{\mapsto} (1-\star)\Big\{\big(\mathbf{d} - \tfrac{3}{4}h\wedge\big)\, g \underbrace{-\tfrac{1}{2}\imath_t g}_{\text{or } -\frac{1}{2}\star\imath_t\,\star\, g}\Big\}} \tag{5.683}$$

The equations on the form side thus look the same as for type IIA. In particular we have

$$\gamma^c_{\gamma\alpha}\,\underline{\nabla}_c \mathfrak{p}^{\alpha\beta}\big|_{\breve{\omega}=\omega} = \underline{\nabla}_c \mathfrak{p}^{\alpha\beta}\big|_{\breve{\omega}=\omega}\cdot\gamma^c_{\beta\gamma} = 0 \quad\Longleftrightarrow\quad \big(\mathbf{d} - \tfrac{3}{4}h\wedge\big)\, g - \tfrac{1}{2}\star\imath_t\star g = 0 \tag{5.684}$$

- <u>(delta|2,1,0)$_{\boldsymbol{\alpha}bc}{}^{\boldsymbol{\delta}}$ ↔((hdelta|2,0,1))$_{\hat{\boldsymbol{\alpha}}bc}{}^{\hat{\boldsymbol{\delta}}}$)dim $\frac{4}{2}$:</u>

$$\begin{aligned}
0 \quad &\overset{!}{=} \quad \underline{\nabla}_{[\boldsymbol{\alpha}}T_{bc]}{}^{\boldsymbol{\delta}} + 2\underline{T}_{[\boldsymbol{\alpha}b|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} - R_{[\boldsymbol{\alpha}bc]}{}^{\boldsymbol{\delta}} = && (5.685)\\
&= \quad \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}T_{bc}{}^{\boldsymbol{\delta}} + \frac{4}{3}\underline{T}_{\boldsymbol{\alpha}[b|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} - \frac{1}{3}R_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} = && (5.686)\\
&= \quad \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}T_{bc}{}^{\boldsymbol{\delta}} + \frac{4}{3}\check{T}_{\boldsymbol{\alpha}[b|}{}^{e}T_{e|c]}{}^{\boldsymbol{\delta}} + \frac{4}{3}\hat{T}_{\boldsymbol{\alpha}[b|}{}^{\hat{\boldsymbol{\varepsilon}}}T_{\hat{\boldsymbol{\varepsilon}}|c]}{}^{\boldsymbol{\delta}} - \frac{1}{3}R_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} = && (5.687)\\
&= \quad \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}T_{bc}{}^{\boldsymbol{\delta}} + \frac{4}{3}\underbrace{\hat{\nabla}_a\Phi\check{T}_{\boldsymbol{\alpha}[b|}{}^{e}}_{=0\text{ for }\check{\Omega}=\hat{\Omega}}T_{e|c]}{}^{\boldsymbol{\delta}} + \frac{4}{3}\tilde{\gamma}_{[b|\,\boldsymbol{\alpha\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \frac{1}{3}R_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} && (5.688)
\end{aligned}$$

$$\boxed{\begin{aligned}
R_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} \quad &= \quad \underline{\nabla}_{\boldsymbol{\alpha}}T_{bc}{}^{\boldsymbol{\delta}}\big|_{\check{\Omega}=\hat{\Omega}} + 4\tilde{\gamma}_{[b|\,\boldsymbol{\alpha\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.689)\\
\hat{R}_{bc\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}} \quad &= \quad \underline{\nabla}_{\hat{\boldsymbol{\alpha}}}\hat{T}_{bc}{}^{\hat{\boldsymbol{\delta}}}\Big|_{\check{\Omega}=\Omega} + 4\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\gamma}}}\tilde{\gamma}_{|c]\,\boldsymbol{\varepsilon\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.690)
\end{aligned}}$$

Plugging in $T_{bc}{}^{\boldsymbol{\delta}} = \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{bc}{}^{\hat{\boldsymbol{\gamma}}}{}_{\hat{\boldsymbol{\delta}}}$yields

$$\begin{aligned}
R_{bc\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}} &= \frac{1}{16}\underline{\nabla}_{\boldsymbol{\alpha}}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\cdot\tilde{\gamma}_{bc}{}^{\hat{\boldsymbol{\gamma}}}{}_{\hat{\boldsymbol{\delta}}} + \\
&\quad + \frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)2\underbrace{\hat{\nabla}_{\boldsymbol{\alpha}}\Phi}_{=0}\tilde{\gamma}_{bc}{}^{\hat{\boldsymbol{\gamma}}}{}_{\hat{\boldsymbol{\delta}}} + \\
&\quad + 4\tilde{\gamma}_{[b|\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = && (5.691)\\
&= \left(\frac{1}{16}\underline{\nabla}_{\boldsymbol{\alpha}}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + \frac{8}{16}\underline{\nabla}_{\boldsymbol{\alpha}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + \frac{8}{16}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\cdot\tilde{\gamma}_{bc}{}^{\hat{\boldsymbol{\gamma}}}{}_{\hat{\boldsymbol{\delta}}} + \\
&\quad + 4\tilde{\gamma}_{[b|\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = && (5.692)\\
&= \left(\frac{1}{16}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - \frac{1}{8}R_{\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}} + \frac{1}{8}R_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\varepsilon}}}{}^{\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}} + \right.\\
&\quad \left. + \hat{F}^{(D)}_{\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + \frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right)\cdot\tilde{\gamma}_{bc}{}^{\hat{\boldsymbol{\gamma}}}{}_{\hat{\boldsymbol{\delta}}} + \\
&\quad + 4\tilde{\gamma}_{[b|\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.693)
\end{aligned}$$

Taking the trace yields

$$\begin{aligned}
-8F^{(D)}_{bc} &= \left(\frac{1}{16}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\underbrace{\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}}_{8\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\delta}}\Phi} - \frac{1}{8}R_{\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}\boldsymbol{\varepsilon}}{}^{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}} + \frac{1}{8}R_{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\varepsilon}}}{}^{\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\varepsilon}}} + \right.\\
&\quad \left. + \hat{F}^{(D)}_{\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}} + \frac{1}{2}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\underbrace{\underline{\nabla}_{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}}}_{8\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\delta}}\Phi}\right)\cdot\tilde{\gamma}_{bc}{}^{\hat{\boldsymbol{\gamma}}}{}_{\hat{\boldsymbol{\delta}}} + \\
&\quad + 4\tilde{\gamma}_{[b|\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\delta}}} && (5.694)
\end{aligned}$$

- <u>(delta|2,0,1)$_{\hat{\boldsymbol{\alpha}}bc}{}^{\boldsymbol{\delta}}$ ↔(hdelta|2,1,0)$_{\boldsymbol{\alpha}bc}{}^{\hat{\boldsymbol{\delta}}}$,dim$\frac{4}{2}$</u>:

$$\begin{aligned}
0 &\overset{!}{=} \underline{\nabla}_{[\hat{\boldsymbol{\alpha}}}T_{bc]}{}^{\boldsymbol{\delta}} + 2\underline{T}_{[\hat{\boldsymbol{\alpha}}b|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} - R_{[\hat{\boldsymbol{\alpha}}bc]}{}^{\boldsymbol{\delta}} = && (5.695)\\
&= \frac{1}{3}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}} + \frac{2}{3}\underline{\nabla}_{[b}T_{c]\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\delta}} + \frac{4}{3}\underline{T}_{\hat{\boldsymbol{\alpha}}[b|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} + \frac{2}{3}\underline{T}_{bc}{}^{E}T_{E\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\delta}} = && (5.696)\\
&= \frac{1}{3}\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}} - \frac{2}{3}\underline{\nabla}_{[b}\left(\tilde{\gamma}_{c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}\right) + \frac{4}{3}\check{T}_{\hat{\boldsymbol{\alpha}}[b|}{}^{e}T_{e|c]}{}^{\boldsymbol{\delta}} + \frac{2}{3}\check{T}_{bc}{}^{e}T_{e\hat{\boldsymbol{\alpha}}}{}^{\boldsymbol{\delta}} = && (5.697)\\
&\overset{\check{\Omega}=\hat{\Omega}}{=} \frac{1}{3}\left.\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}}\right|_{\check{\Omega}=\hat{\Omega}} + \frac{2}{3}\tilde{\gamma}_{[b\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + \frac{4}{3}\hat{T}_{\hat{\boldsymbol{\alpha}}[b|}{}^{e}T_{e|c]}{}^{\boldsymbol{\delta}} + H_{bc}{}^{e}\tilde{\gamma}_{e\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\beta}}} && (5.698)
\end{aligned}$$

or

$$\begin{aligned}
&\overset{\check{\Omega}=\Omega}{=} \frac{1}{3}\left.\underline{\nabla}_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}}\right|_{\check{\Omega}=\Omega} - \frac{2}{3}\gamma^{d}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\Big[\big(\underbrace{\nabla_{[b|}\Phi}_{=0\,(5.529)} + \underbrace{\hat{\nabla}_{[b|}\Phi}_{=0\,(5.529)}\big)G_{d|c]} + \underbrace{\Delta_{[bc]d}}_{-3H_{bcd}}\Big]\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + \\
&\quad + \frac{2}{3}\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - H_{bce}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} = && (5.699)\\
&= \frac{1}{3}\nabla_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}} + \frac{2}{3}\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + H_{bce}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.700)
\end{aligned}$$

$$\boxed{\begin{aligned}
\nabla_{\hat{\boldsymbol{\alpha}}}T_{bc}{}^{\boldsymbol{\delta}} &= -2\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\underline{\nabla}_{|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} - 3H_{bce}\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.701)\\
\hat{\nabla}_{\boldsymbol{\alpha}}\hat{T}_{bc}{}^{\hat{\boldsymbol{\delta}}} &= -2\tilde{\gamma}_{[b|\,\boldsymbol{\alpha}\boldsymbol{\delta}}\underline{\nabla}_{|c]}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} + 3H_{bce}\gamma^{e}_{\boldsymbol{\alpha}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.702)
\end{aligned}}$$

- <u>(delta|3,0,0)$_{abc}{}^{\boldsymbol{\delta}}$ ↔((hdelta|3,0,0)$_{abc}{}^{\hat{\boldsymbol{\delta}}}$)dim$\frac{5}{2}$</u>:

$$\begin{aligned}
0 &\overset{!}{=} \underline{\nabla}_{[a}T_{bc]}{}^{\boldsymbol{\delta}} + 2T_{[ab|}{}^{E}T_{E|c]}{}^{\boldsymbol{\delta}} - R_{[abc]}{}^{\boldsymbol{\delta}} = && (5.703)\\
&= \underline{\nabla}_{[a}T_{bc]}{}^{\boldsymbol{\delta}} + 2\check{T}_{[ab|}{}^{e}T_{e|c]}{}^{\boldsymbol{\delta}} + 2\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.704)
\end{aligned}$$

$$\boxed{\begin{aligned}
\nabla_{[a}T_{bc]}{}^{\boldsymbol{\delta}} &= -3H_{[ab|}{}^{e}T_{e|c]}{}^{\boldsymbol{\delta}} - 2\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.705)\\
\hat{\nabla}_{[a}\hat{T}_{bc]}{}^{\hat{\boldsymbol{\delta}}} &= 3H_{[ab|}{}^{e}\hat{T}_{e|c]}{}^{\hat{\boldsymbol{\delta}}} - 2T_{[ab|}{}^{\boldsymbol{\varepsilon}}\tilde{\gamma}_{|c]\,\boldsymbol{\varepsilon}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} && (5.706)
\end{aligned}}$$

- $\underline{(\mathrm{d}|0,3,0)_{\boldsymbol{\alpha\beta\gamma}}{}^{d} \leftrightarrow((\mathrm{d}|0,0,3)_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}}{}^{d})\dim\tfrac{1}{2}}$:

$$\begin{aligned} 0 &\stackrel{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}\check{T}_{\boldsymbol{\beta\gamma}]}{}^{d} + 2\check{T}_{[\boldsymbol{\alpha\beta}|}{}^{c}\check{T}_{c|\boldsymbol{\gamma}]}{}^{d} - \underbrace{\check{R}_{[\boldsymbol{\alpha\beta\gamma}]}{}^{d}}_{=0} = & (5.707)\\ &= \underline{\nabla}_{[\boldsymbol{\alpha}}\left(\gamma_{\boldsymbol{\beta\gamma}]}{}^{c}f_{c}{}^{d}\right) + 2\gamma^{e}_{[\boldsymbol{\alpha\beta}|}f_{e}{}^{c}\check{T}_{c|\boldsymbol{\gamma}]}{}^{d} = & (5.708)\\ &\stackrel{f_c{}^d=\delta_c^d}{=} \underbrace{\nabla_{[\boldsymbol{\alpha}}\left(\gamma^{d}_{\boldsymbol{\beta\gamma}]}\right)}_{=0} - 2\underbrace{\gamma^{c}_{[\boldsymbol{\alpha\beta}}T_{\boldsymbol{\gamma}]c}{}^{d}}_{=0\,(5.500)} & (5.709)\end{aligned}$$

- $\underline{(\mathrm{d}|0,1,2)_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}}{}^{a} \leftrightarrow((\mathrm{d}|0,2,1)_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta\gamma}}{}^{a})\dim\tfrac{1}{2}}$:

$$\begin{aligned} 0 &\stackrel{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}\check{T}_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}]}{}^{d} + 2\underline{T}_{[\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}|}{}^{C}\check{T}_{C|\hat{\boldsymbol{\gamma}}]}{}^{d} - \check{R}_{[\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}]}{}^{d} = & (5.710)\\ &= \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}\check{T}_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}}{}^{d} + \frac{2}{3}\check{T}_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}}{}^{c}\check{T}_{c\boldsymbol{\alpha}}{}^{d} = & (5.711)\\ &= \frac{2}{3}\gamma_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\gamma}}}{}^{c}\hat{T}_{c\boldsymbol{\alpha}}{}^{d} = 0 & (5.712)\end{aligned}$$

- $\underline{(\mathrm{d}|1,2,0)_{\boldsymbol{\alpha\beta}c}{}^{d} \leftrightarrow((\mathrm{d}|1,0,2)_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c}{}^{d})\mathrm{dim}1}$:

$$\begin{aligned} 0 &\stackrel{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}\check{T}_{\boldsymbol{\beta}c]}{}^{d} + 2\underline{T}_{[\boldsymbol{\alpha\beta}|}{}^{E}T_{E|c]}{}^{d} - \check{R}_{[\boldsymbol{\alpha\beta}c]}{}^{d} = & (5.713)\\ &= \frac{2}{3}\underline{\nabla}_{[\boldsymbol{\alpha}}\check{T}_{\boldsymbol{\beta}]c}{}^{d} + \frac{1}{3}\underline{\nabla}_{c}T_{\boldsymbol{\alpha\beta}}{}^{d} + \frac{2}{3}\underline{T}_{\boldsymbol{\alpha\beta}}{}^{E}\check{T}_{Ec}{}^{d} + \frac{4}{3}\underline{T}_{c[\boldsymbol{\alpha}|}{}^{E}\check{T}_{E|\boldsymbol{\beta}]}{}^{d} - \frac{1}{3}\check{R}_{\boldsymbol{\alpha\beta}c}{}^{d} - \frac{2}{3}\underbrace{R_{c[\boldsymbol{\alpha\beta}]}{}^{d}}_{=0} = & (5.714)\\ &\stackrel{\substack{f_e{}^d=\delta_e^d\\ \check{\Omega}=\Omega}}{=} \frac{2}{3}\nabla_{[\boldsymbol{\alpha}}T_{\boldsymbol{\beta}]c}{}^{d} + \frac{1}{3}\underbrace{\nabla_{c}\gamma^{d}_{\boldsymbol{\alpha\beta}}}_{=0} + \frac{2}{3}\gamma^{e}_{\boldsymbol{\alpha\beta}}\underbrace{T_{ec}{}^{d}}_{\frac{3}{2}H_{ec}{}^{d}} + \frac{4}{3}T_{[\boldsymbol{\alpha}|c}{}^{e}T_{|\boldsymbol{\beta}]e}{}^{d} + \frac{4}{3}\underbrace{T_{c[\boldsymbol{\alpha}|}{}^{\boldsymbol{\varepsilon}}}_{=0}\gamma^{d}_{\boldsymbol{\varepsilon}|\boldsymbol{\beta}]} - \frac{1}{3}R_{\boldsymbol{\alpha\beta}c}{}^{d} & (5.715)\end{aligned}$$

$$\boxed{\begin{aligned} R_{\boldsymbol{\alpha\beta}c}{}^{d} &\stackrel{!}{=} 2\nabla_{[\boldsymbol{\alpha}}T_{\boldsymbol{\beta}]c}{}^{d} + 3\gamma^{e}_{\boldsymbol{\alpha\beta}}H_{ec}{}^{d} + 4T_{[\boldsymbol{\alpha}|c}{}^{e}T_{|\boldsymbol{\beta}]e}{}^{d} & (5.716)\\ \hat{R}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c}{}^{d} &\stackrel{!}{=} 2\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}}\hat{T}_{\hat{\boldsymbol{\beta}}]c}{}^{d} - 3\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}H_{ec}{}^{d} + 4\hat{T}_{[\hat{\boldsymbol{\alpha}}|c}{}^{e}\hat{T}_{|\hat{\boldsymbol{\beta}}]e}{}^{d} & (5.717)\end{aligned}}$$

Taking the trace (using $R_{\boldsymbol{MM}a}{}^{b} = F^{(D)}{}_{\boldsymbol{MM}}\delta^{b}_{a} + R^{(L)}_{\boldsymbol{MM}a}{}^{b}$) yields

$$10F^{(D)}_{\boldsymbol{\alpha\beta}} \stackrel{!}{=} -10\nabla_{[\boldsymbol{\alpha}}\nabla_{\boldsymbol{\beta}]}\Phi, \qquad \mathit{true} \qquad (5.718)$$

Plugging in the torsion constraints yields

$$\boxed{\begin{aligned} R_{\boldsymbol{\alpha\beta}c}{}^{d} &= -\nabla_{[\boldsymbol{\alpha}}\nabla_{\boldsymbol{\beta}]}\Phi\delta_{c}{}^{d} + \gamma_{c}{}^{d}{}_{[\boldsymbol{\alpha}}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\beta}]}\nabla_{\boldsymbol{\delta}}\Phi + 3\gamma^{e}_{\boldsymbol{\alpha\beta}}H_{ec}{}^{d} +\\ &\quad +\gamma_{c}{}^{e}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\gamma}}\Phi\gamma_{e}{}^{d}{}_{|\boldsymbol{\beta}]}{}^{\boldsymbol{\delta}}\nabla_{\boldsymbol{\delta}}\Phi & (5.719)\\ \hat{R}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}c}{}^{d} &= -\hat{\nabla}_{[\hat{\boldsymbol{\alpha}}}\hat{\nabla}_{\hat{\boldsymbol{\beta}}]}\Phi\delta_{c}{}^{d} + \gamma_{c}{}^{d}{}_{[\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\beta}}]}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi - 3\gamma^{e}_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}H_{ec}{}^{d} +\\ &\quad +\gamma_{c}{}^{e}{}_{[\hat{\boldsymbol{\alpha}}|}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\gamma_{e}{}^{d}{}_{|\hat{\boldsymbol{\beta}}]}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi & (5.720)\end{aligned}}$$

This agrees with (5.556) and (5.555).

- $\underline{(\mathrm{d}|1,1,1)_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{d}\mathrm{dim}1}$:

$$\begin{aligned} 0 &\stackrel{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}\check{T}_{\hat{\boldsymbol{\beta}}c]}{}^{d} + 2\underline{T}_{[\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}|}{}^{E}\check{T}_{E|c]}{}^{d} - \check{R}_{[\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c]}{}^{d} = & (5.721)\\ &= \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}\check{T}_{\hat{\boldsymbol{\beta}}c}{}^{d} + \frac{1}{3}\underline{\nabla}_{\hat{\boldsymbol{\beta}}}\check{T}_{c\boldsymbol{\alpha}}{}^{d} + \frac{2}{3}\underline{T}_{c\boldsymbol{\alpha}}{}^{E}\check{T}_{E\hat{\boldsymbol{\beta}}}{}^{d} + \frac{2}{3}\underline{T}_{\hat{\boldsymbol{\beta}}c}{}^{E}\check{T}_{E\boldsymbol{\alpha}}{}^{d} - \frac{1}{3}\check{R}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{d} = & (5.722)\\ &\stackrel{\check{\Omega}=\Omega}{=} \frac{1}{3}\nabla_{\hat{\boldsymbol{\beta}}}T_{c\boldsymbol{\alpha}}{}^{d} + \frac{2}{3}\hat{T}_{c\boldsymbol{\alpha}}{}^{\hat{\boldsymbol{\varepsilon}}}T_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}}{}^{d} + \frac{2}{3}T_{\hat{\boldsymbol{\beta}}c}{}^{e}T_{e\boldsymbol{\alpha}}{}^{d} + \frac{2}{3}T_{\hat{\boldsymbol{\beta}}c}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon\alpha}}{}^{d} - \frac{1}{3}R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{d} = & (5.723)\\ &= \frac{1}{3}\nabla_{\hat{\boldsymbol{\beta}}}T_{c\boldsymbol{\alpha}}{}^{d} - \frac{2}{3}\tilde{\gamma}_{c\,\boldsymbol{\alpha\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} + \frac{2}{3}\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\alpha}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\alpha}}}\gamma^{d}_{\boldsymbol{\varepsilon\alpha}} - \frac{1}{3}R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{d} & (5.724)\end{aligned}$$

$$\boxed{\begin{aligned} R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^{d} &= \nabla_{\hat{\boldsymbol{\beta}}}T_{c\boldsymbol{\alpha}}{}^{d} - 2\tilde{\gamma}_{c\,\boldsymbol{\alpha\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} + 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\gamma^{d}_{\boldsymbol{\varepsilon\alpha}} & (5.725)\\ \hat{R}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}c}{}^{d} &= \hat{\nabla}_{\boldsymbol{\beta}}\hat{T}_{c\hat{\boldsymbol{\alpha}}}{}^{d} - 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\beta}}}\gamma^{d}_{\boldsymbol{\varepsilon\beta}} + 2\tilde{\gamma}_{c\,\boldsymbol{\beta\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} \quad \text{(equivalent)} & (5.726)\end{aligned}}$$

Plugging the explicit expression for $T_{c\boldsymbol{\alpha}}{}^d$ and $\hat{T}_{c\hat{\boldsymbol{\alpha}}}{}^d$ into (5.725) and (5.726) yields[28]

$$\boxed{\begin{aligned} R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^d &= \frac{1}{2}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi\delta_c^d + \frac{1}{2}\gamma_c{}^d{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\gamma}}\Phi - 2\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} + 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\gamma^d_{\boldsymbol{\varepsilon}\boldsymbol{\alpha}} && (5.727)\\ \hat{R}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}c}{}^d &= \frac{1}{2}\hat{\nabla}_{\boldsymbol{\beta}}\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi\delta_c^d + \frac{1}{2}\gamma_c{}^d{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\boldsymbol{\beta}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\beta}}}\gamma^d_{\boldsymbol{\varepsilon}\boldsymbol{\beta}} + 2\tilde{\gamma}_{c\,\boldsymbol{\beta}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} && (5.728)\end{aligned}}$$

Taking the trace of (5.727) yields

$$\begin{aligned} 10F^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} &= 5\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi - 2\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\varepsilon}}}\gamma^c_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} + 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\gamma^c_{\boldsymbol{\varepsilon}\boldsymbol{\alpha}} && (5.729)\\ \Rightarrow\quad F^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} &= \frac{1}{2}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi && (5.730)\end{aligned}$$

This does not give new information as it follows from $\underline{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^C = 0$, $\nabla_{\hat{\boldsymbol{\beta}}}\Phi = 0$ and the algebra $\underline{\nabla}_{[\boldsymbol{\alpha}}\underline{\nabla}_{\hat{\boldsymbol{\beta}}]}\Phi\Big|_{\check{\Omega}=\Omega} = -\,\underline{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^C\Big|_{\check{\Omega}=\Omega}\nabla_C\Phi - F^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$.

- $\underline{(\mathrm{d}|2,1,0)_{\boldsymbol{\alpha}bc}{}^d \leftrightarrow((\mathrm{d}|2,0,1)_{\hat{\boldsymbol{\alpha}}bc}{}^d)\mathrm{dim}\tfrac{3}{2}:}$

$$\begin{aligned} 0 &\overset{!}{=} \underline{\nabla}_{[\boldsymbol{\alpha}}\check{T}_{bc]}{}^d + 2\underline{T}_{[\boldsymbol{\alpha}b|}{}^E\check{T}_{E|c]}{}^d - \check{R}_{[\boldsymbol{\alpha}bc]}{}^d = && (5.731)\\ &= \frac{1}{3}\underline{\nabla}_{\boldsymbol{\alpha}}\check{T}_{bc}{}^d + \frac{2}{3}\underline{\nabla}_{[b}\underbrace{\check{T}_{c]\boldsymbol{\alpha}}{}^d}_{=0\,\text{for }\check{\Omega}=\hat{\Omega}} + \frac{4}{3}\underline{T}_{\boldsymbol{\alpha}[b|}{}^E\check{T}_{E|c]}{}^d + \frac{2}{3}\underline{T}_{bc}{}^E\check{T}_{E\boldsymbol{\alpha}}{}^d - \frac{2}{3}\check{R}_{\boldsymbol{\alpha}[bc]}{}^d = && (5.732)\\ &\overset{\check{\Omega}=\hat{\Omega}}{=} -\frac{1}{2}\hat{\nabla}_{\boldsymbol{\alpha}}H_{bc}{}^d + \frac{4}{3}\hat{T}_{\boldsymbol{\alpha}[b|}{}^{\hat{\boldsymbol{\varepsilon}}}\hat{T}_{\hat{\boldsymbol{\varepsilon}}|c]}{}^d + \frac{2}{3}T_{bc}{}^{\boldsymbol{\varepsilon}}\gamma^d_{\boldsymbol{\varepsilon}\boldsymbol{\alpha}} - \frac{2}{3}\hat{R}_{\boldsymbol{\alpha}[bc]}{}^d && (5.733)\end{aligned}$$

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[28]From this constraint on $R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^d$ we can also derive a further constraint on some spinorial components. Remember that we have $R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \frac{1}{2}F^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \frac{1}{4}R^{(L)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}c}{}^d\gamma^c{}_{d\,\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}$ and therefore

$$R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \tfrac{1}{4}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \tfrac{1}{4}\left(\tfrac{1}{2}\gamma_c{}^d{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\varepsilon}}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi - 2\tilde{\gamma}_{c\,\boldsymbol{\alpha}\boldsymbol{\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}} + 2\tilde{\gamma}_{c\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\gamma^d_{\boldsymbol{\varepsilon}\boldsymbol{\alpha}}\right)\gamma^c{}_{d\,\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}$$

The last terms can be combined and we arrive at

$$\boxed{R_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} = \tfrac{1}{4}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}} + \tfrac{1}{8}\gamma_{cd\boldsymbol{\alpha}}{}^{\boldsymbol{\varepsilon}}\gamma^{cd}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi - \gamma^c_{\boldsymbol{\alpha}\boldsymbol{\varepsilon}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}}\gamma_{cd\,\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}}$$

Next we can compare whether this is consistent with our earlier constraint $R_{\hat{\boldsymbol{\beta}}[\boldsymbol{\alpha}\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} = -\gamma_{\boldsymbol{\alpha}\boldsymbol{\gamma}}{}^e\tilde{\gamma}_{e\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}$:

$$R_{\hat{\boldsymbol{\beta}}[\boldsymbol{\alpha}\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} = \tfrac{1}{4}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{[\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} + \tfrac{1}{8}\gamma_{cd[\boldsymbol{\alpha}}{}^{\boldsymbol{\varepsilon}}\gamma^{cd}{}_{\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi + \gamma^d_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\gamma^c_{\boldsymbol{\varepsilon}[\boldsymbol{\alpha}|}\tilde{\gamma}_{cd\,|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}$$

Being graded antisymmetric in $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, it can be expanded in $\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\gamma}}$ and $\gamma^{a_1\ldots a_5}_{\boldsymbol{\alpha}\boldsymbol{\gamma}}$, where the coefficient for $\gamma^{[5]}$ should vanish and the other coincide with the old expression. Before projecting the coefficients by brute force one can do a first step in this direction by using the identities $\gamma^{ab}{}_{[\boldsymbol{\alpha}|}{}^{\boldsymbol{\varepsilon}}\gamma_{ab|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} = -4\gamma^{\boldsymbol{\varepsilon}\boldsymbol{\delta}}_a\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\gamma}} - 10\delta_{[\boldsymbol{\alpha}}{}^{\boldsymbol{\varepsilon}}\delta_{\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}$ (graded version of (D.166)) and $\gamma^c_{\boldsymbol{\varepsilon}[\boldsymbol{\alpha}|}\tilde{\gamma}_{cd\,|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} = -\frac{1}{2}\gamma^c_{\boldsymbol{\alpha}\boldsymbol{\gamma}}\tilde{\gamma}_{cd\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}} - \frac{1}{2}\tilde{\gamma}_{d\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\delta_{\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}} - \tilde{\gamma}_{d\,\boldsymbol{\varepsilon}[\boldsymbol{\alpha}|}\delta_{|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}$, which are both immediate consequences of the Fierz identity $\gamma^c_{[\boldsymbol{\alpha}\boldsymbol{\beta}|}\gamma_{c\,|\boldsymbol{\gamma}]\boldsymbol{\delta}} = 0$.

$$\begin{aligned} R_{\hat{\boldsymbol{\beta}}[\boldsymbol{\alpha}\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} &= -\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{[\boldsymbol{\alpha}}\Phi\delta_{\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} - \tfrac{1}{2}\gamma^{\boldsymbol{\varepsilon}\boldsymbol{\delta}}_a\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\gamma}}\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi + \\ &\quad -\tfrac{1}{2}\gamma^c_{\boldsymbol{\alpha}\boldsymbol{\gamma}}\tilde{\gamma}_{cd\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\gamma^d_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}} - \tfrac{1}{2}\tilde{\gamma}_{d\,\boldsymbol{\alpha}\boldsymbol{\gamma}}\gamma^d_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}} - \tilde{\gamma}_{d\,\boldsymbol{\varepsilon}[\boldsymbol{\alpha}|}\delta_{|\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}\gamma^d_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\end{aligned}$$

Now let us write the expansion in $\gamma^{[1]}$ and $\gamma^{[5]}$ as $R_{\hat{\boldsymbol{\beta}}[\boldsymbol{\alpha}\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}} = R^{[1]}_{\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\delta}}{}_a\gamma^a_{\boldsymbol{\alpha}\boldsymbol{\gamma}} + R^{[5]}_{\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\delta}}{}_{a_1\ldots a_5}\gamma^{a_1\ldots a_5}_{\boldsymbol{\alpha}\boldsymbol{\gamma}}$. The second term has to vanish, so that the first condition is (projecting with $\gamma^{\boldsymbol{\alpha}\boldsymbol{\gamma}}_{a_1\ldots a_5}$):

$$\boxed{\gamma^{\boldsymbol{\delta}\boldsymbol{\alpha}}_{a_1\ldots a_5}\left(\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi + \tilde{\gamma}_{d\,\boldsymbol{\alpha}\boldsymbol{\varepsilon}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}}\right) = 0}$$

The other coefficient can be projected with $\gamma^{\boldsymbol{\gamma}\boldsymbol{\alpha}}_a$ via $R^{[1]}_{\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\delta}}{}_a = \frac{1}{16}\gamma^{\boldsymbol{\gamma}\boldsymbol{\alpha}}_a R_{\hat{\boldsymbol{\beta}}[\boldsymbol{\alpha}\boldsymbol{\gamma}]}{}^{\boldsymbol{\delta}}$, which should coincide with $-\tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}$. We thus obtain as second condition

$$\begin{aligned} -\tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}} &= -\tfrac{1}{2}\gamma^{\boldsymbol{\varepsilon}\boldsymbol{\delta}}_a\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi - \tfrac{1}{2}\tilde{\gamma}_{ad\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\delta}}\gamma^d_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}} - \tfrac{1}{2}\tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\\ &\quad -\tfrac{1}{16}\gamma^{\boldsymbol{\delta}\boldsymbol{\alpha}}_a\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi - \tfrac{1}{16}\gamma^{\boldsymbol{\delta}\boldsymbol{\alpha}}_a\tilde{\gamma}_{d\,\boldsymbol{\varepsilon}\boldsymbol{\alpha}}\gamma^d_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\varepsilon}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\end{aligned}$$

which can be further simplified to

$$\boxed{\gamma^{\boldsymbol{\delta}\boldsymbol{\alpha}}_a\left(\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi + \tilde{\gamma}_{d\,\boldsymbol{\alpha}\boldsymbol{\rho}}\mathcal{P}^{\boldsymbol{\rho}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}}\right) = 0}$$

For this last equation we can finally use that $\gamma^a_{\boldsymbol{\beta}\boldsymbol{\delta}}\gamma^{\boldsymbol{\delta}\boldsymbol{\alpha}}_a = -10\delta_{\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}}$ which implies that already the bracket itself has to vanish and we get the following constraint on the compensator superfield (and likewise on the dilaton superfield):

$$\boxed{\nabla_{\hat{\boldsymbol{\beta}}}\nabla_{\boldsymbol{\alpha}}\Phi = -\tilde{\gamma}_{d\,\boldsymbol{\alpha}\boldsymbol{\rho}}\mathcal{P}^{\boldsymbol{\rho}\hat{\boldsymbol{\varepsilon}}}\gamma^d_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\beta}}}} \qquad \diamond$$

$$\boxed{\begin{aligned} \hat{R}_{\boldsymbol{\alpha}[bc]}{}^{d} &= -\frac{3}{4}\hat{\nabla}_{\boldsymbol{\alpha}}H_{bc}{}^{d} + 2\tilde{\gamma}_{[b|\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\hat{T}_{\hat{\boldsymbol{\varepsilon}}|c]}{}^{d} + T_{bc}{}^{\boldsymbol{\varepsilon}}\gamma^{d}_{\boldsymbol{\varepsilon\alpha}} & (5.734)\\ R_{\hat{\boldsymbol{\alpha}}[bc]}{}^{d} &= \frac{3}{4}\nabla_{\hat{\boldsymbol{\alpha}}}H_{bc}{}^{d} + 2\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}T_{\boldsymbol{\varepsilon}|c]}{}^{d} + \hat{T}_{bc}{}^{\hat{\boldsymbol{\varepsilon}}}\gamma^{d}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} & (5.735)\end{aligned}}$$

At this point it is convenient to plug the constraints (5.580) and (5.581) into the above equations to obtain slightly simplified expressions

$$\hat{R}_{\boldsymbol{\alpha}[bc]d} = -2T_{d[b|}{}^{\boldsymbol{\varepsilon}}\gamma_{|c]\boldsymbol{\varepsilon\alpha}} + 2\tilde{\gamma}_{[b|\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\hat{T}_{\hat{\boldsymbol{\varepsilon}}|c]d} \tag{5.736}$$

$$R_{\hat{\boldsymbol{\alpha}}[bc]d} = -2\hat{T}_{d[b|}{}^{\hat{\boldsymbol{\varepsilon}}}\gamma_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} + 2\tilde{\gamma}_{[b|\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}T_{\boldsymbol{\varepsilon}|c]d} \tag{5.737}$$

Let us plug the explicit expressions for the torsion components into the first equation:

$$\begin{aligned} \hat{R}_{\boldsymbol{\alpha}[bc]d} &= -\frac{1}{8}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{d[b|\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\gamma_{|c]\boldsymbol{\varepsilon\alpha}} + \\ &\quad -\tilde{\gamma}_{[b|\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\left(\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi G_{|c]d} + \tilde{\gamma}_{|c]d\,\hat{\boldsymbol{\varepsilon}}}{}^{\hat{\boldsymbol{\delta}}}\hat{\nabla}_{\hat{\boldsymbol{\delta}}}\Phi\right) = \qquad (5.738)\\ &= -\frac{1}{8}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{d[b|\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\gamma_{|c]\boldsymbol{\varepsilon\alpha}} + G_{d[b|}\tilde{\gamma}_{|c]\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi \qquad (5.739)\end{aligned}$$

Including the hatted version, we thus get in summary

$$\boxed{\begin{aligned} \hat{R}_{\boldsymbol{\alpha}[bc]d} &= -\frac{1}{8}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{d[b|\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\gamma_{|c]\boldsymbol{\varepsilon\alpha}} + G_{d[b|}\tilde{\gamma}_{|c]\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi & (5.740)\\ R_{\hat{\boldsymbol{\alpha}}[bc]d} &= -\frac{1}{8}\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{d[b|\,\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}\gamma_{|c]\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}} + G_{d[b|}\tilde{\gamma}_{|c]\,\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\varepsilon}}\Phi & (5.741)\end{aligned}}$$

Finally we take the trace of the first equation in the indices c and d

$$\frac{9}{2}\hat{F}^{(D)}_{\boldsymbol{\alpha}b} - \frac{1}{2}\hat{R}^{(L)}_{\boldsymbol{\alpha}db}{}^{d} = -\frac{1}{16}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\delta}}}\tilde{\gamma}_{db\,\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\gamma^{d}_{\boldsymbol{\varepsilon\alpha}} - \frac{9}{2}\tilde{\gamma}_{b\,\boldsymbol{\alpha\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\varepsilon}}}\hat{\nabla}_{\hat{\boldsymbol{\varepsilon}}}\Phi \tag{5.742}$$

with $\hat{F}^{(D)}_{\boldsymbol{\alpha}b} = -\underline{\nabla}_{[\boldsymbol{\alpha}}\hat{\nabla}_{b]}\Phi - \underline{T}_{\boldsymbol{\alpha}b}{}^{C}\hat{\nabla}_{C}\Phi = -\tilde{\gamma}_{b\,\boldsymbol{\alpha\beta}}\mathcal{P}^{\boldsymbol{\beta}\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi$ or eventually:

$$\boxed{\hat{R}^{(L)}_{d\boldsymbol{\alpha}b}{}^{d} = \frac{1}{8}\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{bc\,\hat{\boldsymbol{\varepsilon}}}{}^{\hat{\boldsymbol{\gamma}}}\gamma^{c}_{\boldsymbol{\varepsilon\alpha}}} \tag{5.743}$$

$$\boxed{R^{(L)}_{d\hat{\boldsymbol{\alpha}}b}{}^{d} = \frac{1}{8}\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\varepsilon}\hat{\boldsymbol{\varepsilon}}}\tilde{\gamma}_{bc\,\boldsymbol{\varepsilon}}{}^{\boldsymbol{\gamma}}\gamma^{c}_{\hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{\alpha}}}} \tag{5.744}$$

- <u>(d|3,0,0)$_{abc}{}^{d}$dim2:</u>

$$\begin{aligned} 0 &\overset{!}{=} \underline{\nabla}_{[a}\check{T}_{bc]}{}^{d} + 2\underline{T}_{[ab|}{}^{E}\check{T}_{E|c]}{}^{d} - \check{R}_{[abc]}{}^{d} = & (5.745)\\ &\overset{\check{\Omega}=\Omega}{=} \nabla_{[a}T_{bc]}{}^{d} + 2T_{[ab|}{}^{e}T_{e|c]}{}^{d} + 2T_{[ab|}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}|c]}{}^{d} - R_{[abc]}{}^{d} = & (5.746)\\ &= \frac{3}{2}\nabla_{[a}H_{bc]}{}^{d} + \frac{9}{2}H_{[ab|}{}^{e}H_{e|c]}{}^{d} + 2T_{[ab|}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}|c]}{}^{d} - R_{[abc]}{}^{d} & (5.747)\end{aligned}$$

$$\boxed{\begin{aligned} R_{[abc]}{}^{d} &= \frac{3}{2}\nabla_{[a}H_{bc]}{}^{d} + \frac{9}{2}H_{[ab|}{}^{e}H_{e|c]}{}^{d} + 2T_{[ab|}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}|c]}{}^{d} & (5.748)\\ \hat{R}_{[abc]}{}^{d} &= -\frac{3}{2}\hat{\nabla}_{[a}H_{bc]}{}^{d} + \frac{9}{2}H_{[ab|}{}^{e}H_{e|c]}{}^{d} + 2\hat{T}_{[ab|}{}^{\hat{\boldsymbol{\varepsilon}}}\hat{T}_{\hat{\boldsymbol{\varepsilon}}|c]}{}^{d} & (5.749)\end{aligned}}$$

Taking the trace yields

$$\begin{aligned} 0 &\overset{!}{=} \frac{1}{2}\nabla_{d}H_{ab}{}^{d} + 3\underbrace{H_{d[a|}{}^{e}H_{e|b]}{}^{d}}_{=0} + \frac{2}{3}T_{ab}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}d}{}^{d} + \\ &\quad + \frac{4}{3}T_{d[a|}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}|b]}{}^{d} - \frac{8}{3}F^{(D)}_{ab} + \frac{2}{3}R^{(L)}_{d[ab]}{}^{d} = \qquad (5.750)\\ &= \frac{1}{2}\nabla_{d}H_{ab}{}^{d} - \frac{10}{3}T_{ab}{}^{\boldsymbol{\varepsilon}}\nabla_{\boldsymbol{\varepsilon}}\Phi + \frac{4}{3}T_{d[a|}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}|b]}{}^{d} - \frac{8}{3}F^{(D)}_{ab} + \frac{2}{3}R^{(L)}_{d[ab]}{}^{d} \qquad (5.751)\end{aligned}$$

with $F^{(D)}_{ab} = -\underline{\nabla}_{[a}\underline{\nabla}_{b]}\Phi\Big|_{\check{\Omega}=\Omega} - \underline{T}_{ab}{}^{C}\big|_{\check{\Omega}=\Omega}\nabla_C\Phi = -T_{ab}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\gamma}}\Phi$. We thus get the following trace constraint on the bosonic left-moving and right-moving (via the left-right-symmetry) Lorentz curvature:

$$\boxed{-R^{(L)}_{d[ab]}{}^{d} = \frac{3}{4}\nabla_d H_{ab}{}^{d} - T_{ab}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\gamma}}\Phi + 2T_{d[a|}{}^{\boldsymbol{\varepsilon}}T_{\boldsymbol{\varepsilon}|b]}{}^{d}} \tag{5.752}$$

$$\boxed{-\hat{R}^{(L)}_{d[ab]}{}^{d} = -\frac{3}{4}\hat{\nabla}_d H_{ab}{}^{d} - \hat{T}_{ab}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi + 2\hat{T}_{d[a|}{}^{\hat{\boldsymbol{\varepsilon}}}\hat{T}_{\hat{\boldsymbol{\varepsilon}}|b]}{}^{d}} \tag{5.753}$$

5.D Identities for the scaling field strength

Instead of extracting in a clumsy way the information about the dilaton field strength, we can obtain the information in a more direct way. At some points this should also serve as a check of equations that we have already obtained. From the torsion Bianchi identity (5.253) we cannot easily extract the dilatation part, because the group indices are antisymmetrized. Instead, we will study the algebra of covariant derivatives acting on the compensator field. We start with the constraints

$$\nabla_{\hat{\boldsymbol{\alpha}}}\Phi = \hat{\nabla}_{\boldsymbol{\alpha}}\Phi = \nabla_a\Phi = \hat{\nabla}_a\Phi = 0 \tag{5.754}$$

Remember, that on the compensator field the commutator of covariant derivatives reads

$$\underline{\nabla}_{[A}\check{\nabla}_{B]}\Phi = -\check{T}_{AB}{}^{C}\check{\nabla}_C\Phi - \check{F}^{(D)}_{AB} \tag{5.755}$$

Now we can plug in various indices:

- $\underline{(a,b):}$

$$\begin{aligned}
\underbrace{\underline{\nabla}_{[a}\underline{\nabla}_{b]}\Phi}_{0} &= -\check{T}_{ab}{}^{c}\check{\nabla}_c\Phi - T_{ab}{}^{\boldsymbol{\gamma}}\check{\nabla}_{\boldsymbol{\gamma}}\Phi - \hat{T}_{ab}{}^{\hat{\boldsymbol{\gamma}}}\check{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \check{F}^{(D)}_{ab} = &(5.756)\\
&\overset{\check{\Omega}=\Omega}{=} -T_{ab}{}^{\boldsymbol{\gamma}}\nabla_{\boldsymbol{\gamma}}\Phi - F^{(D)}_{ab} = &(5.757)\\
&= -\frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ab\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\nabla_{\boldsymbol{\gamma}}\Phi - F^{(D)}_{ab} &(5.758)
\end{aligned}$$

$$\boxed{F_{ab} = -\frac{1}{16}\left(\underline{\nabla}_{\hat{\boldsymbol{\gamma}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}} + 8\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\right)\tilde{\gamma}_{ab\hat{\boldsymbol{\delta}}}{}^{\hat{\boldsymbol{\gamma}}}\nabla_{\boldsymbol{\gamma}}\Phi} \tag{5.759}$$

$$\boxed{\hat{F}_{ab} = -\frac{1}{16}\left(\underline{\nabla}_{\boldsymbol{\gamma}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}} + 8\nabla_{\boldsymbol{\gamma}}\Phi\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\right)\tilde{\gamma}_{ab\boldsymbol{\delta}}{}^{\boldsymbol{\gamma}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi} \tag{5.760}$$

- $\underline{(a,\boldsymbol{\beta}) \leftrightarrow (a,\hat{\boldsymbol{\beta}}):}$

$$\begin{aligned}
\underbrace{\underline{\nabla}_{[a}\check{\nabla}_{\boldsymbol{\beta}]}\Phi}_{0\,\text{for }\check{\Omega}=\hat{\Omega}} &= -\check{T}_{a\boldsymbol{\beta}}{}^{c}\check{\nabla}_c\Phi - T_{a\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\check{\nabla}_{\boldsymbol{\gamma}}\Phi - \hat{T}_{a\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}}\check{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \check{F}^{(D)}_{a\boldsymbol{\beta}} = &(5.761)\\
&\overset{\check{\Omega}=\hat{\Omega}}{=} -\hat{T}_{a\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \hat{F}^{(D)}_{a\boldsymbol{\beta}} = &(5.762)\\
&= \tilde{\gamma}_{a\,\boldsymbol{\beta}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \hat{F}^{(D)}_{a\boldsymbol{\beta}} &(5.763)
\end{aligned}$$

$$\boxed{\hat{F}^{(D)}_{a\boldsymbol{\beta}} = \tilde{\gamma}_{a\,\boldsymbol{\beta}\boldsymbol{\delta}}\mathcal{P}^{\boldsymbol{\delta}\hat{\boldsymbol{\gamma}}}\hat{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi, \qquad F^{(D)}_{a\hat{\boldsymbol{\beta}}} = \tilde{\gamma}_{a\,\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\delta}}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\nabla_{\boldsymbol{\gamma}}\Phi} \tag{5.764}$$

For $\hat{\Omega} = \Omega$ instead, we obtain

$$\underbrace{\underline{\nabla}_{[a}\check{\nabla}_{\boldsymbol{\beta}]}\Phi}_{\frac{1}{2}\nabla_a\nabla_{\boldsymbol{\beta}}\Phi\,\text{for }\check{\Omega}=\Omega} \overset{\check{\Omega}=\Omega}{=} -F^{(D)}_{a\boldsymbol{\beta}} \tag{5.765}$$

$$\boxed{F^{(D)}_{a\boldsymbol{\beta}} = \frac{1}{2}\nabla_a\nabla_{\boldsymbol{\beta}}\Phi, \quad \hat{F}^{(D)}_{a\hat{\boldsymbol{\beta}}} = \frac{1}{2}\hat{\nabla}_a\hat{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi} \tag{5.766}$$

- $\underline{(\boldsymbol{\alpha},\boldsymbol{\beta}) \leftrightarrow (\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}}):}$

$$\begin{aligned}
\underbrace{\underline{\nabla}_{[\boldsymbol{\alpha}}\check{\nabla}_{\boldsymbol{\beta}]}\Phi}_{0\text{ for }\check{\Omega}=\hat{\Omega}} &= -\check{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{c}\check{\nabla}_c\Phi - T_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\boldsymbol{\gamma}}\check{\nabla}_{\boldsymbol{\gamma}}\Phi - \hat{T}_{\boldsymbol{\alpha}\boldsymbol{\beta}}{}^{\hat{\boldsymbol{\gamma}}}\check{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \check{F}^{(D)}_{\boldsymbol{\alpha}\boldsymbol{\beta}} = &(5.767)\\
&\overset{\check{\Omega}=\hat{\Omega}}{=} -\hat{F}^{(D)}_{\boldsymbol{\alpha}\boldsymbol{\beta}} &(5.768)
\end{aligned}$$

$$\boxed{\hat{F}_{\boldsymbol{\alpha\beta}} = F_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0} \tag{5.769}$$

- $\underline{(\boldsymbol{\alpha}, \hat{\boldsymbol{\beta}}):}$

$$\underbrace{\nabla_{[\boldsymbol{\alpha}}\check{\nabla}_{\hat{\boldsymbol{\beta}}]}\Phi}_{\frac{1}{2}\hat{\nabla}_{\boldsymbol{\alpha}}\check{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi \text{ for } \check{\Omega}=\hat{\Omega}} = -\check{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{c}\check{\nabla}_c\Phi - T_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\boldsymbol{\gamma}}\check{\nabla}_{\boldsymbol{\gamma}}\Phi - \hat{T}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}{}^{\hat{\boldsymbol{\gamma}}}\check{\nabla}_{\hat{\boldsymbol{\gamma}}}\Phi - \check{F}^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = \tag{5.770}$$

$$\stackrel{\check{\Omega}=\hat{\Omega}}{=} -\hat{F}^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} \tag{5.771}$$

$$\boxed{\hat{F}^{(D)}_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = -\frac{1}{2}\hat{\nabla}_{\boldsymbol{\alpha}}\check{\nabla}_{\hat{\boldsymbol{\beta}}}\Phi, \quad F^{(D)}_{\hat{\boldsymbol{\alpha}}\boldsymbol{\beta}} = -\frac{1}{2}\nabla_{\hat{\boldsymbol{\alpha}}}\nabla_{\boldsymbol{\beta}}\Phi} \tag{5.772}$$

5.D Recovering flat-space action / comment on linearized SUGRA

If all curvature components vanish, all higher components (in the $\vec{\boldsymbol{\theta}}$-expansion) vanish in the extended WZ-gauge due to (H.116) and (H.118) and the remaining bosonic local Lorentz and scale transformations can be used to fix $\underline{\Omega}_{mA}{}^{B}\big| = 0$ such that all connection components vanish. The only torsion components which are forced to be nonzero are $T_{\boldsymbol{\alpha\beta}}{}^{c} = (\mathbf{d}E^c)_{\boldsymbol{\alpha\beta}} = \gamma^c_{\boldsymbol{\alpha\beta}}$ and $T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^{c} = (\mathbf{d}E^c)_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$. A solution which is compatible with the extended WZ-gauge (H.117), (H.119) and (H.120), and which fixes also the remaining bosonic diffeomorphism invariance is given by

$$E_M{}^A = \begin{pmatrix} \delta^a_m & 0 & 0 \\ (\theta^{\boldsymbol{\beta}}\gamma^a_{\boldsymbol{\beta\mu}}) & \delta_{\boldsymbol{\mu}}{}^{\boldsymbol{\alpha}} & 0 \\ (\hat{\theta}^{\hat{\boldsymbol{\beta}}}\gamma^a_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\mu}}}) & 0 & \delta_{\hat{\boldsymbol{\mu}}}{}^{\hat{\boldsymbol{\alpha}}} \end{pmatrix}, \quad E_A{}^M = \begin{pmatrix} \delta^m_a & 0 & 0 \\ -(\theta^{\boldsymbol{\beta}}\gamma^m_{\boldsymbol{\beta\alpha}}) & \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\mu}} & 0 \\ -(\hat{\theta}^{\hat{\boldsymbol{\beta}}}\gamma^m_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\alpha}}}) & 0 & \delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\mu}}} \end{pmatrix} \tag{5.773}$$

The supersymmetric invariant one -forms thus read

$$E^A = \mathbf{d}x^M E_M{}^A = \left(\mathbf{d}x^a + \mathbf{d}\theta^{\boldsymbol{\mu}}\theta^{\boldsymbol{\beta}}\gamma^a_{\boldsymbol{\beta\mu}} + \mathbf{d}\hat{\theta}^{\hat{\boldsymbol{\mu}}}\hat{\theta}^{\hat{\boldsymbol{\beta}}}\gamma^a_{\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\mu}}}\,,\ \mathbf{d}\theta^{\boldsymbol{\alpha}}\,,\ \mathbf{d}\hat{\theta}^{\hat{\boldsymbol{\alpha}}}\right) \tag{5.774}$$

which agrees with (4.3).

The reasoning is similar for the B-field and its field-strength H. The only components of H which are forced to be nonzero are $H_{c\boldsymbol{\alpha\beta}} = -\frac{2}{3}\gamma_{c\boldsymbol{\alpha\beta}}$ and $H_{c\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = \frac{2}{3}\gamma_{c\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$. A simple solution for $H_{CAB} = (\mathbf{d}B)_{CAB} = \nabla_{[C}B_{AB]} + 2T_{[CA|}{}^{D}B_{D|B]}$ (which is compatible with the WZ-like gauge (H.142), (H.146) has the form

$$B_{AB} = \begin{pmatrix} 0 & x^{\boldsymbol{\gamma}}\gamma_{a\,\boldsymbol{\gamma\beta}} & -x^{\hat{\boldsymbol{\gamma}}}\gamma_{a\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}} \\ -x^{\boldsymbol{\gamma}}\gamma_{b\,\boldsymbol{\gamma\alpha}} & 0 & -\left(\gamma^c_{\boldsymbol{\alpha\gamma}}x^{\boldsymbol{\gamma}}\right)\left(x^{\hat{\boldsymbol{\gamma}}}\gamma_{c\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}\right) \\ x^{\hat{\boldsymbol{\gamma}}}\gamma_{b\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}} & \left(\gamma^c_{\boldsymbol{\beta\gamma}}x^{\boldsymbol{\gamma}}\right)\left(x^{\hat{\boldsymbol{\gamma}}}\gamma_{c\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\alpha}}}\right) & 0 \end{pmatrix} \tag{5.775}$$

All other fields which appear in the Lagrangian can be chosen to vanish. The curved-background action (5.98) thus reduces to

$$\begin{aligned} S_0 &= \int d^2z \;\; \frac{1}{2}\Pi^a_z\eta_{ab}\Pi^b_{\bar{z}} + \frac{1}{2}\Pi^A_z B_{AB}\Pi^B_{\bar{z}} + \bar{\partial}\theta^{\boldsymbol{\gamma}}d_{z\boldsymbol{\gamma}} + \partial\hat{\theta}^{\hat{\boldsymbol{\gamma}}}\hat{d}_{\bar{z}\hat{\boldsymbol{\gamma}}} + \\ &\quad +\bar{\partial}\boldsymbol{\lambda}^{\boldsymbol{\beta}}\boldsymbol{\omega}_{z\boldsymbol{\beta}} + \partial\hat{\boldsymbol{\lambda}}^{\hat{\boldsymbol{\beta}}}\hat{\boldsymbol{\omega}}_{\bar{z}\hat{\boldsymbol{\beta}}} + \frac{1}{2}L_{z\bar{z}a}(\boldsymbol{\lambda}\gamma^a\boldsymbol{\lambda}) + \frac{1}{2}\hat{L}_{z\bar{z}a}(\hat{\boldsymbol{\lambda}}\gamma^a\hat{\boldsymbol{\lambda}}) \end{aligned} \tag{5.776}$$

The B-field term takes the explicit form

$$\frac{1}{2}\Pi^A_z B_{AB}\Pi^B_{\bar{z}} = \frac{1}{2}\Pi^a_z\left(B_{a\boldsymbol{\beta}}\Pi^{\boldsymbol{\beta}}_{\bar{z}} + B_{a\hat{\boldsymbol{\beta}}}\Pi^{\hat{\boldsymbol{\beta}}}_{\bar{z}}\right) + \frac{1}{2}\Pi^{\boldsymbol{\alpha}}_z B_{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}\Pi^{\hat{\boldsymbol{\beta}}}_{\bar{z}} - (z \leftrightarrow \bar{z}) = \tag{5.777}$$

$$= \frac{1}{2}\Pi^a_z\left(\theta^{\boldsymbol{\gamma}}\gamma_{a\,\boldsymbol{\gamma\beta}}\Pi^{\boldsymbol{\beta}}_{\bar{z}} - \theta^{\hat{\boldsymbol{\gamma}}}\gamma_{a\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}\Pi^{\hat{\boldsymbol{\beta}}}_{\bar{z}}\right) - \frac{1}{2}\left(\Pi^{\boldsymbol{\alpha}}_z\gamma^c_{\boldsymbol{\alpha\gamma}}\theta^{\boldsymbol{\gamma}}\right)\left(\theta^{\hat{\boldsymbol{\gamma}}}\gamma_{c\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}\Pi^{\hat{\boldsymbol{\beta}}}_{\bar{z}}\right) - (z \leftrightarrow \bar{z}) = \tag{5.778}$$

$$= \frac{1}{2}\Pi^a_z\left(\Pi^{\boldsymbol{\beta}}_{\bar{z}}\theta^{\boldsymbol{\gamma}}\gamma_{a\,\boldsymbol{\gamma\beta}} - \Pi^{\hat{\boldsymbol{\beta}}}_{\bar{z}}\theta^{\hat{\boldsymbol{\gamma}}}\gamma_{a\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}\right) - \frac{1}{2}\left(\theta^{\boldsymbol{\gamma}}\Pi^{\boldsymbol{\alpha}}_z\gamma^c_{\boldsymbol{\alpha\gamma}}\right)\left(\Pi^{\hat{\boldsymbol{\beta}}}_{\bar{z}}\theta^{\hat{\boldsymbol{\gamma}}}\gamma_{c\,\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\beta}}}\right) - (z \leftrightarrow \bar{z}) \tag{5.779}$$

Upon a shift of the grading from the fermionic indices to the rumpfs, this coincides precisely with the form of the WZ-term given in (4.22). Only the antighost field has to be redefined with a minus sign, in order to match the flat-space Lagrangian.

The BRST transformations (5.195)-(5.202) reduce in flat space to

$$\mathbf{s}_0 x^m = \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \gamma^m_{\boldsymbol{\alpha}\boldsymbol{\beta}} \theta^{\boldsymbol{\beta}}, \qquad \mathbf{s}_0 \theta^{\boldsymbol{\mu}} = \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\mu}} \tag{5.780}$$

$$\mathbf{s}_0 \boldsymbol{\omega}_{z\boldsymbol{\alpha}} = d_{z\boldsymbol{\alpha}} \tag{5.781}$$

$$\mathbf{s}_0 d_{z\boldsymbol{\delta}} = -2\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \Pi_z^c \gamma_{c\,\boldsymbol{\alpha}\boldsymbol{\delta}} \tag{5.782}$$

The corresponding hatted equations are obtained for the hatted fields. All other transformations vanish. In particular, the Lagrange multiplier doesn't transform (the complicated $X_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ vanishes in flat space). The pure spinor constraint guarantees the nilpotency of $\mathbf{s}_0$ when acting twice on $d_{z\boldsymbol{\delta}}$. The BRST transformation of the supersymmetric objects reduce to

$$\mathbf{s}_0 \Pi_z^a = 2\boldsymbol{\lambda}^{\boldsymbol{\alpha}} \partial x^{\boldsymbol{\beta}} \gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}}, \qquad \mathbf{s}_0 \partial x^{\boldsymbol{\alpha}} = \partial \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \tag{5.783}$$

We can see the BRST transformation $\mathbf{s}$of curved background as a perturbation around the one in flat background

$$\mathbf{s} \equiv (\mathbf{s}_0 + \boldsymbol{u}) \tag{5.784}$$

From the point of view of the string in flat background with action S_0, the difference $U \equiv S - S_0$ to the action in curved background is simply a vertex operator which should be BRST-invariant. The condition of a conserved BRST current (which enforced the supergravity constraints) corresponds to the invariance $\mathbf{s}S = 0$ of the action, or written as a perturbation:

$$0 = (\mathbf{s}_0 + \boldsymbol{u})(S_0 + U) = \tag{5.785}$$

$$= \underbrace{\mathbf{s}_0 S_0}_{0} + \boldsymbol{u} S_0 + \mathbf{s}_0 U + \boldsymbol{u} U \tag{5.786}$$

At linearized level, we thus have

$$\boldsymbol{u} S_0 + \mathbf{s}_0 U = 0 \tag{5.787}$$

In the antifield formalism (which we did not really discuss in this context), the BRST transformations are generated by the actions themselves (enlarged with an antifield content) via the antibracket. The above equation then reads

$$(U,S_0) + (S_0,U) = 2(S_0,U) = 2\mathbf{s}_0 U \overset{!}{=} 0 \tag{5.788}$$

This explains the well-known fact that the vertex operators of the flat space pure spinor string have to obey linearized supergravity constraints.

Part III

Derived Brackets in Sigma-Models

"Don't make a break, make a bracket" (Kathi S.)

Introduction to the Bracket Part

This part of the thesis is based on the author's paper [16]. See also [68] for a short article which contains some of the basic ideas. In the meantime a paper by Klaus Bering [69] was brought to my attention. Although it follows a different aim, its geometrical setting, especially in its section 5, is very close to the one presented here. Moreover, the geometrical meaning of the variables is nicely presented there, e.g. in its table 7, and can thus serve as a useful supplement to the reading of the present part of the thesis.

There are quite a lot of different geometric brackets floating around in the literature, like Schouten bracket, Nijenhuis bracket or in generalized complex geometry the Dorfman bracket and Courant bracket, to list just some of them. They are often related to integrability conditions for some structures on manifolds. The vanishing of the Nijenhuis bracket of a complex structure with itself, for example, is equivalent to its integrability. The same is true for the Schouten bracket and a Poisson structure. The above brackets can be unified with the concept of derived brackets [70]. Within this concept, they are all just natural extensions of the Lie-bracket of vector fields to higher rank tensor fields.

It is well known that the antibracket appearing in the Lagrangian formalism for sigma models is closely related to the Schouten-bracket in target space. In addition it was recently observed by Alekseev and Strobl that the Dorfman bracket for sums of vectors and one-forms appears naturally in two dimensional sigma models[1] [71]. This was generalized by Bonelli and Zabzine [73] to a derived bracket for sums of vectors and p-forms on a p-brane[2]. These observations lead to the natural question whether there is a general relation between the sigma-model Poisson bracket or antibracket and derived brackets in target space. Working out the precise relation for sigma models with a special field content but undetermined dimension and dynamics, is the major subject of the present part of the thesis.

One of the motivations for this part of the thesis was the application to generalized complex geometry. The importance of the latter in string theory is due to the observation that effective spacetime supersymmetry after compactification requires the compactification manifold to be a generalized Calabi-Yau manifold [74, 72, 6, 5, 75, 76]. Deviations from an ordinary Calabi Yau manifold are due to fluxes and also the concept of mirror symmetry can be generalized in this context. There are numerous other important articles on the subject, like e.g. [77, 78, 79, 80, 81, 82, 83, 84, 85, 86] and many more. A more complete list of references can be found in [76]. A major part of the considerations so far was done from the supergravity point of view. Target space supersymmetry is, however, related to an $N = 2$ supersymmetry on the worldsheet. For this reason the relation between an extended worldsheet supersymmetry and the presence of an integrable generalized complex structure (GCS) was studied in [87] (the reviews [88, 89] on generalized complex geometry have this relation in mind). Zabzine clarified in [90] the relation in a model independent way in a Hamiltonian description and showed that the existence of a second non-manifest worldsheet supersymmetry $\boldsymbol{Q}_2$ in an $N = 1$ sigma-model is equivalent to the existence of an integrable GCS $\mathcal{J}$. It is the observation that the integrability of the GCS $\mathcal{J}$ can be written as the vanishing of a generalized bracket $[\mathcal{J},\mathcal{J}]_B = 0$ which leads to the natural question, whether there is a direct mapping between $[\mathcal{J},\mathcal{J}]_B = 0 \,\&\, \mathcal{J}^2 = -1$ on the one side and $\{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} = 2P$ on the other side. This will be a natural application in subsection 7.2 of the more general preceding considerations about the relation between (super-)Poisson brackets in sigma models with special field content and derived brackets in the target space.

A second interesting application is Zucchini's Hitchin-sigma-model [91]. There are up to now three more papers on that subject [92, 93, 94], but the present discussion refers only to the first one. Zucchini's model is a two dimensional sigma-model in a target space with a generalized complex structure (GCS). The sigma-model is topological when the GCS is integrable, while the inverse does not hold. The condition for the sigma model to be topological is the master equation $(S,S) = 0$. Again we might wonder whether there is a direct mapping between the antibracket and S on the one hand and the geometric bracket and $\mathcal{J}$ on the other hand and it will be shown in subsection 7.1 how this mapping works as an application of the considerations in subsection 6.5. In order to understand more about geometric brackets in general, however, it was necessary to dive into Kosmann-Schwarzbach's review on derived brackets [70] which led to observations that go beyond the application to the integrability of a GCS .

The structure of this part of the thesis is as follows: The general relation between sigma models and derived brackets in target space will be studied in the next section. The necessary geometric setup will be established in 6.1. Although there are no new deep insights in 6.1, the unconventional idea to extend the exterior derivative on forms to multivector valued forms (see (6.34) and (6.37)) will provide a tool to write down a coordinate expression for the general derived bracket between multivector valued forms (6.51) which to my knowledge does not yet exist in literature. The main results in section 6, however, are the propositions 1 on page 128 and 1b on page 139 for the relation between the Poisson-bracket in a sigma-model with special field content and the derived bracket in the target space, and the proposition 3b on page 133 for the relation between the antibracket in a sigma-model and the derived bracket in target space. Proposition 2 on page 130 is just a short quantum

[1] In [71], the non-symmetric bracket is called 'Courant bracket'. Following e.g. Gualtieri [72] or [70], it will be called 'Dorfman bracket' in this thesis, while 'Courant bracket' is reserved for its antisymmetrization (see (B.31) and (B.38)). ⋄

[2] The *Vinogradov bracket* appearing in [73] is just the antisymmetrization of a derived bracket (see footnote 8 on page 164). ⋄

consideration which only works for the particle case. In section 7 the propositions 1b and 3b are finally applied to the two examples which were mentioned above.

Another result is the relation between the generalized Nijenhuis tensor and the derived bracket of $\mathcal{J}$ with itself, given in (**7**.12). The derivation of this can be found in the appendix on page 154. In addition to this, there is a new coordinate form of the generalized Nijenhuis tensor presented in (B.58) on page 153, which might be easier to memorize than the known ones. There is also a short comment in footnote 3 on page 151 on a possible relation to Hull's doubled geometry.

This part of the thesis makes use of only three of the appendices. Appendix A on page 145 summarizes the used conventions, while appendix C on page 159 is an introduction to geometric brackets. Finally, appendix B on page 148 provides some aspects of generalized complex geometry which might be necessary to understand the two applications of above.

Chapter 6

Sigma-model-induced brackets

6.1 Geometric brackets in phase space formulation

In the following some basic geometric ingredients which are necessary to formulate derived brackets will be given. Although there is no sigma model and no physics explicitly involved in this first subsection, the presentation and the techniques will be very suggestive, s.th. there is visually no big change when we proceed after that with considerations on sigma-models.

6.1.1 Algebraic brackets

Consider a real differentiable manifold M. The interior product with a vector field $v = v^k \boldsymbol{\partial}_k$ (in a local coordinate basis) acting on a differential form ρ is a differential operator in the sense that it differentiates with respect to the basis elements of the cotangent space:[1]

$$\imath_v \rho^{(r)} \quad = \quad r \cdot v^k \rho^{(r)}_{k m_1 \ldots m_{r-1}}(x)\, \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_{r-1}} = v^k \frac{\partial}{\partial(\mathbf{d}x^k)} \left(\rho_{m_1 \ldots m_r} \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_r}\right) \tag{6.1}$$

Let us rename[2]

$$\boldsymbol{c}^m \quad \equiv \quad \mathbf{d}x^m \tag{6.2}$$
$$\boldsymbol{b}_m \quad \equiv \quad \boldsymbol{\partial}_m \tag{6.3}$$

The vector v takes locally the form $v = v^m \boldsymbol{b}_m$ and when we introduce a canonical graded Poisson bracket between $\boldsymbol{c}^m$ and $\boldsymbol{b}_m$ via $\{\boldsymbol{b}_m, \boldsymbol{c}^n\} = \delta_m^n$, we get

$$\imath_v \rho \quad = \quad \{v, \rho\} \tag{6.4}$$

Extending also the local x-coordinate-space to a phase space by introducing the conjugate momentum p_m (whose geometric interpretation we will discover soon), we have altogether the (graded) Poisson bracket

$$\{\boldsymbol{b}_m, \boldsymbol{c}^n\} \quad = \quad \delta_m^n = \{\boldsymbol{c}^n, \boldsymbol{b}_m\} \tag{6.5}$$
$$\{p_m, x^n\} \quad = \quad \delta_m^n = -\{x^n, p_m\} \tag{6.6}$$
$$\{A, B\} \quad = \quad A \frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_k} \frac{\partial}{\partial \boldsymbol{c}^k} B + A \frac{\overleftarrow{\partial}}{\partial p_k} \frac{\partial}{\partial x^k} B - (-)^{AB} \left(B \frac{\overleftarrow{\partial}}{\partial \boldsymbol{b}_k} \frac{\partial}{\partial \boldsymbol{c}^k} A + B \frac{\overleftarrow{\partial}}{\partial p_k} \frac{\partial}{\partial x^k} A \right) \tag{6.7}$$

and can write the exterior derivative acting on forms as generated via the Poisson-bracket by an odd phase-space function $\boldsymbol{o}(\boldsymbol{c}, p)$

$$\boldsymbol{o} \quad \equiv \quad \boldsymbol{o}(\boldsymbol{c}, p) \equiv \boldsymbol{c}^k p_k \tag{6.8}$$
$$\left\{\boldsymbol{o}, \rho^{(r)}\right\} \quad = \quad \boldsymbol{c}^k \{p_k, \rho_{m_1 \ldots m_r}(x)\} \boldsymbol{c}^{m_1} \cdots \boldsymbol{c}^{m_r} = \mathbf{d}\rho^{(r)} \tag{6.9}$$

The variables $\boldsymbol{c}^m$,$\boldsymbol{b}_m$,x^m and p_m can be seen as coordinates of $T^*(\Pi TM)$, the cotangent bundle of the tangent bundle with parity inversed fiber.

---

[1] Note, that a convention is used, were the prefactor $\frac{1}{r!}$ which usually comes along with an r-form is absorbed into the definition of the wedge-product. The common conventions can for all equations easily be recovered by redefining all coefficients appropriately, e.g. $\rho_{m_1 \ldots m_r} \to \frac{1}{r!} \rho_{m_1 \ldots m_r}$. ◇

[2] The similarity with ghosts is of course no accident. It is well known (see e.g. [95]) that ghosts in a gauge theory can be seen as 1-forms dual to the gauge-vector fields and the BRST differential as the sum of the Koszul-Tate differential (whose homology implements the restriction to the constraint surface) and the longitudinal exterior derivative along the constraint surface. In that sense the present description corresponds to a topological theory, where all degrees of freedom are gauged away. But we will not necessarily always view $\boldsymbol{c}^m$ as ghosts in the following. So let us in the beginning see $\boldsymbol{c}^m$ just as another name for $\mathbf{d}x^m$. We do not yet assume an underlying sigma-model, i.e. $\boldsymbol{b}_m$ and $\boldsymbol{c}^m$ do not necessarily depend on a worldsheet variable. ◇

Interior product and "quantization"

Given a multivector valued form $K^{(k,k')}$ of form degree k and multivector degree k', it reads in the local coordinate patch with the new symbols

$$K^{(k,k')} \equiv K^{(k,k')}(x,\boldsymbol{c},\boldsymbol{b}) \quad\equiv\quad K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(x)\,\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\boldsymbol{b}_{n_1}\cdots\boldsymbol{b}_{n_{k'}} \equiv K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{6.10}$$

The notation $K(x,\boldsymbol{c},\boldsymbol{b})$ should stress, that K is locally a (smooth on a C^∞ manifold) function of the phase space variables which will later be used for analytic continuation (x will be allowed to take c-number values of a superfunction). The last expression in the above equation introduces a **schematic index notation** which is useful to write down the explicit coordinate form for lengthy expressions. See in the appendix A at page 147 for a more detailed description of its definition. It should, however, be self-explanatory enough for a first reading of the thesis

One can define a natural generalization of the interior product with a vector $\imath_v$ to an **interior product** with a multivector valued form $\imath_K$ acting on some r-form (in fact, it is more like a combination of an interior and an exterior product – see footnote 6 on page 163 –, but we will stick to this name)

$$\begin{aligned}
\imath_{K^{(k,k')}}\rho^{(r)} &\equiv (k')!\begin{pmatrix} r \\ k' \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}}\underbrace{\rho_{l_{k'}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}}_{r} = & (6.11)\\
&= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\left\{\boldsymbol{b}_{n_1},\left\{\cdots,\left\{\boldsymbol{b}_{n_{k'}},\rho^{(r)}\right\}\right\}\right\} & (6.12)\\
&= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\frac{\partial}{\partial\boldsymbol{c}^{n_1}}\cdots\frac{\partial}{\partial\boldsymbol{c}^{n_{k'}}}\rho^{(r)} & (6.13)
\end{aligned}$$

It is a derivative of order k' and thus not a derivative in the usual sense like $\imath_v$. The third line shows the reason for the normalization of the first line, while the second line is added for later convenience. The interior product is commonly used as an **embedding** of the multivector valued forms in the space of differential operators acting on forms, i.e. $K \to \imath_K$, s.th. structures of the latter can be induced on the space of multivector valued forms. In (6.13) the interior product $\imath_K$ can be seen, up to a factor of $\hbar/i$, as the quantum operator corresponding to K, where the form ρ plays the role of a wave function. The natural ordering is here to put the conjugate momenta to the right. We can therefore fix the following "**quantization**" rule (corresponding to $\hat{\boldsymbol{b}} = \frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{c}}$)

$$\hat{K}^{(k,k')} \equiv \left(\frac{\hbar}{i}\right)^{k'}\imath_{K^{(k,k')}} \tag{6.14}$$

$$\text{with } \imath_{K^{(k,k')}} = K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{n_1\ldots n_{k'}}\frac{\partial^{k'}}{\partial\boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_{k'}}} \tag{6.15}$$

The (graded) commutator of two interior products induces an algebraic bracket due to Buttin [96], which is defined via

$$\left[\imath_{K^{(k,k')}},\imath_{L^{(l,,l')}}\right] \equiv \imath_{[K,L]^{\Delta}} \tag{6.16}$$

A short calculation, using the obvious generalization of $\partial_x^n(f(x)g(x)) = \sum_{p=0}^n \begin{pmatrix} n \\ p \end{pmatrix}\partial_x^p f(x)\partial_x^{n-p}g(x)$ leads to

$$\imath_K\imath_L = \sum_{p\geq 0}\imath_{\imath_K^{(p)}L} = \imath_{K\wedge L} + \sum_{p\geq 1}\imath_{\imath_K^{(p)}L} \tag{6.17}$$

where we introduced the **interior product of order** p

$$\begin{aligned}
\imath^{(p)}_{K^{(k,k')}} &\equiv \begin{pmatrix} k' \\ p \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1\ldots l_p}\frac{\partial^p}{\partial\boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_p}} = & (6.18)\\
&= \frac{1}{p!}K\frac{\overleftarrow{\partial}^p}{\partial\boldsymbol{b}_{n_p}\cdots\partial\boldsymbol{b}_{n_1}}\frac{\partial^p}{\partial\boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_p}} & (6.19)\\
\Rightarrow \imath^{(p)}_{K^{(k,k')}}L^{(l,l')} &= (-)^{(k'-p)(l-p)}p!\begin{pmatrix} k' \\ p \end{pmatrix}\begin{pmatrix} l \\ p \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1\ldots l_p}L_{l_p\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} & (6.20)
\end{aligned}$$

which contracts only p of all k' upper indices and therefore coincides with the interior product of above when acting on forms for $p = k'$ and with the wedge product for $p = 0$.

$$\imath^{(k')}_{K^{(k,k')}}\rho = \imath_{K^{(k,k')}}\rho, \qquad \imath^{(0)}_K L = K\wedge L \tag{6.21}$$

Using (6.17) the **algebraic bracket** $[\ ,\ ]^{\Delta}$ defined in (6.16) can thus be written as

$$[K^{(k,k')},L^{(l,l')}]^{\Delta} = \sum_{p\geq 1}\underbrace{\imath_K^{(p)}L - (-)^{(k-k')(l-l')}\imath_L^{(p)}K}_{\equiv[K,L]^{\Delta}_{(p)}} \tag{6.22}$$

(6.20) provides the explicit coordinate form of this algebraic bracket. From (6.19) we recover the known fact that the $p = 1$ term of the algebraic bracket is induced by the Poisson-bracket and therefore is itself an algebraic bracket, called the **big bracket** [70] or **Buttin's algebraic bracket** [96]

$$\boxed{[K, L]^{\Delta}_{(1)} \quad = \quad \imath^{(1)}_K L - (-)^{(k-k')(l-l')}\imath^{(1)}_L K \quad \stackrel{(6.19)}{=} \{K, L\}} \tag{6.23}$$

$$\stackrel{(6.20)}{=} \quad (-)^{(k'-1)(l-1)} k' l\, K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1} L_{l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\ -(-)^{(k-k')(l-l')}(-)^{(l'-1)(k-1)} l' k\, L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1} K_{l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{6.24}$$

For $k' = l' = 1$ it reduces to the Richardson-Nijenhuis bracket (C.63) for vector valued forms. In [70] the big bracket is described as the canonical Poisson structure on $\bigwedge^{\bullet}(T \oplus T^*)$ which matches with the observation in (6.23). The bracket takes an especially pleasant coordinate form for generalized multivectors as is presented in equation (**B**.77) on page 154.

The multivector-degree of the p-th term of the complete algebraic bracket (6.22) is $(k' + l' - p)$, so that we can rewrite (6.16) in terms of "quantum"-operators (6.14) in the following way:

$$\left[\hat{K}^{(k,k')}, \hat{L}^{(l,l')}\right] \quad = \quad \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^p \widehat{[K, L]^{\Delta}_{(p)}} = \tag{6.25}$$

$$= \quad \left(\frac{\hbar}{i}\right)\widehat{\{K, L\}} + \sum_{p\geq 2}\left(\frac{\hbar}{i}\right)^p \widehat{[K, L]^{\Delta}_{(p)}} \tag{6.26}$$

The Poisson bracket is, as it should be, the leading order of the quantum bracket.

6.1.2 Extended exterior derivative and the derived bracket of the commutator

In the previous subsection the commutator of differential operators induced (via the interior product as embedding) an algebraic bracket on the embedded tensors. Also other structures from the operator space can be induced on the tensors. Having the commutator at hand, one can build the **derived bracket** (see footnote 3 on page 162) of the commutator by additionally commuting the first argument with the exterior derivative. Being interested in the induced structure on multivector valued forms, we consider as arguments only interior products with those multivector valued forms

$$[\imath_K, _{\mathbf{d}} \imath_L] \quad \equiv \quad [[\imath_K, \mathbf{d}], \imath_L] \tag{6.27}$$

One can likewise use other differentials to build a derived bracket, e.g. the twisted differential $[\mathbf{d} + H, \ldots]$ with an odd closed form H, which leads to so called twisted brackets. Let us restrict to $\mathbf{d}$ for the moment. The derived bracket is in general not skew-symmetric but it obeys a graded Jacobi-identity and is therefore what one calls a Loday bracket. When looking for new brackets, the Jacobi identity is the property which is hardest to check. A mechanism like above, which automatically provides it is therefore very powerful. The above derived bracket will induce brackets like the Schouten bracket or even the Dorfman bracket of generalized complex geometry on the tensors. In general, however, the interior products are not closed under its action, i.e. the result of the bracket cannot necessarily be written as $\imath_{\tilde{K}}$ for some $\tilde{K}$. An expression for a general bracket on the tensor level, which reduces in the corresponding cases to the well known brackets therefore does not exist. Instead one normally has to derive the brackets in the special cases separately. In the following, however, a natural approach is discussed including the new variable p_m, introduced in (6.6), which leads to an explicit coordinate expression for the general bracket. This expression is of course tensorial only in the mentioned special cases, that is when terms with p_m vanish. This is not an artificial procedure, as the conjugate variable p_m to x^m is always present in sigma-models, and it will in turn explain the geometric meaning of p_m.

The exterior derivative $\mathbf{d}$ acting on forms is usually not defined acting on multivector valued forms (otherwise we could build the derived bracket of the algebraic bracket (6.22) by $\mathbf{d}$ without lifting everything to operators via the interior product). But via $\{\boldsymbol{o}, K^{(k,k')}\}$ we can, at least formally, define a differential on multivector valued forms. The result, however, contains the variable p_k which we have not yet interpreted geometrically. After extending the definition of the interior product to objects containing p_m, we will get the relation $[\mathbf{d}, \imath_K] = \imath_{\{\boldsymbol{o},K\}}$, i.e. $\{\boldsymbol{o}, \ldots\}$ can be seen as an induced differential from the space of operators. For forms $\omega^{(q)}$, this simply reads $[\mathbf{d}, \imath_\omega] = \imath_{\mathbf{d}\omega}$. The definition $\mathbf{d}K \equiv \{\boldsymbol{o}, K\}$ thus seems to be a reasonable extension of the exterior derivative to multivector valued forms. Let us first provide the necessary definitions.

Consider a phase space function, which is of arbitrary order in the variable p_k

$$T^{(t,t',t'')}(x, \boldsymbol{c}, \boldsymbol{b}, p) \quad \equiv \quad T_{m_1\ldots m_t}{}^{n_1\ldots n_{t'}k_1\ldots k_{t''}}(x)\, \boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_t}\boldsymbol{b}_{m_1}\cdots\boldsymbol{b}_{m_{t'}}p_{k_1}\cdots p_{k_{t''}} \tag{6.28}$$

T is symmetrized in $k_1 \ldots k_{t''}$,while it is antisymmetrized in the remaining indices. Using the usual quantization rules $\boldsymbol{b} \to \frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{c}}$ and $p \to \frac{\hbar}{i}\frac{\partial}{\partial x}$ with the indicated ordering (conjugate momenta to the right) while still insisting on (6.14) as the relation between quantum operator and interior product, we get an extended definition of the **interior product** (6.12,6.13):

$$
\begin{aligned}
\imath_{T^{(t,t',t'')}} &\equiv \left(\frac{i}{\hbar}\right)^{t'+t''} \hat{T}^{(t,t',t'')} \equiv && (6.29)\\
&\equiv T_{m_1\ldots m_t}{}^{n_1\ldots n_{t'}k_1\ldots k_{t''}} \boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_t} \frac{\partial^{t'}}{\partial\boldsymbol{c}^{n_1}\cdots\partial\boldsymbol{c}^{n_{t'}}} \frac{\partial^{t''}}{\partial x^{k_1}\cdots\partial x^{k_{t''}}} = && (6.30)\\
\imath_{T^{(t,t',t'')}}\rho^{(r)} &= T_{m_1\ldots m_t}{}^{n_1\ldots n_{t'}k_1\ldots k_{t''}} \boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_t} \left\{\boldsymbol{b}_{n_1},\left\{\cdots,\left\{\boldsymbol{b}_{n_{t'}},\left\{p_{k_1},\left\{\cdots,\left\{p_{k_{t''}},\rho^{(r)}\right\}\right\}\right\}\right\}\right\}\right\} = && (6.31)\\
&= (t')!\begin{pmatrix} r \\ t'\end{pmatrix} T_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{n_1\ldots n_{t'}k_1\ldots k_{t''}} \frac{\partial^{t''}}{\partial x^{k_1}\cdots\partial x^{k_{t''}}}\rho^{(r)}_{n_{t'}\ldots n_1\boldsymbol{m}\ldots\boldsymbol{m}} && (6.32)
\end{aligned}
$$

The operator $\imath_T$ will serve us as an embedding of T (a phase space function, which lies in the extension of the space of multivector valued forms by the basis element p_k) into the space of differential operators acting on forms. Because of the partial derivatives with respect to x, the last line is not a tensor and T in that sense not a well defined geometric object. Nevertheless it can be a building block of a geometrically well defined object, for example in the definition of the **exterior derivative** on multivector valued forms which we suggested above. Namely, if we define[3]

$$
\begin{aligned}
\mathbf{d}K^{(k,k')} &\equiv \left\{\boldsymbol{o},K^{(k,k')}\right\} = && (6.33)\\
&= \partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} - (-)^{k-k'}k'\cdot K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k}p_k && (6.34)
\end{aligned}
$$

We get via our extended embedding (6.32) the nice relation [4]

$$
\begin{aligned}
\imath_{\mathbf{d}K}\rho &= [\mathbf{d},\imath_K]\rho \overset{(C.48)}{=} -(-)^{k-k'}\boldsymbol{\mathcal{L}}_K\rho && (6.35)\\
\text{with}\quad \boldsymbol{\mathcal{L}}_K\rho &= (k')!\begin{pmatrix} r \\ k'-1\end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}}\partial_{l_{k'}}\rho_{l_{k'-1}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}} + \\
&\quad -(-)^{k-k'}(k')!\begin{pmatrix} r \\ k'\end{pmatrix}\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}}\rho_{l_{k'}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}} && (6.36)
\end{aligned}
$$

$\boldsymbol{\mathcal{L}}_K\rho$ is the natural generalization of the Lie derivative with respect to vectors acting on forms, which is given similarly $\boldsymbol{\mathcal{L}}_v\rho = [\imath_v,\mathbf{d}]\rho$. As $\imath_K$ is a higher order derivative, also $\boldsymbol{\mathcal{L}}_K$ is a higher order derivative. Nevertheless, it will be called **Lie derivative with respect to** K in this thesis. Let us again recall this fact: if p_k appears in a combination like $\mathbf{d}K$, there is a well defined geometric meaning and $\mathbf{d}K$ is up to a sign nothing else than the Lie derivative with respect to K, when embedded in the space of differential operators on forms. The commutator with the exterior derivative is a natural differential in the space of differential operators acting on forms, and via the embedding it induces the differential $\mathbf{d}$ on K. It should perhaps be stressed that the above definition of $\mathbf{d}K$ corresponds to an extended action of the exterior derivative which acts also on the basis elements of the tangent space

$$
\mathbf{d}(\boldsymbol{\partial}_m) = p_m \qquad (6.37)
$$

This approach will enable us to give explicit coordinate expressions for the derived bracket of multivector valued forms even in the general case where the result is not a tensor: In the space of differential operators on forms, we have the commutator $[\imath_K,\imath_L]$ and its derived bracket (C.51) $[\imath_K,_{\mathbf{d}}\imath_L] \equiv [[\imath_K,\mathbf{d}],\imath_L]$, while on the space of multivector valued forms we have the algebraic bracket $[K,L]^{\Delta}$ and want to define its derived bracket up to a sign as $[\mathbf{d}K,L]^{\Delta}$. To this end we also have to extend the definition (6.18,6.19) of $\imath^{(p)}$, which appears in the

[3]This can of course be seen as a BRST differential, which is well known to be the sum of the longitudinal exterior derivate plus the Koszul Tate differential. However, as the constraint surface in our case corresponds to the configuration space (p_k would be the first class constraint generating the BRST-transformation), it is reasonable to regard the BRST differential as a natural extension of the exterior derivative of the configuration space. ◇

[4]The exterior derivative on forms has already earlier (6.9) been seen to coincide with the Poisson bracket with $\boldsymbol{o}$, which can be used to demonstrate (6.35):

$$
\begin{aligned}
[\mathbf{d},\imath_K]\rho &= \mathbf{d}(\imath_K\rho) - (-)^{|K|}\imath_K(\mathbf{d}\rho) = \\
&= \{\boldsymbol{o},\imath_K\rho\} - (-)^{|K|}\imath_K\{\boldsymbol{o},\rho\} = \\
&\overset{(6.12)}{=} \partial_{m_1}K_{m_2\ldots m_{k+1}}{}^{n_1\ldots n_{k'}}\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_{k+1}}\left\{\boldsymbol{b}_{n_1},\left\{\boldsymbol{b}_{n_2},\left\{\cdots,\left\{\boldsymbol{b}_{n_{k'}},\rho^{(r)}\right\}\right\}\right\}\right\} + \\
&\quad +(-)^k k'\cdot K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\left\{\underbrace{\{\boldsymbol{o},\boldsymbol{b}_{n_1}\}}_{p_{n_1}},\left\{\boldsymbol{b}_{n_2},\left\{\cdots,\left\{\boldsymbol{b}_{n_{k'}},\rho^{(r)}\right\}\right\}\right\}\right\} \overset{(6.31)}{\underset{(6.34)}{=}} \imath_{\mathbf{d}K}\rho \qquad \diamond
\end{aligned}
$$

explicit expression of the algebraic bracket in (6.22) to objects that contain p_k. This is done in a way that the old equations for the algebraic bracket remain formally the same. So let us define[5]

$$\begin{aligned}
&\imath^{(p)}_{T^{(t,t',t'')}} \equiv \\
&\equiv \sum_{q=0}^{p} \begin{pmatrix} t' \\ q \end{pmatrix} \begin{pmatrix} t'' \\ p-q \end{pmatrix} T_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} i_1\ldots i_q\, ,\, i_{q+1}\ldots i_p k_1\ldots k_{t''-p+q}} p_{k_1}\cdots p_{k_{t''-p+q}} \frac{\partial^p}{\partial \boldsymbol{c}^{i_1}\ldots\partial \boldsymbol{c}^{i_q}\partial x^{i_{q+1}}\ldots\partial x^{i_p}} && (6.38)\\
&= \frac{1}{p!}\sum_{q=0}^{p} \begin{pmatrix} p \\ q \end{pmatrix} T \frac{\overleftarrow{\partial}^p}{\partial p_{i_p}\ldots\partial p_{i_{q+1}}\partial \boldsymbol{b}_{i_q}\ldots\partial \boldsymbol{b}_{i_1}} \frac{\partial^p}{\partial \boldsymbol{c}^{i_1}\ldots\partial \boldsymbol{c}^{i_q}\partial x^{i_{q+1}}\ldots\partial x^{i_p}} && (6.39)
\end{aligned}$$

For $p = t' + t''$ it coincides with the full interior product (6.32): $\imath^{(t'+t'')}_{T^{(t,t',t'')}} = \imath_{T^{(t,t',t'')}}$. In addition we have with this definition (after some calculation) $\imath^{(p)}_{\mathbf{d}T} = [\mathbf{d}, \imath^{(p)}_T]$ and in particular

$$\imath^{(p)}_{\mathbf{d}K} = [\mathbf{d}, \imath^{(p)}_K] \qquad (6.40)$$

and the equations for the algebraic bracket (6.16)-(6.22)) indeed remain formally the same for objects containing p_m

$$\begin{aligned}
[\imath_{T^{(t,t',t'')}}, \imath_{\tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}}] &\equiv \imath_{[T,\tilde{T}]^{\Delta}} && (6.41)\\
\imath_T \imath_{\tilde{T}} &= \sum_{p\geq 0} \imath_{\imath^{(p)}_T \tilde{T}} && (6.42)\\
[T^{(t,t',t'')}, \tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}]^{\Delta} &\equiv \sum_{p\geq 1} \underbrace{\imath^{(p)}_T \tilde{T} - (-)^{(t-t')(\tilde{t}-\tilde{t}')} \imath^{(p)}_{\tilde{T}} T}_{\equiv [T,\tilde{T}]^{\Delta}_{(p)}} && (6.43)\\
[T, \tilde{T}]^{\Delta}_{(1)} &= \left\{T, \tilde{T}\right\} && (6.44)
\end{aligned}$$

which we can again rewrite in terms of “quantum”-operators (6.14) as

$$\begin{aligned}
\left[\hat{T}^{(k,k')}, \hat{\tilde{T}}^{(l,l')}\right] &= \sum_{p\geq 1} \left(\frac{\hbar}{i}\right)^p \widehat{\left[T,\tilde{T}\right]^{\Delta}_{(p)}} = && (6.45)\\
&= \left(\frac{\hbar}{i}\right) \widehat{\left\{T,\tilde{T}\right\}} + \sum_{p\geq 2} \left(\frac{\hbar}{i}\right)^p \widehat{\left[T,\tilde{T}\right]^{\Delta}_{(p)}} && (6.46)
\end{aligned}$$

It should be stressed that – although very useful – $\imath^{(p)}$ is unfortunately NOT a geometric operation any longer in general, in the sense that $\imath^{(p)}_{\mathbf{d}K} L$ and also $\imath^{(p)}_L \mathbf{d}K$ do not have a well defined geometric meaning, although $\mathbf{d}K$ and L have. $\imath_{\mathbf{d}K}\rho$ and $\imath^{(p)}_K L$ are in contrast well defined. $\imath^{(p)}_{\mathbf{d}K} L$, for example, should rather be understood as a building block of a coordinate calculation which combines only in certain combinations (e.g. the bracket $[\,,\,]^{\Delta}$) to s.th. geometrically meaningful.

We are now ready to define the **derived bracket** of the algebraic bracket for multivector valued forms (see footnote 3 on page 162)

$$\begin{aligned}
\left[K^{(k,k')}, L^{(l,l')}\right] &\equiv [K,_{\mathbf{d}} L]^{\Delta} \equiv -(-)^{k-k'} [\mathbf{d}K, L]^{\Delta} = && (6.47)\\
&= \sum_{p\geq 1} -(-)^{k-k'} \imath^{(p)}_{\mathbf{d}K} L + (-)^{(k+1-k')(l-l')+k-k'} \imath^{(p)}_L \mathbf{d}K = && (6.48)\\
&= \sum_{p\geq 1} -(-)^{k-k'} \imath^{(p)}_{\mathbf{d}K} L + (-)^{(k-k'+1)(l-l'+1)} (-)^{l-l'} \imath^{(p)}_{\mathbf{d}L} K + (-)^{(k-k')(l-l')+k-k'} \mathbf{d}(\imath^{(p)}_L K) && (6.49)
\end{aligned}$$

The result is geometrical in the sense that after embedding via the interior product it is a well defined operator acting on forms. This is the case, because due to our extended definitions we have for **all** multivector valued forms the relation

$$[[\imath_K, \mathbf{d}], \imath_L] = \imath_{[K^{(k,k')}, L^{(l,l')}]} \qquad (6.50)$$

and the lefthand side is certainly a well defined geometric object. A considerable effort went into getting a correct coordinate form for the general derived bracket and for that reason, let us quickly have a glance at the

[5] Note that $\sum_{q=0}^{p} \begin{pmatrix} t' \\ q \end{pmatrix} \begin{pmatrix} t'' \\ p-q \end{pmatrix} = \begin{pmatrix} t'+t'' \\ p \end{pmatrix}$ $\diamond$

final result, although it is kind of ugly:[6]

$$\begin{aligned}
[K \boldsymbol{,} L] \quad = \quad & \sum_{p\geq 1} -(-)^{k-k'}(-)^{(k'-p)(l-p)}p!\binom{l}{p}\binom{k'}{p}\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1\ldots l_p}L_{l_p\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{k+k'l+k'+p+pl+pk'}p!\binom{k}{p}\binom{l'}{p}\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}k_p\ldots k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\ldots k_p\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k'l+k'+pl+pk'}p!\binom{k}{p-1}\binom{l'}{p}\partial_l K_{\boldsymbol{m}\ldots\boldsymbol{m}k_{p-1}\ldots k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\ldots k_{p-1}l\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(k'-p)(l-p+1)}p!\binom{l}{p-1}\binom{k'}{p}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1\ldots l_{p-1}k}\partial_k L_{l_{p-1}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(k'-1-p)(l-p)}p!(k'-p)\binom{l}{p}\binom{k'}{p}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1\ldots l_p k}L_{l_p\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}p_k + \\
& -(-)^{k'l+l+pk'+lp}k'\cdot p!\binom{k}{p}\binom{l'}{p}K_{\boldsymbol{m}\ldots\boldsymbol{m}k_p\ldots k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\ldots k_p\boldsymbol{n}\ldots\boldsymbol{n}}p_k
\end{aligned} \tag{6.51}$$

The result is only a tensor, when both terms with p_k on the righthand side vanish, although the complete expression is in general geometrically well-defined when considered to be a differential operator acting on forms via $\imath_{[K,L]}$ as this equals per definition the well-defined $[[\imath_K, \mathbf{d}], \imath_L]$. The above coordinate form reduces in the appropriate cases to vector Lie-bracket, Schouten-bracket, and (up to a total derivative) to the (Fröhlicher)-Nijenhuis-bracket. If one allows as well sums of a vector and a 1-form, we get the Dorfman bracket, and also the sum of a vector and a general form gives a result without p.

Due to our extended definition of the exterior derivative, we can also define the **derived bracket of the big bracket** (the Poisson bracket) via

$$\left[K^{(k,k')} ,_{\mathbf{d}} L^{(l,l')}\right]^{\Delta}_{(1)} \quad \equiv \quad -(-)^{k-k'}\,[\mathbf{d}K, L]^{\Delta}_{(1)} = \tag{6.52}$$

$$= \quad -(-)^{k-k'}\,\{\mathbf{d}K, L\} \tag{6.53}$$

which is just the $p = 1$ term of the full derived bracket with the explicit coordinate expression

$$\begin{aligned}
[K,_{\mathbf{d}} L]^{\Delta}_{(1)} \quad = \quad & -(-)^{k-k'}(-)^{(k'-1)(l-1)}lk'\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1}L_{l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k+k'l+l}kl'\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k'l+l}l'\partial_l K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(k'-1)l}k'K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k}\partial_k L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{k'(l-1)}(k'-1)lk'K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}l_1k}L_{l_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}p_k + \\
& -(-)^{k'l+k'}k'kl'K_{\boldsymbol{m}\ldots\boldsymbol{m}k_1}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{k_1\boldsymbol{n}\ldots\boldsymbol{n}}p_k
\end{aligned} \tag{6.54}$$

$$[K \boldsymbol{,} L] \quad = \quad [K,_{\mathbf{d}} L]^{\Delta}_{(1)} - (-)^{k-k'}\sum_{p\geq 2}[\mathbf{d}K, L]^{\Delta}_{(p)} \tag{6.55}$$

Like the big bracket itself, also its derived bracket takes a very pleasant coordinate form for generalized multivectors (see (**B**.79) on page 154). In contrast to the full derived bracket, we have no guarantee for this derived bracket to be geometrical itself.

[6] The building blocks are

$$\begin{aligned}
\imath^{(p)}_{\mathbf{d}K}L \quad = \quad & (-)^{(k'-p)(l-p)}p!\binom{k'}{p}\binom{l}{p}\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}i_1\ldots i_p}L_{i_p\ldots i_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k-k'}(-)^{(k'-1-p)(l-p)}(p+1)!\binom{k'}{p+1}\binom{l}{p}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}i_1\ldots i_p k}L_{i_p\ldots i_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}p_k + \\
& -(-)^{k-k'}(-)^{(k'-p)(l-p+1)}p!\binom{k'}{p}\binom{l}{p-1}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}i_1\ldots i_{p-1}i_p}\partial_{i_p}L_{i_{p-1}\ldots i_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}}
\end{aligned}$$

$$\begin{aligned}
\imath^{(p)}_{L}\mathbf{d}K \quad = \quad & (-)^{(l'-p)(k+1-p)+p}p!\binom{k}{p}\binom{l'}{p}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k_1\ldots k_p}\partial_{\boldsymbol{m}}K_{k_p\ldots k_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& +(-)^{(l'-p)(k+1-p)}p!\binom{k}{p-1}\binom{l'}{p}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k_1\ldots k_{p-1}l}\partial_l K_{k_{p-1}\ldots k_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\
& -(-)^{k-k'}(-)^{(l'-p)(k-p)}k'\cdot p!\binom{k}{p}\binom{l'}{p}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k_1\ldots k_p}K_{k_p\ldots k_1\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}k}p_k \qquad \diamond
\end{aligned}$$

Let us eventually note how one can easily adjust the extended exterior derivative to the twisted case:

$$[\mathbf{d}+H\wedge\,,\imath_K] \equiv \imath_{\mathbf{d}_H K} \tag{6.56}$$

$$\mathbf{d}_H K = \mathbf{d}K + [H,K]^\Delta = \mathbf{d}K - (-)^{k-k'}\sum_{p\geq 1}\imath_K^{(p)}H \tag{6.57}$$

with H being an odd closed differential form. It should be stressed that $\mathbf{d}+H\wedge$ is not a differential, but on the operator level its commutator $[\mathbf{d}+H\wedge,\ldots]$ is a differential and thus the above defined $\mathbf{d}_H$ is a differential as well.

6.2 Sigma-Models

A sigma model is a field theory whose fields are embedding functions from a world-volume Σ into a target space M, like in string theory. So far there was no sigma-model explicitly involved into our considerations. One can understand the previous subsection simply as a convenient way to formulate some geometry. The phase space introduced there, however, is like the phase space of a (point particle) sigma model with only one world-volume dimension – the time – which is not showing up in the off-shell phase-space. Let us now naively consider the same setting like before as a sigma model with the coordinates x^m depending on some worldsheet coordinates[7] σ^μ. The resulting model has a very special field content, because its anticommuting fields $\boldsymbol{c}^m(\sigma)$ have the same index structure as the embedding coordinate $x^m(\sigma)$. In one and two worldvolume-dimensions, $\boldsymbol{c}^m$ can be regarded as worldvolume-fermions, and this will be used in the stringy application in 7.2. In general worldvolume dimensions, $\boldsymbol{c}^m$ could be seen as ghosts, leading to a topological theory. In any case the dimension of the worldvolume will not yet be fixed, as the described mechanism does not depend on it.

A multivector valued form on a C^∞-manifold M can locally be regarded as an analytic function of $x^m, \mathbf{d}x^m \equiv \boldsymbol{c}^m$ and $\boldsymbol{\partial}_m \equiv \boldsymbol{b}_m$

$$K^{(k,k')}(x,\mathbf{d}x,\boldsymbol{\partial}) = K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(x)\mathbf{d}x^{m_1}\wedge\cdots\wedge\mathbf{d}x^{m_k}\wedge\boldsymbol{\partial}_{n_1}\wedge\cdots\wedge\boldsymbol{\partial}_{n_{k'}} = \tag{6.58}$$

$$\equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(x)\boldsymbol{c}^{m_1}\cdots\boldsymbol{c}^{m_k}\boldsymbol{b}_{n_1}\cdots\boldsymbol{b}_{n_{k'}} = K^{(k,k')}(x,\boldsymbol{c},\boldsymbol{b}) \tag{6.59}$$

For sigma models, $x^m \to x^m(\sigma), p_m \to p_m(\sigma), \boldsymbol{c}^m \to \boldsymbol{c}^m(\sigma)$ and $\boldsymbol{b}_m \to \boldsymbol{b}_m(\sigma)$ become dependent on the worldvolume variables σ^μ. They are, however, for every σ valid arguments of the function K. Frequently only the worldvolume coordinate σ will then be denoted as new argument of K, which has to be understood in the following sense

$$K^{(k,k')}(\sigma) \equiv K^{(k,k')}\left(x(\sigma),\boldsymbol{c}(\sigma),\boldsymbol{b}(\sigma)\right) = K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(x(\sigma)\right)\cdot\boldsymbol{c}^{m_1}(\sigma)\cdots\boldsymbol{c}^{m_k}(\sigma)\boldsymbol{b}_{n_1}(\sigma)\cdots\boldsymbol{b}_{n_{k'}}(\sigma) \tag{6.60}$$

Also functions depending on p_m, like $\mathbf{d}K(x,\boldsymbol{c},\boldsymbol{b},p)$ in (6.34), or more general a function $T^{(t,t',t'')}(x,\boldsymbol{c},\boldsymbol{b},p)$ as in (6.28) are denoted in this way

$$T^{(t,t',t'')}(\sigma) \equiv T^{(t,t',t'')}\left(x(\sigma),\boldsymbol{c}(\sigma),\boldsymbol{b}(\sigma),p(\sigma)\right) \quad \text{(see (6.28))} \tag{6.61}$$

$$\text{e.g. } \mathbf{d}K(\sigma) \equiv \mathbf{d}K\left(x(\sigma),\boldsymbol{c}(\sigma),\boldsymbol{b}(\sigma),p(\sigma)\right) \quad \text{(see (6.34))} \tag{6.62}$$

$$\text{or } \boldsymbol{o}(\sigma) \equiv \boldsymbol{o}\left(\boldsymbol{c}(\sigma),p(\sigma)\right) = \boldsymbol{c}^m(\sigma)p_m(\sigma) \quad \text{(see (6.8))} \tag{6.63}$$

The expression $\mathbf{d}K(\sigma)$ should **NOT** be mixed up with the world-volume exterior derivative of K which will be denoted by $\boldsymbol{d}^{\mathrm{w}}K(\sigma)$.[8] Every operation of the previous section, like $\imath_K^{(p)}L$ or the algebraic or derived brackets leads again to functions of $x,\boldsymbol{c},\boldsymbol{b}$ and sometimes p. Let us use for all of them the notation as above, e.g. for the derived bracket of the big bracket (6.52,6.54)

$$\left[K^{(k,k')},_{\mathbf{d}}L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma) \equiv \left[K^{(k,k')},L^{(l,l')}\right]^{(\Delta)}_{(1)}\left(x(\sigma),\boldsymbol{c}(\sigma),\boldsymbol{b}(\sigma),p(\sigma)\right) \tag{6.64}$$

And even $\mathbf{d}x^m = \boldsymbol{c}^m$ and $\mathbf{d}\boldsymbol{b}_m = p_m$ will be seen as a function (identity) of $\boldsymbol{c}^m$ or $\boldsymbol{b}_m$, s.th. we denote

$$\mathbf{d}x^m(\sigma) \equiv \boldsymbol{c}^m(\sigma) \tag{6.65}$$

$$\mathbf{d}\boldsymbol{b}_m(\sigma) \equiv p_m(\sigma) \tag{6.66}$$

Although $\mathbf{d}$ acts only in the target space on $x,\boldsymbol{b},\boldsymbol{c}$ and p, the above obviously suggests to introduce a differential – say $\mathbf{s}$ – in the new phase space, which is compatible with the target space differential in the sense

$$\mathbf{s}(x^m(\sigma)) = \mathbf{d}x^m(\sigma) \equiv \boldsymbol{c}^m(\sigma) \tag{6.67}$$

$$\mathbf{s}(\boldsymbol{b}_m(\sigma)) = \mathbf{d}\boldsymbol{b}_m(\sigma) \equiv p_m(\sigma) \tag{6.68}$$

[7]The index μ will not include the worldvolume time, when considering the phase space, but it will contain the time in the Lagrangian formalism. As this should be clear from the context, there will be no notational distinction. ◇

[8] It is much better to mix it up with a BRST transformation or with something similar to a worldsheet supersymmetry transformation. We will come to that later in subsection 7.2. To make confusion perfect, it should be added that in contrast it is not completely wrong in subsection 6.5 to mix up the target space exterior derivative with the worldsheet exterior derivative... ◇

We can generate $\mathbf{s}$ with the Poisson bracket in almost the same way as $\mathbf{d}$ before in (6.8):

$$\boldsymbol{\Omega} \equiv \int_\Sigma d^{d_w-1}\sigma \;\; \boldsymbol{o}(\sigma) = \int d^{d_w-1}\sigma \;\; \boldsymbol{c}^m(\sigma)p_m(\sigma), \qquad \mathbf{s}(\ldots) = \{\boldsymbol{\Omega}, \ldots\} \tag{6.69}$$

The Poisson bracket between the conjugate fields gets of course an additional delta function compared to (6.5,6.6).

$$\begin{aligned} \{p_m(\sigma'), x^n(\sigma)\} &= \delta_m^n \delta^{d_w-1}(\sigma' - \sigma) & (6.70)\\ \{\boldsymbol{b}_m(\sigma'), \boldsymbol{c}^n(\sigma)\} &= \delta_m^n \delta^{d_w-1}(\sigma' - \sigma) & (6.71)\end{aligned}$$

The first important (but rather trivial) observation is then that for $K(\sigma)$ being a function of $x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)$ as in (6.60) (and not a functional, which could contain derivatives on or integrations over σ) we have

$$\mathbf{s}(K(\sigma)) = \left(\boldsymbol{c}^m(\sigma)\frac{\partial}{\partial(x^m(\sigma))} + p_m(\sigma)\frac{\partial}{\partial(\boldsymbol{b}_m(\sigma))}\right) K\left(x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)\right) = \mathbf{d}K(\sigma) \tag{6.72}$$

The same is true for more general objects of the form of T in (6.61). Because of this fact the distinction between $\mathbf{d}$ and $\mathbf{s}$ is not very essential, but in subsection 6.5 the replacement of the arguments as in (6.61) will be different and the distinction very essential in order not to get confused.

The relation between Poisson bracket and big bracket (6.23,6.44) gets obviously modified by a delta function

$$\begin{aligned} \left\{K^{(k,k')}(\sigma'), L^{(l,l')}(\sigma)\right\} &= \left[K^{(k,k')}, L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma)\, \delta^{d_w-1}(\sigma' - \sigma) & (6.73)\\ \text{or more general } \left\{T^{(t,t',t'')}(\sigma'), \tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}(\sigma)\right\} &= \left[T^{(t,t',t'')}, \tilde{T}^{(\tilde{t},\tilde{t}',\tilde{t}'')}\right]^{\Delta}_{(1)}(\sigma)\, \delta^{d_w-1}(\sigma' - \sigma) & (6.74)\end{aligned}$$

The relation between the derived bracket (using $\mathbf{s}$) on the lefthand side and the derived bracket (using $\mathbf{d}$) on the righthand side is (omitting the overall sign in the definition of the derived bracket)

$$\left\{\mathbf{s}K^{(k,k')}(\sigma'), L^{(l,l')}(\sigma)\right\} \overset{(6.72)}{=} \left\{\mathbf{d}K^{(k,k')}(\sigma'), L^{(l,l')}(\sigma)\right\} \overset{(6.74)}{=} \left[\mathbf{d}K^{(k,k')}, L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma)\, \delta^{d_w-1}(\sigma' - \sigma) \tag{6.75}$$

The worldvolume coordinates σ remain so far more or less only spectators. In the subsection 6.5, the worldvolume coordinates play a more active part and already in the following subsection a similar role is taken by an anticommuting extension of the worldsheet.

Before we proceed, it should be stressed that the replacement of $x, \boldsymbol{c}, \boldsymbol{b}$ and p by $x(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma)$ and $p(\sigma)$ was just the most naive replacement to do, and it will be a bit extended in the following section until it can serve as a useful tool in an application in 7.2. But in principle, one can replace those variables by any fields with matching index structure and parity (even composite ones) and study the resulting relations between Poisson bracket on the one side and geometric bracket on the other side. Also the differential $\mathbf{s}$ can be replaced for example by the twisted differential or by more general BRST-like transformations. In this way it should be possible to implement other derived brackets, for example those built with the Poisson-Lichnerowicz-differential (see [70]), in a sigma-model description. In 6.5, a different (but also quite canonical) replacement is performed and we will see that the different replacement corresponds to a change of the role of σ and an anticommuting worldvolume coordinate $\boldsymbol{\theta}$ which will be introduced in the following.

6.3 Natural appearance of derived brackets in Poisson brackets of superfields

In the application to worldsheet theories in section 7, there appear superfields, either in the sense of worldsheet supersymmetry or in the sense of de-Rham superfields (see e.g. [97, 91]). Let us view a superfield in general as a method to implement a fermionic transformation of the fields via a shift in a fermionic parameter $\boldsymbol{\theta}$ which can be regarded as fermionic extension of the worldvolume. In our case the fermionic transformation is just the spacetime de-Rham-differential $\mathbf{d}$, or more precisely $\mathbf{s}$, and is not necessarily connected to worldvolume supersymmetry. In fact, in worldvolumes of dimension higher than two, supersymmetry requires more than one fermionic parameter while a single $\boldsymbol{\theta}$ is enough for our purpose to implement $\mathbf{s}$. In two dimensions, however, this single theta can really be seen as a worldsheet fermion (see 7.2). But let us neglect this knowledge for a while, in order to clearly see the mechanism, which will be a bit hidden again, when applied to the supersymmetric case in 7.2.

As just said above, we want to implement with superfields the fermionic transformation $\mathbf{s}$ and not yet a supersymmetry. So let us define in this section a **superfield** as a function of the phase space fields with additional dependence on $\boldsymbol{\theta}$, $Y = Y(x(\sigma), p(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), \boldsymbol{\theta})$, which obeys [9]

$$\begin{aligned} \mathbf{s}Y(x(\sigma), p(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), \boldsymbol{\theta}) &\overset{!}{=} \partial_{\boldsymbol{\theta}} Y(x(\sigma), p(\sigma), \boldsymbol{c}(\sigma), \boldsymbol{b}(\sigma), \boldsymbol{\theta}) & (6.76)\\ \text{with} \qquad & \mathbf{s}x^m(\sigma) = \boldsymbol{c}^m(\sigma), \mathbf{s}\boldsymbol{b}_m(\sigma) = p_m(\sigma) \quad (\mathbf{s}\boldsymbol{\theta} = 0) & (6.77)\end{aligned}$$

[9] If this seems unfamiliar, compare with the case of worldsheet supersymmetry, where one introduces a differential operator $\mathrm{Q}_{\boldsymbol{\theta}} \equiv \partial_{\boldsymbol{\theta}} + \boldsymbol{\theta}\partial_\sigma$ and the definition of a superfield is, in contrast to here, $\delta_{\boldsymbol{\varepsilon}} Y \overset{!}{=} \boldsymbol{\varepsilon}\mathrm{Q}_{\boldsymbol{\theta}} Y$, where $\delta_{\boldsymbol{\varepsilon}}$ is the supersymmetry transformation of the component fields (compare 7.2). ◇

With our given field content it is possible to define two basic conjugate[10] superfields Φ^m and $\boldsymbol{S}_m$ which build up a super-phase-space[11]

$$\begin{aligned}
\Phi^m(\sigma,\boldsymbol{\theta}) &\equiv x^m(\sigma)+\boldsymbol{\theta}\boldsymbol{c}^m(\sigma)=x^m(\sigma)+\boldsymbol{\theta}\mathbf{s}x^m(\sigma) && (6.78)\\
\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{b}_m(\sigma)+\boldsymbol{\theta}p_m(\sigma)=\boldsymbol{b}_m(\sigma)+\boldsymbol{\theta}\mathbf{s}\boldsymbol{b}_m(\sigma) && (6.79)\\
\left\{\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}),\Phi^n(\sigma',\boldsymbol{\theta}')\right\} &= \left\{\boldsymbol{b}_m(\sigma),\boldsymbol{\theta}'\boldsymbol{c}^n(\sigma')\right\}+\boldsymbol{\theta}\left\{p_m(\sigma),x^n(\sigma')\right\}= && (6.80)\\
&= \underbrace{(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{\equiv\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}\delta(\sigma-\sigma')\delta^n_m && (6.81)
\end{aligned}$$

Φ and $\boldsymbol{S}$ are obviously superfields in the above sense

$$\begin{aligned}
\partial_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta}) &= \underbrace{\mathbf{s}x^m(\sigma)}_{\boldsymbol{c}^m(\sigma)}\underbrace{+\boldsymbol{\theta}\mathbf{s}\boldsymbol{c}^m(\sigma)}_{=0}=\mathbf{s}\Phi^m(\sigma,\boldsymbol{\theta}) && (6.82)\\
\partial_{\boldsymbol{\theta}}\boldsymbol{S}_m &= \underbrace{\mathbf{s}\boldsymbol{b}_m(\sigma)}_{p_m(\sigma)}\underbrace{+\boldsymbol{\theta}\mathbf{s}p_m(\sigma)}_{0}=\mathbf{s}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) && (6.83)
\end{aligned}$$

as well as $\mathbf{s}\Phi(\sigma,\boldsymbol{\theta})=\boldsymbol{c}(\sigma)$ and $\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})=p(\sigma)$ are superfields, and every analytic function of those fields will be a superfield again.

We will convince ourselves in this subsection that in the Poisson brackets of general superfields, the derived brackets come with the complete δ-function (of σ and $\boldsymbol{\theta}$) while the corresponding algebraic brackets come with a derivative of the delta-function. The introduction of worldsheet coordinates σ was not yet really necessary for this discussion, but it will be useful for the comparison with the subsequent subsection. Indeed, we do not specify the dimension d_{w} of the worldsheet yet. An argument sigma is representing several worldsheet coordinates σ^μ. It should be stressed again that the differential $\mathbf{d}$should **NOT** be mixed up with the worldsheet exterior derivative $\boldsymbol{d}^{\mathrm{w}}$, which does not show up in this subsection.

Similar as in 6.2, equations (6.60)-(6.66),we will view all geometric objects as functions of local coordinates and replace the arguments not by phase space fields but by the just defined super-phase fields which reduces for $\boldsymbol{\theta}=0$ to the previous case.

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) \equiv T^{(t,t',t'')}\left(\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right)\overset{\boldsymbol{\theta}=0}{=}T^{(t,t',t'')}(\sigma)\quad(\text{see }(6.61))\qquad(6.84)$$

[10] The superfields Φ and $\boldsymbol{S}$ are conjugate with respect to the following **super-Poisson-bracket**

$$\begin{aligned}
\left\{F(\sigma',\boldsymbol{\theta}'),G(\sigma,\boldsymbol{\theta})\right\} &\equiv \int d^{d_{\mathrm{w}}-1}\tilde{\sigma}\int d\tilde{\boldsymbol{\theta}}\quad \left(\delta F(\sigma',\boldsymbol{\theta}')/\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}G(\sigma,\boldsymbol{\theta})-\delta F(\sigma',\boldsymbol{\theta}')/\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}G(\sigma,\boldsymbol{\theta})\right)=\\
&= \int d^{d_{\mathrm{w}}-1}\tilde{\sigma}\int d\tilde{\boldsymbol{\theta}}\quad \left(\delta F(\sigma',\boldsymbol{\theta}')/\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}G(\sigma,\boldsymbol{\theta})-(-)^{FG}\delta G(\sigma',\boldsymbol{\theta}')/\delta\boldsymbol{S}_k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\frac{\delta}{\delta\Phi^k(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}F(\sigma,\boldsymbol{\theta})\right)
\end{aligned}$$

which, however, boils down to taking the ordinary graded Poisson bracket between the component fields (as can be seen in (6.80)). The **functional derivatives** from the left and from the right are defined as usual via

$$\delta_S A\equiv\int d^{d_{\mathrm{w}}-1}\tilde{\sigma}\int d\tilde{\theta}\quad \delta A/\delta S_k(\tilde{\sigma},\tilde{\theta})\cdot\delta S_k(\tilde{\sigma},\tilde{\theta})\equiv\int d^{d_{\mathrm{w}}-1}\tilde{\sigma}\int d\tilde{\theta}\quad \delta S_k(\tilde{\sigma},\tilde{\theta})\cdot\frac{\delta}{\delta S_k(\tilde{\sigma},\tilde{\theta})}A$$

and similarly for Φ, which leads to

$$\begin{aligned}
\frac{\delta}{\delta\boldsymbol{S}_m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}\boldsymbol{S}_n(\sigma,\boldsymbol{\theta}) &= \delta^m_n(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})\delta^{d_{\mathrm{w}}-1}(\sigma-\tilde{\sigma})=-\delta\boldsymbol{S}_n(\sigma,\boldsymbol{\theta})/\boldsymbol{S}_m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})\\
\frac{\delta}{\delta\Phi^m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}\Phi^n(\sigma,\boldsymbol{\theta}) &= \delta^n_m(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta})\delta^{d_{\mathrm{w}}-1}(\sigma-\tilde{\sigma})=\delta\Phi^n(\sigma,\boldsymbol{\theta})/\delta\Phi^m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})
\end{aligned}$$

The functional derivatives can also be split in those with respect to the component fields

$$\frac{\delta}{\delta\boldsymbol{S}_m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}=\frac{\delta}{\delta p_m(\tilde{\sigma})}-\tilde{\boldsymbol{\theta}}\frac{\delta}{\delta\boldsymbol{b}_m(\tilde{\sigma})},\qquad \frac{\delta}{\delta\Phi^m(\tilde{\sigma},\tilde{\boldsymbol{\theta}})}=\frac{\delta}{\delta\boldsymbol{c}^m(\tilde{\sigma})}+\tilde{\boldsymbol{\theta}}\frac{\delta}{\delta x^m(\tilde{\sigma})}\qquad\diamond$$

[11] For Grassmann variables $\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})=\boldsymbol{\theta}'-\boldsymbol{\theta}$ in the following sense

$$\begin{aligned}
\int \mathbf{d}\boldsymbol{\theta}'(\boldsymbol{\theta}'-\boldsymbol{\theta})F(\boldsymbol{\theta}') &= \int \mathbf{d}\boldsymbol{\theta}'(\boldsymbol{\theta}'-\boldsymbol{\theta})\left(F(\boldsymbol{\theta})+(\boldsymbol{\theta}'-\boldsymbol{\theta})\partial_{\boldsymbol{\theta}}F(\boldsymbol{\theta})\right)=\\
&= \int \mathbf{d}\boldsymbol{\theta}'\quad \boldsymbol{\theta}'F(\boldsymbol{\theta})-\boldsymbol{\theta}'\boldsymbol{\theta}\partial_{\boldsymbol{\theta}}F(\boldsymbol{\theta})-\boldsymbol{\theta}\boldsymbol{\theta}'\partial_{\boldsymbol{\theta}}F(\boldsymbol{\theta})=\\
&= F(\boldsymbol{\theta})
\end{aligned}$$

We have as usual

$$\begin{aligned}
\boldsymbol{\theta}\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}) &= \boldsymbol{\theta}(\boldsymbol{\theta}'-\boldsymbol{\theta})=\boldsymbol{\theta}\boldsymbol{\theta}'=\boldsymbol{\theta}'(\boldsymbol{\theta}'-\boldsymbol{\theta})=\\
&= \boldsymbol{\theta}'\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})
\end{aligned}$$

Pay attention to the antisymmetry

$$\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}) = -\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\qquad\diamond$$

For example for a multivector valued form we write

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) \equiv K^{(k,k')}\big(\Phi^m(\sigma,\boldsymbol{\theta}), \underbrace{\mathbf{s}\Phi^m(\sigma,\boldsymbol{\theta})}_{\boldsymbol{c}^m(\sigma)}, \boldsymbol{S}_m(\sigma,\boldsymbol{\theta})\big) = \tag{6.85}$$

$$= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(\Phi(\sigma,\boldsymbol{\theta})\right) \underbrace{\mathbf{s}\Phi^{m_1}(\sigma,\boldsymbol{\theta})}_{\boldsymbol{c}^{m_1}(\sigma)} \ldots \mathbf{s}\Phi^{m_k}(\sigma,\boldsymbol{\theta})\boldsymbol{S}_{n_1}(\sigma,\boldsymbol{\theta})\ldots\boldsymbol{S}_{n_{k'}}(\sigma,\boldsymbol{\theta}) \underset{(6.60)}{\overset{\boldsymbol{\theta}=0}{\equiv}} K^{(k,k')}(\sigma) \tag{6.86}$$

Likewise for all the other examples of 6.2:

$$\text{e.g. } \mathbf{d}K(\sigma,\boldsymbol{\theta}) \equiv \mathbf{d}K\left(\Phi(\sigma,\boldsymbol{\theta}), \mathbf{s}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \tag{6.87}$$

$$\text{or } \boldsymbol{o}(\sigma,\boldsymbol{\theta}) \equiv \boldsymbol{o}\left(\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}), \mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) = \boldsymbol{c}^m(\sigma)p_m(\sigma) = \boldsymbol{o}(\sigma) \tag{6.88}$$

$$\left[K^{(k,k')},_{\mathbf{d}} L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \equiv \left[K^{(k,k')},L^{(l,l')}\right]^{(\Delta)}_{(1)}\left(\Phi(\sigma,\boldsymbol{\theta}), \mathbf{s}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathbf{s}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \underset{(6.64)}{\overset{\boldsymbol{\theta}=0}{\equiv}} \left[K^{(k,k')},L^{(l,l')}\right]^{(\Delta)}_{(1)}(\sigma) \tag{6.89}$$

$$\mathbf{d}x^m(\sigma,\boldsymbol{\theta}) \equiv \mathbf{s}\Phi^m(\sigma,\boldsymbol{\theta}) = \boldsymbol{c}^m(\sigma) \tag{6.90}$$

$$\mathbf{d}\boldsymbol{b}_m(\sigma,\boldsymbol{\theta}) \equiv \mathbf{s}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = p_m(\sigma) \tag{6.91}$$

For functions of the type $T^{(t,t',t'')}(\sigma,\boldsymbol{\theta})$ we clearly have

$$\mathbf{d}T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) = \mathbf{s}\Big(T^{(t,t',t'')}(\sigma,\boldsymbol{\theta})\Big) \tag{6.92}$$

$$\text{in particular } \mathbf{d}K^{(k,k')}(\sigma,\boldsymbol{\theta}) = \mathbf{s}\Big(K^{(k,k')}(\sigma,\boldsymbol{\theta})\Big) \tag{6.93}$$

As all those analytic functions of the basic superfields are superfields (in the sense of 6.76) themselves, $\partial_{\boldsymbol{\theta}}$ can be replaced by $\mathbf{s}$ in a $\boldsymbol{\theta}$-expansion, so that we get the important relation

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) = T^{(t,t',t'')}(\sigma) + \boldsymbol{\theta}\mathbf{d}T^{(t,t',t'')}(\sigma) \tag{6.94}$$

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) = K^{(k,k')}(\sigma) + \boldsymbol{\theta}\mathbf{d}K^{(k,k')}(\sigma) \tag{6.95}$$

This also implies that $\mathbf{d}T(\sigma,\boldsymbol{\theta})$ and in particular $\mathbf{d}K(\sigma,\boldsymbol{\theta})$ do actually not depend on $\boldsymbol{\theta}$:

$$\mathbf{d}K^{(k,k')}(\sigma,\boldsymbol{\theta}) = \mathbf{d}K^{(k,k')}(\sigma) \tag{6.96}$$

Now comes the nice part:

Proposition 1 *For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$ as in (6.10), the following equation holds for the corresponding superfields (6.85)*

$$\boxed{\{K^{(k,k')}(\sigma',\boldsymbol{\theta}'), L^{(l,l')}(\sigma,\boldsymbol{\theta})\} = \delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\cdot \underbrace{[\mathbf{d}K,L]^{\Delta}_{(1)}}_{-(-)^{k-k'}\,[K,_{\mathbf{d}}L]^{\Delta}_{(1)}}(\sigma,\boldsymbol{\theta}) + \underbrace{\partial_{\boldsymbol{\theta}}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{=1}\delta(\sigma-\sigma')[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})} \tag{6.97}$$

where $[K,L]^{\Delta}_{(1)}$ is the big bracket (6.23) (Buttin's algebraic bracket, which was previously just the Poisson bracket, being true now up to a $\delta(\sigma-\sigma')$ only after setting $\boldsymbol{\theta}=\boldsymbol{\theta}'$) and $[K,_{\mathbf{d}}L]^{\Delta}_{(1)}$ is the derived bracket of the big bracket (6.52).

Proof Using (6.95), we can simply plug $K(\sigma',\boldsymbol{\theta}') = K(\sigma') + \boldsymbol{\theta}'\mathbf{d}K(\sigma')$ and $L(\sigma,\boldsymbol{\theta}) = L(\sigma) + \boldsymbol{\theta}\mathbf{d}L(\sigma)$ into the lefthand side:

$$\big\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\big\} =$$

$$= \{K(\sigma'),L(\sigma)\} + \boldsymbol{\theta}'\{\mathbf{d}K(\sigma'),L(\sigma)\} + (-)^{k-k'}\boldsymbol{\theta}\{K(\sigma'),\mathbf{d}L(\sigma)\} + (-)^{k-k'}\boldsymbol{\theta}\boldsymbol{\theta}'\{\mathbf{d}K(\sigma'),\mathbf{d}L(\sigma)\} = \tag{6.98}$$

$$= \{K(\sigma'),L(\sigma)\} + (\boldsymbol{\theta}'-\boldsymbol{\theta})\{\mathbf{d}K(\sigma'),L(\sigma)\} + \boldsymbol{\theta}\mathbf{d}\{K(\sigma'),L(\sigma)\} - \boldsymbol{\theta}\boldsymbol{\theta}'\mathbf{d}\{\mathbf{d}K(\sigma'),L(\sigma)\} = \tag{6.99}$$

$$\overset{(6.23)}{=} \delta(\sigma-\sigma')\Big([K,L]^{\Delta}_{(1)}(\sigma) + \boldsymbol{\theta}\mathbf{d}[K,L]^{\Delta}_{(1)}(\sigma)\Big) + (\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\Big([\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma) + \boldsymbol{\theta}\mathbf{d}[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma)\Big) = \tag{6.100}$$

$$\overset{(6.94)}{=} \delta(\sigma-\sigma')[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) + (\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \qquad \square \tag{6.101}$$

There is yet another way to see that the bracket at the plain delta functions is the derived bracket of the one at the derivative of the delta-function, which will be useful later: Denote the coefficients in front of the delta-functions by $A(K,L)$ and $B(K,L)$:

$$\big\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\big\} = A(K,L)\cdot\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma') + B(K,L)(\sigma,\boldsymbol{\theta})\underbrace{\partial_{\boldsymbol{\theta}}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{=1}\delta(\sigma-\sigma') \tag{6.102}$$

In order to hit the delta-functions, it is enough to integrate over a patch $U(\sigma)$ containing the point parametrized by σ. We can thus extract A and B via

$$\begin{aligned}
A(K,L)(\sigma,\boldsymbol{\theta}) &= \int \mathbf{d}\boldsymbol{\theta}' \int_{U(\sigma)} d^{d_{\mathrm{w}}-1}\sigma' \left\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\right\} = && (6.103)\\
&= \int \mathbf{d}\boldsymbol{\theta}' \int d^{d_{\mathrm{w}}-1}\sigma' \left\{K(\sigma') + \boldsymbol{\theta}'\mathbf{d}K(\sigma'), L(\sigma,\boldsymbol{\theta})\right\} = && (6.104)\\
&= \int d^{d_{\mathrm{w}}-1}\sigma' \{ \underbrace{\mathbf{d}K(\sigma')}_{\stackrel{(6.96)}{=}\mathbf{d}K(\sigma',\boldsymbol{\theta})}, L(\sigma,\boldsymbol{\theta})\} && (6.105)\\
B(K,L)(\sigma,\boldsymbol{\theta}) &= \int \mathbf{d}\boldsymbol{\theta}' \int_{U(\sigma)} d^{d_{\mathrm{w}}-1}\sigma' (\boldsymbol{\theta}'-\boldsymbol{\theta}) \left\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\right\} = && (6.106)\\
&= \int d^{d_{\mathrm{w}}-1}\sigma' \left\{K(\sigma',\boldsymbol{\theta}'), L(\sigma,\boldsymbol{\theta})\right\} |_{\boldsymbol{\theta}'=\boldsymbol{\theta}} && (6.107)\\
\Rightarrow A(K,L) &= B(\mathbf{d}K, L) && (6.108)
\end{aligned}$$

It is thus enough to collect in a direct calculation the terms at the derivative of the delta-function and verify that it leads to the big bracket. □

6.4 Comment on the quantum case

In (6.14) the embedding via the interior product into the space of operators acting on forms was interpreted as quantization . In the presence of world-volume dimensions, the partial derivative as Schroedinger representation for conjugate momenta is no longer appropriate and one has to switch to the functional derivative. Remember

$$\begin{aligned}
\Phi^m(\sigma,\boldsymbol{\theta}) &= x^m(\sigma) + \boldsymbol{\theta}\boldsymbol{c}^m(\sigma), \qquad \mathbf{d}\Phi^m(\sigma,\boldsymbol{\theta}) = \boldsymbol{c}^m(\sigma) = \mathbf{d}\Phi(\sigma) && (6.109)\\
\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) &= \boldsymbol{b}_m(\sigma) + \boldsymbol{\theta}p_m(\sigma), \qquad \mathbf{d}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = p_m(\sigma) = \mathbf{d}\boldsymbol{S}(\sigma) && (6.110)
\end{aligned}$$

The quantization of the superfields in the Schroedinger representation (conjugate momenta as super functional derivatives) is consistent with the quantization of the component fields (see also footnote 10)

$$\begin{aligned}
\hat{\boldsymbol{S}}_m(\sigma,\boldsymbol{\theta}) &\equiv \frac{\hbar}{i}\frac{\delta}{\delta\Phi^m(\sigma,\boldsymbol{\theta})} = \frac{\hbar}{i}\frac{\delta}{\delta\boldsymbol{c}^m(\sigma)} + \boldsymbol{\theta}\frac{\hbar}{i}\frac{\delta}{\delta x^m(\sigma)} && (6.111)\\
\Rightarrow \left[\hat{\boldsymbol{S}}_m(\sigma,\boldsymbol{\theta}), \hat{\Phi}^n(\sigma',\boldsymbol{\theta}')\right] &= \frac{\hbar}{i}\left(\frac{\delta}{\delta\boldsymbol{c}^m(\sigma)} + \boldsymbol{\theta}\frac{\delta}{\delta x^m(\sigma)}\right)\left(x^n(\sigma') + \boldsymbol{\theta}'\boldsymbol{c}^n(\sigma')\right) = && (6.112)\\
&= \frac{\hbar}{i}\delta_m^n\left(\boldsymbol{\theta}-\boldsymbol{\theta}'\right)\delta(\sigma-\sigma') && (6.113)
\end{aligned}$$

The quantization of a multivector valued form, containing several operators $\hat{\boldsymbol{S}}$ at the same worldvolume-point, however, leads to powers of delta functions with the same argument when acting on some wave functional. This is the usual problem in quantum field theory and requires a model dependent regularization and renormalization. We will stay model independent here and therefore will not treat the quantum case for a present worldvolume coordinate σ. Nevertheless it is instructive to study it for absent σ, but keeping $\boldsymbol{\theta}$ and considering "worldline-superfields" of the form

$$\begin{aligned}
\Phi^m(\boldsymbol{\theta}) &= x^m + \boldsymbol{\theta}\boldsymbol{c}^m, \qquad \mathbf{d}\Phi^m(\boldsymbol{\theta}) = \boldsymbol{c}^m && (6.114)\\
\boldsymbol{S}_m(\boldsymbol{\theta}) &= \boldsymbol{b}_m + \boldsymbol{\theta}p_m, \qquad \mathbf{d}\boldsymbol{S}_m(\boldsymbol{\theta}) = p_m && (6.115)
\end{aligned}$$

Quantum operator and commutator simplify to

$$\begin{aligned}
\hat{\boldsymbol{S}}_m(\boldsymbol{\theta}) &\equiv \frac{\hbar}{i}\frac{\delta}{\delta\Phi^m(\boldsymbol{\theta})} = \frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{c}^m} + \boldsymbol{\theta}\frac{\hbar}{i}\frac{\partial}{\partial x^m} && (6.116)\\
\Rightarrow \left[\hat{\boldsymbol{S}}_m(\boldsymbol{\theta}), \hat{\Phi}^n(\boldsymbol{\theta}')\right] &= \frac{\hbar}{i}\delta_m^n\left(\boldsymbol{\theta}-\boldsymbol{\theta}'\right) && (6.117)\\
\left[\hat{\boldsymbol{S}}_m(\boldsymbol{\theta}), \widehat{\mathbf{d}\Phi}^n(\boldsymbol{\theta}')\right] &= \frac{\hbar}{i}\delta_m^n && (6.118)
\end{aligned}$$

In contrast to σ, products of $\boldsymbol{\theta}$-delta functions are no problem.

The important relation $K(\boldsymbol{\theta}) = K + \boldsymbol{\theta}\mathbf{d}K$ (6.95) can be extended to the quantum case as seen when acting on some r-form.

$$\imath_{K(k,k')}\rho^{(r)}(\boldsymbol{\theta}) \stackrel{(6.94)}{=} \imath_K\rho + \boldsymbol{\theta}\mathbf{d}(\imath_K\rho) = \qquad (6.119)$$

$$\begin{aligned}
&\stackrel{(6.35)}{=} && \imath_K\rho + \boldsymbol{\theta}\left(\imath_{\mathbf{d}K}\rho + (-)^{k-k'}\imath_K\mathbf{d}\rho\right) = && (6.120)\\
&= && \imath_K(\boldsymbol{\theta})\left(\rho(\boldsymbol{\theta})\right) && (6.121)\\
\text{with } \imath_K(\boldsymbol{\theta}) &\equiv && \imath_K + \boldsymbol{\theta}\left[\mathbf{d},\imath_K\right] && (6.122)
\end{aligned}$$

In that sense we have (remember $\hat{K} = \left(\frac{\hbar}{i}\right)^{k'}\imath_K$)

$$\begin{aligned}
\hat{K}^{(k,k')}(\boldsymbol{\theta}) &= \hat{K}^{(k,k')} + \boldsymbol{\theta}\widehat{\mathbf{d}K} && (6.123)\\
\text{with } \widehat{\mathbf{d}K} &\stackrel{(6.35)}{=} \left[\mathbf{d},\hat{K}\right] && (6.124)
\end{aligned}$$

where the explicit form of this quantized multivector valued form reads

$$\hat{K}^{(k,k')}(\boldsymbol{\theta}) \equiv \left(\frac{\hbar}{i}\right)^{k'} K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(\Phi(\boldsymbol{\theta})\right)\underbrace{\mathbf{d}\Phi^{m_1}(\boldsymbol{\theta})}_{\boldsymbol{c}^{m_1}}\ldots\mathbf{d}\Phi^{m_k}(\boldsymbol{\theta})\frac{\delta}{\delta\Phi^{n_1}(\boldsymbol{\theta})}\cdots\frac{\delta}{\delta\Phi^{n_{k'}}(\boldsymbol{\theta})} \qquad (6.125)$$

In the derivation of (6.122), $\imath_K$ and ρ both were evaluated at the same $\boldsymbol{\theta}$. Let us eventually consider the general case:

$$\begin{aligned}
\hat{K}^{(k,k')}(\boldsymbol{\theta}')\rho^{(r)}(\boldsymbol{\theta}) &= \left(\hat{K} + \boldsymbol{\theta}'\widehat{\mathbf{d}K}\right)(\rho + \boldsymbol{\theta}\mathbf{d}\rho) = && (6.126)\\
&= \hat{K}\rho + \boldsymbol{\theta}'\widehat{\mathbf{d}K}\rho + (-)^{k-k'}\boldsymbol{\theta}\hat{K}\mathbf{d}\rho + (-)^{k-k'}\boldsymbol{\theta}\boldsymbol{\theta}'\widehat{\mathbf{d}K}\mathbf{d}\rho = && (6.127)\\
&= \hat{K}\rho + \boldsymbol{\theta}\mathbf{d}\left(\hat{K}\rho\right) + (\boldsymbol{\theta}'-\boldsymbol{\theta})\left(\widehat{\mathbf{d}K}\rho + \boldsymbol{\theta}\mathbf{d}\left(\widehat{\mathbf{d}K}\rho\right)\right) && (6.128)
\end{aligned}$$

The relation between quantum operators acting on forms and the interior product therefore becomes modified in comparison to (6.14) and reads

$$\boxed{\hat{K}^{(k,k')}(\boldsymbol{\theta}')\rho^{(r)}(\boldsymbol{\theta}) = \left(\frac{\hbar}{i}\right)^{k'}\left(\imath_K\rho(\boldsymbol{\theta}) + (\boldsymbol{\theta}'-\boldsymbol{\theta})\underbrace{\imath_{\mathbf{d}K}\rho(\boldsymbol{\theta})}_{(-)^{k-k'}\boldsymbol{\mathcal{L}}_K\rho}\right)} \qquad (6.129)$$

Proposition 2 *For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$ as in (6.10), the following equations holds for the corresponding quantized worldline-superfields (6.125) $\hat{K}(\boldsymbol{\theta})$ and $\hat{L}(\boldsymbol{\theta})$:*

$$\begin{aligned}
[\hat{K}^{(k,k')}(\boldsymbol{\theta}'),\hat{L}^{(l,l')}(\boldsymbol{\theta})] &= \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^p\left(\underbrace{\partial_{\boldsymbol{\theta}}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')}_{=1}\widehat{[K,L]^{\Delta}_{(p)}}(\boldsymbol{\theta}) + \delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\widehat{[\mathbf{d}K,L]^{\Delta}_{(p)}}(\boldsymbol{\theta})\right) && (6.130)\\
[\hat{K}^{(k,k')}(\boldsymbol{\theta}'),\hat{L}^{(l,l')}(\boldsymbol{\theta})]\rho(\tilde{\boldsymbol{\theta}}) &= \\
= \left(\frac{\hbar}{i}\right)^{k'+l'}&\left(\imath_{[K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}}) + \delta(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})\imath_{\mathbf{d}[K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}}) + \right.\\
&\left. +\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\left(\imath_{[\mathbf{d}K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}}) + \delta(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})\imath_{\mathbf{d}[\mathbf{d}K,L]^{\Delta}}\rho^{(r)}(\tilde{\boldsymbol{\theta}})\right)\right) && (6.131)
\end{aligned}$$

Again the algebraic bracket (C.44) comes with the derivative of the delta function while the derived bracket (6.47) comes with the plain delta functions. But this time the algebraic bracket is not only the big bracket $[\,,\,]^{\Delta}_{(1)}$, but the full one.

Proof Let us just plug in (6.123) into the lefthand side:

$$\begin{aligned}
[\hat{K}(\boldsymbol{\theta}'),\hat{L}(\boldsymbol{\theta})] &= [\hat{K} + \boldsymbol{\theta}'\widehat{\mathbf{d}K}\,,\,\hat{L} + \boldsymbol{\theta}\widehat{\mathbf{d}L}] = && (6.132)\\
&= [\hat{K},\hat{L}] + \boldsymbol{\theta}'[\widehat{\mathbf{d}K}\,,\,\hat{L}] + (-)^{k-k'}\boldsymbol{\theta}[\hat{K}\,,\,\widehat{\mathbf{d}L}] - (-)^{k-k'}\boldsymbol{\theta}'\boldsymbol{\theta}[\widehat{\mathbf{d}K}\,,\,\widehat{\mathbf{d}L}] = && (6.133)\\
&\stackrel{(6.124)}{=} [\hat{K},\hat{L}] + \boldsymbol{\theta}\left[\mathbf{d},[\hat{K}\,,\,\hat{L}]\right] + (\boldsymbol{\theta}'-\boldsymbol{\theta})\left([\widehat{\mathbf{d}K}\,,\,\hat{L}] + \boldsymbol{\theta}\left[\mathbf{d},[\widehat{\mathbf{d}K}\,,\,\hat{L}]\right]\right) = && (6.134)\\
&= [\hat{K},\hat{L}](\boldsymbol{\theta}) + (\boldsymbol{\theta}'-\boldsymbol{\theta})[\widehat{\mathbf{d}K}\,,\,\hat{L}] && (6.135)
\end{aligned}$$

Remember now the algebraic bracket (C.43)

$$\left[\imath_{K^{(k,k')}},\imath_{L^{(l,,l')}}\right] = \imath_{[K,L]^{\Delta}} = \sum_{p\geq 1}\imath_{[K,L]^{\Delta}_{(p)}} \qquad (6.136)$$

$$\text{with } [K,L]^{\Delta}_{(p)} \equiv \imath^{(p)}_K L - (-)^{(k-k')(l-l')}\imath^{(p)}_L K \tag{6.137}$$

or likewise written in terms of $\hat{K}$ and $\hat{L}$

$$[\hat{K}^{(k,k')},\hat{L}^{(l,l')}] = \sum_{p\geq 1}\left(\frac{\hbar}{i}\right)^p \widehat{[K,L]^{\Delta}_{(p)}} \tag{6.25=\textbf{6}.138}$$

Due to (6.45) we have exactly the same equation for $[\widehat{\mathbf{d}K}, \hat{L}]$. Plugging this back into (6.135) completes the proof of (6.130). The second equation in the proposition is just a simple rewriting, when acting on a form, which enables to combine the p-th terms of algebraic and derived bracket to the complete ones. □

6.5 Analogy for the antibracket

In the previous subsection the target space exterior derivative $\mathbf{d}$ (realized in the σ-model phase-space by $\mathbf{s}$) was induced by the the derivative $\partial_{\boldsymbol{\theta}}$ with respect to the anticommuting coordinate. But thinking of the pullback of forms in the target space to worldvolume-forms, $\mathbf{d}$ can of course also be induced to some extend by the derivative with respect to the bosonic worldvolume coordinates σ^μ (including the time, because we are in the Lagrangian formalism now) or better by the worldvolume exterior derivative $\boldsymbol{d}^{\mathrm{w}}$. To this end, however, we have to make a different identification of the basis elements in tangent- and cotangent-space of the target space with the fields on the worldvolume than before, namely[12]

$$\mathbf{d}x^m \;\to\; \boldsymbol{d}^{\mathrm{w}}x^m(\sigma) = \boldsymbol{d}^{\mathrm{w}}\sigma^\mu\partial_\mu x^m(\sigma), \qquad \boldsymbol{\partial}_m \to \boldsymbol{x}^+_m(\sigma) \tag{\textbf{6}.139}$$

where $\boldsymbol{x}^+_m$ is the antifield of x^m, i.e. the conjugate field to x^m with respect to the antibracket[13]. Let us rename

$$\boldsymbol{\theta}^\mu \equiv \boldsymbol{d}^{\mathrm{w}}\sigma^\mu \tag{\textbf{6}.140}$$

For a target space r-form

$$\rho^{(r)}(x^m,\mathbf{d}x^m) \equiv \rho_{m_1\ldots m_r}(x)\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_r} \tag{\textbf{6}.141}$$

we define (in analogy to (6.85), but indicating that we allow in the beginning only a variation in σ)

$$\rho^{(r)}_{\boldsymbol{\theta}}(\sigma) \equiv \rho^{(r)}(x^m(\sigma),\boldsymbol{d}^{\mathrm{w}}x^m(\sigma)) = \rho_{m_1\ldots m_r}(x(\sigma))\boldsymbol{d}^{\mathrm{w}}x^{m_1}(\sigma)\cdots\boldsymbol{d}^{\mathrm{w}}x^{m_r}(\sigma) \tag{\textbf{6}.142}$$

Attention: this vanishes identically for $r > d_{\mathrm{w}}$ (worldvolume dimension).

The worldvolume exterior derivative then induces the target space exterior derivative in the following sense

$$\boldsymbol{d}^{\mathrm{w}}\rho^{(r)}_{\boldsymbol{\theta}}(\sigma) = (\mathbf{d}\rho^{(r)})_{\boldsymbol{\theta}}(\sigma) \tag{\textbf{6}.143}$$

Again both sides vanish identically for now $r+1 > d_{\mathrm{w}}$, which means that in this way one can calculate with target space fields of form degree not bigger than the worldvolume dimension. If we want to have the same relation for $K^{(k,k')}_{\boldsymbol{\theta}}(\sigma)$ (defined in the analogous way), we have to extend the identification in (**6**.139) by

$$p_m \;\to\; \boldsymbol{d}^{\mathrm{w}}\boldsymbol{x}^+_m(\sigma) \tag{\textbf{6}.144}$$

[12] This identification resembles the one in [71] with $\boldsymbol{\partial}_m \to p_m(z)$ and $\mathbf{d}x^m \to \partial x^m(z)$, or $\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_p} \to \epsilon^{\mu_1\cdots\mu_p}\partial_{\mu_1}x^{m_1}(\sigma)\cdots\partial_{\mu_p}x^{m_p}(\sigma)$ in [73]. It is observed in [71] that the Poisson bracket induces the Dorfman bracket between sums of vectors and 1-forms (in generalized geometry) and in [73] more generally that the Poisson-bracket for the p-brane induces the corresponding bracket between sums of vectors and p-forms (which is called, Vinogradov bracket in [73]). As ∂x^m and p_m are commuting phase space variables, higher rank tensors would automatically be symmetrized (only volume forms, i.e. p-forms on a p-brane, can be implemented, using the epsilon-tensor). Symmetrized tensors and brackets inbetween (e.g. the Schouten bracket for symmetric multivectors) make sense and one could transfer the present analysis to this setting, but in general a natural exterior derivative is missing. Therefore the analysis for the above identifications is done in the antifield-formalism. The appearing derived brackets will also contain the Dorfman bracket and the corresponding bracket for sums of vectors and p-forms and in that sense the present approach is a generalization of the observations above. ◇

[13] The antibracket looks similar to the Poisson-bracket, but their conjugate fields have opposite parity, which leads to a different symmetry (namely that of a Lie-bracket of degree +1 (or -1), i.e. the one in a Gerstenhaber algebra or Schouten-algebra, see footnote 1 of Appendix C)

$$\begin{aligned}
(A,B) &\equiv \int d\tilde{\sigma}^{d_{\mathrm{w}}}\;\Big(\delta A/\boldsymbol{x}^+_k(\tilde{\sigma})\frac{\delta}{\delta x^k(\tilde{\sigma})}B - \delta A/\delta x^k(\tilde{\sigma})\frac{\delta}{\delta\boldsymbol{x}^+_k(\tilde{\sigma})}B\Big) = \\
&= \int d\tilde{\sigma}^{d_{\mathrm{w}}}\;\Big(\delta A/\boldsymbol{x}^+_k(\tilde{\sigma})\frac{\delta}{\delta x^k(\tilde{\sigma})}B - (-)^{(A+1)(B+1)}\delta B/\boldsymbol{x}^+_k(\tilde{\sigma})\frac{\delta}{\delta x^k(\tilde{\sigma})}A\Big) \\
(A,B) &= -(-)^{(A+1)(B+1)}\,(B,A) \\
(\boldsymbol{x}^+_m(\sigma),B) &= \frac{\delta}{\delta x^m(\sigma)}B = -\,(B,\boldsymbol{x}^+_m(\sigma)) \\
(x^m(\sigma),B) &= -\frac{\delta}{\delta\boldsymbol{x}^+_m(\sigma)}B = (-)^B\,(B,x^m(\sigma)) \qquad \diamond
\end{aligned}$$

and get

$$\boldsymbol{d}^{\mathrm{w}} K_{\boldsymbol{\theta}}^{(k,k')}(\sigma) \quad = \quad (\mathbf{d}K^{(k,k')})_{\boldsymbol{\theta}}(\sigma) \tag{6.145}$$

with

$$K_{\boldsymbol{\theta}}^{(k,k')}(\sigma) \quad \equiv \quad K^{(k,k')}\left(x^m(\sigma), \boldsymbol{d}^{\mathrm{w}} x^m(\sigma), \boldsymbol{x}_m^+(\sigma)\right) \tag{6.146}$$

$$(\mathbf{d}K^{(k,k')})_{\boldsymbol{\theta}}(\sigma) \quad \equiv \quad \mathbf{d}K^{(k,k')}\left(x^m(\sigma), \boldsymbol{d}^{\mathrm{w}} x^m(\sigma), \boldsymbol{x}_m^+(\sigma), \boldsymbol{d}^{\mathrm{w}} \boldsymbol{x}_m^+(\sigma)\right) \tag{6.147}$$

The analysis is thus very similar to that of the previous section.

Proposition 3a *For all multivector valued forms $K^{(k,k')}$, $L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$, the following equation holds for the corresponding sigma-model realizations (6.146,6.147)*

$$\boxed{\begin{aligned}(K_{\boldsymbol{\theta}}(\sigma'), L_{\boldsymbol{\theta}}(\sigma)) = \big(\underbrace{[K,_{\mathbf{d}} L]^{\Delta}_{(1)}}\big)_{\boldsymbol{\theta}}(\sigma)\delta^{d_w}(\sigma-\sigma') - (-)^{k-k'}\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_w}(\sigma-\sigma')\big([K,L]^{\Delta}_{(1)}\big)_{\boldsymbol{\theta}}(\sigma)\\ -(-)^{k-k'}[\mathbf{d}K, L]^{\Delta}_{(1)}\end{aligned}} \tag{6.148}$$

Proof The proof is very similar to that one of proposition 3b (6.168) and is therefore omitted at this place. □

Conjugate Superfields With $\boldsymbol{\theta}^{\mu} = \boldsymbol{d}^{\mathrm{w}}\sigma^{\mu}$ we have introduced anticommuting coordinates and it would be nice to extend the anti-bracket of the fields x^m and $\boldsymbol{x}_m^+$ to a super-antibracket of conjugate superfields. Remember, in the previous subsection we had the superfields $\Phi^m = x^m + \boldsymbol{\theta}\boldsymbol{c}^m$ and its conjugate $\boldsymbol{S}_m$. There we had one $\boldsymbol{\theta}$ and two component fields. In general the number of component fields has to exceed the worldvolume dimension d_{w} (the number of $\boldsymbol{\theta}$'s) by one, s.th. we have to introduce a lot of new fields to realize conjugate superfields. But before, let us define the fermionic integration measure $\mu(\boldsymbol{\theta})$ via

$$\int \mu(\boldsymbol{\theta}) f(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}^{d_{\mathrm{w}}}} \cdots \frac{\partial}{\partial \boldsymbol{\theta}^1} f(\boldsymbol{\theta}) = \frac{1}{d_{\mathrm{w}}!}\epsilon^{\mu_1 \ldots \mu_{d_{\mathrm{w}}}} \frac{\partial}{\partial \boldsymbol{\theta}^{\mu_1}} \cdots \frac{\partial}{\partial \boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}}} f(\boldsymbol{\theta}) \tag{6.149}$$

The corresponding d_{w}-dimensional δ-function is

$$\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}' - \boldsymbol{\theta}) \quad \equiv \quad (\boldsymbol{\theta}'^1 - \boldsymbol{\theta}^1)\cdots(\boldsymbol{\theta}'^{d_{\mathrm{w}}} - \boldsymbol{\theta}^{d_{\mathrm{w}}}) = \tag{6.150}$$

$$= \quad \frac{1}{d_{\mathrm{w}}!}\epsilon_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}}(\boldsymbol{\theta}'^{\mu_1} - \boldsymbol{\theta}^{\mu_1})\cdots(\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}}} - \boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}}) = \tag{6.151}$$

$$= \quad \sum_{k=0}^{d_{\mathrm{w}}} \frac{1}{k!(d_{\mathrm{w}}-k)!}\epsilon_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}} \boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_k}\boldsymbol{\theta}^{\mu_{k+1}}\cdots\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}} \tag{6.152}$$

$$\int \mu(\boldsymbol{\theta}')\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}' - \boldsymbol{\theta}) f(\boldsymbol{\theta}') \quad = \quad f(\boldsymbol{\theta}) \tag{6.153}$$

$$\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}' - \boldsymbol{\theta}) \quad = \quad (-)^{d_{\mathrm{w}}}\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta} - \boldsymbol{\theta}') \tag{6.154}$$

For the two conjugate superfields, call them Φ^m and $\boldsymbol{\Phi}_m^+$, we want to have the canonical super anti bracket

$$\left(\boldsymbol{\Phi}_m^+(\sigma', \boldsymbol{\theta}'), \Phi^n(\sigma, \boldsymbol{\theta})\right) = \delta_m^n \delta^{d_{\mathrm{w}}}(\sigma' - \sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}' - \boldsymbol{\theta}) = -\left(\Phi^n(\sigma, \boldsymbol{\theta}), \boldsymbol{\Phi}_m^+(\sigma', \boldsymbol{\theta}')\right) \tag{6.155}$$

From the above considerations about the fermionic delta function it is now clear, how these superfields can be defined (they are known as **de Rham superfields**, because of the interpretation of $\boldsymbol{\theta}^{\mu}$ as $\boldsymbol{d}^{\mathrm{w}}\sigma^{\mu}$; see e.g. [97, 91]):

$$\Phi^m(\sigma, \boldsymbol{\theta}) \quad \equiv \quad x^m(\sigma) + \boldsymbol{x}^m_{\mu_{d_{\mathrm{w}}}}(\sigma)\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}} + x^m_{\mu_{d_{\mathrm{w}}-1}\mu_{d_{\mathrm{w}}}}(\sigma)\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}-1}}\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}} + \ldots + x^m_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}}(\sigma)\boldsymbol{\theta}^{\mu_1}\cdots\boldsymbol{\theta}^{\mu_{d_{\mathrm{w}}}} \tag{6.156}$$

$$\begin{aligned}\boldsymbol{\Phi}_m^+(\sigma', \boldsymbol{\theta}') \quad \equiv \quad & \frac{1}{d_{\mathrm{w}}!}\epsilon_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}}}\boldsymbol{x}_m^+(\sigma') + \frac{1}{(d_{\mathrm{w}}-1)!1!}\epsilon_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}-1}}x_m^{+\,\mu_{d_{\mathrm{w}}}}(\sigma') + \\ & + \frac{1}{(d_{\mathrm{w}}-2)!2!}\epsilon_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}}\boldsymbol{\theta}'^{\mu_1}\cdots\boldsymbol{\theta}'^{\mu_{d_{\mathrm{w}}-2}}\boldsymbol{x}_m^{+\,\mu_{d_{\mathrm{w}}-1}\mu_{d_{\mathrm{w}}}}(\sigma') + \ldots + \frac{1}{d_{\mathrm{w}}!}\epsilon_{\mu_1 \ldots \mu_{d_{\mathrm{w}}}}\boldsymbol{x}_m^{+\,\mu_1 \ldots \mu_{d_{\mathrm{w}}}}(\sigma')\end{aligned} \tag{6.157}$$

The component fields with the matching number of worldsheet indices are conjugate to each other, e.g.

$$\left(\boldsymbol{x}_m^{+\,\mu_1\mu_2}(\sigma'), x^n_{\nu_1\nu_2}(\sigma)\right) \quad = \quad \delta_m^n \delta^{\mu_1\mu_2}_{\nu_1\nu_2}\delta^{d_{\mathrm{w}}}(\sigma - \sigma') \tag{6.158}$$

For the notation with boldface symbols for anticommuting variables, the worldvolume was assumed to be even-dimensional. In this case, one can analytically continue the coordinate form of multivector-valued forms of the form

$$K^{(k,k')}(x,\mathbf{d}x,\boldsymbol{\partial}) \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\mathbf{d}x^{m_1}\wedge\cdots\wedge\mathbf{d}x^{m_k}\wedge\boldsymbol{\partial}_{n_1}\wedge\cdots\wedge\boldsymbol{\partial}_{n_{k'}} \qquad (\mathbf{6}.159)$$

to functions of superfields (in odd worldvolume dimension one would get a symmetrization of the multivector-indices) and redefine $K(\sigma,\boldsymbol{\theta})$ of (6.85) to

$$\begin{aligned} K^{(k,k')}(\sigma,\boldsymbol{\theta}) &\equiv K^{(k,k')}\left(\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta})\right) = & (\mathbf{6}.160)\\ &= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}(\Phi)\boldsymbol{d}^{\mathrm{w}}\Phi^{m_1}\cdots\boldsymbol{d}^{\mathrm{w}}\Phi^{m_k}\boldsymbol{\Phi}^+_{n_1}\cdots\boldsymbol{\Phi}^+_{n_{k'}} & (\mathbf{6}.161)\end{aligned}$$

All other geometric quantities have to be understood in this new sense now:

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) \equiv T^{(t,t',t'')}\left(\Phi(\sigma,\boldsymbol{\theta}),\mathbf{s}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta})\right) \quad \text{(see (6.28))} \qquad (\mathbf{6}.162)$$

To stay with the examples used in (6.84)-(6.91):

$$\begin{aligned} \text{e.g. } \mathbf{d}K(\sigma,\boldsymbol{\theta}) &\equiv \mathbf{d}K\left(\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta})\right) \quad \text{(compare (6.34))} & (\mathbf{6}.163)\\ \text{or } \boldsymbol{o}(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{o}\left(\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta})\right) = \boldsymbol{d}^{\mathrm{w}}\Phi^m(\sigma,\boldsymbol{\theta})\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta}) \quad (\text{compare } \boldsymbol{o}=\boldsymbol{c}^m p_m) & (\mathbf{6}.164)\\ \left[K^{(k,k')},_{\mathbf{d}}L^{(l,l')}\right]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) &\equiv \left[K^{(k,k')},L^{(l,l')}\right]^{(\Delta)}_{(1)}\left(\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta}),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+(\sigma,\boldsymbol{\theta})\right) & (\mathbf{6}.165)\\ \mathbf{d}x^m(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{d}^{\mathrm{w}}\Phi^m(\sigma,\boldsymbol{\theta}) & (\mathbf{6}.166)\\ (\mathbf{d}\boldsymbol{\partial}_m)(\sigma,\boldsymbol{\theta}) \equiv (\mathbf{d}\boldsymbol{b}_m)(\sigma,\boldsymbol{\theta}) &\equiv \boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta}) & (\mathbf{6}.167)\end{aligned}$$

Note that the former relation $K(\sigma,\boldsymbol{\theta}) = K(\sigma)+\boldsymbol{\theta}\mathbf{d}K(\sigma)$ does NOT hold any longer with those new definitions! Nevertheless we get a very similar statement as compared to propositions 2 on page 128:

Proposition 3b *For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$, the following equation holds for even worldvolume-dimension d_w for the corresponding superfields (6.160):*

$$\boxed{\begin{aligned}(K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})) = \delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_w}(\boldsymbol{\theta}'-\boldsymbol{\theta})\underbrace{[K,_{\mathbf{d}}L]^{\Delta}_{(1)}}_{-(-)^{k-k'}[\mathbf{d}K,L]^{\Delta}_{(1)}}(\sigma,\boldsymbol{\theta}) - (-)^{k-k'}\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma-\sigma')\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}'-\boldsymbol{\theta})\,[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})\end{aligned}} \qquad (\mathbf{6}.168)$$

where $[K,L]^{\Delta}_{(1)}$ is the big bracket (6.23) and $[K,_{\mathbf{d}}L]^{\Delta}_{(1)}$ is the derived bracket of the big bracket (6.52).

Note that σ and $\boldsymbol{\theta}$ have switched their roles compared to the previous subsection (6.97), where the algebraic bracket came together with the derivative with respect to $\boldsymbol{\theta}$ of the delta-functions, while now it comes along with ∂_μ of the delta-functions.

Proof Let us use again the second idea in the proof of proposition 2, i.e. first collect the terms with derivatives of the delta function, only to show that one gets the algebraic bracket, and after that argue that the term with plain delta functions is its derived bracket. In doing this, however, we will need to prove an extension of the above proposition to objects like $\mathbf{d}K$ (or more general an object $T^{(t,t',t'')}$ as in (6.28)) that contain the basis element p_m, which is then replaced by $\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m$ as e.g. in (6.163).
(i) The antibracket between two such objects T and $\tilde{T}$ gets contributions to the derivative of the delta-function only from the antibrackets between $\boldsymbol{d}^{\mathrm{w}}\Phi^m$ and $\boldsymbol{\Phi}^+_m$ and between Φ^m and $\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m$ (compare (6.155))

$$\begin{aligned}\left(\boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta}'),\boldsymbol{d}^{\mathrm{w}}\Phi^n(\sigma,\boldsymbol{\theta})\right) &= \delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}'-\boldsymbol{\theta}) & (\mathbf{6}.169)\\ \left(\boldsymbol{d}^{\mathrm{w}}\Phi^n(\sigma',\boldsymbol{\theta}'),\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right) &= \delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}'-\boldsymbol{\theta}) & (\mathbf{6}.170)\\ \left(\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma',\boldsymbol{\theta}'),\Phi^n(\sigma,\boldsymbol{\theta})\right) &= -\delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}'-\boldsymbol{\theta}) & (\mathbf{6}.171)\\ \left(\Phi^n(\sigma',\boldsymbol{\theta}'),\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right) &= -\boldsymbol{\theta}^{\mu}\left(\Phi^n(\sigma',\boldsymbol{\theta}'),\partial_{\mu}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right) = \delta^n_m\boldsymbol{\theta}^{\mu}\partial_{\mu}\delta^{d_{\mathrm{w}}}(\sigma'-\sigma)\delta^{d_{\mathrm{w}}}(\boldsymbol{\theta}'-\boldsymbol{\theta}) & (\mathbf{6}.172)\end{aligned}$$

The last case is the only one where we had to take care of an extra sign stemming from $\boldsymbol{\theta}$ jumping over the graded comma. Comparing this to (6.5), where we had

$$\begin{aligned}\{\boldsymbol{b}_m,\boldsymbol{c}^n\} &= \delta^n_m & (\mathbf{6}.173)\\ \{\boldsymbol{c}^n,\boldsymbol{b}_m\} &= \delta^n_m & (\mathbf{6}.174)\\ \{p_m,x^n\} &= \delta^n_m & (\mathbf{6}.175)\\ \{x^n,p_m\} &= -\delta^n_m & (\mathbf{6}.176)\end{aligned}$$

one recognizes that the only difference is an overall odd factor $\boldsymbol{\theta}^\mu \partial_\mu \delta^{d_w}(\sigma' - \sigma)\delta^{d_w}(\boldsymbol{\theta}' - \boldsymbol{\theta})$ (the delta-function for $\boldsymbol{\theta}$ is an even object for even worldvolume dimension d_w) and an additional minus sign for the lower two lines, but the corresponding indices just get contracted like for the Poisson bracket. After such a bracket of basis elements has been calculated (which happens just between the remaining factors of T (at σ') on the left and the remaining factors of $\tilde{T}$ (at σ) on the right) this overall odd factor has to be pulled to the very left which gives an additional factor of $(-)^{t-t'}$ (in the notation of (6.28)) plus an additional minus sign for the upper two lines which compensates the relative minus sign of before and we get just an overall factor of $-(-)^{t-t'}\boldsymbol{\theta}^\mu \partial_\mu \delta^{d_w}(\sigma' - \sigma)\delta^{d_w}(\boldsymbol{\theta}' - \boldsymbol{\theta})$ in all cases at the very left as compared to the Poisson-bracket. The remaining terms are still partly at σ and partly at σ', but using

$$A(\sigma)B(\sigma')\partial_\mu \delta(\sigma - \sigma') = A(\sigma)\partial_\mu B(\sigma)\delta(\sigma - \sigma') + A(\sigma)B(\sigma)\partial_\mu \delta(\sigma - \sigma') \quad \forall A, B \qquad (\mathbf{6}.177)$$

we can take all remaining factors in $T(\sigma', \boldsymbol{\theta}')$ at σ, while $\boldsymbol{\theta}'$ is set to $\boldsymbol{\theta}$ anyway by the δ-function. We have thus verified one of the coefficients of the complete antibracket:

$$\begin{aligned}(T(\sigma', \boldsymbol{\theta}'), \tilde{T}(\sigma, \boldsymbol{\theta})) &= -(-)^{t-t'}\boldsymbol{\theta}^\mu \partial_\mu \delta^{d_w}(\sigma - \sigma')\delta^{d_w}(\boldsymbol{\theta}' - \boldsymbol{\theta}) \left[T, \tilde{T}\right]^{\Delta}_{(1)} (\sigma, \boldsymbol{\theta}) + \\ &\quad + \delta^{d_w}(\sigma - \sigma')\delta^{d_w}(\boldsymbol{\theta}' - \boldsymbol{\theta})A(\sigma, \boldsymbol{\theta}) \end{aligned} \qquad (\mathbf{6}.178)$$

with $A(\sigma, \boldsymbol{\theta})$ yet to be determined.

(ii) It remains to show that $A(\sigma, \boldsymbol{\theta})$ is a derived expression of $\left[T, \tilde{T}\right]^{\Delta}_{(1)}$. A hint to this fact is already given in (**6**.177), but this is not enough, as there is also a contribution from the $(\Phi^m, \boldsymbol{\Phi}^+_n)$-brackets. In order to get a precise relation between $A(\sigma, \boldsymbol{\theta})$ and $\left[T, \tilde{T}\right]^{\Delta}_{(1)} (\sigma, \boldsymbol{\theta})$, let us see how one can extract them from the complete antibracket. In order to hit the delta functions with the integration, it is enough to integrate over the patch $U(\sigma)$ containing the point which is parametrized by σ^μ. The last term in (**6**.178) is the only one contributing when integrating over σ' and $\boldsymbol{\theta}$

$$A(\sigma, \boldsymbol{\theta}) = \int_{U(\sigma)} \mathbf{d}^{d_w}\sigma' \int \mu(\boldsymbol{\theta}') \quad (T(\sigma', \boldsymbol{\theta}'), \tilde{T}(\sigma, \boldsymbol{\theta})) \qquad (\mathbf{6}.179)$$

That the first term on the righthand side of (**6**.178) does not contribute is not obvious as $U(\sigma)$ might have a boundary. However, for this term one ends up integrating a d_w-dimensional delta-function over a boundary of dimension not higher than $d_w - 1$, so that one is left with an at least one-dimensional delta-function on the boundary which vanishes as the boundary of the open neighbourhood $U(\sigma)$ of σ of course nowhere hits σ.

Extracting the algebraic bracket $\left[T, \tilde{T}\right]^{\Delta}_{(1)}$ is a bit more tricky. One can do it via

$$\text{for any fixed index } \lambda : \left[T, \tilde{T}\right]^{\Delta}_{(1)} (\sigma, \boldsymbol{\theta}) = -(-)^{t-t'} \int_{U(\sigma)} \mathbf{d}^{d_w}\sigma' \int \mu(\boldsymbol{\theta}') \quad \left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}} - 1\right) \frac{\partial}{\partial \boldsymbol{\theta}^\lambda}(T(\sigma', \boldsymbol{\theta}'), \tilde{T}(\sigma, \boldsymbol{\theta})) \qquad (\mathbf{6}.180)$$

The boundary term proportional to $\left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}} - 1\right)\delta^{d_w}(\sigma - \sigma')$ appearing above on the righthand side after partial integration vanishes as σ' in the prefactor is set to σ via the delta function.

The claim is now that $A(\sigma, \boldsymbol{\theta}) = -(-)^{t-t'}\left[\mathbf{d}T, \tilde{T}\right]^{\Delta}_{(1)} (\sigma, \boldsymbol{\theta})$. So let us calculate the righthand side via (**6**.180):

$$\begin{aligned}\left[\mathbf{d}T, \tilde{T}\right]^{\Delta}_{(1)} (\sigma, \boldsymbol{\theta}) &= -(-)^{t+1-t'} \int_{U(\sigma)} \mathbf{d}^{d_w}\sigma' \int \mu(\boldsymbol{\theta}') \quad \left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}} - 1\right) \frac{\partial}{\partial \boldsymbol{\theta}^\lambda}(\mathbf{d}T(\sigma', \boldsymbol{\theta}'), \tilde{T}(\sigma, \boldsymbol{\theta})) = \qquad (\mathbf{6}.181) \\ &= -(-)^{t+1-t'} \int \mathbf{d}^{d_w}\sigma' \int \mu(\boldsymbol{\theta}') \quad \left(\frac{e^{\sigma'\lambda}}{e^{\sigma\lambda}} - 1\right) \frac{\partial}{\partial \boldsymbol{\theta}^\lambda}\boldsymbol{\theta}'^\mu \partial'_\mu(T(\sigma', \boldsymbol{\theta}'), \tilde{T}(\sigma, \boldsymbol{\theta})) \qquad (\mathbf{6}.182)\end{aligned}$$

$(T, \tilde{T})$ contains in both terms a plain δ-function for the fermionic variables $\boldsymbol{\theta}$, so that we can replace $\boldsymbol{\theta}'$ by $\boldsymbol{\theta}$. Integration by parts of ∂'_μ (where possible boundary terms again do not contribute because of the vanishing of the delta function and its derivative on the boundary) delivers the desired result

$$\left[\mathbf{d}T, \tilde{T}\right]^{\Delta}_{(1)} (\sigma, \boldsymbol{\theta}) = -(-)^{t-t'} \int \mathbf{d}^{d_w}\sigma' \int \mu(\boldsymbol{\theta}') \quad (T(\sigma', \boldsymbol{\theta}'), \tilde{T}(\sigma, \boldsymbol{\theta})) = -(-)^{t-t'}A(\sigma, \boldsymbol{\theta}) \qquad (\mathbf{6}.183)$$

This completes the proof of proposition 3b. □

Chapter 7

Applications in string theory or 2d CFT

In the previous section the dimension of the worldvolume was arbitrary or even dimensional. The appearance of derived brackets (including e.g. the Dorfman bracket) is thus not a special feature of a 2-dimensional sigma-model like string theory. There are, however, special features in string theory. Currents in string theory (which have conformal weight one) naturally are sums of 1-forms and vectors, if one takes the identification $\partial_1 x^m(\sigma) \leftrightarrow \mathbf{d}x^m$ and $p_m(\sigma) \leftrightarrow \boldsymbol{\partial}_m$, as in [71] (see footnote 12), e.g. $\partial x^m = \partial_1 x^m - \partial_0 x^m \hat{=} \mathbf{d}x^m - \eta^{mn}\boldsymbol{\partial}_n$. This is closely related to the identification in our previous section in the antifield formalism. In addition, only in two dimensions a single $\boldsymbol{\theta}$ can be interpreted as a worldsheet Weyl spinor (in 1 dimension it can be seen as a Dirac-spinor, but in higher dimensions the interpretation of $\boldsymbol{\theta}$ as worldvolume spinor breaks down). As we ended the last section with the antifield formalism, which therefore is perhaps still more present, let us start this section in the reversed order, beginning with the application in the antifield formalism.

7.1 Poisson sigma-model and Zucchini's "Hitchin sigma-model"

Remember for a moment the Poisson-σ-model [98, 97]. It is a two-dimensional sigma-model ($d_{\mathrm{w}} = 2$) of the form

$$S_0 = \int_\Sigma \boldsymbol{\eta}_m \boldsymbol{d}^{\mathrm{w}} x^m + \frac{1}{2} P^{mn}(x) \boldsymbol{\eta}_m \boldsymbol{\eta}_n \qquad (\mathbf{7}.1)$$

where $\boldsymbol{\eta}_m$ is a worldsheet one-form. This model is topological if and only if the Poisson-structure $P^{mn}(x)$ is integrable, i.e. the Schouten-bracket of P with itself vanishes

$$S_0 \text{ topological} \quad \Longleftrightarrow \quad [P \boldsymbol{,} P] = 0 \qquad (\mathbf{7}.2)$$

It gives on the one hand a field theoretic implementation of Kontsevich's star product [97] and is on the other hand related to string theory via a topological limit (big antisymmetric part in the open string metric), which leads to the relation between string theory and noncommutative geometry.

The necessary ghost fields for the action can be introduced by extending x and η to de Rham superfields as in (**6**.156,**6**.157)

$$\Phi^m(\sigma, \boldsymbol{\theta}) \equiv x^m(\sigma) + \underbrace{\boldsymbol{x}_\mu^m(\sigma)}_{\epsilon_{\mu\nu}\boldsymbol{\eta}^{+\nu n}} \boldsymbol{\theta}^\mu + \underbrace{x^m_{\mu_1\mu_2}(\sigma)}_{-\frac{1}{2}\varepsilon_{\mu_1\mu_2}\beta^{+\,m}} \boldsymbol{\theta}^{\mu_1}\boldsymbol{\theta}^{\mu_2} \qquad (\mathbf{7}.3)$$

$$\boldsymbol{\Phi}^+_m(\sigma', \boldsymbol{\theta}') \equiv \underbrace{\frac{1}{2!}\epsilon_{\mu_1\mu_2}\boldsymbol{x}^{+\,\mu_1\mu_2}_m(\sigma')}_{\equiv \boldsymbol{\beta}_m(\sigma')} + \boldsymbol{\theta}'^{\mu_1} \underbrace{\epsilon_{\mu_1\mu_2} x^{+\,\mu_2}_m(\sigma')}_{\eta_{\mu_1 m}} + \frac{1}{2}\epsilon_{\mu_1\mu_2}\boldsymbol{\theta}'^{\mu_1}\boldsymbol{\theta}'^{\mu_2}\boldsymbol{x}^+_m(\sigma') \qquad (\mathbf{7}.4)$$

One can use Hodge-duality to rename some component fields as indicated. $\boldsymbol{\beta}_m$ is then the ghost field related to the gauge symmetry. The action including ghost fields and antifields simply reads

$$S = \int d^2\sigma \int \mu(\boldsymbol{\theta}) \quad \boldsymbol{\Phi}^+_m \boldsymbol{d}^{\mathrm{w}}\Phi^m + \frac{1}{2}P^{mn}(\Phi)\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_n \qquad (\mathbf{7}.5)$$

The expression under the integral corresponds to the tensor $-\delta_m{}^n \mathbf{d}x^m \wedge \boldsymbol{\partial}_n + \frac{1}{2}P^{mn}\boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n$ and the antibracket in the master-equation (S, S) implements the Schoutenbracket on P, which is a well known relation. Therefore we will concentrate on a second example, which is very similar, but less known.

Zucchini suggested in [91] a 2-dimensional sigma-model which is topological if a generalized complex structure in the target space is integrable (see subsection B.2 on page 149 and B.4 on page 153 to learn more about generalized complex structures). His model is of the form

$$S = \int d^2\sigma \int \mu(\boldsymbol{\theta}) \quad (\boldsymbol{\Phi}^+_m \boldsymbol{d}^{\mathrm{w}}\Phi^m +) \quad \frac{1}{2}P^{mn}(\Phi)\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_n - \frac{1}{2}Q_{mn}(\Phi)\boldsymbol{d}^{\mathrm{w}}\Phi^m\boldsymbol{d}^{\mathrm{w}}\Phi^n - J^n{}_m \boldsymbol{d}^{\mathrm{w}}\Phi^m\boldsymbol{\Phi}^+_n \qquad (\mathbf{7}.6)$$

where P^{mn}, Q_{mn} and $J^m{}_n$ are the building blocks of the generalized complex structure (B.22)

$$\mathcal{J}^M{}_N = \begin{pmatrix} J^m{}_n & P^{mn} \\ -Q_{mn} & -J^n{}_m \end{pmatrix} \qquad (\mathbf{7}.7)$$

The first term of (**7**.6) can be absorbed by a field redefinition as already observed in [92]. Ignoring thus the first term and using our notations of before, S can be rewritten as

$$S = \int d^2\sigma \int \mu(\boldsymbol{\theta}) \quad \frac{1}{2}\mathcal{J}(\Phi, \boldsymbol{d}^{\mathrm{w}}\Phi, \boldsymbol{\Phi}^+) \qquad (\mathbf{7}.8)$$

Calculating the master equation explicitely and collecting the terms which combine to the lengthy tensors for the integrability condition (see (B.60)-(B.63)) is quite cumbersome, so we can enjoy using instead proposition 3b on page 133. For a worldsheet without boundary its integrated version reads

$$\left(\int d^{d_{\mathrm{w}}}\sigma' \int \mu(\boldsymbol{\theta}')K(\sigma',\boldsymbol{\theta}'), \int \mathbf{d}^{d_{\mathrm{w}}}\sigma \int \mu(\boldsymbol{\theta})L(\sigma,\boldsymbol{\theta})\right) = \int d^{d_{\mathrm{w}}}\sigma \int \mu(\boldsymbol{\theta})\,[K,_{\mathbf{d}}L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \qquad (\mathbf{7}.9)$$

which leads to the relation

$$(S,S) = 0 \qquad \Longleftrightarrow \int d^2\sigma \int \mu(\boldsymbol{\theta})\,[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) = 0 \qquad (\mathbf{7}.10)$$

The derived bracket of the big bracket of $\mathcal{J}$ with itself contains already the generalized Nijenhuis tensor (see in the appendix in equation (**B**.81) and in the discussion around)

$$\begin{aligned}
[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} &= \mathcal{N}_{M_1M_2M_3}\mathbf{t}^{M_1}\mathbf{t}^{M_2}\mathbf{t}^{M_3} - 4\mathcal{J}^{JI}\mathcal{J}_{IM}\mathbf{t}^{M}p_J = & (\mathbf{7}.11)\\
&\overset{\mathcal{J}^2=-1}{=} \mathcal{N}_{M_1M_2M_3}\mathbf{t}^{M_1}\mathbf{t}^{M_2}\mathbf{t}^{M_3} + 4\boldsymbol{o} & (\mathbf{7}.12)\\
\mathbf{t}^M &= (\mathbf{d}x^m, \boldsymbol{\partial}_m), \qquad p_J = (p_j, 0) & (\mathbf{7}.13)\\
\boldsymbol{o}(\mathbf{d}x, p) &= \mathbf{d}x^m p_m & (\mathbf{7}.14)
\end{aligned}$$

For $\mathcal{J}^2 = -1$ the last term is proportional to the generator $\boldsymbol{o}$ (remember (6.8)). In (**7**.10), however, it appears with $\mathbf{d}x$ and p replaced by the superfields as in (**6**.164)

$$\boldsymbol{o}(\sigma,\boldsymbol{\theta}) = \boldsymbol{d}^{\mathrm{w}}\Phi^m(\sigma,\boldsymbol{\theta})\boldsymbol{d}^{\mathrm{w}}\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta}) = -\boldsymbol{d}^{\mathrm{w}}\left(\boldsymbol{d}^{\mathrm{w}}\Phi^m(\sigma,\boldsymbol{\theta})\boldsymbol{\Phi}^+_m(\sigma,\boldsymbol{\theta})\right) \qquad (\mathbf{7}.15)$$

which is a total worldsheet derivative and therefore drops during the integration. We are left with the generalized Nijenhuis tensor as a function of superfields

$$\begin{aligned}
\mathcal{N}(\sigma,\boldsymbol{\theta}) &= \mathcal{N}_{M_1M_2M_3}(\Phi)\underline{\mathbf{t}}^{M_1}\underline{\mathbf{t}}^{M_2}\underline{\mathbf{t}}^{M_3} & (\mathbf{7}.16)\\
\text{with } \underline{\mathbf{t}}^M &\equiv (\boldsymbol{d}^{\mathrm{w}}\Phi^m, \boldsymbol{\Phi}^+_m) & (\mathbf{7}.17)
\end{aligned}$$

Written in small indices

$$\begin{aligned}
\mathcal{N}(\sigma,\boldsymbol{\theta}) = {} & \mathcal{N}_{m_1m_2m_3}(\Phi)\underbrace{\boldsymbol{d}^{\mathrm{w}}\Phi^{m_1}\boldsymbol{d}^{\mathrm{w}}\Phi^{m_1}\boldsymbol{d}^{\mathrm{w}}\Phi^{m_1}}_{=0} + 3\mathcal{N}^n{}_{m_1m_2}(\Phi)\boldsymbol{\Phi}^+_n\boldsymbol{d}^{\mathrm{w}}\Phi^{m_1}\boldsymbol{d}^{\mathrm{w}}\Phi^{m_2} + \\
& + 3\mathcal{N}_n{}^{m_1m_2}(\Phi)\boldsymbol{d}^{\mathrm{w}}\Phi^n\boldsymbol{\Phi}^+_{m_1}\boldsymbol{\Phi}^+_{m_2} + \mathcal{N}^{m_1m_2m_3}(\Phi)\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_m\boldsymbol{\Phi}^+_m \qquad (\mathbf{7}.18)
\end{aligned}$$

One realizes that the first term vanishes identically (as mentioned in [91]) and only the remaining three tensors are required to vanish in order to satisfy (**7**.10).

7.2 Relation between a second worldsheet supercharge and generalized complex geometry

In [87] the relation between an extended worldsheet supersymmetry in string theory and the presence of an integrable generalized complex structure was explored. Zabzine clarified in [90] the relation in an model independent way in a Hamiltonian description. The structures appearing there are almost the same that we have discussed before although we have to modify the procedure a little bit due to the interpretation of $\boldsymbol{\theta}$ as a worldsheet spinor.

Consider a sigma-model with 2-dimensional worldvolume (worldsheet) with manifest $N = 1$ supersymmetry on the worldsheet. In the phase space there is only one σ-coordinate left. Let us denote the corresponding superfields, following loosely [90], by

$$\Phi^m(\sigma,\boldsymbol{\theta}) \equiv x^m(\sigma) + \boldsymbol{\theta}\boldsymbol{\lambda}^m(\sigma) \qquad (\mathbf{7}.19)$$

$$\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) \equiv \boldsymbol{\rho}_m(\sigma)+\boldsymbol{\theta}p_m(\sigma) \qquad (\mathbf{7}.20)$$

In comparison to section 6.3, there is a change of notation from $\boldsymbol{c}^m \to \boldsymbol{\lambda}^m$ and $\boldsymbol{b}_m \to \boldsymbol{\rho}_m$ as $\boldsymbol{b}$ and $\boldsymbol{c}$ suggest the interpretation as ghosts which is not true in this case, where $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ are worldsheet fermions. Introduce now, following Zabzine, the generator $\mathrm{Q}_{\boldsymbol{\theta}}$ of the **manifest SUSY** and the corresponding **covariant derivative** $\mathrm{D}_{\boldsymbol{\theta}}$

$$\mathrm{Q}_{\boldsymbol{\theta}} \equiv \partial_{\boldsymbol{\theta}}+\boldsymbol{\theta}\partial_\sigma \qquad (\mathbf{7}.21)$$
$$\mathrm{D}_{\boldsymbol{\theta}} \equiv \partial_{\boldsymbol{\theta}}-\boldsymbol{\theta}\partial_\sigma \qquad (\mathbf{7}.22)$$

with the SUSY algebra

$$[\mathrm{Q}_{\boldsymbol{\theta}},\mathrm{Q}_{\boldsymbol{\theta}}] = 2\partial_\sigma = -[\mathrm{D}_{\boldsymbol{\theta}},\mathrm{D}_{\boldsymbol{\theta}}] \qquad (\mathbf{7}.23)$$
$$[\mathrm{Q}_{\boldsymbol{\theta}},\mathrm{D}_{\boldsymbol{\theta}}] = 0 \qquad (\mathbf{7}.24)$$

$\mathrm{Q}_{\boldsymbol{\theta}}$ is the sum of two nilpotent differential operators, namely $\partial_{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}\partial_\sigma$. Acting on the Superfields Φ^m and $\boldsymbol{S}^m$, they induce the differentials $\mathbf{s}$ and $\tilde{\mathbf{s}}$ on the component fields, which are in turn generated via the Poisson bracket by phase space functions $\boldsymbol{\Omega}$ (the same as (6.69)) and $\tilde{\boldsymbol{\Omega}}$.

$$\boldsymbol{\Omega} \equiv \int d\sigma\, \boldsymbol{\lambda}^k p_k \qquad (\mathbf{7}.25)$$
$$\tilde{\boldsymbol{\Omega}} = -\int d\sigma\, \partial_\sigma x^k \boldsymbol{\rho}_k \qquad (\mathbf{7}.26)$$
$$\mathbf{s}x^m \equiv \{\boldsymbol{\Omega},x^m\} = \boldsymbol{\lambda}^m \leftrightarrow \mathbf{d}x^m, \quad \mathbf{s}\boldsymbol{\rho}_m \equiv \{\boldsymbol{\Omega},\boldsymbol{\rho}_m\} = p_m \leftrightarrow \mathbf{d}(\boldsymbol{\partial}_m), \qquad (\mathbf{7}.27)$$
$$\tilde{\mathbf{s}}\boldsymbol{\lambda}^m \equiv \left\{\tilde{\boldsymbol{\Omega}},\boldsymbol{\lambda}^m\right\} = -\partial_\sigma x^m, \quad \tilde{\mathbf{s}}p_k = -\partial_\sigma\boldsymbol{\rho}_k = \left\{\tilde{\boldsymbol{\Omega}},p_k\right\}, \qquad (\mathbf{7}.28)$$
$$\mathbf{s}\Phi^m = \partial_{\boldsymbol{\theta}}\Phi^m, \quad \mathbf{s}\boldsymbol{S}_m = \partial_{\boldsymbol{\theta}}\boldsymbol{S}_m \qquad (\mathbf{7}.29)$$
$$\tilde{\mathbf{s}}\Phi^m = \boldsymbol{\theta}\partial_\sigma\Phi^m, \quad \tilde{\mathbf{s}}\boldsymbol{S}_m = \boldsymbol{\theta}\partial_\sigma\boldsymbol{S}_m \qquad (\mathbf{7}.30)$$

The Poisson-generator for the SUSY transformations of the component fields induced by[1] $\mathrm{Q}_{\boldsymbol{\theta}}$ is thus the sum of the generators of $\mathbf{s}$ and $\tilde{\mathbf{s}}$

$$\boldsymbol{Q} = \boldsymbol{\Omega}+\tilde{\boldsymbol{\Omega}} = \int d\sigma\, \boldsymbol{\lambda}^k p_k - \partial_\sigma x^k\boldsymbol{\rho}_k = -\int d\sigma\int d\boldsymbol{\theta}\, \mathrm{Q}_{\boldsymbol{\theta}}\Phi^k\boldsymbol{S}_k \qquad (\mathbf{7}.31)$$

In (6.76) superfields were defined via $\partial_{\boldsymbol{\theta}}Y = \mathbf{s}Y$ in order to implement the exterior derivative directly with $\partial_{\boldsymbol{\theta}}$. In that sense Φ, $\boldsymbol{S}$, $\mathbf{d}\Phi$, $\mathbf{d}\boldsymbol{S}$ and all analytic functions of them were superfields. In the context of worldsheet supersymmetry, one prefers of course a supersymmetric covariant formulation. Let us therefore define in this subsection proper **superfields** via

$$Y \text{ is a superfiled} \quad :\Longleftrightarrow \quad \mathrm{Q}_{\boldsymbol{\theta}}Y \stackrel{!}{=} \{\boldsymbol{Q},Y\} = (\mathbf{s}+\tilde{\mathbf{s}})Y \qquad (\mathbf{7}.32)$$

which holds for Φ, $\boldsymbol{S}$,$\mathrm{D}_{\boldsymbol{\theta}}\Phi$, $\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}$, all analytic functions of them (like our analytically continued multivector valued forms) and worldsheet spatial derivatives ∂_σ thereof (but not for e.g. $\mathrm{Q}_{\boldsymbol{\theta}}\Phi$. This means that although we have $\mathrm{Q}_{\boldsymbol{\theta}}\Phi = (\mathbf{s}+\tilde{\mathbf{s}})\Phi$ this does not hold for a second action, i.e. $\mathrm{Q}_{\boldsymbol{\theta}}^2\Phi \neq (\mathbf{s}+\tilde{\mathbf{s}})^2\Phi$, which explains the somewhat confusing fact that the Poisson-generator $\boldsymbol{Q}$ has the opposite sign in the algebra than $\mathrm{Q}_{\boldsymbol{\theta}}$

$$\{\boldsymbol{Q},\boldsymbol{Q}\} = -2P \qquad (\mathbf{7}.33)$$

where we introduced the phase-space generator P for the worldsheet translation induced by ∂_σ

$$P \equiv \int d\sigma \quad \partial_\sigma x^k p_k + \partial_\sigma\boldsymbol{\lambda}^k\boldsymbol{\rho}_k = \int d\sigma\int d\boldsymbol{\theta} \quad \partial_\sigma\Phi^k\boldsymbol{S}_k \qquad (\mathbf{7}.34)$$

The same phenomenon appears for the differentials $\mathbf{s}$ and $\tilde{\mathbf{s}}$. The graded commutator of $\partial_{\boldsymbol{\theta}}$ and $\boldsymbol{\theta}\partial_\sigma$ is the worldsheet derivative $[\partial_{\boldsymbol{\theta}},\boldsymbol{\theta}\partial_\sigma] = \partial_\sigma$, while the algebra for $\mathbf{s}$ and $\tilde{\mathbf{s}}$ has the opposite sign

$$[\mathbf{s},\tilde{\mathbf{s}}]\,Y(\sigma,\boldsymbol{\theta}) = -\partial_\sigma Y(\sigma,\boldsymbol{\theta}) \qquad (\mathbf{7}.35)$$

[1] We have

$$\mathrm{Q}_{\boldsymbol{\theta}}\Phi^m = \boldsymbol{\lambda}^m+\boldsymbol{\theta}\partial_\sigma x^m, \quad \mathrm{Q}_{\boldsymbol{\theta}}\boldsymbol{S}_m = p_m+\boldsymbol{\theta}\partial_\sigma\boldsymbol{\rho}_m$$
$$\mathrm{D}_{\boldsymbol{\theta}}\Phi^m = \boldsymbol{\lambda}^m(\sigma)-\boldsymbol{\theta}\partial_\sigma x^m, \quad \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}_m = p_m-\boldsymbol{\theta}\partial_\sigma\boldsymbol{\rho}_m$$
$$\delta_{\boldsymbol{\varepsilon}}x^m = \boldsymbol{\varepsilon}\boldsymbol{\lambda}^m, \quad \delta_{\boldsymbol{\varepsilon}}\boldsymbol{\lambda}^m = -\boldsymbol{\varepsilon}\partial_\sigma x^m$$
$$\delta_{\boldsymbol{\varepsilon}}\boldsymbol{\rho}_m = \boldsymbol{\varepsilon}p_m, \quad \delta_{\boldsymbol{\varepsilon}}p_m = -\boldsymbol{\varepsilon}\partial_\sigma\boldsymbol{\rho}_m \quad \diamond$$

$$\mathbf{s}\tilde{\mathbf{\Omega}} = \left\{\mathbf{\Omega}, \tilde{\mathbf{\Omega}}\right\} \quad = \quad -P = \tilde{\mathbf{s}}\mathbf{\Omega} \tag{7.36}$$

One major statement in [90] is as follows: Making a general ansatz for a generator of a second, non-manifest supersymmetry, of the form (some signs are adopted to our conventions)

$$\boldsymbol{Q}_2 \quad \equiv \quad \frac{1}{2}\int d\sigma \int d\boldsymbol{\theta} \quad \left(P^{mn}(\Phi)\boldsymbol{S}_m\boldsymbol{S}_n - Q_{mn}(\Phi)\mathrm{D}_{\boldsymbol{\theta}}\Phi^m\mathrm{D}_{\boldsymbol{\theta}}\Phi^n + 2J^m{}_n(\Phi)\boldsymbol{S}_m\mathrm{D}_{\boldsymbol{\theta}}\Phi^n\right) \tag{7.37}$$

and requiring the same algebra as for $\boldsymbol{Q}$ in (7.33)

$$\{\boldsymbol{Q}_2, \boldsymbol{Q}_2\} \quad = \quad -2P \tag{7.38}$$

$$\Big(\{\boldsymbol{Q}, \boldsymbol{Q}_2\} \quad = \quad 0\Big) \tag{7.39}$$

is equivalent to

$$\mathcal{J}^M{}_N \quad \equiv \quad \begin{pmatrix} J^m{}_n & P^{mn} \\ -Q_{mn} & -J^n{}_m \end{pmatrix} \tag{7.40}$$

being an integrable generalized complex structure (see in the appendix B.2 on page 149 and B.4 on page 153). On a worldsheet without boundary, the second condition is actually superfluous, because it is already implemented via the ansatz: The expression in the integral is an analytic function of superfields and therefore a superfield itself. According to (7.32) we can replace at this point the commutator with $\boldsymbol{Q}$ with the action of $\mathrm{Q}_{\boldsymbol{\theta}}$ and get

$$\{\boldsymbol{Q}, \boldsymbol{Q}_2\} = \int d\sigma \int d\boldsymbol{\theta} \quad \mathrm{Q}_{\boldsymbol{\theta}}(\ldots) = \int d\sigma \quad \partial_\sigma(\ldots) = 0 \tag{7.41}$$

For the other condition, the actual supersymmetry algebra (7.38), the aim of the present considerations should now be clear. The generalized complex structure $\mathcal{J}$ itself is a sum of multivector valued forms

$$\mathcal{J} \quad \equiv \quad \mathcal{J}^{MN}(x)\mathbf{t}_M\mathbf{t}_N \equiv P^{mn}(x)\boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n - Q_{mn}(x)\mathbf{d}x^m\mathbf{d}x^n + 2J^m{}_n(x)\boldsymbol{\partial}_m \wedge \mathbf{d}x^n \tag{7.42}$$

which can be seen as a function of x and the basis elements

$$\mathcal{J} = \mathcal{J}(x, \mathbf{d}x, \boldsymbol{\partial}) \tag{7.43}$$

In 6.3 we replaced the arguments of functions like this with "superfields" $x^m \to \Phi^m$, $\mathbf{d}x^m \to \partial_{\boldsymbol{\theta}}\Phi^m$ and $\boldsymbol{\partial}_m \to \boldsymbol{S}_m$. The name superfield might have been misleading, as $\partial_{\boldsymbol{\theta}}\Phi$ is only a superfield in the sense that it implements the target-space exterior derivative via $\partial_{\boldsymbol{\theta}}$, but it is not a superfield in the sense of worldsheet supersymmetry. In a supersymmetric theory one prefers a supersymmetric covariant formulation. Working with $\partial_{\boldsymbol{\theta}}\Phi$ as before is therefore not desirable and we replace $\partial_{\boldsymbol{\theta}}\Phi$ by $\mathrm{D}_{\boldsymbol{\theta}}\Phi$, leading directly to $\boldsymbol{Q}_2$ (7.37) which now can be written as

$$\boldsymbol{Q}_2 \quad = \quad \frac{1}{2}\int d\sigma \int d\boldsymbol{\theta}\, \mathcal{J}\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \tag{7.44}$$

Apart from the change $\partial_{\boldsymbol{\theta}}\Phi \to \mathrm{D}_{\boldsymbol{\theta}}\Phi$ we expect from the previous section that the Poisson bracket of $\boldsymbol{Q}_2$ with itself induces some algebraic and some derived bracket of $\mathcal{J}$ with itself which then corresponds to the integrability condition for $\mathcal{J}$. This is indeed the case, but we first have to study the changes coming from $\partial_{\boldsymbol{\theta}}\Phi \to \mathrm{D}_{\boldsymbol{\theta}}\Phi$. In other words, we need a new formulation of proposition 1 (6.97) in the case of two-dimensional supersymmetry (Proposition 1 is of course still valid, but it is not formulated in a supersymmetric covariant way. It should, however, be applicable to e.g. BRST symmetries). Let us redefine the meaning of $K(\sigma,\boldsymbol{\theta})$ in (6.85) for a multivector valued form $K^{(k,k')}$

$$K^{(k,k')}(\sigma,\boldsymbol{\theta}) \quad \equiv \quad K^{(k,k')}\big(\Phi^m(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta}), \boldsymbol{S}_m(\sigma,\boldsymbol{\theta})\big) = \tag{7.45}$$

$$= K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\left(\Phi(\sigma,\boldsymbol{\theta})\right)\, \mathrm{D}_{\boldsymbol{\theta}}\Phi^{m_1}(\sigma,\boldsymbol{\theta})\ldots\mathrm{D}_{\boldsymbol{\theta}}\Phi^{m_k}(\sigma,\boldsymbol{\theta})\boldsymbol{S}_{n_1}(\sigma,\boldsymbol{\theta})\ldots\boldsymbol{S}_{n_{k'}}(\sigma,\boldsymbol{\theta}) \underset{(6.60)}{\overset{\boldsymbol{\theta}=0}{=}} K^{(k,k')}(\sigma) \tag{7.46}$$

Likewise for all the other examples in (6.84)-(6.91):

$$T^{(t,t',t'')}(\sigma,\boldsymbol{\theta}) \quad \equiv \quad T^{(t,t',t'')}\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \overset{\boldsymbol{\theta}=0}{=} T^{(t,t',t'')}(\sigma) \quad (\text{see (6.61)}) \tag{7.47}$$

$$\text{e.g. } \mathbf{d}K(\sigma,\boldsymbol{\theta}) \quad \equiv \quad \mathbf{d}K\left(\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \boldsymbol{S}(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \tag{7.48}$$

$$\text{or } \boldsymbol{o}(\sigma,\boldsymbol{\theta}) \quad \equiv \quad \boldsymbol{o}\left(\mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}), \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})\right) \overset{(6.8)}{=} \mathrm{D}_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta})\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) \underset{(6.63)}{\overset{\boldsymbol{\theta}=0}{=}} \boldsymbol{o}(\sigma) \tag{7.49}$$

$$
\begin{aligned}
[K^{(k,k')},_{\mathbf{d}} L^{(l,l')}]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) &\equiv [K^{(k,k')},L^{(l,l')}]^{(\Delta)}_{(1)}(\Phi(\sigma,\boldsymbol{\theta}),\mathrm{D}_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),\mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta})) \overset{\boldsymbol{\theta}=0}{\underset{(6.64)}{\equiv}} [K^{(k,k')},L^{(l,l')}]^{(\Delta)}_{(1)}(\sigma) && (7.50)\\
\mathbf{d}x^m(\sigma,\boldsymbol{\theta}) &\equiv \mathrm{D}_{\boldsymbol{\theta}}\Phi^m(\sigma,\boldsymbol{\theta}) = \boldsymbol{\lambda}^m(\sigma) - \boldsymbol{\theta}\partial_\sigma x^m(\sigma) && (7.51)\\
\mathbf{d}\boldsymbol{\partial}_m(\sigma,\boldsymbol{\theta}) &\equiv \mathrm{D}_{\boldsymbol{\theta}}\boldsymbol{S}_m(\sigma,\boldsymbol{\theta}) = p_m(\sigma) - \boldsymbol{\theta}\partial_\sigma\boldsymbol{\rho}_m(\sigma) && (7.52)
\end{aligned}
$$

Expanding K in $\boldsymbol{\theta}$ yields

$$
\begin{aligned}
K^{(k,k')}(\sigma,\boldsymbol{\theta}) &= K^{(k,k')}(\sigma) + \boldsymbol{\theta}\left(\partial_{\boldsymbol{\theta}'}K^{(k,k')}(\sigma,\boldsymbol{\theta}')\Big|_{\boldsymbol{\theta}'=0}\right) = && (7.53)\\
&= K^{(k,k')}(\sigma) + \boldsymbol{\theta}\left(\mathrm{Q}_{\boldsymbol{\theta}'}K^{(k,k')}(\sigma,\boldsymbol{\theta}')\Big|_{\boldsymbol{\theta}'=0}\right) && (7.54)
\end{aligned}
$$

As K is a superfield, we can replace $\mathrm{Q}_{\boldsymbol{\theta}}$ by $\mathbf{s}+\tilde{\mathbf{s}}$

$$
\begin{aligned}
K^{(k,k')}(\sigma,\boldsymbol{\theta}) &= K^{(k,k')}(\sigma) + \boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})K^{(k,k')}(\sigma) = && (7.55)\\
&= K^{(k,k')}(\sigma) + \boldsymbol{\theta}\left((\mathbf{d}+\imath_v)K^{(k,k')}\right)(\sigma)\Big|_{v^k\to-\partial_\sigma x^k} && (7.56)
\end{aligned}
$$

This is the analogue to the non-supersymmetric (6.95) and delivers the exterior derivative which will lead to the appearance of the derived bracket. The relation between $\tilde{\mathbf{s}}$ and the inner product with a vector should perhaps be clarified. Remember that all multivector forms at $\boldsymbol{\theta}=0$, $K^{(k,k')}(\sigma)$, are analytic functions of the component fields $x^m, \boldsymbol{\lambda}^m$ and $\boldsymbol{\rho}_m$. But among those fields, $\tilde{\mathbf{s}}$ acts only on $\boldsymbol{\lambda}^m$ and we can express it with partial derivatives (instead of functional ones) when acting on K:

$$
\tilde{\mathbf{s}}K(\sigma) = -\partial_\sigma x^m \frac{\partial}{\partial\boldsymbol{\lambda}^m}K(x,\boldsymbol{\lambda},\boldsymbol{\rho}) = \imath_v K(\sigma)|_{v^k=-\partial_\sigma x^k} \qquad (7.57)
$$

in the Poisson bracket of $\tilde{\mathbf{s}}K$ with another multivector valued form L at $\boldsymbol{\theta}=0$, nothing acts on $v^k=-\partial_\sigma x^k$ (which would produce a derivative of a delta function), as L does not contain p_k. Therefore we have

$$
\{\tilde{\mathbf{s}}K(\sigma'),L(\sigma)\} = [\imath_v K,L](\sigma)|_{v^k=-\partial_\sigma x^k}\,\delta(\sigma-\sigma') \qquad (7.58)
$$

which we will need below. For superfields we have $Y(\sigma,\boldsymbol{\theta}) = Y(\sigma)+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})Y(\sigma)$. Applying the same to v yields

$$
\begin{aligned}
v^k(\sigma)+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})v^k(\sigma) &= -\partial_\sigma x^k - \boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\partial_\sigma x^k(\sigma) = && (7.59)\\
&= -\partial_\sigma x^k - \boldsymbol{\theta}\partial_\sigma\lambda^k(\sigma) = -\partial_\sigma\Phi^k && (7.60)
\end{aligned}
$$

Proposition 1b *For all multivector valued forms $K^{(k,k')}, L^{(l,l')}$ on the target space manifold, in a local coordinate patch seen as functions of x^m,$\mathbf{d}x^m$ and $\boldsymbol{\partial}_m$, the following equation holds for the corresponding worldsheet-superfields (7.45)*

$$
\boxed{
\begin{aligned}
\{K^{(k,k')}(\sigma',\boldsymbol{\theta}'),L^{(l,l')}(\sigma,\boldsymbol{\theta})\} &= D_{\boldsymbol{\theta}}\left(\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\sigma-\sigma')\right)[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) + \\
&+\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\Big(\underbrace{[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})}_{-(-)^{k-k'}[K,_{\mathbf{d}}L]^{\Delta}_{(1)}} + \underbrace{[\imath_v K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})}_{-(-)^{k-k'}[K,_{\imath_v}L]}\Big|_{v^k=-\partial_\sigma\Phi^k}\Big) \qquad (7.61)
\end{aligned}
}
$$

where e.g. $[\mathbf{d}K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) \equiv [\mathbf{d}K,L]^{\Delta}_{(1)}(\Phi(\sigma,\boldsymbol{\theta}),D_{\boldsymbol{\theta}}\Phi(\sigma,\boldsymbol{\theta}),\boldsymbol{S}(\sigma,\boldsymbol{\theta}),D_{\boldsymbol{\theta}}\boldsymbol{S}(\sigma,\boldsymbol{\theta}))$.
The integrated version for a worldsheet without boundary reads

$$
\boxed{\Big\{\int d\sigma'\int d\boldsymbol{\theta}'K^{(k,k')}(\sigma',\boldsymbol{\theta}'),\int d\sigma\int d\boldsymbol{\theta}\,L^{(l,l')}(\sigma,\boldsymbol{\theta})\Big\} = (\mathbf{s}+\tilde{\mathbf{s}})\int d\sigma\left([K,_{\mathbf{d}}L]^{\Delta}_{(1)} - (-)^{k-k'}[\imath_v K,L]^{\Delta}_{(1)}\Big|_{v^k=-\partial_\sigma x^k}\right)(\sigma)}
$$
$$(7.62)$$

Proof Let us use (7.55) for both multivector valued fields and plug into the lefthand side of (7.61)

$$
\begin{aligned}
&\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\} = \\
&= \{K(\sigma')+\boldsymbol{\theta}'(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma')\,,\,L(\sigma)+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})L(\sigma)\} = && (7.63)\\
&= \{K(\sigma'),L(\sigma)\} + \boldsymbol{\theta}'\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\} + (-)^{k-k'}\boldsymbol{\theta}\{K(\sigma'),(\mathbf{s}+\tilde{\mathbf{s}})L(\sigma)\} + \\
&\quad +(-)^{k-k'}\boldsymbol{\theta}\boldsymbol{\theta}'\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),(\mathbf{s}+\tilde{\mathbf{s}})L(\sigma)\} = && (7.64)\\
&= \{K(\sigma'),L(\sigma)\} + (\boldsymbol{\theta}'-\boldsymbol{\theta})\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\} + \boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\{K(\sigma'),L(\sigma)\} +
\end{aligned}
$$

$$
\begin{aligned}
& +\boldsymbol{\theta}'\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\} - \boldsymbol{\theta}'\boldsymbol{\theta}\{(\mathbf{s}+\tilde{\mathbf{s}})(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\} = && (\mathbf{7}.65)\\
= \; & (1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\{K(\sigma'),L(\sigma)\} + (\boldsymbol{\theta}'-\boldsymbol{\theta})(1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\} + \\
& -\boldsymbol{\theta}'\boldsymbol{\theta}\{\underbrace{[\mathbf{s},\tilde{\mathbf{s}}]}_{-\partial_{\sigma'}}K(\sigma'),L(\sigma)\} = && (\mathbf{7}.66)\\
= \; & \delta(\sigma-\sigma')(1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))[K,L]^{\Delta}_{(1)}(\sigma) + (\boldsymbol{\theta}'-\boldsymbol{\theta})(1+\boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}}))\{(\mathbf{s}+\tilde{\mathbf{s}})K(\sigma'),L(\sigma)\} + \\
& -(\boldsymbol{\theta}'-\boldsymbol{\theta})\boldsymbol{\theta}\partial_\sigma\delta(\sigma-\sigma')[K,L]^{\Delta}_{(1)}(\sigma) && (\mathbf{7}.67)
\end{aligned}
$$

Now let us make use of (**7**.58) and (**7**.60) to arrive at

$$
\begin{aligned}
& \{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\} = \\
= \; & \mathrm{D}_{\boldsymbol{\theta}}\left(\delta(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta(\sigma-\sigma')\right)[K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta}) + \delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\sigma-\sigma')\,[(\mathbf{d}+\imath_v)K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})\Big|_{v^k=-\partial_\sigma\Phi^k} \qquad (\mathbf{7}.68)
\end{aligned}
$$

which is the first equation of the proposition. Integrating over $\boldsymbol{\theta}'$ and σ' results in

$$
\begin{aligned}
\int d\sigma'\int d\boldsymbol{\theta}'\,\{K(\sigma',\boldsymbol{\theta}'),L(\sigma,\boldsymbol{\theta})\} &= [(\mathbf{d}+\imath_v)K,L]^{\Delta}_{(1)}(\sigma,\boldsymbol{\theta})\Big|_{v^k=-\partial_\sigma\Phi^k} = && (\mathbf{7}.69)\\
&= [(\mathbf{d}+\imath_v)K,L]^{\Delta}_{(1)}(\sigma)\Big|_{v^k=-\partial_\sigma x^k} + \boldsymbol{\theta}(\mathbf{s}+\tilde{\mathbf{s}})\,[(\mathbf{d}+\imath_v)K,L]^{\Delta}_{(1)}(\sigma)\Big|_{v^k=-\partial_\sigma x^k} && (\mathbf{7}.70)
\end{aligned}
$$

A second integration picks out the linear part in $\boldsymbol{\theta}$ and adjusting the order of the integrations gives the additional sign in (**7**.62). □

Application to the second supercharge $\boldsymbol{Q}_2$

We are now ready to apply the proposition in the integrated form (**7**.62) to the question of the existence of a second worldsheet supersymmetry $\boldsymbol{Q}_2$. Remember, we want $\{\boldsymbol{Q}_2,\boldsymbol{Q}_2\} = -2P$. Due to the proposition, the lefthand side can be written as

$$
\{\boldsymbol{Q}_2,\boldsymbol{Q}_2\} = \frac{1}{4}(\mathbf{s}+\tilde{\mathbf{s}})\int d\sigma\left([\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} - [\imath_v\mathcal{J},\mathcal{J}]^{\Delta}_{(1)}\Big|_{v=-\partial_\sigma x^k\boldsymbol{\rho}_k}\right)(\sigma) \qquad (\mathbf{7}.71)
$$

For $\mathcal{J}^2=-1$, the second term under the integral simplifies significantly

$$
-\frac{1}{4}\int d\sigma[\imath_v\mathcal{J},\mathcal{J}]^{\Delta}_{(1)}\Big|_{v=-\partial_\sigma x^k\boldsymbol{\rho}_k} = -\int d\sigma\, v^K\mathcal{J}_K{}^L\mathcal{J}_L{}^M\mathbf{t}_M\Big|_{v=-\partial_\sigma x^k\boldsymbol{\rho}_k} = -\int d\sigma\,\partial_\sigma x^k\boldsymbol{\rho}_k = \tilde{\boldsymbol{\Omega}} \qquad (\mathbf{7}.72)
$$

Recalling that

$$
\begin{aligned}
(\mathbf{s}+\tilde{\mathbf{s}})\tilde{\boldsymbol{\Omega}} &= \mathbf{s}\tilde{\boldsymbol{\Omega}} = \tilde{\mathbf{s}}\boldsymbol{\Omega} = (\mathbf{s}+\tilde{\mathbf{s}})\boldsymbol{\Omega} = -P && (\mathbf{7}.73)\\
\text{and } \boldsymbol{\Omega} &= \int d\sigma\,\boldsymbol{o}(\sigma) \quad (\text{see } (6.63)) && (\mathbf{7}.74)
\end{aligned}
$$

we can rewrite (**7**.71) as

$$
\begin{aligned}
\Rightarrow \{\boldsymbol{Q}_2,\boldsymbol{Q}_2\} &= \frac{1}{4}(\mathbf{s}+\tilde{\mathbf{s}})\left(\int d\sigma\,[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} + 4\boldsymbol{\Omega}\right) = && (\mathbf{7}.75)\\
&= \frac{1}{4}(\mathbf{s}+\tilde{\mathbf{s}})\left(\int d\sigma\left([\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} - 4\boldsymbol{o}\right)(\sigma)\right) + 2\underbrace{\tilde{\mathbf{s}}\boldsymbol{\Omega}}_{-P} && (\mathbf{7}.76)
\end{aligned}
$$

The righthand side clearly equals $-2P$ for

$$
[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} - 4\boldsymbol{o} = 0 \qquad (\mathbf{7}.77)
$$

which is again (according to (**B**.113)) just the integrability condition for the generalized almost complex structure $\mathcal{J}$.

Conclusions to the Bracket Part

We have seen two closely related mechanisms in sigma-models with a special field content which lead to the derived bracket of the target space algebraic bracket by the target space exterior derivative. This exterior derivative is implemented in the sigma model in one case via the derivative with respect to a (worldvolume-) Grassmann coordinate and in the other case via the derivative with respect to the worldvolume coordinate itself. In the latter case this derivative has to be contracted with (worldvolume-) Grassmann coordinates in order to be an odd differential. This leads to the problem that higher powers of the basis elements vanish, as soon as the power exceeds the worldvolume dimension as it happens in Zucchini's application. A big number of Grassmann-variables is therefore advantageous in that approach. For the other mechanism one rather prefers to have only one single Grassmann variable as there is no need for any contraction. There is one worldvolume dimension more in the Lagrangian formalism and for that reason it was preferable to apply there the mechanism with worldvolume derivatives and use the other one in the Hamiltonian formalism.

If one does not consider antisymmetric tensors of higher rank, but only vectors or one-forms (or forms of worldvolume-dimension), the partial worldvolume derivative without a Grassmann-coordinate is enough. There is either no need for antisymmetrization or it can be performed with the worldvolume epsilon tensor. The nature of the mechanism remains the same and leads to the observations in [71, 73] that the Poisson bracket implements the Dorfman bracket for sums of vectors and one-forms and the corresponding derived bracket for sums of vectors and p-forms on a p-brane [73]. In that sense, the present part of the thesis is a generalization of those observations.

There remain a couple of things to do. It should be possible to implement in the same manner by e.g. a BRST differential other target space differentials which can depend on some extra-structure and repeat the same analysis. Symmetric tensors then become more interesting as well, because they need such an extra-structure anyway for a meaningful differential. From the string theory point of view, the application of extended worldsheet supersymmetry corresponds to applications in the RNS string. But generalized complex geometry contains the tools to allow RR-fluxes, which are hard to treat in RNS. It would therefore be nice to find some topological limit in a string theory formalism which is extendable to RR-fields, like the Berkovits-string [12], leading to a topological sigma model like Zucchini's, in order to learn more about the correspondence between string theory and generalized complex geometry.

Conclusion

After the conclusions on the bracket part, we would like to recall the general idea of what we did. Apart from the presentation of the explicit worldsheet BRST transformations, the result of the supergravity-constraint calculations from Berkovits' pure spinor string in part II is not new in itself. It is, however, a very important result and our contribution can be seen as an independent check. This is true in particular, as we used different techniques at several points. We established a covariant variation in this setting and derived everything in the Lagrangian formalism, using "inverse Noether". The argumentation and calculation was done in detail, in order to allow checks by others, and also some subtle points like the antighost gauge symmetry where discussed carefully. Also our starting point was more general. Last but not least, the insight from the first part about superspace conventions served as a very powerful tool throughout. The aim of the calculation in part II was to make contact to generalized geometry. The derivation of the generalized Calabi Yau condition has been done so far from the supergravity point of view, and possible quantum or string corrections to this geometry require a worldsheet calculation. We have therefore derived the supergravity transformations of the fermionic background fields which serve as the starting point of these considerations. We did not yet calculate any string corrections, but it could already be of big advantage to know the natural form of the supergravity transformations as they come out from the string and not from old supergravity considerations. In particular we expect to obtain more insight about the geometric role of the RR-fields in the super-geometrical setting. Non-commutativity considerations for the open superstring (e.g. [99, 100, 101]), for example, assign a similar role to the RR-fields in superspace as the B-field has in bosonic space. And the geometry of the latter (with the field strength H either seen as a twist or a torsion), are understood much better.

There are several directions ahead. One could try to establish the tools of generalized (not necessarily complex) geometry already in ten dimensions, before compactification. Having the superstring in mind (embedded in superspace), it would be even more appealing to consider some generalized supergeometry, i.e. structures on $T \oplus T^*$ of the supermanifold. String statements should simplify if one uses a formulation where the structures of interest appear manifestly. In this context it seems also reasonable to switch to a probably mixed first-second order formalism of the pure spinor string in general background. Topological limits of this formalism might lead to something like the Hitchin sigma-model [91] or some supersymmetric version of it. This again could shed light on the geometric role of RR-fields. Similar to the last point would be the introduction of doubled coordinates as suggested by Hull[102, 103, 104, 105]. Generalized complex geometry and this doubled geometry seem to be very closely related. Deriving the first via supersymmetry conditions in a formalism with doubled coordinates certainly could clarify this relation.

For all these considerations, our insight about brackets and sigma-models and the relation to the integrability of generalized complex geometry that we obtained in the last part of this thesis will be very useful. What we learned about superspace conventions should even be useful for everybody working with superspace.

Appendix

Appendix A

Notations and Conventions

Within the thesis, a lot of different types of tensors have to be denoted. The choices and sometimes some logic behind, will be presented here.

The bracket part (III) (including appendices B and C) differs a bit in the notation from the rest, as it does not treat a superspace. In any case we denote bosonic target space coordinates via x^m. In the bracket part, however, world-volume-coordinates are denoted by σ^μ, while in the worldsheet coordinates in the rest are most often chosen to be complex $(z,\bar{z})$. At some places we write the real coordinates σ^ξ with an worldsheet index ξ or ζ, in order to distinguish it from the curved spinorial indices $\mu, \nu, \ldots$. Our metric signature is 'mostly plus':$\eta_{ab} = \mathrm{diag}\,(-1, 1, \ldots, 1)$.

Superspace In the superspace parts we have $x^M \equiv (x^m, \boldsymbol{\theta}^\mu, \hat{\boldsymbol{\theta}}^{\hat{\mu}})$, where $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are anticommuting coordinates with the dimension 16 of a Majorana Weyl spinor in ten dimensions. The hatted index should include both versions of superspace: IIA (with $\hat{\boldsymbol{\theta}}^{\hat{\mu}} = \hat{\boldsymbol{\theta}}_\mu$) and IIB (with $\hat{\boldsymbol{\theta}}^{\hat{\mu}} = \hat{\boldsymbol{\theta}}^\mu$). The grading of the coordinate x^M depends on the index. We therefore prefer to write $x^M \equiv (x^m, x^{\boldsymbol{\mu}}, x^{\hat{\boldsymbol{\mu}}})$. Writing the fermionic indices boldface is just a reminder and will not be substantial. A vielbein $E_M{}^A$ will transform curved indices (from the middle of the alphabet) into flat indices (from the beginning of the alphabet) and vice verse, e.g. for the pullbacks of the supersymmetric invariant form $\Pi_z^A = \partial x^M E_M{}^A$. The entries then have a corresponding index structure with letters from the beginning of the alphabet: $\Pi_z^A = (\Pi_z^a, \Pi_z^{\boldsymbol{\alpha}}, \Pi_z^{\hat{\boldsymbol{\alpha}}})$. When we want to combine the spinorial indices only, we write $x^{\boldsymbol{\mathcal{M}}} \equiv (x^{\boldsymbol{\mu}}, x^{\hat{\boldsymbol{\mu}}})$ or $\boldsymbol{\theta}^{\mathcal{M}} \equiv (\boldsymbol{\theta}^\mu, \hat{\boldsymbol{\theta}}^{\hat{\mu}})$ or $\Pi_z^{\boldsymbol{\mathcal{A}}} \equiv (\Pi_z^{\boldsymbol{\alpha}}, \Pi_z^{\hat{\boldsymbol{\alpha}}})$. If we want to omit the indices, (e.g. in functions of the coordinates) we write $\vec{\vec{x}}$ for x^M, $\vec{x}$ for x^m, $\vec{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}^{\mathcal{M}}$, $\boldsymbol{\theta}$ for $\boldsymbol{\theta}^{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\theta}}$ for $\hat{\boldsymbol{\theta}}^{\hat{\boldsymbol{\mu}}}$.

Notation for tensors in the bracket part In the bracket-part, we mainly denote target space vector-fields by $a, b, \ldots$ or $v, w, \ldots$, 1-forms by small Greek letters $\alpha, \beta, \ldots$ and generalized $T \oplus T^*$-vectors by $\mathfrak{a}, \mathfrak{b}, \ldots$ or $\mathfrak{v}, \mathfrak{w}, \ldots$. For an explicit split in vector and 1-form, the letters from the beginning of the alphabet are better suited, as there is a better correspondence between Latin and Greek symbols or one can visually better distinguish between Latin and Greek symbols. Compare e.g. $\mathfrak{a} = a + \alpha$ and $\mathfrak{v} = v + (?\nu)$.
Higher order forms will be in general denoted by $\alpha^{(p)}, \beta^{(q)}, \ldots$ or $\omega^{(p)}, \eta^{(q)}, \rho^{(r)}, \ldots$. There will be exceptions, however , for specific forms like the B-field $B = B_{mn}\mathbf{d}x^m \wedge \mathbf{d}x^n$. Following this logic, we will also denote multivectors (tensors with antisymmetric upper indices) by small letters, indicating their multivector-degree in brackets: $a^{(p)}, b^{(q)}, \ldots$ or $v^{(p)}, w^{(q)}, \ldots$. There are again exceptions, e.g. a Poisson structure will often be denoted by $P = P^{mn}\boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n$. The most horrible exception is the one of the beta-transformation, which is denoted by a large beta β^{mn} in (B.47), in order to distinguish it from forms.

Tensors of mixed type will be denoted by capital letters where we denote in brackets first the number of lower indices and then the number of upper indices, e.g. $T^{(p,q)}$. Most of the time, we treat multivector valued forms, e.g. the lower indices as well as the upper indices are antisymmetrized. The letters denoting form degree and multivector degree will often be adapted to the letter of the tensor, e.g. $K^{(k,k')}, L^{(l,l')}, \ldots$
Attention: k and l are also used as dummy indices! Sometimes (I'm sorry for that) the same letter appears with different meanings. However, in those situations the dummy indices will carry indices which might even be one of the degrees k or k', e.g. $K_{\ldots}{}^{k_1 \ldots k_{k'}} L_{k_{k'} \ldots k_1 \ldots}{}^{\cdots}$.

Working all the time with graded algebras with a graded symmetric product (the wedge product), everything in this thesis has to be understood as **graded**. I.e. with commutator we mean the graded commutator and with the Poisson bracket the graded Poisson bracket. They will not be denoted differently than the non-graded operations. Relevant for the sign rules is the **total degree** which we define to be form degree minus the multivector degree. In the field language, it corresponds to the total ghost number which is the pure ghost number minus the antighost number. It will be denoted in the bracket part by

$$|\, K^{(k,k')} \,| \quad = \quad k - k' \tag{A.1}$$

In the rest of the thesis, $|\ldots|$ will only denote the parity, i.e. $+1$ for commuting and -1 for anticommuting variables. As only degrees or parities appear in the exponent of a minus sign, a simplified notation is used there

$$(-)^A \equiv (-1)^{|A|}, \quad (-)^{A+B} \equiv (-)^{|A|+|B|}, \quad (-)^{AB} \equiv (-)^{|A||B|} \quad \forall A, B \tag{A.2}$$

Poisson bracket and derivatives For the Poisson bracket, the following (less common) sign convention is chosen:

$$\{p_m, x^n\} = \delta_m^n = -\{x^n, p_m\} \tag{A.3}$$
$$\{b_m, c^n\} = \delta_m^n = -(-)^{bc}\{c^n, b_m\} \tag{A.4}$$

Derivatives with respect to x^m are denoted by $\frac{\partial}{\partial x^m} f \equiv \partial_m f \equiv f_{,m}$. For graded variables left and right derivatives are denoted respectively by

$$\frac{\partial f}{\partial \boldsymbol{c}} \equiv \frac{\partial}{\partial \boldsymbol{c}} f(\boldsymbol{c}) \equiv \frac{\vec{\partial}}{\partial \boldsymbol{c}} f(\boldsymbol{c}), \qquad \partial f(\boldsymbol{c})/\partial \boldsymbol{c} \equiv f \frac{\overleftarrow{\partial}}{\partial \boldsymbol{c}} \tag{A.5}$$

The corresponding notations are used for functional derivatives $\frac{\delta}{\delta \boldsymbol{c}(\sigma)}$.

Boldface philosophy and antisymmetrizations With respect to the wedge product, the basis element $\boldsymbol{\partial}_m$ is an odd object ($\boldsymbol{\partial}_m \wedge \boldsymbol{\partial}_n = -\boldsymbol{\partial}_n \wedge \boldsymbol{\partial}_m$). The partial derivative ∂_k acting on some coefficient function, however, is an even operator (it does not change the parity as long as it is not contracted with a basis element $\mathbf{d}x^k$). That is why we denote the odd basis element $\boldsymbol{\partial}_m$ and $\mathbf{d}x^m$ as well as the odd exterior derivative $\mathbf{d}$ with boldface symbols. The interior product itself does not carry a grading in the sense that $|\imath_K \rho| = |K| + |\rho|$, while for the Lie derivative $\boldsymbol{\mathcal{L}}_K = [\imath_K, \mathbf{d}]$ the $\boldsymbol{\mathcal{L}}$ carries a grading in the sense $|\boldsymbol{\mathcal{L}}_K \rho| = |K| + |\rho| + 1$. That is why the Lie derivative is denoted with a boldface $\boldsymbol{\mathcal{L}}$ which is also very good to distinguish it from generalized multivectors $\mathcal{K}, \mathcal{L}, \ldots$. The philosophy of writing odd objects in boldface style is also extended to the combined basis element

$$\mathbf{t}_M \equiv (\boldsymbol{\partial}_m, \mathbf{d}x^m), \quad \mathbf{t}^M \equiv (\mathbf{d}x^m, \boldsymbol{\partial}_m) \tag{A.6}$$

and to the comma in the derived bracket $[\,\boldsymbol{,}\,]$ in contrast to the commutator $[\,,\,]$. This should be, however, just a reminder. It will be obvious for other reasons, which bracket is meant. But we do **not** extend this philosophy to vectors and 1-forms, where it would be consistent (but too much effort) to write the vectors and basis elements in boldface style and the coefficients in standard style. We will instead write the vector in the same style as the coefficient $a = a_m \mathbf{d}x^m$.

A square bracket is used as usual to denote the antisymmetrization of, say p, indices (including a normalization factor $\frac{1}{p!}$). A vertical line is used to exclude some indices from antisymmetrization. An extreme example would be

$$A^{[ab|cd|e|fg|hi]} \tag{A.7}$$

where A is antisymmetrized only in a, b, e, h and i, but not in c, d, f and g. Normally we use only expressions like $A^{[ab|cd|efg]}$, where a, b, e, f and g are antisymmetrized.

Wedge product A significant difference from usual conventions is that for multivectors, forms and generalized multivectors we include the normalization of the factor already in the definition of the wedge product

$$\mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_n} \equiv \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_n} \equiv \mathbf{d}x^{[m_1} \otimes \ldots \otimes \mathbf{d}x^{m_n]} \equiv \sum_P \frac{1}{n!} \mathbf{d}x^{m_{P(1)}} \otimes \ldots \otimes \mathbf{d}x^{m_{P(n)}} \tag{A.8}$$

$$\boldsymbol{\partial}_{m_1} \cdots \boldsymbol{\partial}_{m_n} \equiv \boldsymbol{\partial}_{m_1} \wedge \cdots \wedge \boldsymbol{\partial}_{m_n} \equiv \boldsymbol{\partial}_{[m_1} \otimes \cdots \otimes \boldsymbol{\partial}_{m_n]} \equiv \sum_P \frac{1}{n!} \boldsymbol{\partial}_{m_{P(1)}} \otimes \cdots \otimes \boldsymbol{\partial}_{m_{P(n)}} \tag{A.9}$$

$$\mathbf{t}_{M_1} \ldots \mathbf{t}_{M_n} \equiv \mathbf{t}_{M_1} \wedge \ldots \wedge \mathbf{t}_{M_n} \equiv \mathbf{t}_{[M_1} \otimes \ldots \otimes \mathbf{t}_{M_n]} \equiv \sum_P \frac{1}{n!} \mathbf{t}_{M_{P(1)}} \otimes \ldots \otimes \mathbf{t}_{M_{P(n)}} \tag{A.10}$$

(where we sum over all permutations P), such that we omit the usual factor of $\frac{1}{p!}$ in the coordinate expression of a p-form, or a p-vector

$$\alpha^{(p)} \equiv \alpha_{m_1 \ldots m_p} \mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_p} \equiv \alpha_{m_1 \ldots m_p} \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_p} \tag{A.11}$$
$$v^{(p)} \equiv v^{m_1 \ldots m_p} \partial_{m_1} \wedge \ldots \wedge \partial_{m_p} \tag{A.12}$$

Readers who prefer the $\frac{1}{p!}$, can easily reintroduce it in every equation by replacing e.g. the coefficient functions $v^{m_1\ldots m_p} \to \frac{1}{p!}v^{m_1\ldots m_p}$. The equation for the Schouten bracket (C.10), for example, would change as follows:

$$\left[v^{(p)}, w^{(q)}\right]^{m_1\ldots m_{p+q-1}} = pv^{[m_1\ldots m_{p-1}|k}\partial_k w^{|m_p\ldots m_{p+q-1}]} - qv^{[m_1\ldots m_p|}{}_{,k}w^{k\,|m_{p+1}\ldots m_{p+q-1}]} \quad \text{(A.13)}$$

$$\to \frac{1}{(p+q-1)!}\left[v^{(p)}, w^{(q)}\right]^{m_1\ldots m_{p+q-1}} = \frac{1}{(p-1)!}\frac{1}{q!}v^{[m_1\ldots m_{p-1}|k}\partial_k w^{|m_p\ldots m_{p+q-1}]} + -\frac{1}{p!}\frac{1}{(q-1)!}v^{[m_1\ldots m_p|}{}_{,k}w^{k\,|m_{p+1}\ldots m_{p+q-1}]} \quad \text{(A.14)}$$

Schematic index notation For longer calculations in coordinate form it is useful to introduce the following notation, where every boldface index is assumed to be contracted with the corresponding basis element (at the same position of the index), s.th. the indices are automatically antisymmetrized.

$$\omega^{(p)} = \omega_{m_1\ldots m_p}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_p} \equiv \omega_{\boldsymbol{m}\ldots\boldsymbol{m}} \quad \text{(A.15)}$$

$$a^{(p)} = a^{n_1\ldots n_p}\boldsymbol{\partial}_{n_1}\wedge\ldots\boldsymbol{\partial}_{n_p} \equiv a^{\boldsymbol{n}\ldots\boldsymbol{n}} \quad \text{(A.16)}$$

$$\mathcal{K}^{(p)} = \mathcal{K}_{M_1\ldots M_p}\mathbf{t}^{M_1}\ldots\mathbf{t}^{M_p} \equiv \mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} = \quad \text{(A.17)}$$

$$= \mathcal{K}^{M_1\ldots M_p}\mathbf{t}_{M_1}\ldots\mathbf{t}_{M_p} \equiv \mathcal{K}^{\boldsymbol{M}\ldots\boldsymbol{M}} \quad \text{(A.18)}$$

or for products of tensors e.g.

$$\omega_{\boldsymbol{m}\ldots\boldsymbol{m}}\eta_{\boldsymbol{m}\ldots\boldsymbol{m}} \equiv \omega_{[m_1\ldots m_p}\eta_{m_{p+1}\ldots m_{p+q}]}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_{p+q}} = \quad \text{(A.19)}$$

$$= \omega_{m_1\ldots m_p}\eta_{m_{p+1}\ldots m_{p+q}}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_{p+q}} = (-)^{pq}\eta_{\boldsymbol{m}\ldots\boldsymbol{m}}\omega_{\boldsymbol{m}\ldots\boldsymbol{m}} \quad \text{(A.20)}$$

A boldface index might be hard to distinguish from an ordinary one, but this notation is nevertheless easy to recognize, as normally several coinciding indices appear (which are not summed over as they are at the same position). Similarly, for multivector valued forms we define[1]

$$K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}}\mathbf{d}x^{m_1}\wedge\ldots\wedge\mathbf{d}x^{m_k}\otimes\boldsymbol{\partial}_{m_1}\wedge\ldots\wedge\boldsymbol{\partial}_{m_{k'}} \quad \text{(A.21)}$$

$$K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}p}L_{p\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'-1}p}L_{pm_1\ldots m_{l-1}}{}^{n_1\ldots n_{l'}}\mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_{k+l-1}}\otimes\boldsymbol{\partial}_{m_1}\cdots\boldsymbol{\partial}_{m_{k'+l'-1}} \quad \text{(A.22)}$$

[1]Upper and lower signs are thus treated independently. For calculational reasons this is not the best way to do. We can interpret every boldface index on the lefthand side of (A.22) as a basis element sitting at the position of the index, so that the order of the basis elements on the lefthand side is first $k\times\mathbf{d}x^m$, $(k'-1)\boldsymbol{\partial}_m$, $(l-1)\times\mathbf{d}x^m$ and $l'\times\boldsymbol{\partial}_m$, s.th., in order to get the order of the righthand side, we have to interchange $(k'-1)\boldsymbol{\partial}_m$ with $(l-1)\times\mathbf{d}x^m$, which gives a sign factor of $(-)^{(k'-1)(l-1)}$. This is a natural sign factor which appears all the way in the equations, which could be easily absorbed into the definition. However, we wanted to keep the sign factors explicitly in the equations in order to keep the notation as self-explaining as possible and not confuse the reader too much. ◇

Appendix B

Generalized Complex Geometry

For introductions into Hitchin's [74] generalized complex geometry (GCG) see e.g. Zabzine's review [88] or Gualtieri's thesis [72]. In the appendix of [106] there is another nice introduction with emphasis on the pure spinor formulation of GCG. For a survey of compactification with fluxes and its relation to GCG see Graña's review [76].

B.1 Basics

In **generalized geometry** one is looking at structures (e.g. a complex structure) on the direct sum of tangent and cotangent bundle $T \oplus T^*$. Let us call a section of this bundle a **generalized vector** (field) or synonymously **generalized 1-form**, which is the sum of a vector field and a 1-form

$$\mathfrak{a} = a + \alpha = \tag{B.1}$$

$$= a^m \boldsymbol{\partial}_m + \alpha_m \mathbf{d}x^m \tag{B.2}$$

Using the **combined basis elements**

$$\mathbf{t}_M \equiv (\boldsymbol{\partial}_m, \mathbf{d}x^m) \tag{B.3}$$

a generalized vector $\mathfrak{a}$ can be written as

$$\mathfrak{a} = \mathfrak{a}^M \mathbf{t}_M \tag{B.4}$$

$$\mathfrak{a}^M = (a^m, \alpha_m) \tag{B.5}$$

There is a **canonical metric** $\mathcal{G}$ on $T \oplus T^*$

$$\langle \mathfrak{a}, \mathfrak{b} \rangle \equiv \alpha(b) + \beta(a) = \tag{B.6}$$

$$= \alpha_m b^m + \beta_m a^m \equiv \tag{B.7}$$

$$\equiv \mathfrak{a}^M \mathcal{G}_{MN} \mathfrak{b}^N \tag{B.8}$$

with

$$\mathcal{G}_{MN} \equiv \begin{pmatrix} 0 & \delta_m^n \\ \delta_n^m & 0 \end{pmatrix} \tag{B.9}$$

which has **signature** (d,-d) (if d is the dimension of the base manifold). The above definition differs by a factor of 2 from the most common one. We prefer, however, to have an inverse metric of the same form

$$\mathcal{G}^{MN} \equiv \left(\mathcal{G}^{-1}\right)^{MN} = \begin{pmatrix} 0 & \delta_n^m \\ \delta_m^n & 0 \end{pmatrix} \tag{B.10}$$

As it is constant, we can always pull it through partial derivatives. Using this metric to lower and raise indices just interchanges vector and form component. We can equally rewrite $\mathfrak{a}$ in (B.4) with a basis with upper capital indices and the vector coefficients with lower indices

$$\mathbf{t}^M \equiv (\mathbf{d}x^m, \boldsymbol{\partial}_m) \tag{B.11}$$

$$\mathfrak{a} = \mathfrak{a}_M \mathbf{t}^M \tag{B.12}$$

$$\mathfrak{a}_M = (\alpha_m, a^m) \tag{B.13}$$

Note that in the present text there is no existence of any metric on the tangent bundle assumed. Therefore we cannot raise or lower small indices. In cases where 1-form and vector have a similar symbol, the position of the small index therefore uniquely determines which is which (e.g. ω_m and w^m).

In addition to the canonical metric $\mathcal{G}_{MN}$ there is also a **canonical antisymmetric 2-form** $\mathcal{B}$, s.th. $\alpha(b) - \beta(a) = \mathfrak{a}^M \mathcal{B}_{MN} \mathfrak{b}^N$ with coordinate form

$$\mathcal{B}_{MN} \equiv \begin{pmatrix} 0 & -\delta_m^n \\ \delta_n^m & 0 \end{pmatrix} \tag{B.14}$$

Raising the indices with $\mathcal{G}^{MN}$ yields

$$\mathcal{B}^M{}_N = \begin{pmatrix} \delta_n^m & 0 \\ 0 & -\delta_m^n \end{pmatrix} = -B_N{}^M \tag{B.15}$$

$$\mathcal{B}^{MN} = \begin{pmatrix} 0 & \delta_n^m \\ -\delta_m^n & 0 \end{pmatrix} \tag{B.16}$$

We can thus use $\mathcal{B}$ and $\mathcal{G}$ to construct **projection operators** $\mathcal{P}_{\mathcal{T}}$ and $\mathcal{P}_{\mathcal{T}^*}$ to tangent and cotangent space

$$\mathcal{P}_{\mathcal{T}}{}^M{}_N \equiv \frac{1}{2}\left(\delta^M{}_N + B^M{}_N\right) = \begin{pmatrix} \delta_n^m & 0 \\ 0 & 0 \end{pmatrix} \tag{B.17}$$

$$\mathcal{P}_{\mathcal{T}^*}{}^M{}_N \equiv \frac{1}{2}\left(\delta^M{}_N - B^M{}_N\right) = \begin{pmatrix} 0 & 0 \\ 0 & \delta_m^n \end{pmatrix} \tag{B.18}$$

$$\mathcal{P}_{\mathcal{T}}\mathfrak{a} = a, \qquad \mathcal{P}_{\mathcal{T}^*}\mathfrak{a} = \alpha \tag{B.19}$$

B.2 Generalized almost complex structure

A **generalized almost complex structure** is a linear map from $T \oplus T^*$ to itself which squares to minus the identity-map, i.e. in components

$$\mathcal{J}^M{}_K \mathcal{J}^K{}_N = -\delta_N^M \tag{B.20}$$

It is called a **generalized complex structure** if it is integrable (see subsection B.4). It should be **compatible** with our canonical metric $\mathcal{G}$ which means that it should behave like multiplication with i in a Hermitian scalar product of a complex vector space[1]

$$\langle \mathfrak{v}, \mathcal{J}\mathfrak{w}\rangle = -\langle \mathcal{J}\mathfrak{v}, \mathfrak{w}\rangle \iff (\mathcal{G}\mathcal{J})^T = -\mathcal{G}\mathcal{J} \iff \mathcal{J}_{MN} = -\mathcal{J}_{NM} \tag{B.21}$$

This property is also known as **antihermiticity** of $\mathcal{J}$. Because of (B.21), $\mathcal{J}$ can be written as

$$\mathcal{J}^M{}_N = \begin{pmatrix} J^m{}_n & P^{mn} \\ -Q_{mn} & -J^n{}_m \end{pmatrix} \qquad \mathcal{J}_{MN} = \begin{pmatrix} -Q_{mn} & -J^n{}_m \\ J^m{}_n & P^{mn} \end{pmatrix} \tag{B.22}$$

where P^{mn} and Q_{mn} are antisymmetric matrices, and (B.20) translates into

$$J^2 - PQ = -\mathbb{1} \tag{B.23}$$

$$JP - PJ^T = 0 \tag{B.24}$$

$$-QJ + J^T Q = 0 \tag{B.25}$$

Here it becomes obvious that the generalized complex structure contains the case of an ordinary almost complex structure J with $J^2 = -1$ for $Q = P = 0$ as well as the case of an almost symplectic structure of a non-degenerate 2-form Q with existing inverse $PQ = \mathbb{1}$ for $J = 0$. In addition to those algebraic constraints, the integrability of the generalized almost complex structure gives further differential conditions (see subsection B.4) which boil down in the two special cases to the integrability of the ordinary complex structure or to the integrability of the symplectic structure.

Because of $\mathcal{J}^2 = -\mathbb{1}$, $\mathcal{J}$ has eigenvalues $\pm i$. The corresponding eigenvectors span the space of **generalized holomorphic vectors** L or generalized antiholomorphic vectors $\bar{L}$ respectively. This provides a natural splitting of the complexified bundle

$$(T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L} \tag{B.26}$$

The **projector** Π to the space of eigenvalue $+i$ (namely L) can be be written as

$$\Pi \equiv \frac{1}{2}\left(\mathbb{1} - i\mathcal{J}\right) \tag{B.27}$$

[1] In a complex vector space with Hermitian scalar product $\langle a, b\rangle = \overline{\langle b, a\rangle}$ we have $\langle a, ib\rangle = -\langle ia, b\rangle$. ◇

while the projector to $\bar{L}$ is just the complex conjugate $\bar{\Pi} = \frac{1}{2}(\mathbb{1} + i\mathcal{J}) = G^{-1}\Pi^T G$. Indeed, for any generalized vector field $\mathfrak{v}$ we have

$$\mathcal{J}\Pi\mathfrak{v} = i\Pi\mathfrak{v} \tag{B.28}$$

L and $\bar{L}$ are what one calls **maximally isotropic subspaces**, i.e. spaces which are *isotropic*

$$\langle \mathfrak{v}, \mathfrak{w} \rangle = 0 \quad \forall \mathfrak{v}, \mathfrak{w} \in L \tag{B.29}$$

(this is because $\Pi^T G \Pi = \mathcal{G}\bar{\Pi}\Pi = 0$) and which have half the dimension of the complete bundle. As the canonical metric $\langle \cdots \rangle$ is nondegenerate, this is the maximal possible dimension for isotropic subbundles.

B.3 Dorfman and Courant bracket

Something which seems to be a bit unnatural in this whole business in the beginning is the introduction of the Courant bracket, which is the antisymmetrization of the so-called Dorfman-bracket. The **Dorfman bracket** in turn is the natural generalization of the Lie bracket from the point of view of derived brackets (C.51)[2]

$$\begin{aligned}
[[\imath_{\mathfrak{a}}, \mathbf{d}], \imath_{\mathfrak{b}}] &= \imath_{[\mathfrak{a},\mathfrak{b}]} && \text{(B.30)}\\
\text{where } [\mathfrak{a},\mathfrak{b}] &\equiv [a,b] + \boldsymbol{\mathcal{L}}_a\beta - \boldsymbol{\mathcal{L}}_b\alpha + \mathbf{d}(\imath_b\alpha) = && \text{(B.31)}\\
&= [a,b] + \boldsymbol{\mathcal{L}}_a\beta - \imath_b(\mathbf{d}\alpha) = && \text{(B.32)}\\
&= \boldsymbol{\mathcal{L}}_a\mathfrak{b} - \imath_b(\mathbf{d}\alpha) && \text{(B.33)}
\end{aligned}$$

To get a homogeneous coordinate expression, we define

$$\partial_M \equiv (\partial_m, 0) \quad \Rightarrow \partial^M = (0, \partial_m) \tag{B.34}$$

[2] The twisted Dorfman bracket is defined similarly via

$$[[\imath_{\mathfrak{a}}, \mathbf{d} + H\wedge], \imath_{\mathfrak{b}}] \equiv \imath_{[\mathfrak{a},\mathfrak{b}]_H}$$

Remembering that $H\wedge = \imath_H$ and using $[\imath_a, \imath_H] = \imath_{[a,H]^\Delta} = \imath_{\imath_a^{(1)} H}$, we get

$$[\mathfrak{a},\mathfrak{b}]_H \equiv [a,b] - \imath_b\imath_a H \qquad \diamond$$

The Dorfman bracket can then be written as[3]

$$[\mathfrak{a},\mathfrak{b}]^M = \mathfrak{a}^K\partial_K\mathfrak{b}^M + \left(\partial^M\mathfrak{a}_K - \partial_K\mathfrak{a}^M\right)\mathfrak{b}^K \tag{B.35}$$

$$\text{or } [\mathfrak{a},\mathfrak{b}]_M = \mathfrak{a}^K\partial_K\mathfrak{b}_M + 2\partial_{[M}\mathfrak{a}_{K]}\mathfrak{b}^K \tag{B.36}$$

Apart from the term in the middle $\partial^M\mathfrak{a}_K$, (B.35) looks formally the same as the Lie bracket of vector fields (C.1). The Dorfman bracket is in general not antisymmetric but it obeys a **Jacobi-identity** (Leibniz from the left) of the form

$$[\mathfrak{a},[\mathfrak{b},\mathfrak{c}]] = [[\mathfrak{a},\mathfrak{b}],\mathfrak{c}] + [\mathfrak{b},[\mathfrak{a},\mathfrak{c}]] \tag{B.37}$$

Although the Dorfman bracket is all we need, most of the literature on generalized complex geometry so far works with its antisymmetrization, which is called **Courant bracket**

$$[\mathfrak{a},\mathfrak{b}]_- \equiv [a,b] + \boldsymbol{\mathcal{L}}_a\beta - \boldsymbol{\mathcal{L}}_b\alpha + \frac{1}{2}\mathbf{d}(\imath_b\alpha - \imath_a\beta) \tag{B.38}$$

$$[\mathfrak{a},\mathfrak{b}]_{-M} = \mathfrak{a}^K\partial_K\mathfrak{b}_M - \partial_K\mathfrak{a}_M\mathfrak{b}^K + \frac{1}{2}\left(\partial_M\mathfrak{a}_K\mathfrak{b}^K - \mathfrak{a}^K\partial_M\mathfrak{b}_K\right) \tag{B.39}$$

and which does not obey any Jacobi identity. As it is much simpler to go from Dorfman to Courant, than the other way round, we will only work with the Dorfman bracket. On any isotropic subspace ($\imath_b\alpha + \imath_a\beta = 0$) the two coincide anyway, i.e. they become a Lie bracket, obeying Jacobi and being antisymmetric.

We call a transformation a **symmetry of the bracket** when the bracket of two vectors transforms in the same way as the vectors

$$[(\mathfrak{b}+\delta\mathfrak{b}),(\mathfrak{c}+\delta\mathfrak{c})] = [\mathfrak{b},\mathfrak{c}] + \delta[\mathfrak{b},\mathfrak{c}] \tag{B.40}$$

$$\delta[\mathfrak{b},\mathfrak{c}] = [\delta\mathfrak{b},\mathfrak{c}] + [\mathfrak{b},\delta\mathfrak{c}] + [\delta\mathfrak{b},\delta\mathfrak{c}] \tag{B.41}$$

I.e. infinitesimal symmetry transformations (where the last term drops) have to obey a product rule. Similar as for the Lie-bracket of vector fields, infinitesimal transformations are generated by the bracket itself. Let us call the corresponding derivative, in analogy to the Lie derivative, the **Dorfman derivative** of a generalized vector with respect to a generalized vector.

$$\delta\mathfrak{b} = \boldsymbol{\mathcal{D}}_{\mathfrak{a}}\mathfrak{b} \equiv [\mathfrak{a},\mathfrak{b}] \tag{B.42}$$

These transformations are therefore, due to the Jacobi-identity (B.37) always symmetries of the bracket. From (B.33) we can see that the Dorfman derivative consists of a usual Lie derivative and second part which acts only on the vector part of $\mathfrak{b}$ by contracting it with the exact 2-form $\mathbf{d}\alpha$

$$\boldsymbol{\mathcal{D}}_a\mathfrak{b} = \boldsymbol{\mathcal{L}}_a\mathfrak{b} \tag{B.43}$$

$$\boldsymbol{\mathcal{D}}_\alpha b = -\imath_b(\mathbf{d}\alpha) = b^m(\partial_n\alpha_m - \partial_m\alpha_n)\mathbf{d}x^n \tag{B.44}$$

In fact, it is enough for the 2-form to be closed, in order to get a symmetry. If we replace $-\mathbf{d}\alpha$ by a *closed 2-form B*, the transformation is known as **B-transform**

$$\delta_B b = \imath_b B \tag{B.45}$$

[3]It is perhaps interesting to note that this notation of the partial derivative with capital index suggests the extension to a derivative with respect to some dual coordinate

$$\partial^m \equiv \partial_{\tilde{x}_m}$$

We could understand this as coordinates of a dual manifold whose tangent space coincides in some sense with the cotangent space of the original space and vice versa. This might be connected to Hull's doubled geometry [105, 103, 104, 102, 107].

To see that such an ad-hoc extension of the Dorfman bracket is not completely unfounded, note that there is a more general notion of a Dorfman bracket (or Courant bracket) in the context of Lie-bialgebroids (for a definition see e.g. [72, p.32,20]). There we have two Lie algebroids L and L^* which are dual with respect to some inner product and which both carry some Lie bracket. (For T and T^*, only T carries a Lie bracket in the beginning. For a non-trivial Lie bracket of forms on T^* we need some extra structure like e.g. a Poisson structure which would lead to the Koszul bracket on forms.) The Lie bracket on L induces a differential $\mathbf{d}$ on L^* and the Lie bracket on L^* induces a differential $\mathbf{d}^*$ on L. The definition for the Dorfman bracket on the Lie bialgebroid $L \oplus L^*$ is then

$$\begin{aligned}[\mathfrak{a},\mathfrak{b}] \equiv\ & [a,b] + \boldsymbol{\mathcal{L}}_a\beta - \boldsymbol{\mathcal{L}}_b\alpha + \mathbf{d}(\imath_b\alpha) + \\ & + [\alpha,\beta] + \boldsymbol{\mathcal{L}}_\alpha b - \boldsymbol{\mathcal{L}}_\beta a + \mathbf{d}^*(\imath_\beta a)\end{aligned}$$

The first line is the part we are used to from our usual Dorfman bracket on $T \oplus T^*$, while second line is the corresponding part coming from the nontrivial structure on L^*. Taking now $L = T$, $L^* = T^*$ and assuming that $[\alpha,\beta]$ and $\boldsymbol{\mathcal{L}}_\alpha$ and $\mathbf{d}^*$ are a Lie bracket, Lie derivative and exterior derivative built in the ordinary way, but with the new partial derivative w.r.t. the dual coordinates ∂^m, the coordinate form of the Dorfman bracket remains exactly the one of (B.35,B.36), but with $\partial_M = (\partial_m, 0)$ replaced by $\partial_M = (\partial_m, \partial^m)$. ◇

Finally, we should note that the B-transform is part of the $O(d,d)$-transformations, i.e. the transformations which leave the canonical metric invariant. As usual for orthogonal groups the infinitesimal generators are antisymmetric when the second index is pulled down with the corresponding metric. The generators of an $O(d,d)$-transformation can therefore be written as [72, p.6]

$$\Omega_{MN} = \begin{pmatrix} B_{mn} & -A_m{}^n \\ A_n{}^m & \beta^{mn} \end{pmatrix} \tag{B.46}$$

$$\Omega^M{}_N = \begin{pmatrix} A_n{}^m & \beta^{mn} \\ B_{mn} & -A_m{}^n \end{pmatrix} \tag{B.47}$$

In addition to the B-transform, acting with Ω on a generalized vector induces the so-called **beta-transform** on the 1-form component[4] as well as $Gl(d)$-transformations of vector and 1-form component via A. For constant tensors, the Lie-derivative is just a $Gl(d)$ transformation. Therefore both symmetries of the Dorfman bracket are symmetries of the canonical metric $\mathcal{G}$ as well. For this reason the canonical metric is invariant under the **Dorfman derivative $\boldsymbol{\mathcal{D}}_{\mathfrak{v}}$** with respect to a generalized vector $\mathfrak{v}$, which we define on generalized rank p tensors using (B.35) in a way that it acts via Leibniz on tensor products (like the Lie derivative) and as a directional derivative on scalars

$$(\boldsymbol{\mathcal{D}}_{\mathfrak{v}}\mathcal{T})^{M_1\ldots M_p} \equiv \mathfrak{v}^K\partial_K\mathcal{T}^{M_1\ldots M_p} + \sum_i (\partial^{M_i}\mathfrak{v}_K - \partial_K\mathfrak{v}^{M_i})\mathcal{T}^{M_1\ldots M_{i-1}KM_{i+1}\ldots M_p} \tag{B.48}$$

$$\boldsymbol{\mathcal{D}}_{\mathfrak{v}}(\mathcal{A}\otimes\mathcal{B}) = \boldsymbol{\mathcal{D}}_v\mathcal{A}\otimes\mathcal{B} + \mathcal{A}\otimes\boldsymbol{\mathcal{D}}_v\mathcal{B} \tag{B.49}$$

$$\boldsymbol{\mathcal{D}}_{\mathfrak{v}}(\phi) = \mathfrak{v}^K\partial_K\phi = v^k\partial_k\phi \tag{B.50}$$

Acting on the canonical metric, one recovers the fact, that the Dorfman derivative contains the isometries of the metric

$$\boldsymbol{\mathcal{D}}_{\mathfrak{v}}\mathcal{G} = 2(\partial^{M_1}\mathfrak{v}_K - \partial_K\mathfrak{v}^{M_1})\mathcal{G}^{KM_2} = 0 \tag{B.51}$$

Comparing the role of Lie-derivative and Dorfman-derivative, the B-transform should be understood as an extension of diffeomorphisms. In string theory it shows up in the Buscher-rules for T-duality ([108, 109]) and can perhaps be better understood geometrically via Hull's doubled geometry [105, 103, 104] (compare to footnote 3). The beta-transform is not a symmetry of the Dorfman bracket as it stands. However, if we introduce dual coordinates as suggested in footnote 3, the beta-transform would show up in the symmetry-transformations of the extended Dorfman bracket generated by itself.[5]

On an isotropic subspace L (e.g. the generalized holomorphic subspace) Courant- and Dorfman-bracket coincide and have the properties of a Lie bracket. It is therefore possible to define a Schouten bracket on generalized multivectors on $\bigwedge^\bullet L$ which have e.g. only generalized holomorphic indices (compare [72, p.21]). If we use again the notation with repeated boldface indices

$$\mathcal{A}^{(p)} \equiv \mathcal{A}_{\boldsymbol{M}\ldots\boldsymbol{M}} \equiv \mathcal{A}_{M_1\ldots M_p}\mathfrak{t}^{M_1}\cdots\mathfrak{t}^{M_2} \tag{B.52}$$

we get as coordinate form for this **Dorfman-Schouten bracket**

$$\left[\mathcal{A}^{(p)},\mathcal{B}^{(q)}\right] = p\mathcal{A}^{\boldsymbol{M}\ldots\boldsymbol{M}K}\partial_K\mathcal{B}^{\boldsymbol{M}\ldots\boldsymbol{M}} + q\left(p\partial^{\boldsymbol{M}}\mathcal{A}_K{}^{\boldsymbol{M}\ldots\boldsymbol{M}} - \partial_K\mathcal{A}^{\boldsymbol{M}\ldots\boldsymbol{M}}\right)\mathcal{B}^{K\boldsymbol{M}\ldots\boldsymbol{M}} \tag{B.53}$$

In the first term in the bracket on the righthand side, the $\partial^{\boldsymbol{M}}$ can as well be shifted with a minus sign to $\mathcal{B}$, because in $\bigwedge^\bullet L$ we have only isotropic indices in the sense that

$$\mathcal{A}^{\boldsymbol{M}\ldots\boldsymbol{M}}{}_K\mathcal{B}^{K\boldsymbol{M}\ldots\boldsymbol{M}} = 0 \tag{B.54}$$

For this reason, the Dorfman-Schouten bracket has really the required skew-symmetry of a Schouten-bracket

$$\left[\mathcal{A}^{(p)},\mathcal{B}^{(q)}\right] = -(-)^{(q+1)(p+1)}\left[\mathcal{B}^{(q)},\mathcal{A}^{(p)}\right] \tag{B.55}$$

On $\bigwedge^\bullet L$ this bracket coincides with the derived bracket of the big bracket, as the extra term with p_M in (**B**.79) vanishes because of (B.54).

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[4] The letter β for the beta-transformations does not really fit into the philosophy of the present notations, where we use small Greek letters for 1-forms (or sometimes p-forms) only, but not for multivectors. As the transformation is, however, commonly known as beta-transformation, we use a large β, in order to distinguish it from the one-forms β, which are floating around. ◇

[5] Taking the Dorfman bracket of footnote 3, we get as Dorfman derivative of a generalized vector $\mathfrak{c}$ instead of (B.43,B.44) the extended transformation

$$\boldsymbol{\mathcal{D}}_a\mathfrak{c} \equiv \boldsymbol{\mathcal{L}}_a\mathfrak{c} - \imath_\gamma(\mathbf{d}^* a)$$
$$\boldsymbol{\mathcal{D}}_\alpha\mathfrak{c} \equiv -(\imath_c\mathbf{d}\alpha) + \boldsymbol{\mathcal{L}}_\alpha\mathfrak{c}$$

I.e. the first line is extended by a beta-transformation of γ with $\beta = -\mathbf{d}^* a$ and the B-transform of α ($B = -\mathbf{d}\alpha$) in the second line is extended by a Lie derivative with respect to α. ◇

B.4 Integrability

Integrability for an ordinary complex structure means that there exist in any chart $\dim_M/2$ holomorphic vector fields (with respect to the almost complex structure) which can be integrated to holomorphic coordinates z^a in this chart of the manifold and make it a complex manifold. Those vector fields are then just $\partial/\partial z^a$. Those coordinate differentials have vanishing Lie bracket among each other (partial derivatives commute). In turn, every set of vectors with vanishing Lie bracket can be integrated to coordinates. The existence of such a set of integrable holomorphic vector fields is guaranteed when the holomorphic subbundle is closed under the Lie bracket, i.e. the Lie bracket of two holomorphic vector fields is again a holomorphic vector field.

As the Dorfman bracket restricted to the generalized holomorphic subbundle $L \subset (T \oplus T^*) \otimes \mathbb{C}$ has the properties of a Lie bracket, we can demand exactly the same for generalized holomorphic vectors as above for holomorphic ones. The condition for the generalized complex structure to be integrable is thus that the generalized holomorphic subbundle L is closed under the Dorfman bracket, i.e. in terms of the projectors

$$\bar{\Pi}\,[\Pi\mathfrak{v},\Pi\mathfrak{w}] = 0 \tag{B.56}$$

$$\iff [\mathfrak{v},\mathfrak{w}] - [\mathcal{J}\mathfrak{v},\mathcal{J}\mathfrak{w}] + \mathcal{J}\,[\mathcal{J}\mathfrak{v},\mathfrak{w}] + \mathcal{J}\,[\mathfrak{v},\mathcal{J}\mathfrak{w}] = 0 \tag{B.57}$$

In the following two sub-subsections we will show that this is equivalent to the vanishing of a **generalized Nijenhuis-tensor** [72, p.25] of the coordinate form[6,7]

$$\boxed{\frac{1}{4}\mathcal{N}^{M_1M_2M_3} \equiv \mathcal{J}^{[M_1|K}\partial_K\mathcal{J}^{|M_2M_3]} + \mathcal{J}^{[M_1|K}\mathcal{J}_K{}^{|M_2,M_3]} \stackrel{!}{=} 0} \tag{B.58}$$

Recalling that

$$\mathcal{J}^{MN} = \begin{pmatrix} P^{mn} & J^m{}_n \\ -J^n{}_m & -Q_{mn}\end{pmatrix}, \qquad \mathcal{J}_M{}^N = \begin{pmatrix} -J^n{}_m & -Q_{mn} \\ P^{mn} & J^m{}_n\end{pmatrix}, \qquad \partial^M = (0, \partial_m) \tag{B.59}$$

we can rewrite this condition in ordinary tensor components, just to compare it with the conditions given in literature (for the antisymmetrization of the capital indices we take into account that in the last term of (B.58) the indices M_1 and M_2 are automatically antisymmetrized because of $\mathcal{J}^2 = -1$):

$$\frac{1}{4}\mathcal{N}^{m_1m_2m_3} = P^{[m_1|k}\partial_k P^{|m_2m_3]} \stackrel{!}{=} 0 \tag{B.60}$$

$$\frac{1}{4}\mathcal{N}_n{}^{m_1m_2} = \frac{1}{3}\left(-J^k{}_n\partial_k P^{[m_1m_2]} + 2P^{[m_1|k}\partial_k J^{|m_2]}{}_n - P^{[m_1|k}J^{|m_2]}{}_{k,n} + J^{[m_1|}{}_k P^{k|m_2]}{}_{,n}\right) \stackrel{!}{=} 0 \tag{B.61}$$

$$\frac{1}{4}\mathcal{N}^n{}_{m_1m_2} = \frac{1}{3}\left(-P^{nk}\partial_k Q_{[m_1m_2]} + 2J^k{}_{[m_1|}\partial_k J^n{}_{|m_2]} + 2J^n{}_k J^k{}_{[m_1,m_2]} - 2P^{nk}Q_{k[m_1,m_2]}\right) \stackrel{!}{=} 0 \tag{B.62}$$

$$\frac{1}{4}\mathcal{N}_{m_1m_2m_3} = J^k{}_{[m_1|}\partial_k Q_{|m_2m_3]} + J^k{}_{[m_1|}Q_{k|m_2,m_3]} - Q_{[m_1|k}J^k{}_{|m_2,m_3]} \stackrel{!}{=} 0 \tag{B.63}$$

If we compare those expressions with the tensors A, B, C and D given in (2.16) of [91, p.7], we recognize (replacing Q by $-Q$) that our first line is just $\frac{1}{3}A$, the second line is $-\frac{1}{3}B$ (using (B.24)), the third $\frac{1}{3}C$ and the fourth line is $-\frac{1}{3}D$. There, in turn, it is claimed that the expressions are equivalent to those originally given in (3.16)-(3.19) of [87, p.7].

---

[6] This looks formally like the generalized Schouten bracket (e.g. [72, p.21]) on $\bigwedge^\bullet L$ (with L being the generalized holomorphic bundle) of $\mathcal{J}$ with itself (see also the statement below (**B**.79)), but it is not, as $\mathcal{J}$ has neither holomorphic nor antiholomorphic indices

$$\Pi\mathcal{J} = i\Pi \neq \mathcal{J}$$
$$\bar{\Pi}\mathcal{J} = -i\Pi \neq \mathcal{J}$$

In fact, we get zero if we contract both indices with the holomorphic projector

$$\Pi^N{}_L\Pi^M{}_K\mathcal{J}^{KL} = \Pi\mathcal{J}\Pi^T = i\Pi\bar{\Pi} = 0$$

The same happens for two antiholomorphic projectors. But we can project one index with an holomorphic projector and the other one with an antiholomorphic one. This yields

$$\bar{\Pi}^N{}_L\Pi^M{}_K\mathcal{J}^{KL} = \Pi\mathcal{J}\Pi = i\Pi$$

Up to a constant prefactor the bracket of Π with Π coincides with the bracket of $\mathcal{J}$ with $\mathcal{J}$. And like for the ordinary complex structure, where we have the Nijenhuis bracket of the complex structure with itself, which has one index in T and the second in T^*, we could here take Π with one index in L and the other in $\bar{L}$ and regard the bracket as generalized Nijenhuis bracket of Π with itself. ◇

[7] If instead the twisted Dorfman bracket (see footnote 2) is used, one gets the integrability condition for a twisted generalized complex structure with a twisted generalized Nijenhuis tensor. Consider the closed three form $H = H_{M_1M_2M_3}\mathbf{t}^{M_1}\mathbf{t}^{M_2}\mathbf{t}^{M_3}$ with $H_{m_1m_2m_3}$ the only nonvanishing components. The twisted generalized Nijenhuis tensor then reads

$$\mathcal{N}^H_{M_1M_2M_3} = \mathcal{N}_{M_1M_2M_3} + 6H_{M_1M_2M_3} - 18\mathcal{J}_{M_1}{}^K H_{KM_2L}\mathcal{J}^L{}_{M_3}$$

Like (B.60)-(B.63) this twisted generalized Nijenhuis tensor as well matches with the tensors given in [91] if one redefines $H_{mnk} \to \frac{1}{3!}H_{mnk}$. ◇

B.4.1 Coordinate based way to derive the generalized Nijenhuis-tensor

In this sub-subsection we will see that calculations with capital-index notation is rather convenient. So we simply calculate (B.57) brute force by using the explicit coordinate formula for the Dorfman-bracket

$$[\mathfrak{v},\mathfrak{w}]^M = \mathfrak{v}^K\partial_K\mathfrak{w}^M + \left(\partial^M\mathfrak{v}_K - \partial_K\mathfrak{v}^M\right)\mathfrak{w}^K \tag{B.35=B.64}$$

The brackets of interest are:

$$\begin{aligned}
[\mathfrak{v},\mathcal{J}\mathfrak{w}]^N &= \mathfrak{v}^K\partial_K\mathcal{J}^N{}_L\mathfrak{w}^L + \mathcal{J}^N{}_L\mathfrak{v}^K\partial_K\mathfrak{w}^L + \left(\partial^N\mathfrak{v}_K - \partial_K\mathfrak{v}^N\right)(\mathcal{J}\mathfrak{w})^K && (\mathbf{B}.65)\\
(\mathcal{J}[\mathfrak{v},\mathcal{J}\mathfrak{w}])^M &= \underline{\mathfrak{v}^K\mathcal{J}^M{}_N\partial_K\mathcal{J}^N{}_L\mathfrak{w}^L} - \mathfrak{v}^K\partial_K\mathfrak{w}^M + \mathcal{J}^M{}_N\left(\partial^N\mathfrak{v}_K - \partial_K\mathfrak{v}^N\right)(\mathcal{J}\mathfrak{w})^K && (\mathbf{B}.66)\\
[\mathcal{J}\mathfrak{v},\mathfrak{w}]^N &= \mathcal{J}^K{}_L\mathfrak{v}^L\partial_K\mathfrak{w}^N + \left(\partial^N\mathcal{J}_{KL} - \partial_K\mathcal{J}^N{}_L\right)\mathfrak{v}^L\mathfrak{w}^K + \left(\mathcal{J}_K{}^L\partial^N\mathfrak{v}_L - \mathcal{J}^N{}_L\partial_K\mathfrak{v}^L\right)\mathfrak{w}^K && (\mathbf{B}.67)\\
(\mathcal{J}[\mathcal{J}\mathfrak{v},\mathfrak{w}])^M &= \mathcal{J}^M{}_N(\mathcal{J}\mathfrak{v})^K\partial_K\mathfrak{w}^N + \underline{\mathcal{J}^M{}_N\left(\partial^N\mathcal{J}_{KL} - \partial_K\mathcal{J}^N{}_L\right)\mathfrak{v}^L\mathfrak{w}^K} + \\
&\quad -(\mathcal{J}\mathfrak{w})^L\mathcal{J}^M{}_N\partial^N\mathfrak{v}_L + \partial_K\mathfrak{v}^M\mathfrak{w}^K && (\mathbf{B}.68)\\
[\mathcal{J}\mathfrak{v},\mathcal{J}\mathfrak{w}]^M &= \mathcal{J}^K{}_N\mathfrak{v}^N\partial_K\mathcal{J}^M{}_L\mathfrak{w}^L + \mathcal{J}^K{}_N\mathfrak{v}^N\mathcal{J}^M{}_L\partial_K\mathfrak{w}^L + \\
&\quad \left(\partial^M\mathcal{J}_{KN}\mathfrak{v}^N - \partial_K\mathcal{J}^M{}_N\mathfrak{v}^N\right)\mathcal{J}^K{}_L\mathfrak{w}^L + \left(\mathcal{J}_{KN}\partial^M\mathfrak{v}^N - \mathcal{J}^M{}_N\partial_K\mathfrak{v}^N\right)\mathcal{J}^K{}_L\mathfrak{w}^L = && (\mathbf{B}.69)\\
&= (\mathcal{J}\mathfrak{v})^K\mathcal{J}^M{}_L\partial_K\mathfrak{w}^L - \mathcal{J}^M{}_N\partial_K\mathfrak{v}^N(\mathcal{J}\mathfrak{w})^K + \\
&\quad +\underline{\left(\mathcal{J}^K{}_L\partial^M\mathcal{J}_{KN} + 2\mathcal{J}^K{}_{[N|}\partial_K\mathcal{J}^M{}_{|L]}\right)\mathfrak{v}^N\mathfrak{w}^L} + \partial^M\mathfrak{v}_L\mathfrak{w}^L && (\mathbf{B}.70)
\end{aligned}$$

The underlined terms sum up in the complete expression to the generalized Nijenhuis tensor, while the rest cancels

$$\begin{aligned}
0 &\stackrel{!}{=} [\mathfrak{v},\mathfrak{w}]^M - [\mathcal{J}\mathfrak{v},\mathcal{J}\mathfrak{w}]^M + (\mathcal{J}[\mathcal{J}\mathfrak{v},\mathfrak{w}])^M + (\mathcal{J}[\mathfrak{v},\mathcal{J}\mathfrak{w}])^M = && (\mathbf{B}.71)\\
&= \left(2\mathcal{J}^M{}_K\partial_{[N}\mathcal{J}^K{}_{L]} - \mathcal{J}^K{}_L\partial^M\mathcal{J}_{KN} + \mathcal{J}^{MK}\partial_K\mathcal{J}_{LN} - 2\mathcal{J}^K{}_{[N|}\partial_K\mathcal{J}^M{}_{|L]}\right)\mathfrak{v}^N\mathfrak{w}^L = && (\mathbf{B}.72)\\
&= \mathfrak{v}_N\left(3\mathcal{J}^{[M|}{}_K\mathcal{J}^{K|L,N]} + 3\mathcal{J}^{[N|K}\partial_K\mathcal{J}^{|ML]}\right)\mathfrak{w}_L = && (\mathbf{B}.73)\\
&= \frac{3}{4}\mathfrak{v}_N\mathcal{N}^{NML}\mathfrak{w}_L && (\mathbf{B}.74)
\end{aligned}$$

B.4.2 Derivation via derived brackets

Eventually we want to see directly how the generalized Nijenhuis tensor is connected to derived brackets. We will use our insight from the subsections 6.1.1 and 6.1.2. Remember, our basis $\mathbf{t}^M = (\mathbf{d}x^m, \boldsymbol{\partial}_m)$ was identified with the conjugate (ghost-)variables $\mathbf{t}^M \equiv (\boldsymbol{c}^m, \boldsymbol{b}_m)$. One can define generalized multi-vector fields of the form

$$\mathcal{K}^{(\mathrm{K})} \equiv \mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} \equiv \mathcal{K}_{M_1\ldots M_{\mathrm{K}}}\mathbf{t}^{M_1}\cdots\mathbf{t}^{M_{\mathrm{K}}} \tag{B.75}$$

They are in fact just sums of multivector valued forms:

$$\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} = \sum_{k=0}^{\mathrm{K}}\binom{\mathrm{K}}{k}\mathcal{K}_{\underbrace{\boldsymbol{m}\ldots\boldsymbol{m}}_{k}}{}^{\overbrace{\boldsymbol{n}\ldots\boldsymbol{n}}^{\mathrm{K}-k}} \equiv \sum_{k=0}^{\mathrm{K}} K^{(k,\mathrm{K}-k)} \tag{B.76}$$

The big bracket, or Buttin's algebraic bracket is then just the canonical Poisson bracket

$$\begin{aligned}
[\mathcal{K},\mathcal{L}]^{\Delta}_{(1)} &\equiv \mathrm{KL}\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\mathcal{L}_{I\boldsymbol{M}\ldots\boldsymbol{M}} = \{\mathcal{K},\mathcal{L}\} && (\mathbf{B}.77)\\
\{\mathbf{t}_M,\mathbf{t}_N\} &= \mathcal{G}_{MN} && (\mathbf{B}.78)
\end{aligned}$$

The coordinate expression for its derived bracket (compare to (6.52,6.54)) reads

$$\begin{aligned}
(-)^{\mathrm{K}-1}\left[\mathbf{d}\mathcal{K}^{(\mathrm{K})},\mathcal{L}^{(\mathrm{L})}\right]^{\Delta}_{(1)} &= \mathrm{K}\cdot\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\partial_I\mathcal{L}_{\boldsymbol{M}\ldots\boldsymbol{M}} - (-)^{(\mathrm{K}+1)(\mathrm{L}+1)}\mathrm{L}\cdot\mathcal{L}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\partial_I\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}} + \\
&\quad +(-)^{\mathrm{K}-1}\mathrm{KL}\partial_{\boldsymbol{M}}\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{I}\mathcal{L}_{I\boldsymbol{M}\ldots\boldsymbol{M}} + \mathrm{K}(\mathrm{K}-1)\,\mathrm{L}\mathcal{K}_{\boldsymbol{M}\ldots\boldsymbol{M}}{}^{IJ}\mathcal{L}_{I\boldsymbol{M}\ldots\boldsymbol{M}}p_J && (\mathbf{B}.79)
\end{aligned}$$

with $p_J \equiv (p_j, 0)$ and $\partial_I \equiv (\partial_i, 0)$. In the case were both $\mathcal{K}$ and $\mathcal{L}$ only have generalized holomorphic indices, the p-term drops and this expression should coincide with the Schouten-bracket on $\bigwedge^{\bullet} L$ for the holomorphic Lie-algebroid L (see e.g. [72, p.21] and footnote 6). For two rank-two objects, like the generalized complex structure $\mathcal{J}$, this reduces to

$$[\mathcal{K},_{\mathbf{d}}\mathcal{L}]^{\Delta}_{(1)} = 2\cdot\mathcal{K}_{\boldsymbol{M}}{}^{I}\partial_I\mathcal{L}_{\boldsymbol{M}\boldsymbol{M}} + 2\cdot\mathcal{L}_{\boldsymbol{M}}{}^{I}\partial_I\mathcal{K}_{\boldsymbol{M}\boldsymbol{M}} - 4\partial_{\boldsymbol{M}}\mathcal{K}_{\boldsymbol{M}}{}^{I}\mathcal{L}_{I\boldsymbol{M}} + 4\mathcal{K}^{IJ}\mathcal{L}_{I\boldsymbol{M}}p_J \tag{B.80}$$

which reads for two coinciding tensors $\mathcal{J}$

$$[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} = 4\cdot\mathcal{J}_{\boldsymbol{M}}{}^{I}\partial_{I}\mathcal{J}_{\boldsymbol{MM}} - 4\partial_{\boldsymbol{M}}\mathcal{J}_{\boldsymbol{M}}{}^{I}\mathcal{J}_{I\boldsymbol{M}} - 4\mathcal{J}^{JI}\mathcal{J}_{I\boldsymbol{M}}p_{J} = \qquad (\mathbf{B}.81)$$

$$\overset{(B.58)}{\underset{\mathcal{J}^2=-1}{=}} \mathcal{N}_{\boldsymbol{M}\ldots\boldsymbol{M}} + 4\underbrace{p_{\boldsymbol{M}}\mathbf{t}^{\boldsymbol{M}}}_{=\boldsymbol{o}\ (6.8)} \qquad (\mathbf{B}.82)$$

where $\boldsymbol{o} = \mathbf{d}x^k p_k = -\mathbf{d}(\mathbf{d}x^k \wedge \boldsymbol{\partial}_k)$. We will verify this relation between the generalized Nijenhuis tensor and the derived bracket in the following calculation, where we calculate $\mathcal{N}$ using the big bracket ($\mathbf{B}$.77) all the time. This bracket is like a matrix multiplication if one of the objects has only one index. We will use this fact frequently for the multiplication of $\mathcal{J}$ with a vector

$$\mathcal{J}\mathfrak{v} \equiv \mathcal{J}^{M}{}_{N}\mathfrak{v}^{N}\mathbf{t}_{M} = \frac{1}{2}\{\mathcal{J},\mathfrak{v}\} \qquad (\mathbf{B}.83)$$

$$\Rightarrow \{\mathcal{J},\{\mathcal{J},\mathfrak{v}\}\} = 4\mathcal{J}^2\mathfrak{v} = -4\mathfrak{v} = \{\{\mathfrak{v},\mathcal{J}\},\mathcal{J}\} \qquad (\mathbf{B}.84)$$

$$\{\{\mathfrak{v},\mathcal{J}\},\{\mathcal{J},\mathfrak{w}\}\} = -4\mathfrak{v}^{K}\mathfrak{w}_{K} = -4\{\mathfrak{v},\mathfrak{w}\} \qquad (\mathbf{B}.85)$$

If both objects are of higher rank, however, antisymmetrization of the remaining indices modifies the result. We thus have to be careful with the following examples

$$\{\mathcal{J},\mathcal{J}\} = 4\mathcal{J}_{\boldsymbol{M}}{}^{K}\mathcal{J}_{K\boldsymbol{M}} = -4\mathcal{G}_{\boldsymbol{MM}} = 0 \quad \text{(! because of antisymmetrization)} \qquad (\mathbf{B}.86)$$

$$\{\mathcal{J},\{\mathcal{J},\mathbf{d}\mathfrak{v}\}\} = \mathcal{J}_{\boldsymbol{M}}{}^{K}\mathcal{J}_{[K|}{}^{L}(\mathbf{d}\mathfrak{v})_{L|\boldsymbol{M}]} \neq -4\mathbf{d}\mathfrak{v} \qquad (\mathbf{B}.87)$$

As mentioned earlier, the Dorfman bracket (B.31) used in our integrability condition is just the derived bracket of the algebraic bracket. I.e. we have

$$[\mathfrak{v},\mathfrak{w}] = [\mathbf{d}\mathfrak{v},\mathfrak{w}]^{\Delta} = \qquad (\mathbf{B}.88)$$

$$= [\mathbf{d}\mathfrak{v},\mathfrak{w}]^{\Delta}_{(1)} + \underbrace{\sum_{p\geq 2}[\mathbf{d}\mathfrak{v},\mathfrak{w}]^{\Delta}_{(p)}}_{=0} = \qquad (\mathbf{B}.89)$$

$$= \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} \qquad (\mathbf{B}.90)$$

where the differential $\mathbf{d}$ has to be understood in the extended sense of (6.9,6.33), namely as Poisson-bracket with the BRST-like generator

$$\boldsymbol{o} = \mathbf{t}^{M}p_{M} = \boldsymbol{c}^{m}p_{m} \overset{\text{locally}}{=} \mathbf{d}(x^{m}p_{m}) = -\mathbf{d}(\boldsymbol{c}^{m}\boldsymbol{b}_{m}) \qquad (\mathbf{B}.91)$$

$$p_{M} \equiv (p_{m},0) \qquad (\mathbf{B}.92)$$

$$\mathbf{d}\mathfrak{v} \equiv \{\boldsymbol{o},\mathfrak{v}\} = \partial_{\boldsymbol{M}}v_{\boldsymbol{M}} + \mathfrak{v}^{K}p_{K} \qquad (\mathbf{B}.93)$$

where p_m is the conjugate variable to x^m. We can now rewrite the integrability condition (B.57) as

$$\{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\{\mathcal{J},\mathfrak{w}\}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathfrak{w}\}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} \overset{!}{=} 0 \qquad (\mathbf{B}.94)$$

Remember that the Poisson bracket is a graded one, and $\mathfrak{v},\mathfrak{w}$ and $\mathbf{d}$ are odd, while $\mathcal{J}$ is even.

Let us now start with applying Jacobi to the second term of ($\mathbf{B}$.94)

$$-\frac{1}{4}\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\{\mathcal{J},\mathfrak{w}\}\} = -\frac{1}{4}\{\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} - \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathfrak{w}\}\} \qquad (\mathbf{B}.95)$$

so that we get

$$0 \overset{!}{=} \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\{\mathbf{d}\{\mathcal{J},\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} = \qquad (\mathbf{B}.96)$$

$$= \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\{\{\mathbf{d}\mathcal{J},\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} - \frac{1}{4}\{\{\{\mathcal{J},\mathbf{d}\mathfrak{v}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} = \qquad (\mathbf{B}.97)$$

$$= \{\mathbf{d}\mathfrak{v},\mathfrak{w}\} - \frac{1}{4}\{\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\{\{\mathbf{d}\mathfrak{v},\mathcal{J}\},\mathcal{J}\},\mathfrak{w}\} + \frac{1}{4}\{\mathcal{J},\{\mathbf{d}\mathfrak{v},\{\mathcal{J},\mathfrak{w}\}\}\} \qquad (\mathbf{B}.98)$$

It would be nice to separate $\mathfrak{w}$ completely by moving it for the last term into the last bracket like in the first three terms. We thus consider only the last term for a moment and calculate it in two different ways (first using

Jacobi for second and third bracket and after that using Jacobi for first and second bracket):

$$\frac{1}{4}\{\mathcal{J},\{\mathfrak{dv},\{\mathcal{J},\mathfrak{w}\}\}\} \overset{1.}{=} \frac{1}{4}\{\mathcal{J},\{\{\mathfrak{dv},\mathcal{J}\},\mathfrak{w}\}\}+\frac{1}{4}\{\mathcal{J},\{\mathcal{J},\{\mathfrak{dv},\mathfrak{w}\}\}\} = \tag{B.99}$$

$$= \frac{1}{4}\{\mathcal{J},\{\{\mathfrak{dv},\mathcal{J}\},\mathfrak{w}\}\}-\{\mathfrak{dv},\mathfrak{w}\} \tag{B.100}$$

$$\overset{2.}{=} \frac{1}{4}\{\{\mathcal{J},\mathfrak{dv}\},\{\mathcal{J},\mathfrak{w}\}\}+\frac{1}{4}\{\mathfrak{dv},\{\mathcal{J},\{\mathcal{J},\mathfrak{w}\}\}\} = \tag{B.101}$$

$$= \frac{1}{4}\{\mathcal{J},\{\{\mathcal{J},\mathfrak{dv}\},\mathfrak{w}\}\}+\frac{1}{4}\{\{\{\mathcal{J},\mathfrak{dv}\},\mathcal{J}\},\mathfrak{w}\}-\{\mathfrak{dv},\mathfrak{w}\} = \tag{B.102}$$

$$= -\frac{1}{4}\{\mathcal{J},\{\{\mathfrak{dv},\mathcal{J}\},\mathfrak{w}\}\}+\{\mathfrak{dv},\mathfrak{w}\}-2\{\mathfrak{dv},\mathfrak{w}\}+\frac{1}{4}\{\{\{\mathcal{J},\mathfrak{dv}\},\mathcal{J}\},\mathfrak{w}\} \tag{B.103}$$

Comparing both calculations yields

$$\frac{1}{4}\{\mathcal{J},\{\mathfrak{dv},\{\mathcal{J},\mathfrak{w}\}\}\} = -\frac{1}{8}\{\{\mathcal{J},\{\mathcal{J},\mathfrak{dv}\}\},\mathfrak{w}\}-\{\mathfrak{dv},\mathfrak{w}\} \tag{B.104}$$

We can plug this back in (**B**.98) and leave away the outer bracket with $\mathfrak{w}$:

$$0 \overset{!}{=} \mathfrak{dv}-\frac{1}{4}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\}+\frac{1}{4}\{\{\mathfrak{dv},\mathcal{J}\},\mathcal{J}\}-\frac{1}{8}\{\mathcal{J},\{\mathcal{J},\mathfrak{dv}\}\}-\mathfrak{dv} = \tag{B.105}$$

$$= -\frac{1}{4}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\}+\frac{1}{8}\{\{\mathfrak{dv},\mathcal{J}\},\mathcal{J}\} = \tag{B.106}$$

$$= -\frac{1}{8}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\}+\frac{1}{8}\{\mathbf{d}\{\mathfrak{v},\mathcal{J}\},\mathcal{J}\} = \tag{B.107}$$

$$= -\frac{1}{8}\{\{\mathfrak{v},\mathbf{d}\mathcal{J}\},\mathcal{J}\}+\frac{1}{8}\mathbf{d}\{\{\mathfrak{v},\mathcal{J}\},\mathcal{J}\}+\frac{1}{8}\{\{\mathfrak{v},\mathcal{J}\},\mathbf{d}\mathcal{J}\} = \tag{B.108}$$

$$= -\frac{1}{8}\{\mathfrak{v},\{\mathbf{d}\mathcal{J},\mathcal{J}\}\}-\frac{1}{2}\mathfrak{dv} = \tag{B.109}$$

$$= \frac{1}{8}\Big(\{[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)},\mathfrak{v}\}-4\mathfrak{dv}\Big) = \tag{B.110}$$

$$= \frac{1}{8}\{[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)}-4\boldsymbol{o},\mathfrak{v}\} \tag{B.111}$$

where we used

$$\mathfrak{dv} = \{\boldsymbol{o},\mathfrak{v}\} \tag{B.112}$$

The integrability condition is thus (explaining the normalization of $\mathcal{N}$ of above) as promised in (**B**.82)

$$\boxed{\mathcal{N}\equiv[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)}-4\boldsymbol{o}\overset{!}{=}0} \tag{B.113}$$

The derived bracket $[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)}$ indeed contains the term $4\boldsymbol{o}=4\mathbf{t}^{M}p_{M}$ which therefore is exactly cancelled.

Precisely the same calculation can be performed by calculating with the complete algebraic bracket $[\,,\,]^{\Delta}$ instead of the Poisson-bracket, its first order part. Similarly to above, we have

$$\mathcal{J}\mathfrak{v} \equiv \frac{1}{2}[\mathcal{J},\mathfrak{v}]^{\Delta} \tag{B.114}$$

$$\Rightarrow [\mathcal{J},[\mathcal{J},\mathfrak{v}]^{\Delta}]^{\Delta} = 4\mathcal{J}^{2}\mathfrak{v}=-4\mathfrak{v} \tag{B.115}$$

In combination with (**B**.88) this is enough to redo the same calculation and get as integrability condition (using $[\mathcal{J},\mathcal{J}]\equiv-[\mathbf{d}\mathcal{J},\mathcal{J}]^{\Delta}$)

$$\boxed{\mathcal{N}\equiv[\mathcal{J},\mathcal{J}]-4\boldsymbol{o}\overset{!}{=}0} \tag{B.116}$$

which also proves that the derived bracket bracket of the big bracket (which is not necessarily geometrically well defined) coincides in this case with the complete derived bracket

$$[\mathcal{J},_{\mathbf{d}}\mathcal{J}]^{\Delta}_{(1)} = [\mathcal{J},\mathcal{J}] \tag{B.117}$$

As discussed in (C.53) and (C.55), throwing away the $\mathbf{d}$-closed part corresponds to taking Buttin's bracket instead of the derived one. Remember that $\boldsymbol{o}=\mathbf{d}x^{k}p_{k}=-\mathbf{d}(\mathbf{d}x^{k}\wedge\boldsymbol{\partial}_{k})$, s.th. $\mathbf{d}\boldsymbol{o}=0$. We can thus equally write

$$\mathcal{N} = [\mathcal{J},\mathcal{J}]_{B} \tag{B.118}$$

B.5 SO(d,d) pure spinors

There exists an alternative description of a generalized complex structure and its integrability with the help of pure spinors (see e.g. [72, p.8] or in section 3 of [106]). "Spinor" here refers to the special orthonormal group $SO(d,d)$ (d being the dimension of the manifold M) of transformations on $T \oplus T^*$ which leave the canonical metric $\langle \ldots , \ldots \rangle$ or $\mathcal{G}_{MN}$ (which has signature (d,d)) invariant. It turns out that $T \oplus T^*$ itself, embedded via

$$\imath_{X+\alpha}\rho \quad \equiv \quad \underbrace{\imath_X \rho}_{\equiv X \llcorner \rho} + \alpha \wedge \rho, \qquad X \in TM, \quad \alpha \in T^*M, \quad \rho \in \wedge^\bullet T^*M \tag{B.119}$$

into the space of endomorphisms of $\wedge^\bullet T^*M$ (formal sum of differential forms on M), forms a representation of the Clifford algebra. The spinors are thus differential forms $\rho \in \wedge^\bullet T^*M$ and the gamma "matrices" $\mathbf{\Gamma}^M$ are up to a normalization factor just the interior products $\imath_{\mathbf{t}^M} = \frac{i}{\hbar}\hat{\mathbf{t}}_M$ with respect to the basis elements $\mathbf{t}^M = (\mathbf{d}x^m, \boldsymbol{\partial}_m) \equiv (\boldsymbol{c}^m, \boldsymbol{b}_m)$, i.e. $\mathbf{\Gamma}^M = \{\sqrt{2}\imath_{\boldsymbol{\partial}_m}, \sqrt{2}\imath_{\mathbf{d}x^n} \equiv \mathbf{d}x^n \wedge\}$. Indeed, the graded commutator (i.e. anticommutator) of the basis elements reads $[\imath_{\boldsymbol{\partial}_m}, \mathbf{d}x^n \wedge] = \delta_m^n$ and therefore[8]

$$\left[\mathbf{\Gamma}^M, \mathbf{\Gamma}^N\right] = 2\mathcal{G}^{MN} \tag{B.120}$$

For general elements of the algebra (generalized vectors) $\mathfrak{v} = \mathfrak{v}_M \mathbf{t}^M$, $\mathfrak{w} = \mathfrak{w}_N \mathbf{t}^N$, the Clifford algebra becomes as usual $[\imath_\mathfrak{v}, \imath_\mathfrak{w}] = 2\langle \mathfrak{v}, \mathfrak{w} \rangle$.

One can further define a chirality matrix $\Gamma^\#$. It is characterized by the properties that it squares to 1 and anticommutes with all other $\mathbf{\Gamma}$-matrices. Usually it is proportional to the product of all $\mathbf{\Gamma}$-matrices, but this is only true in a basis where $\mathcal{G}_{MN}$ is diagonal. In our basis $\mathbf{t}_M$ it is off-diagonal. The definition of the $\mathbf{\Gamma}$-matrices as $\tilde{\mathbf{\Gamma}}^M = \left\{\imath_{(\mathbf{d}x^m - \boldsymbol{\partial}_m)}, \imath_{(\mathbf{d}x^m + \boldsymbol{\partial}_m)}\right\}$ thus would be more appropriate in this context. The overall sign is a matter of taste and we choose it such that the eigenvalues of rank r forms in (**B**.125) do not depend on the dimension. The chirality matrix is then given by

$$\begin{aligned}
\Gamma^\# \quad &\equiv \quad (-)^d \prod_{k=0}^{d-1} \imath_{(\mathbf{d}x^k - \boldsymbol{\partial}_k)} \imath_{(\mathbf{d}x^k + \boldsymbol{\partial}_k)} = && \text{(B.121)}\\
&= \quad (-)^d \left(\imath_{\mathbf{d}x^0}\imath_{\boldsymbol{\partial}_0} - \imath_{\boldsymbol{\partial}_0}\imath_{\mathbf{d}x^0}\right) \cdots \left(\imath_{\mathbf{d}x^{d-1}}\imath_{\boldsymbol{\partial}_{d-1}} - \imath_{\boldsymbol{\partial}_{d-1}}\imath_{\mathbf{d}x^{d-1}}\right) = && \text{(B.122)}\\
&= \quad (-)^d \left(2\imath_{\mathbf{d}x^0}\imath_{\boldsymbol{\partial}_0} - 1\right) \cdots \left(2\imath_{\mathbf{d}x^{d-1}}\imath_{\boldsymbol{\partial}_{d-1}} - 1\right) = && \text{(B.123)}\\
&= \quad (-)^d \prod_{k=0}^{d-1} \left(2\mathfrak{n}_{\mathbf{d}x^k} - 1\right) && \text{(B.124)}
\end{aligned}$$

where $\mathfrak{n}_{\mathbf{d}x^k} \equiv \sum \imath_{\mathbf{d}x^k}\imath_{\boldsymbol{\partial}_k}$ counts the number of $\mathbf{d}x^k$ (with fixed k) of the differential form $\rho^{(r)}$ on which $\Gamma^\#$ is acting. This number can be (in each term of the expansion in basis elements) either zero or one, because $(\mathbf{d}x^k)^2 = 0$. The terms $(2\mathfrak{n}_{\mathbf{d}x^k} - 1)$ are therefore either -1 (if $\mathbf{d}x^k$ does not appear) or 1 (if it appears). In a form $\rho^{(r)}$ of rank r, there are of course in any term of the expansion r basis elements $\mathbf{d}x^k$ which appear and $d - r$ which do not appear. We thus have

$$\Gamma^\# \rho^{(r)} \quad = \quad (-)^d (-1)^{d-r} \rho^{(r)} = (-1)^r \rho^{(r)} \tag{B.125}$$

The chiral and antichiral spinors (those with eigenvalues $+1$ or -1) therefore correspond to even and odd forms respectively.

A *pure spinor* is defined to be a spinor which is annihilated by half of the gamma matrices. (The same was true for the pure spinor in the Berkovits context, although it is not obvious due to the formulation via the quadratic constraint $\boldsymbol{c}\gamma^m\boldsymbol{c} = 0$.) :

$$\rho \text{ is pure} : \iff \begin{array}{c} L_\rho \equiv \{\mathfrak{a} \in (T^*M \oplus TM) \otimes \mathbb{C} | \quad i_\mathfrak{a}\rho = 0\} \\ \text{is of dimension } d = \dim M \end{array} \tag{B.126}$$

In other words, the Clifford action of $(T \oplus T^*)$ is maximally light-like. How is this related to an almost generalized complex structure $\mathcal{J}$? The structure $\mathcal{J}$ induces a splitting of $(T^*M \oplus TM) \otimes \mathbb{C}$ into a subbundle of eigenvalue i and another one of eigenvalue $-i$:

$$\begin{aligned}
(T^*M \oplus TM) \otimes \mathbb{C} \quad &= \quad L_\mathcal{J} \oplus L_\mathcal{J}^* \\
L_\mathcal{J} \quad &\equiv \quad \{\mathfrak{a} \in (T^*M \oplus TM) \otimes \mathbb{C} | \, \mathcal{J}(\mathfrak{a}) = i\mathfrak{a}\}
\end{aligned} \tag{B.127}$$

[8] Note that one can think of $\imath_{\partial_m}$ as $\frac{\partial}{\partial \mathbf{d}x^m}$. Another observation is that the Poisson bracket of the $T \oplus T^*$ basis elements also forms a Clifford-algebra

$$\left\{\mathbf{t}^M, \mathbf{t}^N\right\} \quad = \quad \mathcal{G}^{MN} \qquad \diamond$$

Setting $L_{\mathcal{J}} \stackrel{!}{=} L_{\rho_{\mathcal{J}}}$ induces a map from generalized complex structures to pure spinors and one can prove that it is well-defined and one-to-one (up to a rescaling of the pure spinor) [72]. The previosly discussed (twisted) integrability condition can also be refomulated in the pure spinor language. Integrability of $L_{\mathcal{J}}$ is closed under the action of the (twisted) Dorfman bracket. $\mathfrak{a}, \mathfrak{b} \in L_{\rho_{\mathcal{J}}} \Rightarrow [\mathfrak{a},\mathfrak{b}] = [[\imath_{\mathfrak{a}}, \mathbf{d}], \imath_{\mathfrak{b}}] \in L_{\rho_{\mathcal{J}}}$. In other words $[[\imath_{\mathfrak{a}}, \mathbf{d} + H\wedge], \imath_{\mathfrak{b}}]\rho_{\mathcal{J}} = 0 \quad \forall \mathfrak{a}, \mathfrak{b}$ with $\imath_{\mathfrak{a}}\rho_{\mathcal{J}} = \imath_{\mathfrak{b}}\rho_{\mathcal{J}} = 0$. Writing the graded commutator explicitely and using $\imath_{\mathfrak{a}}\rho_{\mathcal{J}} = \imath_{\mathfrak{b}}\rho_{\mathcal{J}} = 0$, this becomes [106]

$$\mathcal{J} \quad \text{is twisted integrable} \quad :\Longleftrightarrow \quad \imath_{\mathfrak{b}}\imath_{\mathfrak{a}}\mathbf{d}_H\rho_{\mathcal{J}} \equiv \imath_{\mathfrak{b}}\imath_{\mathfrak{a}}\left(\mathbf{d} + H\wedge\right)\rho_{\mathcal{J}} = 0 \quad \forall \mathfrak{a}, \mathfrak{b} \in L_{\rho_{\mathcal{J}}} \tag{B.128}$$

One can think of $\rho_{\mathcal{J}}$ as a Clifford vacuum and of the elements of $L_{\rho_{\mathcal{J}}}$ as annihilation operators. The creation operators then lie in $L^*_{\mathcal{J}}$ and $\mathbf{d}_H\rho_{\mathcal{J}}$ must be at most at creator level two. However, as any creator changes parity, and $\mathbf{d}\rho$ is of opposite parity than ρ itself, it can only be at odd creator-levels, i.e. level one. The above condition is thus equivalent to

$$\mathcal{J} \quad \text{is twisted integrable} \quad :\Longleftrightarrow \mathbf{d}_H\rho_{\mathcal{J}} = \imath_{\mathfrak{c}}\rho_{\mathcal{J}} \quad \text{for some } \mathfrak{c} \in L^*_{\mathcal{J}} \tag{B.129}$$

Appendix C

Derived Brackets

Mathematics in this section is based on the review article on derived brackets by Kosmann-Schwarzbach [70]. The presentation, however, will be somewhat different and in addition to (or sometimes instead of) the abstract definitions coordinate expressions will be given.

C.1 Lie bracket of vector fields, Lie derivative and Schouten bracket

This first subsection is intended to give a feeling, why the Schouten bracket is a very natural extension of the Lie bracket of vector fields. It is a good example to become more familiar with the subject, before we become more general in the subsequent subsections, but it can be skipped without any harm (note however the notation introduced before (C.13)).

Consider the ordinary **Lie-bracket of vector fields** which turns the tangent space of a manifold into a Lie algebra or the tangent bundle into a Lie algebroid and which takes in a local coordinate basis the familiar form

$$[v,w]^m = v^k\partial_k w^m - w^k\partial_k v^m \tag{C.1}$$

We will convince ourselves in the following that numerous other common differential brackets are just natural extensions of this bracket and can be regarded as one and the same bracket. Such a generalized bracket is e.g. useful to formulate integrability conditions and it can serve via the Jacobi identity as a powerful tool in otherwise lengthy calculations . In addition it shows up naturally in some sigma-models as is discussed in section 6.

Given the Lie-bracket of vector fields, it seems natural to extend it to higher rank tensor fields by demanding a Leibniz rule on tensor products of the form $[v,w_1\otimes w_2] = [v,w_1]\otimes w_2 + w_1\otimes[v,w_2]$. Remembering that the Lie-bracket of two vector fields is just the Lie derivative of one vector field with respect to the other

$$[v,w] = \boldsymbol{\mathcal{L}}_v w \tag{C.2}$$

the **Lie derivative** of a general tensor $T = T_{m_1\ldots m_p}^{n_1\ldots n_q}\mathbf{d}x^{m_1}\otimes\ldots\otimes\mathbf{d}x^{m_p}\otimes\boldsymbol{\partial}_{n_1}\otimes\cdots\otimes\boldsymbol{\partial}_{n_q}$ with respect to a vector field v can be seen as a first extension of the Lie bracket:

$$[v,T] \equiv \boldsymbol{\mathcal{L}}_v T \tag{C.3}$$

$$[v,T]_{m_1\ldots m_p}^{n_1\ldots n_q} = v^k\partial_k T_{m_1\ldots m_p}^{n_1\ldots n_q} - \sum_i \partial_k v^{n_i} T_{m_1\ldots m_p}^{n_1\ldots n_{i-1}k\,n_{i+1}\ldots n_q} + \sum_j \partial_{m_j} v^k T^{n_1\ldots n_q}_{m_1\ldots m_{j-1}k\,m_{j+1}\ldots m_p} \tag{C.4}$$

The Lie derivative obeys (as a derivative should) the **Leibniz rule**

$$[v,T_1\otimes T_2] = [v,T_1]\otimes T_2 + T_1\otimes[v,T_2] \tag{C.5}$$

In fact, giving as input only the Lie derivative of a scalar ϕ, namely the directional derivative $[v,\phi]\equiv v^k\partial_k\phi$, and the Lie bracket of vector fields (C.1), the Lie derivative of general tensors (C.4) is determined by the Leibniz-rule. Insisting on antisymmetry of the bracket, we have to define

$$[T,v] \equiv -[v,T] \tag{C.6}$$

Indeed, it can be checked that the above definitions lead to a valid Jacobi-identity of the form

$$[v,[w,T]] = [[v,w],T] + [w,[v,T]] \quad \text{for arbitrary tensors } T \tag{C.7}$$

which is perhaps better known in the form

$$[\boldsymbol{\mathcal{L}}_v,\boldsymbol{\mathcal{L}}_w]T = \boldsymbol{\mathcal{L}}_{[v,w]}T \tag{C.8}$$

We have now vectors acting via the bracket on general tensors, but tensors only acting on vectors via (C.6) . It is thus natural to use Leibniz again to define the action of tensors on tensors. To make a long story short, this is not possible for general tensors. It is possible, however, for tensors with only upper indices which are either antisymmetrized (**multivectors**) or symmetrized (**symmetric multivectors**). We will concentrate in this paper on tensors with antisymmetrized indices (the reason being the natural given differential for forms which also have antisymmetrized indices), but the symmetric case makes perfect sense and at some points we will give short comments. (See e.g. [110] for more information on the Schouten bracket of symmetric tensor fields.)

Given two **multivector fields** (note that the prefactor $1/p!$ is intentionally missing (see page 146).

$$v^{(p)} \equiv v^{m_1\ldots m_p}\boldsymbol{\partial}_{m_1}\wedge\ldots\wedge\boldsymbol{\partial}_{m_p}, \qquad w^{(q)} \equiv w^{m_1\ldots m_q}\boldsymbol{\partial}_{m_1}\wedge\ldots\wedge\boldsymbol{\partial}_{m_q} \tag{C.9}$$

their Schouten(-Nijenhuis) bracket, or **Schouten bracket** for short, is given in a local coordinate basis by

$$\left[v^{(p)},w^{(q)}\right]^{m_1\ldots m_{p+q-1}} = pv^{[m_1\ldots m_{p-1}|k}\partial_k w^{|m_p\ldots m_{p+q-1}]} - qv^{[m_1\ldots m_p|}{}_{,k}w^{k\,|m_{p+1}\ldots m_{p+q-1}]} \tag{C.10}$$

Realizing that the Lie-derivative (C.4) of a multivector field $w^{(q)}$ with respect to a vector $v^{(1)}$ is

$$\left[v,w^{(q)}\right]^{n_1\ldots n_q} = v^k\partial_k w^{n_1\ldots n_q} - q\partial_k v^{[n_1|}w^{k\,|n_2\ldots n_q]} \tag{C.11}$$

one recognizes that (C.10) is a natural extension of this, obeying a Leibniz rule, which we will write down below in (C.18). However, as the coordinate form of generalized brackets will become very lengthy at some point, we will first introduce some **notation** which is more schematic, although still exact. Namely we imagine that every **boldface index $\boldsymbol{m}$** is an ordinary index m contracted with the corresponding basis vector $\boldsymbol{\partial}_m$ at the position of the index:

$$v^{(p)} = v^{m_1\ldots m_p}\boldsymbol{\partial}_{m_1}\wedge\ldots\wedge\boldsymbol{\partial}_{m_p} \equiv v^{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.12}$$

This saves us the writing of the basis vectors as well as the enumeration or manual antisymmetrization of the indices. As a boldface index might be hard to distinguish from an ordinary one, we will use this notation only for several indices, s.th. we get repeated indices $\boldsymbol{m}\ldots\boldsymbol{m}$ which are easily to recognize (and are not summed over, as they are at the same vertical position). See in the appendix A on page 147 for a more detailed explanation. The Schouten bracket then reads

$$\left[v^{(p)},w^{(q)}\right] = pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - qv^{\boldsymbol{m}\ldots\boldsymbol{m}}{}_{,k}w^{k\,\boldsymbol{m}\ldots\boldsymbol{m}} = \tag{C.13}$$

$$= pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - (-)^{p(q-1)}qw^{k\,\boldsymbol{m}\ldots\boldsymbol{m}}v^{\boldsymbol{m}\ldots\boldsymbol{m}}{}_{,k} = \tag{C.14}$$

$$= pv^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - (-)^{(p-1)(q-1)}qw^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k v^{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.15}$$

In the last line it becomes obvious that the bracket is **skew-symmetric** in the sense of a Lie algebra of degree[1] -1:

$$\left[v^{(p)},w^{(q)}\right] = -(-)^{(p-1)(q-1)}\left[w^{(q)},v^{(p)}\right] \tag{C.16}$$

[1]A **Lie bracket** $\left[\;,_{(n)}\;\right]$ **of degree** n in a graded algebra increases the degree (which we denote by $|\ldots|$) by n

$$\left|\left[A,_{(n)}B\right]\right| = |A| + |B| + n$$

It can be understood as an ordinary graded Lie-bracket, when we redefine the grading $\|\ldots\| \equiv |\ldots| + n$, such that the Lie bracket itself does not carry a grading any longer

$$\left\|\left[A,_{(n)}B\right]\right\| = \|A\| + \|B\|$$

The symmetry properties are thus (**skew symmetry of degree** n)

$$\left[A,_{(n)}B\right] = -(-)^{(|A|+n)(|A|+n)}\left[B,_{(n)}A\right]$$

and it obeys the usual graded Jacobi-identity (with shifted degrees)

$$\left[A,_{(n)}\left[B,_{(n)}C\right]\right] = \left[\left[A,_{(n)}B\right],_{(n)}C\right] + (-)^{(|A|+n)(|A|+n)}\left[B,_{(n)}\left[A,_{(n)}C\right]\right]$$

In addition there might be a Poisson-relation with respect to some other product which respects the original grading. To be consistent with both gradings, this relation has to read

$$\left[A,_{(n)}B\cdot C\right] = \left[A,_{(n)}B\right]\cdot C + (-)^{(|A|+n)|B|}B\cdot\left[A,_{(n)}C\right]$$

This is consistent with $B\cdot C = (-)^{|B||C|}C\cdot B$ on the one hand and the skew symmetry of the bracket on the other hand. One can imagine the grading of the bracket to sit at the position of the comma.

For the bracket of multivectors we have as degree the vector degree. Later, when we will have tensors of mixed type (vector and form), we will use the form degree minus the vector degree as total degree. Then the Schouten-bracket is of degree +1, which should not confuse the reader. ◇

It obeys the corresponding **Jacobi identity**

$$\left[v_1^{(p_1)},\left[v_2^{(p_2)},v_3^{(p_3)}\right]\right] = \left[\left[v_1^{(p_1)},v_2^{(p_2)}\right],v_3^{(p_3)}\right]+(-)^{(p_1-1)(p_2-1)}\left[v_2^{(p_2)},\left[v_1^{(p_1)},v_3^{(p_3)}\right]\right] \tag{C.17}$$

Our starting point was to extend the bracket in a way that it acts via Leibniz on the wedge product. A Lie algebra which has a second product on which the bracket acts via Leibniz is known as Poisson algebra. However, here the bracket has degree -1 (it reduces the multivector degree by one) while the wedge product has no degree (the degree of the wedge product of multivectors is just the sum of the degrees). According to footnote 1, we have to adjust the Leibniz rule. The resulting algebra for Lie brackets of degree -1 is known as **Gerstenhaber algebra** or in this special case **Schouten algebra** (which is the standard example for a Gerstenhaber algebra). The **Leibniz rule** is

$$\left[v_1^{(p_1)},v_2^{(p_2)}\wedge v_3^{(p_3)}\right] = \left[v_1^{(p_1)},v_2^{(p_2)}\right]\wedge v_3^{(p_3)}+(-)^{(p_1-1)p_2}v_2^{(p_2)}\wedge\left[v_1^{(p_1)},v_3^{(p_3)}\right] \tag{C.18}$$

The standard example in field theory for a Poisson algebra is the phase space equipped with the Poisson bracket or the commutator of operators or matrices.[2] The Schouten algebra is naturally realized by the **antibracket** of the BV antifield formalism (see subsection 6.5).

C.2 Embedding of vectors into the space of differential operators

The Leibniz rule is not the only concept to generalize the vector Lie bracket to higher rank tensors. The major difficulty in the definition of brackets between higher rank tensors is the Jacobi-identity, which should hold for them. It is therefore extremely useful to have a mechanism which automatically guarantees the Jacobi identity. A way to get such a mechanism is to **embed** the tensors into some space of differential operators, as for the operators we have the commutator as natural Lie bracket which might in turn induce some bracket on the tensors we started with. Vector fields e.g. naturally act on differential forms via the **interior product**

$$\imath_v\omega^{(p)} \equiv p\cdot v^k\omega_{k\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.19}$$

This can be seen as the embedding of vector fields in the space of differential operators acting on forms, because the interior product with respect to a vector is a graded derivative with the grading -1 of the vector (we take as total degree the form degree minus the multivector degree, which for a vector is just -1)

$$\imath_v\left(\omega^{(p)}\wedge\eta^{(q)}\right) = \imath_v\omega^{(p)}\wedge\eta^{(q)}+(-)^q\omega^{(p)}\wedge\imath_v\eta^{(q)} \tag{C.20}$$

Taking the idea of above we can take the commutator of two interior products. We note, however, that it only induces a trivial (always vanishing) bracket on the vectorfields

$$[\imath_v,\imath_w] = 0=\imath_0 \tag{C.21}$$

As the interior product (C.19) does not include any partial derivative on the vector-coefficient, it was clear from the beginning that this ansatz does not lead to the Lie bracket of vector fields or any generalization of it. We have to bring the exterior derivative into the game, in our notation

$$\mathbf{d}\omega^{(p)}=\partial_{\boldsymbol{m}}\omega_{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.22}$$

There are two ways to do this

- *Change the embedding:* Instead of embedding the vectors via the interior product acting on forms, we can embed them via the Lie-derivative acting on forms. When acting on forms, the Lie derivative can be written as the (graded) commutator of interior product and exterior derivative

$$\boldsymbol{\mathcal{L}}_v = [\imath_v,\mathbf{d}] \tag{C.23}$$
$$\boldsymbol{\mathcal{L}}_v\omega^{(p)} = v^k\partial_k\omega_{\boldsymbol{m}\ldots\boldsymbol{m}}+p\cdot\partial_{\boldsymbol{m}}v^k\omega_{k\,\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.24}$$

  Indeed, using the Lie derivative as embedding $v\mapsto\boldsymbol{\mathcal{L}}_v$, the commutator of Lie derivatives induces the Lie bracket of vector fields (a special case of (C.8)

$$[\boldsymbol{\mathcal{L}}_v,\boldsymbol{\mathcal{L}}_w] = \boldsymbol{\mathcal{L}}_{[v,w]} \tag{C.25}$$

[2] In fact, working with totally symmetric multivector fields would have lead to a Poisson algebra instead of a Gerstenhaber algebra. ◇

- *Change the bracket:* In the space of differential operators acting on forms, the commutator is the most natural Lie bracket. However, the existence of a nilpotent odd operator acting on our algebra, namely the commutator with the exterior derivative, enables the construction of what is called a **derived bracket**[3].

$$[\imath_v,_{\mathbf{d}}\imath_w] \equiv [[\imath_v, \mathbf{d}], \imath_w] \tag{C.26}$$

This derived bracket (which is in this case a Lie bracket again, as we are considering the abelian subalgebra of interior products of vector fields) indeed induces the Lie bracket of vector fields when we use the interior product as embedding

$$[\imath_v,_{\mathbf{d}}\imath_w] = \imath_{[v,w]} \tag{C.27}$$

The above equations plus two additional ones are the well known **Cartan formulae**

$$[\imath_v, \imath_w] = 0 = [\mathbf{d}, \mathbf{d}] \tag{C.28}$$
$$\boldsymbol{\mathcal{L}}_v = [\imath_v, \mathbf{d}] \tag{C.29}$$
$$[\boldsymbol{\mathcal{L}}_v, \mathbf{d}] = 0 \tag{C.30}$$
$$[\boldsymbol{\mathcal{L}}_v, \boldsymbol{\mathcal{L}}_w] = \boldsymbol{\mathcal{L}}_{[v,w]} \tag{C.31}$$
$$\big[\underbrace{[\imath_v, \mathbf{d}]}_{\boldsymbol{\mathcal{L}}_v}, \imath_w]\big] = \imath_{[v,w]} \tag{C.32}$$

(C.25) can be rewritten, using Jacobi's identity and $[\mathbf{d}, \mathbf{d}] = 0$, as

$$[[[\imath_v, \mathbf{d}], \imath_w], \mathbf{d}] = \left[\imath_{[v,w]}, \mathbf{d}\right] \tag{C.33}$$

Starting from (C.27), one thus arrives at (C.25) by simply taking the commutator with **d**. We will therefore concentrate in the following on the second possibility, using the derived bracket, as the first one can be deduced from it. Let us just mention that the generalization in the spirit of the derived bracket (C.27) (or more precise its skew-symmetrization) is known as **Vinogradov bracket** [113, 114] (see footnote 8), while the generalization in the spirit of (C.25) is known as **Buttin's bracket** [96].

C.3 Derived bracket for multivector valued forms

Let us now consider a much more general case, namely the space of multivector valued forms, i.e. tensors which are antisymmetric in the upper as well as in the lower indices. With the Schouten bracket we have a bracket for multivectors, which are antisymmetric in all (upper) indices. There exists as well a bracket for vector valued forms, namely tensors with one upper index and arbitrary many antisymmetrized lower indices. This bracket (which we have not yet discussed) is the (Fröhlicher-) Nijenhuis bracket (see (C.67)), which shows up in the integrability condition for almost complex structures. Multivector valued forms have arbitrary many antisymmetrized upper and arbitrary antisymmetrized lower indices and thus contain both cases. The antisymmetrization appears quite naturally in field theory (we give only a few remarks about completely symmetric indices, which appear as well, but which will not be subject of this paper). It makes also sense to define brackets on sums of tensors of different type (e.g. the Dorfman bracket for generalized complex geometry). Those brackets are then simply given by linearity.

[3] Given a bracket $\left[\ ,_{(n)}\ \right]$ of degree n (not necessarily a Lie bracket. It can be as well a **Loday bracket** where the skew-symmetry property as compared to footnote 1 is missing, but the Jacobi identity still holds) and a differential D (derivation of degree 1 and square 0), its **derived bracket** [111, 112, 70] (which is of degree $n+1$) is defined by

$$\left[a,_{(\mathrm{D})}\, b\right] = (-)^{n+a+1}\left[\mathrm{D}a,_{(n)}\, b\right]$$

We put the subscript (D) at the position of the comma, to indicate that the grading of D is sitting there. The strange sign is just to make the definition nicer for the most frequent case of an interior derivation, where $\mathrm{D}a = \left[d,_{(n)}\, a\right]$ with d some element of the algebra with degree $\mid d \mid= 1-n$ and $\left[d,_{(n)}\, d\right] = 0$, s.th. we have

$$[a,_d\, b] = \left[\left[a,_{(n)}\, d\right],_{(n)}\, b\right]$$

The derived bracket is then again a Loday bracket (of degree $n+1$) and obeys the corresponding Jacobi-identity (that is always the nontrivial part). If a, b are elements of a commuting subalgebra ($[a,_{(n)}\, b] = 0$), the derived bracket even is skew-symmetric and thus a Lie bracket of degree $n+1$.

In the case at hand we start with a Lie bracket of degree 0 (the commutator) and take as interior derivation the commutator with the exterior derivative $[\mathbf{d}, \ldots]$. Note that the exterior derivative itself is a derivative on forms, but not on the space of differential operators on forms. Therefore we need the commutator. ◇

So let us consider two multivector valued forms (we denote the number of lower indices and the number of upper indices in this order via superscripts)[4]

$$K^{(k,k')} \equiv K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \equiv K_{m_1\ldots m_k}{}^{n_1\ldots n_{k'}} \mathbf{d}x^{m_1}\cdots\mathbf{d}x^{m_k}\otimes\boldsymbol{\partial}_{n_1}\cdots\boldsymbol{\partial}_{n_{k'}} \tag{C.34}$$

$$L^{(l,l')} \equiv L_{\underbrace{\boldsymbol{m}\ldots\boldsymbol{m}}_{l}}{}^{\underbrace{\boldsymbol{n}\ldots\boldsymbol{n}}_{l'}} \tag{C.35}$$

Note the use of the schematic index notation, which we used for upper indices already in subsection C.1 and which is explained in the appendix A on page 147. Following the ideas of above, we want to embed those vector valued forms in some space of differential operators. As we have upper as well as lower indices now, it is less clear why we should choose the space of operators acting on forms and not on some other tensors for the embedding. However, the space of forms is the only one where we have a natural exterior derivative without using any extra structure[5]. Therefore we will define again a natural embedding into the space of differential operators acting on forms as a generalization of the interior product. Namely, we will act with a multivector valued form K on a form ρ by just contracting all upper indices with form-indices and antisymmetrizing the remaining lower indices s.th. we get again a form as result. The formal definition goes in two steps. First one defines the interior product with multivectors. For a decomposable multivector $v^{(p)} = v_1\wedge\ldots\wedge v_p$ set

$$\imath_{v_1\wedge\ldots\wedge v_p}\rho^{(r)} \equiv \imath_{v_1}\cdots\imath_{v_p}\rho^{(r)} \tag{C.36}$$

This fixes the interior product for a generic multivector uniquely (contracting all indices with form-indices). The next step is to define for a multivector valued form $K^{(k,k')} = \eta^{(k)}\wedge v^{(k')}$ which is decomposable in a form and a multivector, that it acts on a form by first acting with the multivector as above and then wedging the result with the form

$$\imath_{\eta^{(k)}\wedge v^{(k')}}\rho \equiv \eta^{(k)}\wedge\imath_{v^{(k)}}\rho = (-)^{k'k}\imath_{v^{(k')}\wedge\eta^{(k)}}\rho \tag{C.37}$$

It is kind of a normal ordering that $\imath_{v^{(k')}}$ acts first:

$$\imath_\eta\imath_v = \imath_{\eta^{(k)}\wedge v^{(k')}} = (-)^{kk'}\imath_{v^{(k')}\wedge\eta^{(k)}} \neq \imath_v\imath_\eta \tag{C.38}$$

For a generic multivector valued form, the above definitions fix the following coordinate form of the **interior product**[6] with a multivector valued form

$$\imath_{K^{(k,k')}}\rho^{(r)} \equiv (k')!\begin{pmatrix} r\\ k'\end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}}\rho_{\underbrace{l_{k'}\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}_{r}} \tag{C.39}$$

So we are just contracting all the upper indices of K with an appropriate number of indices of the form and are wedging the remaining lower indices. The origin of the combinatorial prefactor is perhaps more transparent in the phase space formulation (6.13) in subsection 6.1. For multivectors $v^{(p)}$ and $w^{(q)}$ the operator product of $\imath_{v^{(p)}}$ and $\imath_{w^{(q)}}$ induces, due to (C.36) simply the wedge product of the multivectors

$$\imath_{v^{(p)}}\imath_{w^{(q)}} = \imath_{v^{(p)}\wedge w^{(q)}} \tag{C.40}$$

But for general multivector-valued forms we have instead[7]

$$\imath_{K^{(k,k')}}\imath_{L^{(l,l')}} = \sum_{p=0}^{k'}\imath_{\imath^{(p)}_K L} = \imath_{K\wedge L} + \sum_{p=1}^{k'}\imath_{\imath^{(p)}_K L} \tag{C.41}$$

[4]One can certainly map a tensor $K_m{}^n\mathbf{d}x^m\otimes\boldsymbol{\partial}_n$ to one where the basis elements are antisymmetrized $K_m{}^n\mathbf{d}x^m\wedge\boldsymbol{\partial}_n \overset{\text{see page 146}}{\equiv} \frac{1}{2}K_m{}^n\mathbf{d}x^m\otimes\boldsymbol{\partial}_n - \frac{1}{2}K_m{}^n\boldsymbol{\partial}_n\otimes\mathbf{d}x^m$ and vice versa. In the field theory applications we will always get a complete antisymmetrization. This mapping is the reason why we take care for the horizontal positions of the indices. It should just indicate the order of the basis elements which was chosen for the mapping. ◇

[5]One can define an exterior derivative – the **Lichnerowicz-Poisson differential** – on the space of multivectors as well (via the Schouten bracket), but for this we need an integrable Poisson structure: $\mathbf{d}_P N^{(q)} \equiv [P^{(2)},N^{(q)}]$, with $[P^{(2)},P^{(2)}] = 0$ ◇

[6]The name 'interior product' is misleading in the sense that the operation is (for decomposable tensors) a composition of interior and exterior wedge product. It will, however, in the generalizations of Cartan's formulae play the role of the interior product. We will therefore stick to this name. We can also see it as a short name for 'interior product of maximal order' in the sense that all upper indices are contracted as opposed to an interior 'product of order p', where we contract only p upper indices. 'Order' is in the sense of the order of a derivative. While $\imath_v$ is a derivative for any vector v, the general interior product acts like a higher order derivative. ◇

[7]The product of interior products in (C.41) induces a noncommutative product (star product) for the multivector-valued forms, whose commutator is the algebraic bracket, namely

$$K * L \equiv \sum_{p\geq 0}\imath^{(p)}_K L$$
$$[K,L]^{\Delta} = K*L - (-)^{(k-k')(l-l')}L*K \qquad \diamond$$

with

$$\imath^{(p)}_{K^{(k,k')}} L^{(l,l')} \equiv (-)^{(k'-p)(l-p)} p! \begin{pmatrix} k' \\ p \end{pmatrix} \begin{pmatrix} l \\ p \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \tag{C.42}$$

For $p = k'$, $\imath^{(p)}_K$ reduces to the interior product (C.39). Both are in general not a derivative any longer. $\imath^{(p)}$ is, however, a p-th order derivative, as contracting p indices means taking the p-th derivative with respect to p basis elements (see 6.18 in subsection 6.1). Our embedding $\imath_{K^{(k,k')}}$ in (C.39) is therefore a k'-th order derivative. For $p = 0$ on the other hand, $\imath^{(p)}_K$ is just a wedge product with K

While for vectors the commutator of two interior products (C.21) did only induce a trivial bracket on vectors, which is the same for multivectors due to (C.40), this is different for multivector-valued forms.

$$[\imath_{K^{(k,k')}}, \imath_{L^{(l,l')}}] = \imath_{[K,L]^{\Delta}} \tag{C.43}$$

$$[K,L]^{\Delta} \equiv \sum_{p\geq 1} \underbrace{\imath^{(p)}_K L - (-)^{(k-k')(l-l')} \imath^{(p)}_L K}_{\equiv [K,L]^{\Delta}_{(p)}} = \tag{C.44}$$

$$\begin{aligned} &= \sum_{p\geq 1} (-)^{(k'-p)(l-p)} p! \begin{pmatrix} k' \\ p \end{pmatrix} \begin{pmatrix} l \\ p \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} L_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\ &\quad -(-)^{(k-k')(l-l')} (-)^{(l'-p)(k-p)} p! \begin{pmatrix} l' \\ p \end{pmatrix} \begin{pmatrix} k \\ p \end{pmatrix} L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1\ldots l_p} K_{l_p\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \end{aligned} \tag{C.45}$$

where we introduced an **algebraic bracket** $[K,L]^{\Delta}$ in the second line, which is is due to Buttin [96], and which is a generalization of the Nijenhuis-Richardson bracket for vector-valued forms (C.63). As it was induced via the embedding from the graded commutator, it has the same properties, i.e. it is graded antisymmetric and obeys the graded Jacobi identity. Actually, the term with lowest p, so $[K,L]^{\Delta}_{(p=1)}$, is itself an algebraic bracket, which appears in subsection 6.1.1 as canonical Poisson bracket. It is known under the name **Buttin's algebraic bracket** ([96], denoted in [70] by $[\,,\,]^0_B$) or as **big bracket**

$$[K,L]^{\Delta}_{(1)} = \imath^{(1)}_K L - (-)^{(k-k')(l-l')} \imath^{(1)}_L K = \tag{C.46}$$

$$\begin{aligned} &= (-)^{(k'-1)(l-1)} k' l \cdot K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} L_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} + \\ &\quad -(-)^{(k-k')(l-l')} (-)^{(l'-1)(k-1)} l' k \cdot L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n} l_1} K_{l_1 \boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}\ldots\boldsymbol{n}} \end{aligned} \tag{C.47}$$

But as for the vector fields in subsection C.2, we are rather interested in the derived bracket of $[K,L]^{\Delta}$, or at the bracket induced via an embedding based on the Lie derivative. An obvious generalization of the Lie derivative is the commutator $[\imath_K, \mathbf{d}]$, which will be a derivative of the same order as $\imath_K$ and therefore is not a derivative in the sense that it obeys the Leibniz rule. Although it is common to use this generalization, I am not aware of an appropriate name for it. Let us just call it the **Lie derivative with respect to** K (being a derivative of order k')

$$\boldsymbol{\mathcal{L}}_{K^{(k,k')}} \equiv [\imath_{K^{(k,k')}}, \mathbf{d}] \tag{C.48}$$

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{K^{(k,k')}} \rho &= (k')! \begin{pmatrix} r+1 \\ k' \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \partial_{[l_{k'}} \rho_{l_{k'-1}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}]} + \\ &\quad -(-)^{k-k'} (k')! \begin{pmatrix} r \\ k' \end{pmatrix} \partial_{\boldsymbol{m}} \left(K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \rho_{l_{k'}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}} \right) = \end{aligned} \tag{C.49}$$

$$\begin{aligned} &= (k')! \begin{pmatrix} r \\ k'-1 \end{pmatrix} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \partial_{l_{k'}} \rho_{l_{k'-1}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}} + \\ &\quad -(-)^{k-k'} (k')! \begin{pmatrix} r \\ k' \end{pmatrix} \partial_{\boldsymbol{m}} K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l_1\ldots l_{k'}} \rho_{l_{k'}\ldots l_1 \boldsymbol{m}\ldots\boldsymbol{m}} \end{aligned} \tag{C.50}$$

The Lie derivative above is an ingredient to calculate the **derived bracket** (remember footnote 3 on page 162) which is given by[8]

$$[\imath_K,_{\mathbf{d}} \imath_L] \equiv [[\imath_K, \mathbf{d}], \imath_L] \equiv \imath_{[K,L]} \quad \text{if possible} \tag{C.51}$$

[8] The **Vinogradov bracket** [114, 113] (see also [70]) is a bracket in the space of all graded endomorphisms in the space of differential forms $\Omega^{\bullet}(M)$

$$[a,b]_V = \frac{1}{2} \left([[a,d], b] - (-)^b [a,[b,d]] \right) \quad \forall a,b \in \Omega^{\bullet}(M)$$

It is the skew symmetrization of a derived bracket. The embedding of the multivector valued forms into the endomorphisms $\Omega^{\bullet}(M)$ via the interior product which we consider is neither closed under the Vinogradov bracket nor under the derived bracket in the general case. ◇

One should distinguish the derived bracket on the level of operators on the left from the derived bracket on the tensors $[K,L]$ on the right. Only in special cases the result of the commutator on the left can be written as the interior product of another tensorial object which then can be considered as the derived bracket with respect to the algebraic bracket $[\,,\,]^{\Delta}$. Therefore one normally does not find an explicit general expression for this derived bracket in literature. In 6.1.2, however, the meaning of exterior derivative and interior product are extended in order to be able to write down an explicit general coordinate expression (6.51) which reduces in the mentioned special cases to the well known results (see e.g. C.4.2).

Closely related to the derived bracket in (C.51) of above is **Buttin's differential bracket**, given by

$$[\boldsymbol{\mathcal{L}}_K, \boldsymbol{\mathcal{L}}_L] \equiv \boldsymbol{\mathcal{L}}_{[K,L]_B} \quad \text{if possible} \tag{C.52}$$

Because of $[\mathbf{d},\mathbf{d}] = 0$ and $\boldsymbol{\mathcal{L}}_K = [\imath_K, \mathbf{d}]$ we have (using Jacobi)

$$[\boldsymbol{\mathcal{L}}_K, \boldsymbol{\mathcal{L}}_L] = [[\imath_K, _{\mathbf{d}}\imath_L], \mathbf{d}] = [[\imath_K, _{\mathbf{d}}\imath_L], \mathbf{d}] \overset{!}{=} [\imath_{[K,L]_B}, \mathbf{d}] \tag{C.53}$$

Comparing with (C.51) s.th. in cases where $[K,L]$ exists, the brackets have to coincide up to a closed term, or locally a total derivative

$$\imath_{[K,L]} = \imath_{[K,L]_B} + [\mathbf{d}, \ldots] \tag{C.54}$$

Using again the extended definition of exterior derivative and interior product of 6.1.2, this relation can be rewritten as

$$[K,L] = [K,L]_B + \mathbf{d}(\ldots) \tag{C.55}$$

The Nijenhuis bracket (C.74) is the major example for this relation.

C.4 Examples

C.4.1 Schouten(-Nijenhuis) bracket

Let us shortly review the Schouten bracket under the new aspects. For multivectors $v^{(p)}, w^{(q)}$ the algebraic bracket vanishes

$$[\imath_{v^{(p)}}, \imath_{w^{(q)}}] = 0 \tag{C.56}$$

The **Schouten bracket** $\left[v^{(p)},w^{(q)}\right]$ coincides with the derived bracket as well as with Buttin's differential bracket, i.e. we have

$$[[\imath_{v^{(p)}}, \mathbf{d}], \imath_{w^{(q)}}] = \imath_{\left[v^{(p)},w^{(q)}\right]} \tag{C.57}$$

$$[\boldsymbol{\mathcal{L}}_{v^{(p)}}, \boldsymbol{\mathcal{L}}_{w^{(q)}}] = \boldsymbol{\mathcal{L}}_{\left[v^{(p)},w^{(q)}\right]} \tag{C.58}$$

Its coordinate form – given already before in (C.15) – is

$$\left[v^{(p)},w^{(q)}\right] = p v^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k w^{\boldsymbol{m}\ldots\boldsymbol{m}} - (-)^{(p-1)(q-1)} q w^{\boldsymbol{m}\ldots\boldsymbol{m}k}\partial_k v^{\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.59}$$

The vector Lie bracket is a special case of the Schouten bracket as well as of the Nijenhuis bracket.

C.4.2 (Fröhlicher-)Nijenhuis bracket and its relation to the Richardson-Nijenhuis bracket

Consider vector valued forms, i.e. tensors of the form

$$K^{(k,1)} \equiv K_{m_1\ldots m_k}{}^n \mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_k} \wedge \boldsymbol{\partial}_n \cong K_{m_1\ldots m_k}{}^n \mathbf{d}x^{m_1} \wedge \cdots \wedge \mathbf{d}x^{m_k} \otimes \boldsymbol{\partial}_n \tag{C.60}$$

The algebraic bracket of two such tensors, defined via the graded commutator (note that $\mid \imath_K \mid = \mid K \mid = k-1$)

$$[\imath_K, \imath_L] = \imath_{[K,L]^{\Delta}} \tag{C.61}$$

consists only of the first term in the expansion, because we have only one upper index to contract.

$$\left[K^{(k,1)}, L^{(l,1)}\right]^{\Delta} = \left[K^{(k,1)}, L^{(l,1)}\right]^{\Delta}_{(1)} = \imath^{(1)}_K L - (-)^{(k-1)(l-1)}\imath^{(1)}_L K = \tag{C.62}$$

$$\overset{(C.47)}{=} l\, K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j L_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} - (-)^{(k-1)(l-1)} k\, L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^j K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} \tag{C.63}$$

It is thus just the big bracket or Buttin's algebraic bracket but in this case it is known as **Richardson-Nijenhuis-bracket**.

The Lie derivative of a form with respect to K (in the sense of (C.48)) is because of $k' = 1$ really a (first order) derivative and takes the form

$$\mathcal{L}_{K^{(k,1)}} \equiv [\imath_{K^{(k,1)}}, \mathbf{d}] \tag{C.64}$$

$$\mathcal{L}_{K^{(k,1)}}\rho^{(r)} = K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l}\partial_l\rho_{\boldsymbol{m}\ldots\boldsymbol{m}} + (-)^k r\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{l}\rho_{l\boldsymbol{m}\ldots\boldsymbol{m}} \tag{C.65}$$

The **(Froehlicher-)Nijenhuis bracket** is defined as the unique tensor $[K,L]_N$, s.th.

$$[\mathcal{L}_K,\mathcal{L}_L] = \mathcal{L}_{[K,L]_N} \tag{C.66}$$

It is therefore an example of Buttin's differential bracket. Its explicit coordinate form reads

$$\begin{aligned}[K,L]_N \equiv\ & K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}\partial_j L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + (-)^k l\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}L_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + \\ & -(-)^{kl}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}\partial_j K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} - (-)^{kl}(-)^l k\partial_{\boldsymbol{m}}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} \end{aligned} \tag{C.67}$$

$$= \text{"}\mathcal{L}_K L - (-)^{kl}\mathcal{L}_L K\text{"} \tag{C.68}$$

A different point of view on the Nijenhuis bracket is via the **derived bracket** on the level of the differential operators acting on forms:

$$[\imath_K,_{\mathbf{d}}\imath_L] \equiv [[\imath_K,\mathbf{d}],\imath_L] \tag{C.69}$$

It induces the Nijenhuis-bracket only up to a total derivative (the Lie-derivative-term)

$$[\imath_K,_{\mathbf{d}}\imath_L] \equiv \imath_{[K,L]_N} - (-)^{k(l-1)}\mathcal{L}_{\imath_L K} \tag{C.70}$$

Using the extended definition of the exterior derivative in the sense of (6.37) and of the interior product (6.32), one can write the Lie derivative as an interior product (see 6.35) $\mathcal{L}_{\imath_L K} = -(-)^{l+k}\imath_{\mathbf{d}(\imath_L K)}$ and $[[\imath_K,\mathbf{d}],\imath_L] = (-)^k[\imath_{\mathbf{d}K},\imath_L] = (-)^k\imath_{[\mathbf{d}K,L]^\Delta}$, so that we can rewrite (C.70) as

$$[K,L] \equiv [K,L]_N + (-)^{(k-1)l}\mathbf{d}(\imath_L K) \tag{C.71}$$

$$\text{with } [K,L] \equiv (-)^k[\mathbf{d}K,L]^\Delta \tag{C.72}$$

In that sense, $[K,L]$ is the derived bracket of the Richardson Nijenhuis bracket while the Nijenhuis bracket differs by a total derivative. The explicit coordinate form can be read off from (6.49,6.51) (with only the $p=1$ term surviving)

$$[K,L] = (-)^k\imath^{(1)}_{\mathbf{d}K}L + (-)^{kl}(-)^l\imath^{(1)}_{\mathbf{d}L}K + (-)^{(k-1)l}\mathbf{d}(\imath^{(p)}_L K) = \tag{C.73}$$

$$\begin{aligned} =\ & K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}\partial_j L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + (-)^k l\partial_{\boldsymbol{m}}K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}L_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + \\ & -(-)^{kl}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}\partial_j K_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} - (-)^{kl}(-)^l k\partial_{\boldsymbol{m}}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}} + \\ & +(-)^{(k-1)l}\mathbf{d}\big(\underbrace{kL_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}}}_{\imath_L K}\big) \end{aligned} \tag{C.74}$$

where the last part is non-tensorial due to the appearance of the basis element p_i (see subsection 6.1.2):

$$\mathbf{d}(\imath_L K) = \mathbf{d}\big(kL_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}}\big) = k\partial_{\boldsymbol{m}}\left(L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{\boldsymbol{n}}\right) - (-)^{l+k}L_{\boldsymbol{m}\ldots\boldsymbol{m}}{}^{j}K_{j\boldsymbol{m}\ldots\boldsymbol{m}}{}^{i}p_i \tag{C.75}$$

The remaining part coincides with the coordinate form of the **Nijenhuis bracket** as given in (C.67).

One can nicely summarize the algebra of graded derivations on forms as

$$\begin{aligned}&\left[\mathcal{L}_{K_1^{(k_1)}} + \imath_{L_1^{(l_1)}}, \mathcal{L}_{K_2^{(k_2)}} + \imath_{L_2^{(l_2)}}\right] = \\ &= \mathcal{L}_{[K_1,K_2]_N + \imath_{L_1}K_2 - (-)^{(l_2-1)k_1}\imath_{L_2}K_1} + \imath_{[K_1,L_2]_N - (-)^{(l_1-1)k_2}[K_2,L_1]_N + [L_1,L_2]^\Delta}\end{aligned} \tag{C.76}$$

Appendix D

Gamma-Matrices in 10 Dimensions

D.1 Clifford algebra, Fierz identity and more for the Dirac matrices

In the following we will collect some general relations for Dirac-Γ-matrices in d dimensions. In contrast to the rest of this document, we are not using graded conventions in most of this appendix. In other words, the spinorial indices are not understood to carry a grading and we are thus using neither graded summation conventions nor the graded equal sign. The reason is that a lot of people (me included) are used to calculate with Γ-matrices in ordinary conventions, and it therefore seemed to be simpler for me to translate only the results into the graded conventions. This does not mean, however, that calculating in the graded conventions would be more complicated. Let us give two examples, how to translate the results. Remember first that in northwest-southeast (NW) $\delta_\alpha^\beta = \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = -\delta^{\boldsymbol{\beta}}{}_{\boldsymbol{\alpha}}$. The equation $\delta_\alpha^\alpha = 16$ therefore becomes $16 = \delta_\alpha^\alpha = \sum_\alpha \delta_\alpha^\alpha = \sum_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}} = -\sum_{\boldsymbol{\alpha}} (-)^{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}} = -\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}}$. When there are naked indices, we also have to take into account the graded equal sign, which compares the order of the indices in each term: $\gamma^c_{\alpha\beta} = \gamma^c_{\beta\alpha}$ becomes $\gamma^c_{\boldsymbol{\alpha\beta}} = (-)^{\boldsymbol{\alpha\beta}} \gamma^c_{\boldsymbol{\beta\alpha}} = -\gamma^c_{\boldsymbol{\beta\alpha}}$.

Remember the form of the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1} \qquad \Longleftrightarrow \Gamma^{(a}\Gamma^{b)} = \eta^{ab}\mathbb{1} \tag{D.1}$$

Define as ususal $\Gamma^{a_1\ldots a_p} \equiv \Gamma^{[a_1} \cdots \Gamma^{a_p]}$. The set $\{\Gamma^I\} \equiv \{\mathbb{1}, \Gamma^a, \Gamma^{a_1 a_2}, \ldots, \Gamma^{a_1 \ldots a_{10}}\}$ then builds a basis of $Gl(2^{[d/2]})$ where $2^{[d/2]}$ is the dimension of the representation space.

Product of antisymmetrized products of Γ-matrices One can in particular expand any product of antisymmetrized gamma matrices in the basis $\{\Gamma^I\}$:

$$\Gamma^{a_1\ldots a_p}\Gamma^{b_1\ldots b_q} = \sum_{k=0}^{min\{p,q\}} k! \begin{pmatrix} p \\ k \end{pmatrix} \begin{pmatrix} q \\ k \end{pmatrix} \eta^{[a_p|\,[b_1|}\eta^{|a_{p-1}|\,|b_2|} \cdots \eta^{|a_{p+1-k}|\,|b_k|}\Gamma^{|a_1\ldots a_{p-k}]\,|b_{k+1}\ldots b_q]} \tag{D.2}$$

The antisymmetrization brackets on the righthand side shall indicate that all the a_i's and all the b_i's are independently antisymmetrized. The expressions become quite lengthy, if one spells out the antisymmetrization explicitely. Let us write down the first terms only, using the notation where a check ˇ above an index means that this index is omitted:[1]

$$\begin{aligned}
\Gamma^{a_1\ldots a_k}\Gamma^{b_1\ldots b_l} = \; & \Gamma^{a_1\ldots a_k b_1\ldots b_l} + \sum_{i=1}^{k}\sum_{j=1}^{l}(-)^{k-i+j-1}\eta^{a_i b_j}\Gamma^{a_1\ldots\check{a}_i\ldots a_k b_1\ldots\check{b}_j\ldots b_l} + \\
& + \sum_{i_1=1}^{k}\sum_{j_1=1}^{l}\sum_{i_2=1}^{i_1-1}\Big(\sum_{j_2=1}^{j_1-1}\underbrace{(-)^{k-i_1+j_1-1+k-1-i_2+j_2-1}}_{-(-)^{2k+i_1+i_2+j_1+j_2}}\eta^{a_{i_1}b_{j_1}}\eta^{a_{i_2}b_{j_2}}\Gamma^{a_1\ldots\check{a}_{i_2}\ldots\check{a}_{i_1}\ldots a_k b_1\ldots\check{b}_{j_2}\ldots\check{b}_{j_1}\ldots b_l} + \\
& + \sum_{j_2=j_1+1}^{l}\underbrace{(-)^{k-i_1+j_1-1+k-1-i_2+j_2-2}}_{(-)^{2k+i_1+i_2+j_1+j_2}}\eta^{a_{i_1}b_{j_1}}\eta^{a_{i_2}b_{j_2}}\Gamma^{a_1\ldots\check{a}_{i_2}\ldots\check{a}_{i_1}\ldots a_k b_1\ldots\check{b}_{j_1}\ldots\check{b}_{j_2}\ldots b_l}\Big) + \ldots
\end{aligned} \tag{D.3}$$

For some applications the precise coefficients are not important, and a schematic version is enough. Let us denote $\Gamma^{a_1\ldots a_k}$ schematically simply by $\Gamma^{[k]}$. Neglecting all coefficients, we can write

$$\boxed{\Gamma^{[k]}\Gamma^{[l]} \propto \Gamma^{[|k-l|]} + \Gamma^{[|k-l|+2]} + \ldots + \Gamma^{[k+l]}} \tag{D.4}$$

[1] For the proof of (D.2) one can simply study independently the cases of how many indices a_i and b_i coincide. For a nonvanishing lefthand side all the a's are different and all the b's are different. If even none of the a's coincides with one of the b's, we have simply $\Gamma^{a_1\ldots a_k}\Gamma^{b_1\ldots b_l} = \Gamma^{a_1\ldots a_k b_1\ldots b_l}$. If $a_1 = b_1$ and all others are different, we have $\Gamma^{a_1\ldots a_k}\Gamma^{b_1\ldots b_l} = (-)^{k-1}\eta^{a_1 b_1}\Gamma^{a_2\ldots a_k b_2\ldots b_l}$. If two indices coincide, e.g. $a_1 = b_1, a_2 = b_2$, then we have $\Gamma^{a_1\ldots a_k}\Gamma^{b_1\ldots b_l} = (-)^{k-1+k-2}\eta^{a_1 b_1}\eta^{a_2 b_2}\Gamma^{a_3\ldots a_k b_3\ldots b_l}$. And so on... ◇

Some simpler cases are of particular interest for us:

$$\Gamma^{a_1}\Gamma^{b_1\ldots b_l} = \Gamma^{a_1b_1\ldots b_l} + l\cdot\eta^{a_1[b_1}\Gamma^{b_2\ldots b_l]} \tag{D.5}$$

$$\Gamma^{a_1a_2}\Gamma^{b_1\ldots b_l} = \Gamma^{a_1a_2b_1\ldots b_l} - l\cdot\eta^{a_1[b_1|}\Gamma^{a_2|b_2\ldots b_l]} + l\cdot\eta^{a_2[b_1|}\Gamma^{a_1|b_2\ldots b_l]} - l(l-1)\eta^{a_1[b_1|}\eta^{a_2|b_2}\Gamma^{b_3\ldots b_l]} \tag{D.6}$$

$$\begin{aligned}\Gamma^{a_1a_2}\Gamma^{b_1b_2} &= \Gamma^{a_1a_2b_1b_2} + \eta^{a_2b_1}\Gamma^{a_1b_2} + \eta^{a_1b_2}\Gamma^{a_2b_1} - \eta^{a_1b_1}\Gamma^{a_2b_2} - \eta^{a_2b_2}\Gamma^{a_1b_1} + \\ &\quad + \eta^{a_1b_2}\eta^{a_2b_1} - \eta^{a_1b_1}\eta^{a_2b_2}\end{aligned} \tag{D.7}$$

Contracting (D.5) with Γ_{a_1} from the left yields

$$(d-l)\Gamma^{b_1\ldots b_l} = \Gamma_{a_1}\Gamma^{a_1b_1\ldots b_l} \tag{D.8}$$

Acting instead from the righthand side yields

$$\begin{aligned}\Gamma^a\Gamma^{b_1\ldots b_l}\Gamma_a &= \Gamma^{ab_1\ldots b_l}\Gamma_a + l\eta^{a[b_1}\Gamma^{b_2\ldots b_l]}\Gamma_a = \\ &= (-)^l(d-2l)\cdot\Gamma^{b_1\ldots b_l}\end{aligned} \tag{D.9}$$

In particular for $l=0$ and $l=1$, we have

$$\Gamma^a\Gamma_a = d \tag{D.10}$$

$$\Gamma^a\Gamma^b\Gamma_a = -(d-2)\cdot\Gamma^b \tag{D.11}$$

For even dimensions the righthand side of (D.9) vanishes for $l = d/2$. We will need this fact for ten dimensions:

$$\boxed{\Gamma^a\Gamma^{b_1\ldots b_5}\Gamma_a = 0} \quad \text{for } d=10 \tag{D.12}$$

Chirality matrix as a "Hodge star" Remember the definition and the basic properties of the chirality matrix in even dimensions:

$$\Gamma^\# \equiv \sqrt{-\epsilon_{(d)}}\Gamma^0\cdots\Gamma^{d-1} = \frac{1}{d!}\sqrt{-\epsilon_{(d)}}\epsilon_{c_1\ldots c_d}\Gamma^{c_1\ldots c_d}, \quad \text{with} \left\{\begin{array}{c}\epsilon_{01\ldots(d-1)} \equiv 1 \\ \epsilon_{(d)} \equiv (-)^{d(d-1)/2} = (-)^{[d/2]}\end{array}\right. \tag{D.13}$$

$$(\Gamma^\#)^2 = 1\!\!1 \tag{D.14}$$

$$\{\Gamma^a,\Gamma^\#\} = 0 \quad \forall a\in\{0,1,\ldots,d-1\}, \qquad \text{for even } d, \qquad \Gamma^\# = \pm 1\!\!1 \quad \text{for odd } d \tag{D.15}$$

The sign $\epsilon_{(d)}$ is the sign that one obtains when reversing the order of d indices of an antisymmetric object. Likewise if we have an antisymmetric object with an arbitrary number p of indices, reversing the order yields the sign $\epsilon_{(p)} \equiv (-)^{\sum_{k=0}^{(p-1)}k} = (-)^{p(p-1)/2} = (-)^{[p/2]}$. It takes the explicit values

d	0	1	2	3	4	5	6	7	8	9	10	11
$\epsilon_{(d)} = (-)^{[d/2]}$	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1

(D.16)

and has the properties

$$\epsilon_{(p+q)} = (-)^{pq}\epsilon_{(p)}\epsilon_{(q)}, \quad \epsilon_{(p)}^2 = 1, \quad \epsilon_{(2p)} = (-)^p, \quad \epsilon_{(-p)} = \underbrace{\epsilon_{(p+1)}}_{-\epsilon_{(p-1)}} = (-)^p\epsilon_{(p)}, \quad \epsilon_{(d-p)} = \epsilon_{(d)}\epsilon_{(p)}(-)^{p(d-p)} \tag{D.17}$$

The prefactor $\sqrt{-\epsilon_{(d)}}$ in the definition of the chirality matrix guarantees the fact that it squares to the unity. For half of the dimensions the square root is ill-defined, because $-\epsilon_{(d)}$ is negative. It should simply be understood via $\sqrt{-1} = i$, i.e. $\sqrt{-\epsilon_{(d)}} \equiv i^{\frac{1}{2}(1+\epsilon_{(d)})} \overset{d=10}{=} 1$. Of course, a redefinition of $\Gamma^\#$ with an overall (perhaps d-dependent) sign does not change its properties and might be useful in certain situations. Because $\Gamma^\#$ squares to $1\!\!1$, it can have eigenvalues ± 1. The corresponding eigenvectors are chiral and antichiral spinors. For odd dimension, when $\Gamma^\#$ coincides with unity, there is only the eigenvalue 1 and there is no such split.

There is a natural isomorphism between the antisymmetrized product of Γ-matrices $\Gamma^{a_1\ldots a_p}$ and the wedge product of the cotangent basis elements (vielbeins) $e^{a_1}\wedge\ldots\wedge e^{a_p}$. The multiplication with the chirality matrix on the one side then corresponds to the application of the Hodge star on the other. It maps p-forms to $(d-p)$-forms in the following sense:

$$\begin{aligned}\Gamma^\#\Gamma^{a_1\ldots a_p} &= \frac{1}{d!}\sqrt{-\epsilon_{(d)}}\epsilon_{c_d\ldots c_1}\Gamma^{c_d\ldots c_1}\Gamma^{a_1\ldots a_p} = \\ &\overset{(D.2)}{=} \frac{1}{d!}\sqrt{-\epsilon_{(d)}}\epsilon_{c_d\ldots c_1}p!\begin{pmatrix} p \\ p\end{pmatrix}\begin{pmatrix} d \\ p\end{pmatrix}\eta^{c_1a_1}\cdots\eta^{c_pa_p}\Gamma^{c_dc_{d-1}\ldots c_{p+1}} = \\ &= \frac{1}{(d-p)!}\sqrt{-\epsilon_{(d)}}\Gamma^{c_d\ldots c_{p+1}}\epsilon_{c_d\ldots c_{p+1}}{}^{a_p\ldots a_1}\end{aligned} \tag{D.18}$$

Up to a sign $(-)^{p(d-p)}$ ($(-)^p$ for even d and 1 for odd d) the same result is obtained when acting from the right, s.t. we can summarize

$$\boxed{\Gamma^{a_1\ldots a_p}\Gamma^{\#} = \epsilon_{(p)}\sqrt{-\epsilon_{(d)}}\frac{1}{(d-p)!}\epsilon^{a_1\ldots a_p}{}_{c_1\ldots c_{d-p}}\Gamma^{c_1\ldots c_{d-p}} = (-)^{(d-p)p}\Gamma^{\#}\Gamma^{a_1\ldots a_p}} \tag{D.19}$$

The above calculation is also true if we are in odd dimensions where $\Gamma^{\#}$ is the unity. The antisymmetrized products $\Gamma^{a_1\cdots a_p}$ do then not correspond to $e^{a_1}\wedge\ldots\wedge e^{a_p}$, but (at least in dimensions where $-\epsilon_{(d)}=1$, i.e. $d\in\{3,7,11,\ldots\}$) to self dual forms $e^{a_1}\wedge\ldots\wedge e^{a_p}+\star(e^{a_1}\wedge\ldots\wedge e^{a_p})$ (see intermezzo below for the discussion of the Hodge star). The same will be true in the even dimensions $d\in\{2,6,10\}$ for the chiral blocks $\gamma^{a_1\cdots a_p}$ that will be discussed in particular for $d=10$ later. In order to understand better the correspondence between the multiplication with $\Gamma^{\#}$ and the Hodge star operation, let us give a short review of the latter.

Intermezzo on Clifford map and Hodge star operator

In order to avoid confusion about prefactors, note first that we use a definition of the wedge product that absorbs the normalization factor $\frac{1}{p!}$ which is therefore absent at other places:

$$\omega^{(p)} = \omega_{m_1\ldots m_p}\mathbf{d}x^{m_1}\wedge\ldots\wedge\mathbf{d}x^{m_p} \tag{D.20}$$

Replacing $\omega_{m_1\ldots m_p}\longrightarrow\frac{1}{p!}\omega_{m_1\ldots m_p}$ everywhere leads to the equations in the standard convention.

In even dimensions d there is a natural isomorphism, the **Clifford map**, from bispinors (which can be expanded in the complete basis of antisymmetrized products of Γ-matrices) and the formal sum of p-forms in $\bigwedge^{\bullet}T^*M\equiv\oplus_p\bigwedge^p T^*M$. The basis elements map simply as

$$/^{-1}:\quad \Gamma^{a_1\ldots a_p}\mapsto e^{a_1}\cdots e^{a_p}\equiv e^{a_1}\wedge\ldots\wedge e^{a_p} \tag{D.21}$$

where $e^a=\mathbf{d}x^m e_m{}^a$ is an orthonormal vielbein-basis. Its inverse map is often denoted by a slash

$$/:\quad e^{a_1}\cdots e^{a_p} \mapsto \Gamma^{a_1\ldots a_p} \tag{D.22}$$

$$\rho=\sum_p\rho_{a_1\ldots a_p}e^{a_1}\cdots e^{a_p} \mapsto \not\rho\equiv\sum_p\rho_{a_1\ldots a_p}\Gamma^{a_1\ldots a_p} \tag{D.23}$$

See in particular [6, 5, 115, 116, 81] for frequent use of this map in the context of generalized complex geometry. Operations on the one side can then be translated to the other. There is in particular the multiplication with the chirality matrix on the bispinor side which corresponds more or less to the Hodge star operator on the other side. The 'more or less' statement depends on how exactly one defines the Hodge star, and we will simply define it in such a way, that it corresponds exactly to the multiplication with the chirality matrix, at least with the multiplication from the righthand side.

The Hodge star operation on a manifold with metric maps p-forms to $(n-p)$-forms using the metric and the ε-tensor[2]

$$\varepsilon_{m_1\ldots m_d}\equiv\sqrt{|\,g\,|}\epsilon_{m_1\ldots m_d},\quad \epsilon_{0\ldots d-1}\equiv 1 \tag{D.24}$$

[2] In the following we will use some identities for the epsilon-symbol and for the antisymmetrized Kronecker-delta, which we would like to recall. Remember first the definition of the antisymmetrized Kronecker symbols

$$\delta^{c_1\ldots c_n}_{d_1\ldots d_n} \equiv \delta^{c_1}_{[d_1}\cdots\delta^{c_n}_{d_n]}$$

If we contract one index pair, we arrive at

$$\delta^{c_1\ldots c_{n-1}c_n}_{d_1\ldots d_{n-1}c_n} = \frac{d-(n-1)}{n}\delta^{c_1\ldots c_{n-1}}_{d_1\ldots d_{n-1}}$$

Contracting several indices leads to

$$\boxed{\delta^{c_1\ldots c_{n-p}a_1\ldots a_p}_{d_1\ldots d_{n-p}a_1\ldots a_p}=\frac{\binom{d-n+p}{p}}{\binom{n}{p}}\delta^{c_1\ldots c_{n-p}}_{d_1\ldots d_{n-p}}}$$

In particular, if all indices are contracted ($p=n$) or if the original number of indices matches the dimension ($n=d$), we end up with

$$\delta^{a_1\ldots a_p}_{a_1\ldots a_p}=\begin{pmatrix} d\\ p\end{pmatrix},\qquad \delta^{c_1\ldots c_{d-p}a_1\ldots a_p}_{d_1\ldots d_{d-p}a_1\ldots a_p}=\begin{pmatrix} d\\ p\end{pmatrix}^{-1}\delta^{c_1\ldots c_{d-p}}_{d_1\ldots d_{d-p}}$$

(see also [117, p.456]). The last identities are important to derive the identities for the Levi-Civita symbol ϵ. The first observation is that we have

$$\epsilon_{a_1\ldots a_d}\epsilon^{b_1\ldots b_d} = -d!\delta^{b_1\ldots b_d}_{a_1\ldots a_d}$$

Both sides are completely antisymmetric in all a and all b. It is therefore enough to check the validity for $(a_1,\ldots,a_d)=(b_1,\ldots,b_d)=(0,\ldots,d-1)$. The minus sign is coming from the different definition of the ϵ-symbol with upper sign, i.e. $\epsilon_{0\ldots d-1}=-\epsilon^{0\ldots d-1}=1$.

where $\epsilon_{m_1 \ldots m_d}$ is the totally antisymmetric Levi Civita symbol. Let us define the same symbol with upper indices with a different sign, i.e. as $\epsilon^{0\ldots d-1} \equiv -1$ (corresponding to the ε-tensor in flat Minkowski spacetime where raising a zero-index yields the minus). Using that $\det g^{-1} = \epsilon_{m_1 \ldots m_d} g^{m_1 0} \cdots g^{m_d d-1}$ the ε-tensor with upper indices takes the familiar form

$$\varepsilon^{m_1 \ldots m_d} = \frac{1}{\sqrt{|g|}} \epsilon^{m_1 \ldots m_d}, \quad \epsilon^{0 \ldots d-1} \equiv -1 \tag{D.25}$$

The definition of the Hodge star on a manifold with metric $\star : \bigwedge^p T^*M \to \bigwedge^{d-p} T^*M$ has some ambiguity in the sign, depending on which behaviour one prefers $\star$ to have. For us it will be most convenient to define it simply in the way as $\Gamma^\#$ acts (at least for even dimensions). One still has the freedom to decide whether it should correspond to an action from the left or from the right, which differs by a factor of $(-)^{(d-p)p}$ according to (D.19). We choose the Hodge star corresponding to multiplication of $\Gamma^\#$ from the right as given in (D.19). The dimension dependent prefactor $\sqrt{-\epsilon_{(d)}}$, however, will not be included, because it is complex in some dimensions (but fortunately equal one in 10 dimensions) and the definition of the Hodge dual should make sense for real manifolds. We therefore define

$$\star \left(\mathbf{d}x^{k_1} \wedge \ldots \wedge \mathbf{d}x^{k_p} \right) \equiv \frac{\epsilon_{(p)}}{(d-p)!} \varepsilon^{k_1 \ldots k_p}{}_{m_1 \ldots m_{d-p}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{d-p}} \tag{D.26}$$

$$(\star \omega^{(p)})_{m_1 \ldots m_{d-p}} = \frac{(-)^{p(d-p)} \epsilon_{(p)}}{(d-p)!} \varepsilon_{m_1 \ldots m_{d-p}}{}^{k_1 \ldots k_p} \omega^{(p)}_{k_1 \ldots k_p} \tag{D.27}$$

The sign prefactor $\epsilon_{(p)} = (-)^{p(p-1)/2}$ is usually not present in the old definitions in the literature. At some places (e.g. in [81]) the Hodge star is defined such that it coincides with multiplication of $\Gamma^\#$ from the left. This corresponds to a redefinition of our Hodge star by $(-)^{p(d-p)}$. Let us denote with

$$\tilde{\omega}^{(p)} \equiv \omega^{m_1 \ldots m_p} \boldsymbol{\partial}_{m_1} \wedge \ldots \wedge \boldsymbol{\partial}_{m_p} \tag{D.28}$$

the multivector that arises when raising all the indices of the differential form $\omega^{(p)}$ with the metric g^{mn} and remember the definition of the interior product (C.39) with respect to multivector fields:

$$\imath_{\tilde{\omega}^{(p)}} \rho^{(r)} \equiv \frac{r!}{(r-p)!} \underbrace{\omega^{l_1 \ldots l_p} \rho_{l_p \ldots l_1 m_1 \ldots m_{r-p}}}_{\epsilon_{(p)} \omega^{l_1 \ldots l_p} \rho_{l_1 \ldots l_p m_1 \ldots m_{r-p}}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p}}, \quad \imath_{\tilde{\omega}^{(p)}} \rho^{(r)} = 0 \text{ for } p > r \tag{D.29}$$

Using (D.17) and the identities for the ε-tensor given in footnote 2 on the previous page, we obtain the following

Using the above formula for contractions of the antisymmetrized Kronecker-delta, we obtain

$$\boxed{\epsilon_{a_1 \ldots a_{d-p} c_1 \ldots c_p} \epsilon^{b_1 \ldots b_{d-p} c_1 \ldots c_p} = -p!(d-p)! \delta^{b_1 \ldots b_{d-p}}_{a_1 \ldots a_{d-p}}}$$

This equation remains the same if replace the Levi Civita symbol ϵ with the ε-tensor (D.24) and (D.25), as the normalization factors cancel. $\diamond$

relations for the Hodge star operator[3]

$$\star^2 \quad = \quad -\epsilon_{(d)} \tag{D.30}$$

$$(\star 1)_{m_1\ldots m_d} \quad = \quad \frac{1}{d!}\varepsilon_{m_1\ldots m_d} \tag{D.31}$$

$$\star(\omega^{(p)}\wedge\eta^{(q)}) \quad = \quad (\imath_{\tilde\omega^{(p)}}\star\eta^{(q)}) = (-)^{pq}(\imath_{\tilde\eta^{(q)}}\star\omega^{(p)}), \quad \text{for } p+q\leq d \tag{D.32}$$

This implies $\star(\omega^{(p)}\wedge\star\eta^{(q)}) = -\epsilon_{(d)}(\imath_{\tilde\omega^{(p)}}\eta^{(q)})$ $(p\leq q)$ and $\star(\star\omega^{(p)}\wedge\eta^{(q)}) = -\epsilon_{(d)}(-)^{(d-p)q}(\imath_{\tilde\eta^{(q)}}\omega^{(p)})$ $(q\leq p)$ and in particular for $p=q$

$$(\omega^{(p)}\wedge\star\eta^{(p)})_{m_1\ldots m_d} \quad = \quad -\epsilon_{(d)}(\imath_{\tilde\omega^{(p)}}\eta^{(p)})\frac{1}{d!}\varepsilon_{m_1\ldots m_d} \tag{D.33}$$

$$(\star\omega^{(p)}\wedge\eta^{(p)}) \quad = \quad -\epsilon_{(d)}(-)^{(d-p)p}(\imath_{\tilde\eta^{(q)}}\omega^{(p)})\frac{1}{d!}\varepsilon_{m_1\ldots m_d} \tag{D.34}$$

Note that wedge product and inner product play both the role as an embedding $\imath$ of forms or vectors into the space of endomorphisms acting on forms. Thus the equation (D.32) can be written as $\star(\imath_{\omega^{(p)}}\eta^{(q)}) = (\imath_{\tilde\omega^{(p)}}\star\eta^{(q)})$. In turn, the same equation acted upon with an overall $\star$ and in addition with η replaced by $\star\eta$ and $\tilde\omega$ renamed as v becomes $(\imath_{\tilde v^{(p)}}\star\eta^{(q)}) = \star(\imath_{v^{(p)}}\eta^{(q)})$ (where $\tilde v$ is the p-form obtained from the p-vector v by lowering all indices). For decomposable multivector valued forms $\omega^{(p)}\otimes v^{(k)}$ (with ω a p-form and v a k-multivector) the embedding is defined as $\imath_{\omega\otimes v} = \imath_\omega\imath_v = \omega\wedge\imath_v$ (see (C.39) on page 163). We thus obtain

$$\star\left(\imath_{\omega^{(p)}\otimes v^{(k)}}\eta^{(q)}\right) \quad = \quad \imath_{\tilde\omega^{(p)}}\star(\imath_{v^{(k)}}\eta^{(q)}) = \imath_{\tilde\omega^{(p)}}\imath_{\tilde v^{(k)}}\star\eta^{(q)} \tag{D.35}$$

The order of the operators on the righthand side is not the “normal order”. The wedge product acts before the interior product, while the definition of the embedding of a multivector valued form is the other way round. In order to write it as an embedding again, we need to apply the commutator which yields the algebraic bracket $[\imath_{\tilde\omega^{(p)}},\imath_{\tilde v^{(k)}}]\equiv\imath_{[\tilde\omega^{(p)},\tilde v^{(k)}]}$ (see (C.43)). The above righthand side then becomes $\imath_{((-)^{pk}\tilde v^{(k)}\otimes\tilde\omega^{(p)}+[\tilde\omega^{(p)},\tilde v^{(k)}]^\triangle)}\star\eta^{(q)}$. For general multivector valued forms $K^{(k,k')}$ of form-degree k and multivector degree k' we therefore cannot set $\star(\imath_{K^{(k,k')}}\eta^{(q)})$ equal to $\imath_{\tilde K^{(k',k)}}\star\eta^{(q)}$, although this would be tempting. Instead, we get in the schematic index notation of page A

$$\star\left(\imath_{K^{(k,k')}}\eta^{(q)}\right) = (k)!\begin{pmatrix} d-q+k' \\ k\end{pmatrix} K^{l_1\ldots l_k}{}_{[l_k\ldots l_1\boldsymbol{m}\ldots\boldsymbol{m}}(\star\eta)^{(d-q)}_{\boldsymbol{m}\ldots\boldsymbol{m}]} \tag{D.36}$$

Only for multivector valued forms with vanishing contractions (e.g. for a torsion which is completely antisymmetric after pulling down one index) the righthand side reduces to $\imath_{\tilde K^{(k',k)}}\star\eta^{(q)}$, where $\tilde K^{(k',k)}$ is obtained from $K^{(k,k')}$ by raising all k form indices and lowering all k' multivector indices with the metric.

Finally we can use (D.32) formally also to calculate the action of $\star\mathbf{d}\star$, if we consider the exterior derivative as wedge product $\mathbf{d}\wedge$. In flat space and Cartesian coordinates, there is no contribution from the action of the derivative on the metric and we arrive formally at $\star(\mathbf{d}\wedge\star\eta^{(q)}) = -\epsilon_{(d)}(\imath_{\tilde{\mathbf{d}}}\eta^{(q)})$, or explicitely $(\star\mathbf{d}\star\eta^{(q)})_{m_1\ldots m_{q-1}} = -q\epsilon_{(d)}\partial^k\eta^{(q)}_{km_1\ldots m_{q-1}}$. In curved space this result gets covariantized to

$$(\star\mathbf{d}\star\eta^{(q)})_{m_1\ldots m_{q-1}} = -q\epsilon_{(d)}\nabla^{(LC)\,k}\eta^{(q)}_{km_1\ldots m_{q-1}} \tag{D.37}$$

[3]Because of the uncommon definition of the Hodge star, we'll provide here the equations also for a redefined $\star$. Let us replace the sign factor $(-)^{p(d-p)}\epsilon_{(p)} = (-)^{p(d-p)+p(p-1)/2}$ in the definition (D.27) of the Hodge star by some arbitrary d and p dependent sign factor $\epsilon_{(d,p)}$

$$(\star\omega^{(p)})_{m_1\ldots m_{d-p}} \equiv \frac{\epsilon_{(d,p)}}{(d-p)!}\varepsilon_{m_1\ldots m_{d-p}}{}^{k_1\ldots k_p}\omega^{(p)}_{k_1\ldots k_p}$$

where some natural choices for $\epsilon_{(d,p)}$ are $1, (-)^{p(d-p)}, \epsilon_{(p)}$ and $(-)^{p(d-p)}\epsilon_{(p)}$. The last one corresponds to our definition, while the second is quite common in the literature. With this more general ansatz we have

$$(\star 1)_{m_1\ldots m_d} \quad = \quad \frac{\epsilon_{(d,0)}}{d!}\varepsilon_{m_1\ldots m_d}$$

$$\star^2 \quad = \quad -(-)^{p(d-p)}\epsilon_{(d,p)}\epsilon_{(d,d-p)}$$

$$\star(\omega^{(p)}\wedge\eta^{(q)}) \quad = \quad (-)^{pq+p(d-p)}\epsilon_{(d,p+q)}\epsilon_{(d,q)}(-)^{p(p-1)/2}\imath_{\tilde\omega^{(p)}}\star\eta^{(q)} = (-)^{q(d-q)}\epsilon_{(d,p+q)}\epsilon_{(d,p)}(-)^{q(q-1)/2}\imath_{\tilde\eta^{(q)}}\omega^{(p)}$$

In particular for $\epsilon_{(d,p)} = (-)^{p(d-p)}$ one obtains the more familiar equations

$$(\star 1)_{m_1\ldots m_d} \quad = \quad \frac{1}{d!}\varepsilon_{m_1\ldots m_d}$$

$$\star^2 \quad = \quad -(-)^{p(d-p)}$$

$$\omega^{(p)}\wedge\star\eta^{(p)} \quad = \quad -\imath_{\epsilon_{(p)}\tilde\omega^{(p)}}\eta^{(p)}\frac{1}{d!}\varepsilon_{m_1\ldots m_d}$$

where the last equation follows from $\star(\omega^{(p)}\wedge\eta^{(q)}) = (-)^{pq}\imath_{\epsilon_{(p)}\tilde\omega^{(p)}}\star\eta^{(q)} = \imath_{\epsilon_{(q)}\tilde\eta^{(q)}}\star\omega^{(p)}$ with $\eta^{(q)}$ replaced by $\star\eta^{(p)}$. The nice feature of our present definition (with $\epsilon_{(d,p)} = (-)^{p(d-p)}\epsilon_{(p)}$) is that the expression for $\star^2$ in (D.30) does not depend on the form degree. ◇

where the Levi-Civita connection arises from the action of the divergence on the metric ($\frac{1}{\sqrt{|g|}}\partial^k(\sqrt{|g|}\rho_k) = \nabla^{(LC)k}\rho_k$). Note that for a Levi-Civita connection the covariant antisymmetrized derivative $\nabla^{(LC)}_{[m_0}\omega_{m_1\ldots m_p]}$ reduces to the exterior derivative $\partial_{[m_0}\omega_{m_1\ldots m_p]}$ because of the symmetry of the connection. This is not true any longer, if a torsion is present. In that case it makes sense to define a different exterior derivative via

$$(\boldsymbol{\nabla}\omega^{(p)})_{m_0\ldots m_p} \equiv \nabla_{[m_0}\omega_{m_1\ldots m_p]} = (\mathbf{d}\omega^{(p)})_{m_0\ldots m_p} - pT_{[m_0 m_1|}{}^k\omega_{k|m_2\ldots m_p]} \tag{D.38}$$

$$\text{or } \boldsymbol{\nabla} \equiv \mathbf{d} - \imath_T \tag{D.39}$$

The relation for $\star\mathbf{d}\star$ then turns into

$$(\star\boldsymbol{\nabla}\star\eta^{(q)})_{m_1\ldots m_{q-1}} = -q\epsilon_{(d)}\nabla^k\eta^{(q)}_{km_1\ldots m_{q-1}} \tag{D.40}$$

Apart from the Hodge duality (induced by $\Gamma^{\#}$-multiplication) there are other interesting operations on the bispinor side which get translated to the form side via $/^{-1}$ (D.21). E.g. the matrix multiplications with a Γ-matrix either from the left or from the right translate due to (D.5) into

$$\Gamma^a\cdot \not\rho \overset{/^{-1}}{\mapsto} e^a\wedge\rho + \eta^{ab}\underbrace{\frac{\partial}{\partial e^b}\rho}_{\imath_{e_b}\rho} = \tag{D.41}$$

$$\overset{(D.32)}{=} e^a\wedge\rho - \epsilon_{(d)}\star(e^a\wedge\star\rho) \tag{D.42}$$

$$\not\rho\cdot\Gamma^a \overset{/^{-1}}{\mapsto} \rho\wedge e^a + \eta^{ab}\partial\rho/\partial e^b = \tag{D.43}$$

$$= (-)^r e^a\wedge\rho + (-)^{r-1}\eta^{ab}\underbrace{\frac{\partial}{\partial e^b}\rho}_{\imath_{e_b}\rho} = \tag{D.44}$$

$$= (-)^r\left(e^a\wedge\rho + \epsilon_{(d)}\star(e^a\wedge\star\rho)\right) \tag{D.45}$$

The form degree r in the last line makes strictly speaking only sense if $\rho = \rho^{(r)}$ is a form of definite degree. If it is instead a formal sum, r should be understood as an operator (acting on ρ) whose eigenvalues are the form degrees (i.e. $e^a\frac{\partial}{\partial e^a}$).

In order to obtain the action of the Dirac operator on the first or on the second index of a bispinor, the above equations can be contracted with a covariant derivative ∇_a (whose connection is compatible with the metric η^{ab}, the Γ-matrices and the vielbein-components, i.e. leaves each of them invariant):

$$\underbrace{\Gamma^a\nabla_a}_{\not\nabla_a}\cdot\not\rho \mapsto \boldsymbol{\nabla}\rho - \epsilon_{(d)}\star\boldsymbol{\nabla}\star\rho \tag{D.46}$$

$$\nabla_a\not\rho\cdot\Gamma^a \mapsto \sum_r(-)^r\left(\boldsymbol{\nabla}\rho^{(r)} + \epsilon_{(d)}\star\boldsymbol{\nabla}\star\rho^{(r)}\right) \tag{D.47}$$

Vanishing of both expressions on the bispinor side yields (because of the different relative signs in the brackets of both results) $\boldsymbol{\nabla}\rho = \star\boldsymbol{\nabla}\star\rho = 0$, which for vanishing torsion corresponds to $\mathbf{d}\rho = \star\mathbf{d}\star\rho = 0$. Let us try to recover $\mathbf{d}$ and $\star\mathbf{d}\star$ also in the case with torsion. According to (G.23) or (G.27) any connection which is compatible with the metric can be written as

$$\Gamma_{mn}{}^k = \Gamma^{(LC)\,k}_{mn} + T_{mn}{}^k + T^k{}_{m|n} - T_n{}^k{}_{|m} \tag{D.48}$$

$$\omega_{ca}{}^b = \omega^{(LC)\,b}_{ca} + T_{ca}{}^b + 2T^b{}_{(c|a)} \tag{D.49}$$

so that

$$\underbrace{r\nabla^c\rho^{(r)}_{ca_1\ldots a_{r-1}}}_{-\epsilon_{(d)}(\star\boldsymbol{\nabla}\star\rho^{(r)})} = \underbrace{r\nabla^{(LC)\,c}\rho^{(r)}_{ca_1\ldots a_{r-1}}}_{-\epsilon_{(d)}(\star\mathbf{d}\star\rho^{(r)})_{a_1\ldots a_{r-1}}} + \underbrace{2rT^{cd}{}_c\rho^{(r)}_{da_1\ldots a_{r-1}} - r(r-1)T^{bc}{}_{[a_1|}\rho^{(r)}_{cb|a_2\ldots a_{r-1}]}}_{\epsilon_{(d)}(\star\imath_T\star\rho^{(r)})_{a_1\ldots a_{r-1}}} \tag{D.50}$$

As indicated below the brackets, the same result is obtained via $\star\boldsymbol{\nabla}\star\rho = \star(\mathbf{d}\star\rho - \imath_T\star\rho)$ and then using (D.36) for $\star(\imath_T\star\rho)$, considering T as a vector valued 2-form.

As a next step we should study the effect of multiplying the bispinor with another bispinor which again can be expanded in antisymmetrized products $\Gamma^{b_1\cdots b_p}$ of Γ-matrices. Using (D.2), we obtain

$$\not\omega^{(p)}\,\not\rho^{(r)} = \sum_{k=0}^{min\{p,r\}} k!\begin{pmatrix} p \\ k\end{pmatrix}\begin{pmatrix} r \\ k\end{pmatrix}\omega_{a_1\ldots a_{p-k}}{}^{c_k\ldots c_1}\rho^{(r)}_{c_1\ldots c_k a_{p-k+1}\ldots a_{p+r-2k}}\Gamma^{a_1\ldots a_{p+r-2k}} \tag{D.51}$$

The $\Gamma^{a_1\ldots a_{p+r-2k}}$'s get mapped to $e^{a_1}\cdots e^{a_{p+r-2q}}$ by $/^{-1}$. For forms which are not of definite degree, the result can then be written as

$$\not{\omega}\ \not{\rho} \quad \overset{/^{-1}}{\mapsto} \quad \sum_{k\geq 0}\frac{1}{k!}\omega\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}\rho \tag{D.52}$$

which defines the **Clifford multiplication** between forms. The Clifford multiplication of two self dual forms is either 0 or another self-dual form:

$$\not{\omega}^{(p)}\tfrac{1}{2}(\mathbb{1}+\Gamma^{\#})\ \not{\rho}^{(r)}\tfrac{1}{2}(\mathbb{1}+\Gamma^{\#}) = \begin{cases} \not{\omega}^{(p)}\ \not{\rho}^{(r)}\tfrac{1}{2}(\mathbb{1}+\Gamma^{\#}) & \text{for } r \text{ even} \\ 0 \quad \text{for } r \text{ odd} \end{cases} \tag{D.53}$$

Note finally that the matrix-commutator on the bispinor side naturally defines an (algebraic) bracket on the form-side

$$[\not{\omega},\not{\rho}] \overset{/^{-1}}{\mapsto} \sum_{k\geq 0}\frac{1}{k!}\Big(\omega\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}\rho-\rho\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}\omega\Big) \tag{D.54}$$

Although this is a valid and consistent map, it is not the most natural object from the form point of view. On the lefthand side we have the possibility to think of the gamma matrices as fermionic supermatrices as suggested in section 2.7 on page 26 and consider the graded commutator which would include an additional sign $(-)^{pr}$ in front of the second term for forms $\omega^{(p)}$ and $\rho^{(r)}$ of definite degree. Then one can use that $\omega^{(p)}\frac{\overleftarrow{\partial}}{\partial e^a} = -(-)^p\frac{\partial}{\partial e^a}\omega^{(p)}$ and therefore $\omega^{(p)}\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}} = (-)^{kp+k}\epsilon_{(k)}\frac{\partial}{\partial e^{a_k}}\cdots\frac{\partial}{\partial e^{a_1}}\omega^{(p)}$ in order to interchange the position of ω and ρ and arrives at

$$\underbrace{[\not{\omega},\not{\rho}]}_{\text{with odd }\Gamma'\text{s}} \quad \overset{/^{-1}}{\mapsto} \quad \sum_{k\geq 0}\left(1-(-)^k\right)\frac{1}{k!}\omega\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_k}}\eta^{a_1b_1}\cdots\eta^{a_kb_k}\frac{\partial}{\partial e^{b_k}}\cdots\frac{\partial}{\partial e^{b_1}}\rho = \tag{D.55}$$

$$= \sum_{k\geq 0}\frac{2}{(2k+1)!}\omega\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\cdots\frac{\overleftarrow{\partial}}{\partial e^{a_{2k+1}}}\eta^{a_1b_1}\cdots\eta^{a_{2k+1}b_{2k+1}}\frac{\partial}{\partial e^{b_{2k+1}}}\cdots\frac{\partial}{\partial e^{b_1}}\rho \tag{D.56}$$

This contains as a special case the anticommutator of the gamma-matrices themselves

$$\left\{\Gamma^a,\Gamma^b\right\} \mapsto 2e^a\frac{\overleftarrow{\partial}}{\partial e^{a_1}}\eta^{a_1b_1}\frac{\partial}{\partial e^{b_1}}e^b = 2\eta^{ab} \tag{D.57}$$

The Hodge star as defined in the previous intermezzo corresponds to a multiplication with $\sqrt{-\epsilon_{(d)}}\Gamma^{\#}$ from the right. It would of course be possible to absorb the prefactor in the definition of $\Gamma^{\#}$. This, however, would spoil $(\Gamma^{\#})^2 = 1$ in general dimensions. Let us now continue with the discussion of the properties of the chirality matrix. From (D.19) we obtain in particular

$$\begin{aligned}\Gamma^{\#}\Gamma^{a_1\ldots a_p}\otimes\Gamma_{a_p\ldots a_1}\Gamma^{\#} &= (-)^{p(d-p)}\Gamma^{\#}\Gamma^{a_1\ldots a_p}\otimes\Gamma^{\#}\Gamma_{a_p\ldots a_1} = \\ &= (-)^{p(d-p)}\left(\frac{\sqrt{-\epsilon_{(d)}}}{(d-p)!}\right)^2\epsilon_{c_d\ldots c_{p+1}}{}^{a_p\ldots a_1}\Gamma^{c_d\ldots c_{p+1}}\otimes\epsilon_{b_d\ldots b_{p+1}a_1\ldots a_p}\Gamma^{b_d\ldots b_{p+1}}\end{aligned} \tag{D.58}$$

Using $\epsilon^{c_d\cdots c_{p+1}a_p\ldots a_1}\epsilon_{b_d\ldots b_{p+1}a_1\ldots a_p} = -\epsilon_{(p)}p!(d-p)!\eta_{c_d\ldots c_{p+1},b_d\ldots b_{p+1}}$ (see footnote 2) we get

$$\Gamma^{\#}\Gamma^{a_1\ldots a_p}\otimes\Gamma_{a_p\ldots a_1}\Gamma^{\#} \quad = \quad (-)^{p(d-p)}\epsilon_{(d)}\epsilon_{(p)}\frac{p!}{(d-p)!}\Gamma_{b_d\ldots d_{p+1}}\otimes\Gamma^{b_d\ldots d_{p+1}} \tag{D.59}$$

Reversing the order of the indices of one of the Γ's on the righthand side of the equation (contributing a factor $\epsilon_{(d-p)} = \epsilon_{(d)}\epsilon_{(p)}(-)^{p(d-p)}$), we arrive at

$$\boxed{\Gamma^{\#}\Gamma^{a_1\ldots a_p}\otimes\Gamma_{a_p\ldots a_1}\Gamma^{\#} = \frac{p!}{(d-p)!}\Gamma^{b_1\ldots b_{d-p}}\otimes\Gamma_{b_{d-p}\ldots b_1}} \tag{D.60}$$

In particular in ten dimensions and for $p=5$, we obtain

$$\Gamma^{\#}\Gamma^{a_1\ldots a_5}\otimes\Gamma_{a_5\ldots a_1}\Gamma^{\#} = \Gamma_{b_1\ldots b_5}\otimes\Gamma^{b_1\ldots b_5} \qquad \text{for } d=10 \tag{D.61}$$

Trace The trace of all antisymmetrized products of Gamma-matrices vanishes in even dimensions:

$$\begin{aligned} \mathrm{tr}\,\Gamma^{a_1\ldots a_{2k+1}} = \mathrm{tr}\,\Gamma^{a_1\ldots a_{2k+1}}\Gamma^{\#}\Gamma^{\#} \overset{\text{even } d}{=} \pm\mathrm{tr}\,\Gamma^{\#}\Gamma^{a_1\ldots a_{2k+1}}\Gamma^{\#} \quad &\Rightarrow \quad \mathrm{tr}\,\Gamma^{a_1\ldots a_{2k+1}} = 0 \\ \mathrm{tr}\,\Gamma^{a_1\ldots a_{2k}} = \pm\mathrm{tr}\,\Gamma^{a_{2k}a_1\ldots a_{2k-1}} \quad &\Rightarrow \quad \mathrm{tr}\,\Gamma^{a_1\ldots a_{2k}} = 0 \end{aligned}$$

$$\boxed{\mathrm{tr}\,\Gamma^{a_1\ldots a_p} = 0} \quad \forall p \geq 1 \quad \text{for even } d \tag{D.62}$$

Fierz identity (see e.g. [118]) The Fierz identity is simply a completeness relation. Given a basis $\{|e^k>\}$ of a vector space, define its dual basis via $<e_k||e^l> = \delta^l_k$. The completeness relation then reads

$$\sum_k |e^k><e_k| = \mathbb{1} \tag{D.63}$$

In our case the vector space is the space of all $2^{[d/2]}\times 2^{[d/2]}$-matrices and in **even dimensions** the antisymmetrized products of Γ-matrices form a basis of it: $\{\mathbb{1},\Gamma^a,\Gamma^{a_1a_2},\ldots,\Gamma^{a_1\ldots a_d}\}\equiv\{\Gamma^I\}$. In odd dimensions this is still a generating set, but not linearly independent. The dual basis to $\{\Gamma^I\}$ in even dimensions is simply given by $2^{-d/2}\cdot\{\mathbb{1},\Gamma_a,\Gamma_{a_2a_1},\ldots,\Gamma_{a_d\cdots a_1}\}\equiv\{\Gamma_I\}$ (acting on the original basis by contracting all spinor indices). One can convince oneself that we have indeed (using $\mathrm{tr}\,\Gamma^{a_1\ldots a_p}=0$)

$$\begin{aligned} 2^{-d/2}\delta^{\underline{\alpha}}_{\underline{\beta}}\delta^{\underline{\beta}}_{\underline{\alpha}} &= 1 &\text{(D.64)}\\ \frac{2^{-d/2}}{p!}\Gamma_{a_p\ldots a_1}{}^{\underline{\alpha}}{}_{\underline{\beta}}\Gamma^{b_1\ldots b_q}{}^{\underline{\beta}}{}_{\underline{\alpha}} &= \delta^q_p\delta^{b_1\ldots b_p}_{a_1\ldots a_p} \equiv \delta^q_p\delta^{b_1}_{[a_1}\cdots\delta^{b_p}_{a_p]} &\text{(D.65)} \end{aligned}$$

The completeness relation or **Fierz identity** thus reads

$$\boxed{\sum_{p=0}^{d}\frac{2^{-d/2}}{p!}\Gamma^{a_1\ldots a_p}{}^{\underline{\alpha}}{}_{\underline{\beta}}\Gamma_{a_p\ldots a_1}{}^{\underline{\gamma}}{}_{\underline{\delta}} = \delta^{\underline{\alpha}}_{\underline{\delta}}\delta^{\underline{\gamma}}_{\underline{\beta}}} \tag{D.66}$$

Using (D.60) it can be rewritten as

$$\boxed{\sum_{p=0}^{d/2-1}\frac{2^{-d/2}}{p!}\Big(\Gamma^{a_1\ldots a_p}{}^{\underline{\alpha}}{}_{\underline{\beta}}\Gamma_{a_p\ldots a_1}{}^{\underline{\gamma}}{}_{\underline{\delta}} + (-)^p(\Gamma^{a_1\ldots a_p}\Gamma^{\#})^{\underline{\alpha}}{}_{\underline{\beta}}(\Gamma_{a_p\ldots a_1}\Gamma^{\#})^{\underline{\gamma}}{}_{\underline{\delta}}\Big) + \frac{2^{-d/2}}{(d/2)!}\Gamma^{a_1\ldots a_{d/2}}{}^{\underline{\alpha}}{}_{\underline{\beta}}\Gamma_{a_{d/2}\ldots a_1}{}^{\underline{\gamma}}{}_{\underline{\delta}} = \delta^{\underline{\alpha}}_{\underline{\delta}}\delta^{\underline{\gamma}}_{\underline{\beta}}} \tag{D.67}$$

which further simplifies when contracted with chiral spinors for which $\Gamma^{\#}\to\mathbb{1}$. The identities (D.66) and equivalently (D.67) can be rewritten in various ways. One appearance of the Fierz identity which is of particular interest, is to contract the identity (D.66) with $\Gamma^c{}^{\underline{\tilde\alpha}}{}_{\underline{\alpha}}\Gamma_c{}^{\underline{\tilde\gamma}}{}_{\underline{\gamma}}$ which yields (after relabeling in the result $\underline{\tilde\alpha}\to\underline{\alpha}$, $\underline{\tilde\gamma}\to\underline{\gamma}$)

$$\sum_{p=0}^{d}\frac{2^{-d/2}}{p!}(\underbrace{\Gamma^c\Gamma^{a_1\ldots a_p}}_{\Gamma^{ca_1\cdots a_p}+p\eta^{c[a_1}\Gamma^{a_2\cdots a_p]}})^{\underline{\alpha}}{}_{\underline{\beta}}(\underbrace{\Gamma_c\Gamma_{a_p\ldots a_1}}_{\Gamma_{ca_p\ldots a_1}+p\eta_{c[a_p}\Gamma_{a_{p-1}\cdots a_1]}})^{\underline{\gamma}}{}_{\underline{\delta}} = \Gamma^c{}^{\underline{\alpha}}{}_{\underline{\delta}}\Gamma_c{}^{\underline{\gamma}}{}_{\underline{\beta}} \tag{D.68}$$

Some relabeling yields

$$\sum_{p=0}^{d}\frac{(-)^p}{2^{d/2}p!}(d-2p)\,(\Gamma^{a_1\ldots a_p})^{\underline{\alpha}}{}_{\underline{\beta}}(\Gamma_{a_p\ldots a_1})^{\underline{\gamma}}{}_{\underline{\delta}} = \Gamma^c{}^{\underline{\alpha}}{}_{\underline{\delta}}\Gamma_c{}^{\underline{\gamma}}{}_{\underline{\beta}} \tag{D.69}$$

Finally we can use again (D.60), in order to arrive at

$$\sum_{p=0}^{d/2-1}\frac{(-)^p}{2^{d/2}p!}(d-2p)\Big((\Gamma^{a_1\ldots a_p})^{\underline{\alpha}}{}_{\underline{\beta}}(\Gamma_{a_p\ldots a_1})^{\underline{\gamma}}{}_{\underline{\delta}} - (-)^p(\Gamma^{a_1\ldots a_p}\Gamma^{\#})^{\underline{\alpha}}{}_{\underline{\beta}}(\Gamma_{a_p\ldots a_1}\Gamma^{\#})^{\underline{\gamma}}{}_{\underline{\delta}}\Big) = \Gamma^c{}^{\underline{\alpha}}{}_{\underline{\delta}}\Gamma_c{}^{\underline{\gamma}}{}_{\underline{\beta}} \tag{D.70}$$

Contracting the identity **with chiral spinors** $\Psi^{\underline{\beta}} = (\psi^\beta, 0)$ and $\Phi^{\underline{\delta}} = (\phi^\delta, 0)$ leads to

$$\boxed{\sum_{p=1,\text{ odd}}^{d/2-1}\frac{2\,(d-2p)}{2^{d/2}p!}(\Gamma^{a_1\ldots a_p}\Psi)^{\underline{\alpha}}(\Gamma_{a_p\ldots a_1}\Phi)^{\underline{\gamma}} = -(-)^{\Phi\Psi}(\Gamma^c\Phi)^{\underline{\alpha}}(\Gamma_c\Psi)^{\underline{\gamma}}} \tag{D.71}$$

$d = 4, 6:$

$$(\Gamma^c\Psi)^{\underline{\alpha}}(\Gamma_c\Phi)^{\underline{\gamma}} = -(-)^{\Phi\Psi}(\Gamma^c\Phi)^{\underline{\alpha}}(\Gamma_c\Psi)^{\underline{\gamma}} \tag{D.72}$$

$d = 10$:

$$\frac{1}{2}(\Gamma^{a_1}\Psi)^{\underline{\alpha}}(\Gamma_{a_p}\Phi)^{\underline{\gamma}} + \frac{1}{24}(\Gamma^{a_1a_2a_3}\Psi)^{\underline{\alpha}}(\Gamma_{a_3a_2a_1}\Phi)^{\underline{\gamma}} = -(-)^{\Phi\Psi}(\Gamma^c\Phi)^{\underline{\alpha}}(\Gamma_c\Psi)^{\underline{\gamma}} \quad \text{(D.73)}$$

In 10 dimensions this can be further rewritten, using the symmetry properties of the gamma matrices in their fermionic indices. We will come back to that in subsection D.3.4.

D.2 Explicit 10d-representation

In the following we will give an explicit representation of the Dirac-Γ-matrices in 10 dimensions which we are using throughout this document. The presentation is based on the one given in the appendix of [9].

D.2.1 D=(2,0): Pauli-matrices (2x2)

We start with the 3 Pauli matrices

$$\tau^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(D.74)}$$

$$\begin{aligned} \tau^i\tau^j &= i\epsilon^{ijk}\tau^k + \delta^{ij}\mathbb{1} && \text{(D.75)} \\ [\tau^i, \tau^j] &= 2i\epsilon^{ijk}\tau^k && \text{(D.76)} \\ \{\tau^i, \tau^j\} &= 2\delta^{ij}\mathbb{1} && \text{(D.77)} \\ \operatorname{tr}\tau^i &= 0, \quad \det(\sigma^i) = -1 && \text{(D.78)} \\ (\tau^i)^\dagger &= \tau^i && \text{(D.79)} \end{aligned}$$

D.2.2 D=(3,1), 4x4

Define $\gamma^k \equiv \tau^k \otimes \tau^2$, $\gamma^4 \equiv \mathbb{1} \otimes \tau^1$,$\gamma^5 \equiv \mathbb{1} \otimes \tau^3$. The tensor product can be understood in different ways when writing down the resulting matrices. We understand it as plugging the lefthand matrix into the righthand one:

$$\gamma^k \equiv \begin{pmatrix} 0 & -i\tau^k \\ i\tau^k & 0 \end{pmatrix}, \quad \gamma^4 \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \equiv i\gamma^0, \quad \gamma^5 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \text{(D.80)}$$

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2\delta^{\mu\nu}\mathbb{1} && \text{(D.81)} \\ \operatorname{tr}(\gamma^\mu) &= 0 && \text{(D.82)} \\ (\gamma^\mu)^\dagger &= \gamma^\mu && \text{(D.83)} \\ \gamma^1\gamma^2\gamma^3\gamma^4 &= \begin{pmatrix} 0 & -i\tau^1\tau^2\tau^3 \\ i\tau^1\tau^2\tau^3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \gamma^5 && \text{(D.84)} \end{aligned}$$

γ^2, γ^4 and γ^5 are real and symmetric, while γ^1 and γ^3 are imaginary and antisymmetric.

D.2.3 D=(7,0), 8x8

We can define seven purely imaginary 8×8 matrices λ^i as follows:

$$\begin{aligned} \lambda^i &= \{\gamma^2 \otimes \tau^2, \gamma^4 \otimes \tau^2, \gamma^5 \otimes \tau^2, \gamma^1 \otimes \mathbb{1}, \gamma^3 \otimes \mathbb{1}, i\gamma^2\gamma^4\gamma^5 \otimes \tau^1, i\gamma^2\gamma^4\gamma^5 \otimes \tau^3\} && \text{(D.85)} \\ & \text{with } i\gamma^2\gamma^4\gamma^5 = i\tau^2 \otimes \tau^2\tau^1\tau^3 = \tau^2 \otimes \mathbb{1} = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \{\lambda^i, \lambda^j\} &= 2\delta^{ij}\mathbb{1} && \text{(D.86)} \\ \operatorname{tr}(\lambda^i) &= 0 && \text{(D.87)} \\ (\lambda^i)^\dagger &= \lambda^i && \text{(D.88)} \\ \lambda^1\cdots\lambda^6 &= (\gamma^2\gamma^4\gamma^5\gamma^1\gamma^3 i\gamma^2\gamma^4\gamma^5) \otimes \tau^2\tau^1 = -(\gamma^1\gamma^3) \otimes \tau^3 = (i\tau^2 \otimes \mathbb{1}) \otimes \tau^3 = ii\gamma^2\gamma^4\gamma^5 \otimes \tau^3 = i\lambda^7 && \text{(D.89)} \end{aligned}$$

D.2.4 D=(8,0), 16x16

Now we can define 8 real symmetric 16×16 matrices $\sigma^\mu \equiv \{\lambda^i \otimes \tau^2, \mathbb{1} \otimes \tau^1\}$

$$\sigma^i \equiv \begin{pmatrix} 0 & -i\lambda^i \\ i\lambda^i & 0 \end{pmatrix}, \quad \sigma^8 \equiv \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{D.90}$$

$$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}\mathbb{1} \tag{D.91}$$
$$(\sigma^\mu)^\dagger = \sigma^\mu \tag{D.92}$$
$$\mathrm{tr}\,(\sigma^\mu) = 0 \tag{D.93}$$
$$\chi \equiv \sigma^1 \cdots \sigma^8 = \lambda^1 \cdots \lambda^7 \otimes \tau^2\tau^1 = \mathbb{1} \otimes \tau^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{D.94}$$

D.2.5 D=(9,1),32x32

Finally we define the real Dirac-matrices for 10-dimensional Minkowski-space as $\Gamma^a \equiv \left\{\mathbb{1} \otimes i\tau^2, \sigma^\mu \otimes \tau_1, \chi \otimes \tau_1\right\}$

$$\Gamma^0 \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \equiv -i\Gamma^{10}, \quad \Gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \Gamma^9 \equiv \begin{pmatrix} 0 & \chi \\ \chi & 0 \end{pmatrix} \tag{D.95}$$

$$\Gamma^{a\,\underline{\alpha}}{}_{\underline{\beta}} \equiv \begin{pmatrix} 0 & \gamma^{a\,\alpha\beta} \\ \gamma^a_{\alpha\beta} & 0 \end{pmatrix}, \quad \text{with } \gamma^{a\,\alpha\beta} \equiv \{\delta^{\alpha\beta}, \sigma^{\mu\,\alpha}{}_\beta, \chi^\alpha{}_\beta\}, \quad \gamma^a_{\alpha\beta} \equiv \{-\delta_{\alpha\beta}, \sigma^{\mu\,\alpha}{}_\beta, \chi^\alpha{}_\beta\} \tag{D.96}$$

The small γ^a (chiral gamma matrices) are thus all real and symmetric! The Dirac matrices obey

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1} \tag{D.97}$$
$$\Gamma^\# \equiv \Gamma^0 \cdots \Gamma^9 = i\Gamma^1 \cdots \Gamma^{10} = \sigma^1 \cdots \sigma^8 \chi \otimes i\tau^2(\tau^1)^9 = \mathbb{1} \otimes \tau^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{D.98}$$
$$(\Gamma^\#)^2 = \mathbb{1}, \quad \Gamma^\#\Gamma^a = -\Gamma^a\Gamma^\# \tag{D.99}$$
$$(\Gamma^a)^\dagger = \Gamma^a, \quad (\Gamma^\#)^\dagger = \Gamma^\# \tag{D.100}$$
$$\mathrm{tr}\,\Gamma^a = 0, \quad \mathrm{tr}\,\Gamma^\# = 0 \tag{D.101}$$

Intertwiners The unitary intertwiners A, B and C are defined via

$$(\Gamma^a)^\dagger = A\Gamma^a A^\dagger, \qquad -(\Gamma^a)^* = B^\dagger\Gamma^a B, \qquad -(\Gamma^a)^T = C^\dagger\Gamma^a C \tag{D.102}$$

We can choose

$$A_{\underline{\alpha\beta}} = -\Gamma^0\Gamma^\# = \begin{pmatrix} 0 & \delta^\beta_\alpha \\ \delta^\alpha_\beta & 0 \end{pmatrix} \tag{D.103}$$
$$B = \Gamma^\# \tag{D.104}$$
$$C = BA^\dagger = -\Gamma^\#\Gamma^0\Gamma^\# = \Gamma^0 \tag{D.105}$$

The Dirac conjugate is $\bar{\psi} \equiv \psi^\dagger A$. In the Lorentz-covariant expression $\bar{\psi}\Gamma^m\phi$, there appears therefore the combination

$$(A\Gamma^a)_{\underline{\alpha\beta}} = \begin{pmatrix} \gamma^a_{\alpha\beta} & 0 \\ 0 & \gamma^{a\,\alpha\beta} \end{pmatrix}, \quad \gamma^a_{\alpha\beta} \text{ sym and real} \tag{D.106}$$

The other conjugate is the charge conjugate spinor $\psi^c \equiv C\bar{\psi}^T = CA^T\psi^* = B\psi^* = \Gamma^\#\psi^*$.

D.3 Clifford algebra, Fierz identity and more for the chiral blocks in 10 dimensions

Above we have defined

$$\Gamma^{a\,\underline{\alpha}}{}_{\underline{\beta}} = \begin{pmatrix} 0 & \gamma^{a\,\alpha\beta} \\ \gamma^a_{\alpha\beta} & 0 \end{pmatrix} \tag{D.107}$$

The Clifford algebra for the Γ's reads in terms of the smallo γ's:

$$\gamma^{(a|\,\alpha\gamma}\gamma^{|b)}_{\gamma\beta} = \eta^{ab}\delta^\alpha_\beta \tag{D.108}$$
$$\gamma^{(a|\,\alpha\beta}\gamma^{|b)}_{\beta\alpha} = 16\eta^{ab} \tag{D.109}$$

D.3.1 Product of antisymmetrized products of gamma-matrices

Antisymmetrized products of Γ's are block-diagonal for even number of factors and block-offdiagonal for odd number of factors[4]. The chiral blocks read:

$$\begin{aligned}
\gamma^{a_1\ldots a_{2k}\,\alpha}{}_{\beta} &\equiv \gamma^{[a_1|\,\alpha\gamma_1}\gamma^{|a_2|}_{\gamma_1\gamma_2}\cdots\gamma^{|a_{2k}]}_{\gamma_{2k-1}\beta} = (-)^k\gamma^{a_1\ldots a_{2k}}{}_{\beta}{}^{\alpha} && \text{(D.110)}\\
\gamma^{a_1\ldots a_{2k+1}}_{\alpha\beta} &= (-)^k\gamma^{a_1\ldots a_{2k+1}}_{\beta\alpha}, \qquad \gamma^{a_1\ldots a_{2k+1}\,\alpha\beta} = (-)^k\gamma^{a_1\ldots a_{2k+1}\,\beta\alpha} && \text{(D.111)}
\end{aligned}$$

The schematic expansion of antisymmetrized products of Γ-matrices given in (D.4) has the same form for the chiral blocks, if we suppress the index structure:

$$\boxed{\gamma^{[k]}\gamma^{[l]} \propto \gamma^{[|k-l|]} + \gamma^{[|k-l|+2]} + \ldots + \gamma^{[k+l]}} \qquad \text{(D.112)}$$

Indeed, without the spinorial indices, even the exact equations (including the correct prefactors) look identically for the small γ's:

$$\gamma^{a_1\ldots a_p}\gamma^{b_1\ldots b_q} = \sum_{k=0}^{min\{p,q\}} k!\begin{pmatrix} p\\ k\end{pmatrix}\begin{pmatrix} q\\ k\end{pmatrix}\eta^{[a_p|\,[b_1|}\eta^{|a_{p-1}|\,|b_2|}\cdots\eta^{|a_{p+1-k}|\,|b_k|}\gamma^{|a_1\ldots a_{p-k}]\,|b_{k+1}\ldots b_q]} \qquad \text{(D.113)}$$

In particular we have

$$\begin{aligned}
\gamma^{a_1}\gamma^{b_1\ldots b_l} &= \gamma^{a_1b_1\ldots b_l} + l\cdot\eta^{a_1[b_1}\gamma^{b_2\ldots b_l]}, \qquad \gamma^{b_1\ldots b_l}\gamma^{a_1} = \gamma^{b_1\ldots b_l a_1} + l\cdot\gamma^{[b_1\ldots b_{l-1}}\eta^{b_l]a_1} && \text{(D.114)}\\
\gamma^{a_1a_2}\gamma^{b_1\ldots b_l} &= \gamma^{a_1a_2b_1\ldots b_l} - l\cdot\eta^{a_1[b_1|}\gamma^{a_2|b_2\ldots b_l]} + l\cdot\eta^{a_2[b_1|}\gamma^{a_1|b_2\ldots b_l]} +\\
&\quad -l(l-1)\eta^{a_1[b_1|}\eta^{a_2|b_2}\gamma^{b_3\ldots b_l]} && \text{(D.115)}\\
\gamma^{a_1a_2}\gamma^{b_1b_2} &= \gamma^{a_1a_2b_1b_2} - 2\eta^{a_1[b_1|}\gamma^{a_2|b_2]} + 2\eta^{a_2[b_1|}\gamma^{a_1|b_2]} - 2\eta^{a_1[b_1|}\eta^{a_2|b_2]} =\\
&= \gamma^{a_1a_2b_1b_2} + \eta^{a_2b_1}\gamma^{a_1b_2} + \eta^{a_1b_2}\gamma^{a_2b_1} - \eta^{a_1b_1}\gamma^{a_2b_2} - \eta^{a_2b_2}\gamma^{a_1b_1} +\\
&\quad +\eta^{a_1b_2}\eta^{a_2b_1} - \eta^{a_1b_1}\eta^{a_2b_2} && \text{(D.116)}
\end{aligned}$$

Reintroducing the spinorial indices for the last line yields (remember that we do not use our graded conventions in this part of the appendix):

$$\begin{aligned}
\gamma^{a_1a_2}{}_{\alpha}{}^{\gamma}\gamma^{b_1b_2}{}_{\gamma}{}^{\beta} &= \gamma^{a_1a_2b_1b_2}{}_{\alpha}{}^{\beta} + \eta^{a_2b_1}\gamma^{a_1b_2}{}_{\alpha}{}^{\beta} + \eta^{a_1b_2}\gamma^{a_2b_1}{}_{\alpha}{}^{\beta} - \eta^{a_1b_1}\gamma^{a_2b_2}{}_{\alpha}{}^{\beta} - \eta^{a_2b_2}\gamma^{a_1b_1}{}_{\alpha}{}^{\beta} +\\
&\quad +\eta^{a_1b_2}\eta^{a_2b_1}\delta_{\alpha}^{\beta} - \eta^{a_1b_1}\eta^{a_2b_2}\delta_{\alpha}^{\beta} && \text{(D.117)}
\end{aligned}$$

If we regard $\gamma^{a_1a_2}{}_{\alpha}{}^{\gamma}$ as a matrix with collected indices (a_1,α) and (a_2,γ), we can use the above equation also to construct an inverse to this matrix: Contracting a_2 and b_1, we obtain

$$\gamma^{a_1}{}_{c\alpha}{}^{\gamma}\gamma^{cb_2}{}_{\gamma}{}^{\beta} = 8\gamma^{a_1b_2}{}_{\alpha}{}^{\beta} + 9\eta^{a_1b_2}\delta_{\alpha}^{\beta} \qquad \text{(D.118)}$$

and therefore

$$\frac{1}{9}\gamma^{a_1}{}_{c\alpha}{}^{\gamma}\left(\gamma^{cb_2}{}_{\gamma}{}^{\beta} - 8\eta^{cb_2}\delta_{\gamma}^{\beta}\right) = \eta^{a_1b_2}\delta_{\alpha}^{\beta} \qquad \text{(D.119)}$$

If two indices in (D.117) are contracted, it turns into

$$\gamma^{ab}{}_{\alpha}{}^{\gamma}\gamma_{ba\gamma}{}^{\beta} = 90\delta_{\alpha}^{\beta} \qquad \text{(D.120)}$$

The equations (D.118) and (D.120) are special cases of the following equations (which are in turn a direct consequence of (D.114) and (D.115)):

$$\begin{aligned}
\gamma_{b_1}\gamma^{b_1\ldots b_l} &= l\cdot\delta^{[b_1}_{b_1}\gamma^{b_2\ldots b_l]} = (11-l)\gamma^{b_2\ldots b_l}, \qquad \gamma^{b_1\ldots b_l}\gamma_{b_l} = (11-l)\gamma^{b_1\ldots b_{l-1}} && \text{(D.121)}\\
\gamma^{a_1}{}_{b_1}\gamma^{b_1\ldots b_l} &= (10-l)\cdot\gamma^{a_1b_2\ldots b_l} + (11-l)\,(l-1)\eta^{a_1[b_2}\gamma^{b_3\ldots b_l]} && \text{(D.122)}\\
\gamma_{b_2b_1}\gamma^{b_1\ldots b_l} &= (11-l)\,(12-l)\,\gamma^{b_3\ldots b_l} && \text{(D.123)}
\end{aligned}$$

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[4] For example, the product of two gamma-matrices reads

$$\begin{aligned}
\Gamma^{a_1a_2\,\underline{\alpha}}{}_{\underline{\beta}} &\equiv \Gamma^{[a_1|\,\underline{\alpha}}{}_{\underline{\gamma}}\Gamma^{|a_1]\,\underline{\gamma}}{}_{\underline{\beta}} =\\
&= \begin{pmatrix} \gamma^{[a_1|\,\alpha\gamma}\gamma^{|a_2]}_{\gamma\beta} & 0\\ 0 & \gamma^{[a_1}_{\alpha\gamma}\gamma^{a_2]\,\gamma\beta} = -\gamma^{[a_1|\,\beta\gamma}\gamma^{|a_2]}_{\gamma\alpha}\end{pmatrix} \equiv \begin{pmatrix} \gamma^{a_1a_2\,\alpha}{}_{\beta} & 0\\ 0 & \gamma^{a_1a_2}{}_{\alpha}{}^{\beta}\end{pmatrix}\\
\gamma^{a_1a_2\,\alpha}{}_{\beta} &= -\gamma^{a_1a_2}{}_{\beta}{}^{\alpha}\\
\gamma^{[0]\,\alpha}{}_{\beta} &\equiv \delta^{\alpha}_{\beta} \quad \text{(no index-grading here!)} \qquad \diamond
\end{aligned}$$

D.3.2 Hodge duality

In the intermezzo on page 169, we had defined the Hodge star operator such that it coincides with the multiplication of $\Gamma^{\#}$ from the right. Remember

$$\Gamma^{\#\,\underline{\alpha}}{}_{\underline{\beta}} \equiv \Gamma^{0\ldots 9\,\underline{\alpha}}{}_{\underline{\beta}} = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \tag{D.124}$$

$$\Gamma^{\#}\Gamma^{a_1\ldots a_p} = \frac{1}{(10-p)!}(-)^{p(p+1)/2}\epsilon^{a_1\ldots a_p}{}_{c_1\ldots c_{10-p}}\Gamma^{c_1\ldots c_{10-p}} = \frac{1}{(10-p)!}\Gamma^{c_{10}\ldots c_{p+1}}\epsilon_{c_{10}\ldots c_{p+1}}{}^{a_p\ldots a_1} \tag{D.125}$$

The chiral blocks of $\Gamma^{\#}$coincide either with plus or minus the unit matrix:

$$\gamma^{\#\,\alpha}{}_{\beta} \equiv \gamma^{0\ldots 9\,\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} = \frac{1}{10!}\epsilon_{c_1\ldots c_{10}}\gamma^{c_1\ldots c_{10}\,\alpha}{}_{\beta} \quad \text{with } \epsilon_{01\ldots 9} \equiv 1 \tag{D.126}$$

$$\gamma^{\#\,\beta}_{\alpha} \equiv \gamma^{0\ldots 9}{}_{\alpha}{}^{\beta} = -\delta^{\beta}_{\alpha} = \frac{1}{10!}\epsilon_{c_1\ldots c_{10}}\gamma^{c_1\ldots c_{10}}{}_{\alpha}{}^{\beta} \tag{D.127}$$

Any chiral block $\gamma^{[p]}$ of $\Gamma^{[p]}$ is therefore always equal (not only "Hodge-dual") to a $\gamma^{[10-p]}$:

$$\gamma^{a_1\ldots a_{2k}\,\alpha}{}_{\beta} = \frac{1}{(10-2k)!}(-)^k\epsilon^{a_1\ldots a_{2k}}{}_{c_1\ldots c_{10-2k}}\gamma^{c_1\ldots c_{10-2k}\,\alpha}{}_{\beta} = \frac{1}{(10-2k)!}\gamma^{c_{10}\ldots c_{2k+1}\,\alpha}{}_{\beta}\epsilon_{c_{10}\ldots c_{2k+1}}{}^{a_{2k}\ldots a_1} \tag{D.128}$$

$$-\gamma^{a_1\ldots a_{2k}}{}_{\alpha}{}^{\beta} = \frac{1}{(10-2k)!}(-)^k\epsilon^{a_1\ldots a_{2k}}{}_{c_1\ldots c_{10-2k}}\gamma^{c_1\ldots c_{10-2k}}{}_{\alpha}{}^{\beta} = \frac{1}{(10-2k)!}\gamma^{c_{10}\ldots c_{2k+1}}{}_{\alpha}{}^{\beta}\epsilon_{c_{10}\ldots c_{2k+1}}{}^{a_{2k}\ldots a_1} \tag{D.129}$$

$$\gamma^{a_1\ldots a_{2k+1}\,\alpha\beta} = \frac{1}{(9-2k)!}(-)^{(k+1)}\epsilon^{a_1\ldots a_{2k+1}}{}_{c_1\ldots c_{9-2k}}\gamma^{c_1\ldots c_{9-2k}\,\alpha\beta} = \frac{1}{(9-2k)!}\gamma^{c_{10}\ldots c_{2k+2}\,\alpha\beta}\epsilon_{c_{10}\ldots c_{2k+2}}{}^{a_{2k+1}\ldots a_1} \tag{D.130}$$

$$-\gamma_{\alpha\beta}^{a_1\ldots a_{2k+1}} = \frac{1}{(9-2k)!}(-)^{(k+1)}\epsilon^{a_1\ldots a_{2k+1}}{}_{c_1\ldots c_{9-2k}}\gamma_{\alpha\beta}^{c_1\ldots c_{9-2k}} = \frac{1}{(9-2k)!}\gamma_{\alpha\beta}^{c_{10}\ldots c_{2k+2}}\epsilon_{c_{10}\ldots c_{2k+2}}{}^{a_{2k+1}\ldots a_1} \tag{D.131}$$

In particular this leads to a self duality constraint for $\gamma^{[5]}$:

$$\gamma^{a_1\ldots a_5\,\alpha\beta} = -\frac{1}{5!}\epsilon^{a_1\ldots a_5}{}_{c_1\ldots c_5}\gamma^{c_1\ldots c_5\,\alpha\beta} \tag{D.132}$$

$$\gamma_{\alpha\beta}^{a_1\ldots a_5} = \frac{1}{5!}\epsilon^{a_1\ldots a_5}{}_{c_1\ldots c_5}\gamma_{\alpha\beta}^{c_1\ldots c_5} \tag{D.133}$$

This is the same behaviour as for the $\Gamma^{[p]}$'s themselves in odd dimensions, where $\Gamma^{\#}$ coincides with the unit matrix. This means that a bispinor with two chiral indices cannot just be seen as a sum of odd (same chirality) or even (opposite chirality) forms, but as a self-dual sum of odd an even forms. This is also further discussed in the intermezzo on RR-fields on page 104.

For the five-form we had $\Gamma^{\#}\Gamma^{a_1\ldots a_5}\otimes\Gamma_{a_5\ldots a_1}\Gamma^{\#} = \Gamma_{d_1\ldots d_5}\otimes\Gamma^{d_1\ldots d_5}$, which turns into $-\gamma^{a_1\ldots a_5\,\alpha\beta}\gamma^{\gamma\delta}_{a_5\ldots a_1} = \gamma^{\alpha\beta}_{d_1\ldots d_5}\gamma^{d_1\ldots d_5\,\gamma\delta}$ and $-\gamma^{a_1\ldots a_5}_{\alpha\beta}\gamma_{a_5\ldots a_1\,\gamma\delta} = \gamma_{d_1\ldots d_5\,\alpha\beta}\gamma^{d_1\ldots d_5}_{\gamma\delta}$ and thus

$$\boxed{\gamma^{a_1\ldots a_5\,\alpha\beta}\gamma^{\gamma\delta}_{a_5\ldots a_1} = \gamma^{a_1\ldots a_5}_{\alpha\beta}\gamma_{a_5\ldots a_1\,\gamma\delta} = 0} \tag{D.134}$$

D.3.3 Vanishing of gamma-traces and projectors for the gamma-matrix expansion

For any even p $(2 \le p \le 8)$ we have

$$\gamma^{a_1\ldots a_p\,\alpha}{}_{\alpha} = 0, \quad 2 \le p \le 8,\ p \text{ even} \tag{D.135}$$

The reason is that there is no invariant constant tensor with p antisymmetrized indices apart from the ϵ-tensor for $p = 10$ and the Kronecker delta for $p = 0$:

$$\gamma^{a_1\ldots a_{10}}{}_{\alpha}{}^{\alpha} = -\gamma^{a_1\ldots a_{10}\,\alpha}{}_{\alpha} = 16\epsilon^{a_1\ldots a_{10}}, \qquad \gamma^{[0]}{}_{\alpha}{}^{\alpha} \equiv \gamma^{[0]\,\alpha}{}_{\alpha} \equiv \delta^{\alpha}_{\alpha} = 16 \tag{D.136}$$

With the same argument we get $\gamma^a_{\alpha\beta}\gamma^{\alpha\beta}_b \propto \delta^a_b$ and fixing the proportionality by taking the trace yields

$$\gamma^a_{\alpha\beta}\gamma^{\beta\alpha}_b = 16\delta^a_b \tag{D.137}$$

Alternatively this can be derived from $\gamma^a_{\alpha\beta}\gamma^{b\,\beta\gamma} = \eta^{ab}\delta^{\gamma}_{\alpha} + \gamma^{ab}{}_{\alpha}{}^{\gamma}$ (the Clifford algebra for the chiral blocks and thus a special case of (D.113)) together with (D.135). In the same manner we get for all other forms (using (D.113) and (D.135))

$$\gamma^{a_1\ldots a_p}_{\alpha\beta}\gamma^{\beta\alpha}_{b_p\ldots b_1} = 16p!\delta^{a_1\ldots a_p}_{b_1\ldots b_p} \quad \text{for } p \in \{1,3\} \tag{D.138}$$

$$\gamma^{a_1\ldots a_5}_{\alpha\beta}\gamma^{\beta\alpha}_{b_5\ldots b_1} = 16\epsilon^{a_1\ldots a_5}{}_{b_5\ldots b_1} + 16\cdot 5!\delta^{a_1\ldots a_5}_{b_1\ldots b_5} \tag{D.139}$$

$$\gamma^{a_1\ldots a_p\,\alpha}{}_{\beta}\gamma_{b_p\ldots b_1}{}^{\beta}{}_{\alpha} = 16p!\delta^{a_1\ldots a_p}_{b_1\ldots b_p} \quad \text{for } p \in \{2,4\} \tag{D.140}$$

The extra term in the $\gamma^{[5]}\gamma_{[5]}$ contraction on the righthand side of the second line is due to the fact that the trace of $\gamma^{[10]}$ does not vanish according to (D.136). Any other contraction, where the number of bosonic indices does not match, vanishes

$$\mathrm{tr}\,(\gamma^{[p]}\gamma^{[q]}) \quad = \quad 0 \quad \text{for } p \neq q, \text{ and } p,q \leq 5 \tag{D.141}$$

The results of above can be used to project to the coefficients of γ-matrix expansions:

$$\begin{aligned} A^{\alpha\beta} &= A_a\gamma^{a\,\alpha\beta} + A_{a_1a_2a_3}\gamma^{a_1a_2a_3\,\alpha\beta} + A_{a_1\ldots a_5}\gamma^{a_1\ldots a_5\,\alpha\beta}, \\ &\quad \text{with } A_{a_1\ldots a_p} = \frac{1}{16p!}\gamma_{a_p\ldots a_1\,\beta\alpha}A^{\alpha\beta} \text{ for } p\in\{1,3\} \text{ and } A_{a_1\ldots a_5} = \frac{1}{32\cdot 5!}\gamma_{a_5\ldots a_1\,\beta\alpha}A^{\alpha\beta} \end{aligned} \tag{D.142}$$

$$\begin{aligned} D_{\alpha\beta} &= D_a\gamma^a_{\alpha\beta} + D_{a_1a_2a_3}\gamma^{a_1a_2a_3}_{\alpha\beta} + D_{a_1\ldots a_5}\gamma^{a_1\ldots a_5}_{\alpha\beta}, \\ &\quad \text{with } D_{a_1\ldots a_p} = \frac{1}{16p!}\gamma^{\beta\alpha}_{a_p\ldots a_1}D_{\alpha\beta} \text{ for } p\in\{1,3\} \text{ and } D_{a_1\ldots a_5} = \frac{1}{32\cdot 5!}\gamma^{\beta\alpha}_{a_5\ldots a_1}D_{\alpha\beta} \end{aligned} \tag{D.143}$$

$$B^\alpha{}_\beta = B_{[0]}\delta^\alpha_\beta + B_{a_1a_2}\gamma^{a_1a_2\,\alpha}{}_\beta + B_{a_1a_2a_3a_4}\gamma^{a_1a_2a_3a_4\,\alpha}{}_\beta, \qquad B_{a_1\ldots a_p} = \frac{1}{16p!}\gamma_{a_p\ldots a_1}{}^\beta{}_\alpha B^\alpha{}_\beta \tag{D.144}$$

$$C_\alpha{}^\beta = C_{[0]}\delta^\beta_\alpha + C_{a_1a_2}\gamma^{a_1a_2}{}_\alpha{}^\beta + C_{a_1a_2a_3a_4}\gamma^{a_1a_2a_3a_4}{}_\alpha{}^\beta, \qquad C_{a_1\ldots a_p} = \frac{1}{16p!}\gamma_{a_p\ldots a_1\beta}{}^\alpha C_\alpha{}^\beta \tag{D.145}$$

For the first two expansions it was used that due to the restrictions (D.132) and (D.133) on $\gamma^{[5]}$, the corresponding expansion coefficients can always be chosen to obey (anti) self-duality constraints of the form

$$A_{a_1\ldots a_5} \quad = \quad -\frac{1}{5!}A_{c_1\ldots c_5}\epsilon^{c_1\ldots c_5}{}_{a_1\ldots a_5} \tag{D.146}$$

$$D_{a_1\ldots a_5} \quad = \quad \frac{1}{5!}D_{c_1\ldots c_5}\epsilon^{c_1\ldots c_5}{}_{a_1\ldots a_5} \tag{D.147}$$

which lead together with (D.139) to an extra factor of two and thus to a normalization factor $\frac{1}{32}$ instead of $\frac{1}{16}$ for $p=5$.

D.3.4 Chiral Fierz

Remember

$$\sum_{p=0}^{10}\frac{1}{32p!}\Gamma^{a_1\ldots a_p\,\underline{\alpha}}{}_{\underline{\beta}}\Gamma_{a_p\ldots a_1}{}^{\underline{\gamma}}{}_{\underline{\delta}} = \delta^{\underline{\alpha}}_{\underline{\delta}}\delta^{\underline{\gamma}}_{\underline{\beta}} \tag{D.148}$$

or

$$\sum_{p=0}^{4}\frac{1}{32p!}\left(\Gamma^{a_1\ldots a_p\,\underline{\alpha}}{}_{\underline{\beta}}\Gamma_{a_p\ldots a_1}{}^{\underline{\gamma}}{}_{\underline{\delta}} + (\Gamma^\#\Gamma^{a_1\ldots a_p})^{\underline{\alpha}}{}_{\underline{\beta}}(\Gamma_{a_p\ldots a_1}\Gamma^\#)^{\underline{\gamma}}{}_{\underline{\delta}}\right) + \frac{1}{32\cdot 5!}\Gamma^{a_1\ldots a_5\,\underline{\alpha}}{}_{\underline{\beta}}\Gamma_{a_5\ldots a_1}{}^{\underline{\gamma}}{}_{\underline{\delta}} = \delta^{\underline{\alpha}}_{\underline{\delta}}\delta^{\underline{\gamma}}_{\underline{\beta}} \tag{D.149}$$

We want to make a distinction of the different cases corresponding to the chiral indices:

$$\sum_{p\in\{0,2,4\}}\frac{1}{16p!}\left(\gamma^{a_1\ldots a_p\,\alpha}{}_\beta\gamma_{a_p\ldots a_1}{}^\gamma{}_\delta\right) \quad = \quad \delta^\alpha_\delta\delta^\gamma_\beta \tag{D.150}$$

$$0\cdot + \frac{-4}{4}\sum_{p\in\{1,3\}}\frac{1}{16p!}\gamma^{a_1\ldots a_p\,\alpha\beta}\gamma_{a_p\ldots a_1}{}^{\gamma\delta} + \frac{1}{32\cdot 5!}\underbrace{\gamma^{a_1\ldots a_5\,\alpha\beta}\gamma_{a_5\ldots a_1}{}^{\gamma\delta}}_{=0} \quad = \quad 0 \tag{D.151}$$

$$0\cdot\sum_{p\in\{1,3\}}\frac{1}{16p!}\gamma^{a_1\ldots a_p}{}_{\alpha\beta}\gamma_{a_p\ldots a_1\,\gamma\delta} + \frac{1}{32\cdot 5!}\underbrace{\gamma^{a_1\ldots a_5}{}_{\alpha\beta}\gamma_{a_5\ldots a_1\,\gamma\delta}}_{=0} \quad = \quad 0 \tag{D.152}$$

$$\sum_{p\in\{1,3\}}\frac{1}{16p!}\gamma^{a_1\ldots a_p\,\alpha\beta}\gamma_{a_p\ldots a_1\,\gamma\delta} + \frac{1}{32\cdot 5!}\gamma^{a_1\ldots a_5\,\alpha\beta}\gamma_{a_5\ldots a_1\,\gamma\delta} \quad = \quad \delta^\alpha_\delta\delta^\beta_\gamma \tag{D.153}$$

Only the first and the last give nontrivial information.

$$\delta^\alpha_\beta\delta^\gamma_\delta + \frac{1}{2}\gamma^{a_1a_2\,\alpha}{}_\beta\gamma_{a_2a_1}{}^\gamma{}_\delta + \frac{1}{4!}\gamma^{a_1a_2a_3a_4\,\alpha}{}_\beta\gamma_{a_4a_3a_2a_1}{}^\gamma{}_\delta \quad = \quad 16\delta^\alpha_\delta\delta^\gamma_\beta \tag{D.154}$$

$$\gamma^{a\,\alpha\beta}\gamma_{a\,\gamma\delta} + \frac{1}{3!}\gamma^{a_1a_2a_3\,\alpha\beta}\gamma_{a_3a_2a_1\,\gamma\delta} + \frac{1}{2\cdot 5!}\gamma^{a_1\ldots a_5\,\alpha\beta}\gamma_{a_5\ldots a_1\,\gamma\delta} \quad = \quad 16\delta^\alpha_\delta\delta^\beta_\gamma \tag{D.155}$$

Contracting γ,δ in (D.154) yields $16\delta^\alpha_\beta = 16\delta^\alpha_\beta$, contracting γ,β instead, yields[5]

$$\begin{aligned}
\delta^\alpha_\delta + \frac{1}{2}\gamma^{a_1a_2\,\alpha}{}_\gamma\gamma_{a_2a_1}{}^\gamma{}_\delta + \frac{1}{4!}\gamma^{a_1a_2a_3a_4\,\alpha}{}_\gamma\gamma_{a_4a_3a_2a_1}{}^\gamma{}_\delta &= (16)^2\delta^\alpha_\delta && \text{(D.156)}\\
\underbrace{\gamma^{a\,\alpha\beta}\gamma_{a\,\beta\delta}}_{10\delta^\alpha_\delta} + \frac{1}{3!}\gamma^{a_1a_2a_3\,\alpha\beta}\gamma_{a_3a_2a_1\,\beta\delta} + \frac{1}{2\cdot 5!}\gamma^{a_1\ldots a_5\,\alpha\beta}\gamma_{a_5\ldots a_1\,\beta\delta} &= (16)^2\delta^\alpha_\delta && \text{(D.157)}
\end{aligned}$$

We can also contract (D.154) with $\gamma^b_{\alpha\rho}\gamma_{b\,\gamma\sigma}$ to arrive at

$$0 = \gamma^b_{\beta\rho}\gamma_{b\,\delta\sigma} + \frac{1}{2}\underbrace{\gamma^{a_1a_2\,\alpha}{}_\beta\gamma^b_{\alpha\rho}}_{\gamma^{[3]}+\gamma^{[1]}}\underbrace{\gamma_{b\,\gamma\sigma}\gamma_{a_2a_1}{}^\gamma{}_\delta}_{\gamma_{[3]}+\gamma_{[1]}} + \frac{1}{4!}\underbrace{\gamma^{a_1a_2a_3a_4\,\alpha}{}_\beta\gamma^b_{\alpha\rho}}_{\gamma^{[5]}+\gamma^{[3]}}\underbrace{\gamma_{b\,\gamma\sigma}\gamma_{a_4a_3a_2a_1}{}^\gamma{}_\delta}_{\gamma_{[5]}+\gamma_{[3]}} - 16\gamma^b_{\delta\rho}\gamma_{b\,\beta\sigma} \qquad \text{(D.158)}$$

Now we use that $\gamma^{[3]}$is antisymmetric in $\beta\rho$ and that $\gamma^{[5]}\gamma_{[5]} = 0$ (mixed terms like $\gamma^{[5]}\gamma_{[3]}$ also vanish, because some η are contracted with antisymmetric indices of $\gamma^{[5]}$). Symmetrizing the above equation in $\beta\rho$ yields

$$\begin{aligned}
0 &= \gamma^b_{\beta\rho}\gamma_{b\,\delta\sigma} + 2\eta^{b[a_1}\gamma^{a_2]}_{\rho\beta}\eta_{b[a_2}\gamma_{a_1]\,\sigma\delta} - 16\gamma^b_{\delta(\rho|}\gamma_{b\,|\beta)\sigma} = \\
&= \gamma^b_{\beta\rho}\gamma_{b\,\delta\sigma} + 2\delta^{[a_1}_{a_2}\gamma^{a_2]}_{\rho\beta}\gamma_{a_1\,\sigma\delta} - 16\gamma^b_{\delta(\rho|}\gamma_{b\,|\beta)\sigma} = \\
&= \gamma^b_{\beta\rho}\gamma_{b\,\delta\sigma} + \delta^{a_1}_{a_2}\gamma^{a_2}_{\rho\beta}\gamma_{a_1\,\sigma\delta} - \delta^{a_2}_{a_2}\gamma^{a_1}_{\rho\beta}\gamma_{a_1\,\sigma\delta} - 16\gamma^b_{\delta(\rho|}\gamma_{b\,|\beta)\sigma} = \\
&= \gamma^b_{\beta\rho}\gamma_{b\,\delta\sigma} + \gamma^a_{\rho\beta}\gamma_{a\,\sigma\delta} - 10\gamma^{a_1}_{\rho\beta}\gamma_{a_1\,\sigma\delta} - 16\gamma^b_{\delta(\rho|}\gamma_{b\,|\beta)\sigma} = \\
&= -8\gamma^b_{\beta\rho}\gamma_{b\,\delta\sigma} - 16\gamma^b_{\delta(\rho|}\gamma_{b\,|\beta)\sigma} && \text{(D.159)}
\end{aligned}$$

$$\boxed{\gamma^b_{(\beta\rho|}\gamma_{b\,|\delta)\sigma} = 0} \qquad \text{(D.160)}$$

We could have used directly equation (D.73) to derive this result. This is a very important identity because it is so simple and can be used to derive many other identities. One example will be useful for us in the main part. Consider the contraction of the bosonic indices of two $\gamma^{[2]}$'s:

$$\begin{aligned}
\gamma^{ab\,\alpha}{}_\beta\gamma_{ab}{}^\gamma{}_\delta &= \left(\gamma^{a\,\alpha\rho}\gamma^b_{\rho\beta} - \eta^{ab}\delta^\alpha_\beta\right)\left(\gamma^{\gamma\sigma}_a\gamma_{b\,\sigma\delta} - \eta_{ab}\delta^\gamma_\delta\right) = && \text{(D.161)}\\
&= \gamma^{a\,\alpha\rho}\gamma^{\gamma\sigma}_a\gamma^b_{\rho\beta}\gamma_{b\,\sigma\delta} - \gamma^{a\,\alpha\rho}\gamma_{a\,\rho\beta}\delta^\gamma_\delta - \gamma^{b\,\gamma\sigma}\gamma_{b\,\sigma\delta}\,\delta^\alpha_\beta + 10\delta^\alpha_\beta\delta^\gamma_\delta && \text{(D.162)}
\end{aligned}$$

In order to make use of (D.160) we symmetrize the lower spinorial indices and obtain

$$\begin{aligned}
\gamma^{ab\,\alpha}{}_{(\beta|}\gamma_{ab}{}^\gamma{}_{|\delta)} &= \left(\gamma^{a\,\alpha\rho}\gamma^b_{\rho\beta} - \eta^{ab}\delta^\alpha_\beta\right)\left(\gamma^{\gamma\sigma}_a\gamma_{b\,\sigma\delta} - \eta_{ab}\delta^\gamma_\delta\right) = && \text{(D.163)}\\
&= \gamma^{a\,\alpha\rho}\gamma^{\gamma\sigma}_a\underbrace{\gamma^b_{\rho(\beta|}\gamma_{b\,|\delta)\sigma}}_{-\frac{1}{2}\gamma^b_{\beta\delta}\gamma_{b\,\rho\sigma}\ (D.160)} - \underbrace{\gamma^{a\,\alpha\rho}\gamma_{a\,\rho(\beta}}_{10\delta^\alpha_{(\beta}}\delta^\gamma_{\delta)} - \underbrace{\gamma^{b\,\gamma\sigma}\gamma_{b\,\sigma(\delta}}_{10\delta^\gamma_{(\delta}}\delta^\alpha_{\beta)} + 10\delta^\alpha_{(\beta}\delta^\gamma_{\delta)} = && \text{(D.164)}\\
&= -\frac{1}{2}\underbrace{\gamma^{a\,\alpha\rho}\gamma_{b\,\rho\sigma}\gamma^{\sigma\gamma}_a}_{-8\gamma^{\alpha\gamma}_b\ (D.9)}\gamma^b_{\beta\delta} - 10\delta^\alpha_{(\beta}\delta^\gamma_{\delta)} && \text{(D.165)}
\end{aligned}$$

We can thus express $\gamma^{[2]}\gamma_{[2]}$ by $\gamma^{[1]}\gamma_{[1]}$ and Kronecker deltas

$$\boxed{\gamma^{ab\,\alpha}{}_{(\beta|}\gamma_{ab}{}^\gamma{}_{|\delta)} = 4\gamma^{\alpha\gamma}_a\gamma^a_{\beta\delta} - 10\delta^\alpha_{(\beta}\delta^\gamma_{\delta)}} \qquad \text{(D.166)}$$

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[5]As a consitency check we can in addition contract α,δ and get for the first Fierz

$$\begin{aligned}
16 + 16\frac{1}{2}2!\delta^{a_1a_2}_{a_1a_2} + 16\frac{1}{4!}4!\delta^{a_1\cdots a_4}_{a_1\cdots a_4} &= (16)^3\\
1 + \underbrace{\binom{10}{2}}_{45} + \underbrace{\binom{10}{4}}_{210} &= (16)^2 = 256
\end{aligned}$$

and for the second one

$$10 + \underbrace{\binom{10}{3}}_{120} + \frac{1}{2}\underbrace{\binom{10}{5}}_{252} = 256 \qquad \diamond$$

Appendix E

Noether

E.1 Noether's theorem and the inverse Noether method

Most of the following presentation is based on [95, p.67f, p.95], although somewhat modified. Consider an action of the quite general form

$$S[\phi^{\mathcal{I}}_{\rm all}] \quad \equiv \quad \int d^n\sigma \quad \mathcal{L}(\phi^{\mathcal{I}}_{\rm all}, \partial_\mu \phi^{\mathcal{I}}_{\rm all}, \partial_{\mu_1}\partial_{\mu_2}\phi^{\mathcal{I}}_{\rm all}, \ldots) \tag{E.1}$$

In most of the applications there appear no higher derivatives than $\partial_\mu \phi^{\mathcal{I}}_{\rm all}$. Let us treat global and local symmetries at the same time and consider a symmetry transformation with infinitesimal transformation parameter $\rho(\sigma)$. The transformation can be expanded in derivatives of the transformation parameter:

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{\rm all} \quad \equiv \quad \underbrace{\rho^a \delta_a \phi^{\mathcal{I}}_{\rm all}}_{\delta^0_{(\rho)}\phi^{\mathcal{I}}_{\rm all}} + \underbrace{\partial_\mu \rho^a \delta^\mu_a \phi^{\mathcal{I}}_{\rm all}}_{\delta^1_{(\rho)}\phi^{\mathcal{I}}_{\rm all}} + \underbrace{\partial_{\mu_1}\partial_{\mu_2}\rho^a \delta^{\mu_1\mu_2}_a \phi^{\mathcal{I}}_{\rm all}}_{\delta^2_{(\rho)}\phi^{\mathcal{I}}_{\rm all}} + \ldots \tag{E.2}$$

In order to define properly the variational derivatives for this more general case, consider first the variation of the Lagrangian[1]

$$\begin{aligned}
\delta\mathcal{L} \quad = \quad & \delta\phi^{\mathcal{I}}_{\rm all}\left(\frac{\partial \mathcal{L}}{\partial \phi^{\mathcal{I}}_{\rm all}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{\mathcal{I}}_{\rm all})} + \partial_{\mu_1}\partial_{\mu_2}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi^{\mathcal{I}}_{\rm all})} - \ldots\right) + \\
& + \partial_\mu\left(\delta\phi^{\mathcal{I}}_{\rm all}\cdot\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{\mathcal{I}}_{\rm all})} + \sum_{k\geq 2}\sum_{i=0}^{k-1}(-)^i \partial_{\nu_1}\ldots\partial_{\nu_{k-1-i}}\delta\phi^{\mathcal{I}}_{\rm all}\cdot\partial_{\nu_{k-i}}\ldots\partial_{\nu_{k-1}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\nu_1}\ldots\partial_{\nu_{k-1}}\phi^{\mathcal{I}}_{\rm all})}\right)
\end{aligned} \tag{E.3}$$

The total derivative term reduces to a boundary term in the variation of the action, while the remaining term defines the variational derivative. As the boundary of a boundary vanishes, one can further partially integrate

[1] In (E.3) we have reformulated the variations containing derivatives of the fields $\phi^{\mathcal{I}}_{\rm all}$ using schematically the following iterated 'partial integration':

$$\begin{aligned}
\partial^k a\cdot b \quad &= \quad \partial\left(\partial^{k-1}a\cdot b\right) - \partial^{k-1}a\cdot\partial b = \\
&= \quad \partial\left(\partial^{k-1}a\cdot b\right) - \partial\left(\partial^{k-2}a\cdot\partial b\right) + \partial^{k-2}a\cdot\partial^2 b = \\
&= \quad \partial\left[\partial^{k-1}a\cdot b - \partial^{k-2}a\cdot\partial b + \ldots + (-)^{k-1}a\cdot\partial^{k-1}b\right] + (-)^k a\cdot\partial^k b = \\
&= \quad \partial\left[\sum_{i=0}^{k-1}(-)^i\partial^{k-1-i}a\cdot\partial^i b\right] + (-)^k a\cdot\partial^k b
\end{aligned}$$

This equation is applicable in (E.3), because the indices of the partial derivatives are all contracted and symmetrized and therefore behave like one-dimensional derivatives. In our case the above formula takes the explicit form

$$\begin{aligned}
&\delta(\partial_{\mu_1}\ldots\partial_{\mu_k}\phi^{\mathcal{I}}_{\rm all})\cdot\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\phi^{\mathcal{I}}_{\rm all})} = \\
&= \quad \partial_\mu\Big[\sum_{i=0}^{k-1}(-)^i\partial_{\nu_1}\ldots\partial_{\nu_{k-i-1}}\delta\phi^{\mathcal{I}}_{\rm all}\cdot\partial_{\nu_{k-i}}\ldots\partial_{\nu_{k-1}}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\nu_1}\ldots\partial_{\nu_{k-1}}\phi^{\mathcal{I}}_{\rm all})}\Big] + (-)^k\delta\phi^{\mathcal{I}}_{\rm all}\cdot\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\ldots\partial_{\mu_k}\phi^{\mathcal{I}}_{\rm all})} \qquad \diamond
\end{aligned}$$

the boundary term in order to obtain a convenient form that determines the boundary conditions:[2]

$$\delta S = \int_\Sigma d^n\sigma \quad \delta\phi^{\mathcal{I}}_{\text{all}} \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi^{\mathcal{I}}_{\text{all}}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{\mathcal{I}}_{\text{all}})} + \partial_{\mu_1}\partial_{\mu_2}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\partial_{\mu_2}\phi^{\mathcal{I}}_{\text{all}})} - \ldots\right)}_{\equiv \frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}} +$$
$$+\int_{\partial\Sigma} \delta\phi^{\mathcal{I}}_{\text{all}} \underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{\mathcal{I}}_{\text{all}})} - 2\partial_{\mu_2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\mu_2}\phi^{\mathcal{I}}_{\text{all}})} + 3\partial_{\mu_2}\partial_{\mu_3}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\mu_2}\partial_{\mu_3}\phi^{\mathcal{I}}_{\text{all}})} - \ldots\right)}_{(bc)^\mu_{\mathcal{I}}} \times$$
$$\times \frac{1}{(n-1)!}\epsilon_{\mu\nu_1\ldots\nu_{n-1}} \mathbf{d}\boldsymbol{\sigma}^{\nu_1}\wedge\cdots\wedge\mathbf{d}\boldsymbol{\sigma}^{\nu_{n-1}} \qquad \text{(E.4)}$$

A general variation $\delta\phi^{\mathcal{I}}_{\text{all}}$ determines via $\delta S = 0$ the equations of motion $\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)} = 0$ (and the boundary conditions $n_\mu(bc)^\mu_{\mathcal{I}} = 0$ with n_μ the normal one form), while for a symmetry transformation $\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$ the variation of the action has to vanish off-shell. Then the variation of the Lagrangian has to be a divergence independent from the equations of motion:

$$\delta_{(\rho)}\mathcal{L} \stackrel{!}{=} \partial_\mu\mathcal{K}^\mu_{(\rho)} \qquad \text{with} \quad n_\mu\mathcal{K}^\mu_{(\rho)}\Big|_{\partial\Sigma} = 0 \qquad \text{(E.5)}$$

The symmetry variation of the Lagrangian is thus on the one hand equal to a divergence and on the other hand (according to (E.3)) equal to the equations of motion plus another divergence. One can therefore define an object whose divergence is proportional to the equations of motion. So let us define the current

$$j^\mu_{(\rho)} \equiv \delta\phi^{\mathcal{I}}_{\text{all}}\cdot\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{\mathcal{I}}_{\text{all}})} + \sum_{k\geq 1}\sum_{i=0}^{k}(-)^i\partial_{\nu_1}\ldots\partial_{\nu_{k-i}}\delta\phi^{\mathcal{I}}_{\text{all}}\cdot\partial_{\nu_{k-i+1}}\ldots\partial_{\nu_k}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\nu_1}\ldots\partial_{\nu_k}\phi^{\mathcal{I}}_{\text{all}})} - \mathcal{K}^\mu_{(\rho)} \qquad \text{(E.6)}$$

Note that $\mathcal{K}^\mu_{(\rho)}$ is determined only up to off-shell divergence free terms. The same is of course true for the current. Using this definition, we can deduce from the above (E.3) that

$$\boxed{\partial_\mu j^\mu_{(\rho)} = -\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}} \qquad \text{(E.7)}$$

This equation shows one direction of Noether's theorem:

Theorem 2 (Noether) *To every transformation $\delta_{(\rho)}\phi^{\mathcal{I}}_{all}$ which leaves the action S invariant, there is an on-shell divergence-free current $j^\mu_{(\rho)}$ whose explicit form is given in (E.6). Its off-shell divergence is given in (E.7). The such defined Noether current is unique up to trivially conserved terms of the form $\partial_\nu S^{[\nu\mu]}$.*

In turn, for any given on-shell divergence-free current $\tilde{j}^\mu$ (see (E.8)), which is furthermore itself on-shell neither vanishing nor trivial, there is a corresponding nonzero symmetry transformation $\delta\phi^{\mathcal{I}}_{all}$ of the form given in (E.12) .

[2]Stokes' theorem reads

$$\int_{\Sigma^{(n)}} \mathbf{d}\omega = \int_{\partial\Sigma}\omega^{(n-1)}$$

For any Σ that can be covered by one single coordinate patch, we can write

$$\int_\Sigma \mathbf{d}\boldsymbol{\sigma}^{\mu_1}\wedge\ldots\wedge\mathbf{d}\boldsymbol{\sigma}^{\mu_n}\partial_{[\mu_1}\omega_{\mu_2\ldots\mu_n]} = \int_{\partial\Sigma}\mathbf{d}\boldsymbol{\sigma}^{\mu_1}\wedge\ldots\wedge\mathbf{d}\boldsymbol{\sigma}^{\mu_{n-1}}\omega_{\mu_1\ldots\mu_{n-1}}$$

where on the righthand side the coordinate differentials $\mathbf{d}\boldsymbol{\sigma}^\mu$ have to be understood as pullbacks $\mathbf{d}\boldsymbol{\tau}^i\partial_i\sigma^\mu(\tau)$ on the boundary.

For the integral of a divergence term like

$$\int_\Sigma d^n\sigma\,\partial_\mu v^\mu \equiv \int_\Sigma \mathbf{d}\boldsymbol{\sigma}^1\wedge\ldots\wedge\mathbf{d}\boldsymbol{\sigma}^n\,\partial_\mu v^\mu$$

we can use the fact that

$$\mathbf{d}\boldsymbol{\sigma}^1\wedge\ldots\wedge\mathbf{d}\boldsymbol{\sigma}^n\,\partial_\mu v^\mu = \mathbf{d}\omega$$

with

$$\omega \equiv \frac{1}{(n-1)!}v^\mu\epsilon_{\mu\mu_1\ldots\mu_{n-1}}\mathbf{d}\boldsymbol{\sigma}^{\mu_1}\wedge\ldots\wedge\mathbf{d}\boldsymbol{\sigma}^{\mu_{n-1}}$$

Applying Stokes then leads to

$$\int_\Sigma d^n\sigma\,\partial_\mu v^\mu = \int_{\partial\Sigma}\frac{1}{(n-1)!}v^\mu\epsilon_{\mu\mu_1\ldots\mu_{n-1}}\mathbf{d}\boldsymbol{\sigma}^{\mu_1}\wedge\ldots\wedge\mathbf{d}\boldsymbol{\sigma}^{\mu_{n-1}} \qquad \diamond$$

Remark: The equation (E.7) for the off-shell divergence can serve for reconstructing the symmetry transformations for a given current. In the Hamiltonian formalism , the current (or better the charge) generates the transformations via the Poisson bracket. In the Lagrangian formalism one can simply calculate all functional derivatives $\frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}}$ (i.e. the equations of motion) and try to express the divergence of the current as a linear combination of them. This method – let's call it **inverse Noether** – determines the transformations up to trivial gauge transformations (see e.g. [95, p.69]) and we are using it frequently in the main part, in particular to derive the BRST transformations.

Proof of the theorem: We have already shown the first part (every symmetry transformation induces a conserved current) by deriving (E.7). The uniqueness up to trivial terms follows from the algebraic Poincaré lemma. This does not yet show the inverse. For a given on-shell divergence-free current $\tilde{j}^\mu$ we do not necessarily have the form (E.7), but its off-shell divergence can also depend on derivatives of the equations of motion:

$$\partial_\mu \tilde{j}^\mu = -y^{\mathcal{I}}_{(0)} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} - y^{\mathcal{I}\mu_1}_{(1)} \partial_{\mu_1} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} - \ldots - y^{\mathcal{I}\mu_N \ldots \mu_1}_{(N)} \partial_{\mu_1} \ldots \partial_{\mu_N} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \tag{E.8}$$

However, one can always redefine the current such that we get the form (E.7). This is achieved by performing the iterated 'partial integration' of footnote 1 on page 181. We have schematically

$$y^{\mathcal{I}}_{(k)} \partial^k \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} = \partial \left[\sum_{i=0}^{k-1} (-)^i \partial^i y^{\mathcal{I}}_{(k)} \cdot \partial^{k-1-i} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \right] + (-)^k \partial^k y^{\mathcal{I}}_{(k)} \cdot \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \tag{E.9}$$

We can then rewrite schematically the divergence of the current as follows

$$\begin{aligned} \partial_\mu \tilde{j}^\mu &= -\sum_{k=0}^{N} y^{\mathcal{I}}_{(k)} \partial^k \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} = \\ &= -\partial \left[\sum_{k=1}^{N} \sum_{i=0}^{k-1} (-)^i \partial^i y^{\mathcal{I}}_{(k)} \cdot \partial^{k-1-i} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \right] - \sum_{k=0}^{N} (-)^k \partial^k y^{\mathcal{I}}_{(k)} \cdot \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \end{aligned} \tag{E.10}$$

To summarize, if we define

$$j^\mu \equiv \tilde{j}^\mu + \sum_{k=1}^{N} \sum_{i=0}^{k-1} (-)^i \partial_{\mu_1} \ldots \partial_{\mu_i} y^{\mathcal{I}\mu\, \mu_1 \ldots \mu_{k-1}}_{(k)} \cdot \partial_{\mu_{i+1}} \ldots \partial_{\mu_{k-1}} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}} \tag{E.11}$$

$$\delta \phi^{\mathcal{I}}_{\text{all}} \equiv \sum_{k=0}^{N} (-)^k \partial_{\mu_1} \ldots \partial_{\mu_k} y^{\mathcal{I}\, \mu_1 \ldots \mu_k}_{(k)} \tag{E.12}$$

we get $\partial j^\mu = -\delta \phi^{\mathcal{I}}_{\text{all}} \frac{\delta S}{\delta \phi^{\mathcal{I}}_{\text{all}}}$ and thus discover that the above defined $\delta \phi^{\mathcal{I}}_{\text{all}}$ is a symmetry transformation. We assumed that the current was on-shell neither vanishing nor trivial, while we redefined it with on-shell zero terms only. Therefore the new current will not be trivial and its divergence is off-shell non-zero. The symmetry transformations constructed above are therefore (at least off-shell) non-zero as well. This completes the proof. □

We should add that an on-shell vanishing current does not in general imply vanishing transformations. In fact all Noether currents of gauge transformations are vanishing on-shell. The gauge transformations will be discussed in the following, where one discovers that the equations of motion are not independent but are related via the Noether identities. Going back to our construction of the transformations from an arbitrarily conserved current one can make use of these dependencies instead of only redefining the current. This avoids ending up with an identically vanishing current after the redefinitions.

E.2 Noether identities and on-shell vanishing gauge currents

Equation (E.7) is valid for any symmetry transformation, global as well as local ones. For local ones, however, the relation has to hold for any local parameter ρ^a which is much more restrictive and allows to extract additional information. Let us assume that there is some highest component $j_a^{\mu_N\, \mu_{N-1} \ldots \mu_1}$, or in other words $\exists N$, s.t. $j_a^{\mu_k\, \mu_{k-1} \ldots \mu_1} = 0 \quad \forall k > N$. The expansion of $j^\mu_{(\rho)}$ in derivatives of the transformation parameter ρ takes the form

$$j^\mu_{(\rho)} \equiv \rho^a j^\mu_a + \partial_{\mu_1} \rho^a j_a^{\mu\, \mu_1} + \ldots + \partial_{\mu_1} \ldots \partial_{\mu_{N-1}} \rho^a j_a^{\mu\, \mu_1 \ldots \mu_{N-1}} \tag{E.13}$$

Now we plug this expansion and the one of $\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$ (E.2) into the equation for the current-divergence (E.7):

$$\begin{aligned} &\rho^a\partial_\mu j^\mu_a + \partial_{\mu_1}\rho^a\left(j^{\mu_1}_a + \partial_\mu j^{\mu\mu_1}_a\right) + \partial_{\mu_1}\partial_{\mu_2}\rho^a\left(j^{(\mu_1\mu_2)}_a + \partial_\mu j^{\mu\mu_1\mu_2}_a\right) + \ldots = \\ &= \ -\rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \partial_{\mu_1}\rho^a\delta^{\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \partial_{\mu_1}\partial_{\mu_2}\rho^a\delta^{\mu_1\mu_2}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \ldots \end{aligned} \tag{E.14}$$

Depending on whether we have a local or global symmetry, we get a number of recursive relations:

$$\boxed{\partial_{\mu_1}j^{\mu_1}_a \ = \ -\delta_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \qquad \text{if } \rho^a \neq 0} \tag{E.15}$$

$$\partial_{\mu_2}j^{\mu_2\mu_1}_a \ = \ -j^{\mu_1}_a - \delta^{\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \qquad \text{if } \partial_{\mu_1}\rho^a \neq 0 \tag{E.16}$$

$$\partial_{\mu_3}j^{\mu_3\mu_2\mu_1}_a \ = \ -j^{(\mu_2\mu_1)}_a - \delta^{\mu_2\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \qquad \text{if } \partial_{\mu_1}\partial_{\mu_2}\rho^a \neq 0 \tag{E.17}$$

$$\ddots$$

$$\partial_{\mu_N}j^{\mu_N\,\mu_{N-1}\cdots\mu_1}_a \ = \ -j^{(\mu_{N-1}\,\mu_{N-2}\cdots\mu_1)}_a - \delta^{\mu_{N-1}\cdots\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \qquad \text{if } \partial_{\mu_1}\ldots\partial_{\mu_{N-1}}\rho^a \neq 0 \tag{E.18}$$

$$0 \ = \ -j^{(\mu_N\,\mu_{N-1}\cdots\mu_1)}_a - \delta^{\mu_N\cdots\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \qquad \text{if } \partial_{\mu_1}\ldots\partial_{\mu_N}\rho^a \neq 0 \tag{E.19}$$

The first equation (E.15) is present already for a global symmetry and corresponds to the Noether's theorem for global symmetries. If the transformation parameters are instead local and arbitrary, the complete set of equations is forced. Taking then the divergence of the second equation, the double divergence of the third and so on, and adding them with appropriate signs, we can remove all currents from the equations and arrive at a version of the **Noether's identities**:

$$\delta_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \partial_{\mu_1}\left(\delta^{\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\right) + \ldots + (-)^{N+1}\partial_{\mu_1}\ldots\partial_{\mu_{N+1}}\left(\delta^{\mu_{N+1}\cdots\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\right) \ = \ 0 \tag{E.20}$$

From the recursive equations above, one can also obtain an interesting statement about the current of a gauge symmetry (compare [95, p.95]):

Proposition 6 *: The Noether current of a gauge symmetry vanishes on-shell up to trivially conserved terms (see (E.21)). In turn, if a given global symmetry transformation has an on-shell vanishing current (see (E.35)), then one can extend the transformation to a local one (see (E.40)).*

Proof Start with a given gauge symmetry $\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$ and its corresponding current $j^\mu_{(\rho)}$ with the expansion given in (E.13), which defines the number N of the highest derivative on ρ. We want to show that the current of a local symmetry is of the form

$$j^\mu_{(\rho)} \ = \ \sum_{k=0}^{N}\lambda^{\mu\mathcal{I}\mu_1\ldots\mu_k}_{(\rho)}\partial_{\mu_1}\ldots\partial_{\mu_k}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} + t^\mu_{(\rho)} \tag{E.21}$$

for some coefficients $\lambda^{\mu\mathcal{I}\mu_1\ldots\mu_k}_{(\rho)}$ and with a term t^μ whose divergence vanishes off-shell: $\partial_\mu t^\mu_{(\rho)} \equiv 0$. (Due to the algebraic Poincaré lemma, this means that there is some antisymmetric tensor $S^{[\mu\nu]}_{(\rho)}$ such that $t^\mu_{(\rho)} = \partial_\nu S^{[\mu\nu]}_{(\rho)}$.)

In order to reduce the length of the equations, define first[3]

$$E^{\mu_k\cdots\mu_1}_a \ \equiv \ \delta^{\mu_k\cdots\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}, \qquad E^{\mu_k\cdots\mu_1}_a = E^{(\mu_k\cdots\mu_1)}_a \tag{E.22}$$

$$A^{\mu_{k+1}\,\mu_k\cdots\mu_1}_a \ \equiv \ j^{\mu_{k+1}\,\mu_k\cdots\mu_1}_a - j^{(\mu_{k+1}\,\mu_k\cdots\mu_1)}_a, \quad A^{\mu_{k+1}\,\mu_k\cdots\mu_1}_a = A^{\mu_{k+1}\,(\mu_k\cdots\mu_1)}_a, \quad A^{(\mu_{k+1}\,\mu_k\cdots\mu_1)}_a = 0 \tag{E.23}$$

[3]Note that from

$$k\cdot j^{(\mu_k\,\mu_{k-1}\cdots\mu_1)}_a \ = \ j^{\mu_k\,\mu_{k-1}\cdots\mu_1}_a + (k-1)j^{(\mu_{k-1}\,\mu_{k-2}\cdots\mu_1)\mu_k}_a$$

one can deduce

$$j^{\mu_k\,\mu_{k-1}\cdots\mu_1}_a - j^{(\mu_k\,\mu_{k-1}\cdots\mu_1)}_a \ = \ \frac{2}{k}\sum_{i=1}^{k-1}j^{[\mu_k|\,\mu_{k-1}\cdots|\mu_i]\cdots\mu_1}_a \qquad \diamond$$

The first object is symmetric in all indices and the second is symmetric in the last k indices and vanishes when symmetrized in all indices. Using this notation, we can rewrite the recursive equations (E.16)-(E.19) in the following form

$$\begin{aligned} j_a^{\mu_1} &= -E_a^{\mu_1} - \partial_{\mu_2} j_a^{\mu_2\mu_1} &\quad (E.24)\\ j_a^{\mu_2\,\mu_1} &= A_a^{\mu_2\,\mu_1} - E_a^{\mu_2\mu_1} - \partial_{\mu_3} j_a^{\mu_3\mu_2\mu_1} &\quad (E.25)\\ &\ddots \\ j_a^{\mu_{N-1}\,\mu_{N-2}\cdots\mu_1} &= A_a^{\mu_{N-1}\,\mu_{N-2}\cdots\mu_1} - E_a^{\mu_{N-1}\cdots\mu_1} - \partial_{\mu_N} j_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} &\quad (E.26)\\ j_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} &= A_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} - E_a^{\mu_N\cdots\mu_1} &\quad (E.27)\end{aligned}$$

This set of equations can now formally be solved for all components of the current, starting from the N-th equation. We end up with

$$\begin{aligned} j_a^{\mu_1} &= -\partial_{\mu_2} A_a^{\mu_2\,\mu_1} + \partial_{\mu_2}\partial_{\mu_3} A_a^{\mu_3\,\mu_2\mu_1} - \partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4} A_a^{\mu_4\,\mu_3\mu_2\mu_1} + \ldots + \\ &\quad -E_a^{\mu_1} + \partial_{\mu_2} E_a^{\mu_2\mu_1} - \partial_{\mu_2}\partial_{\mu_3} E_a^{\mu_3\,\mu_2\mu_1} + \partial_{\mu_2}\partial_{\mu_3}\partial_{\mu_4} E_a^{\mu_4\,\mu_3\mu_2\mu_1} - \ldots &\quad (E.28)\\ j_a^{\mu_2\,\mu_1} &= A_a^{\mu_2\,\mu_1} - \partial_{\mu_3} A_a^{\mu_3\,\mu_2\mu_1} + \partial_{\mu_3}\partial_{\mu_4} A_a^{\mu_4\,\mu_3\mu_2\mu_1} - \ldots + \\ &\quad -E_a^{\mu_2\mu_1} + \partial_{\mu_3} E_a^{\mu_3\,\mu_2\mu_1} - \partial_{\mu_3}\partial_{\mu_4} E_a^{\mu_4\,\mu_3\mu_2\mu_1} + \ldots &\quad (E.29)\\ &\ddots \\ j_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} &= A_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} - \partial_{\mu_{k+1}} A_a^{\mu_{k+1}\,\mu_k\cdots\mu_1} + \ldots + (-)^{N-k}\partial_{\mu_{k+1}}\ldots\partial_{\mu_N} A_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} + \\ &\quad -E_a^{\mu_k\cdots\mu_1} + \partial_{\mu_{k+1}} E_a^{\mu_{k+1}\cdots\mu_1} - \ldots - (-)^{N-k}\partial_{\mu_{k+1}}\ldots\partial_{\mu_N} E_a^{\mu_N\cdots\mu_1} &\quad (E.30)\\ &\ddots \\ j_a^{\mu_{N-1}\,\mu_{N-2}\cdots\mu_1} &= A_a^{\mu_{N-1}\,\mu_{N-2}\cdots\mu_1} - \partial_{\mu_N} A_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} - E_a^{\mu_{N-1}\cdots\mu_1} + \partial_{\mu_N} E_a^{\mu_N\cdots\mu_1} &\quad (E.31)\\ j_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} &= A_a^{\mu_N\,\mu_{N-1}\cdots\mu_1} - E_a^{\mu_N\cdots\mu_1} &\quad (E.32)\end{aligned}$$

In order to obtain the complete current $j^{\mu_1}_{(\rho)}$ we have to contract the k-th term $j_a^{\mu_1\,\mu_k\cdots\mu_2}$ (with interchanged $\mu_1 \leftrightarrow \mu_k$!) with $\partial_{\mu_2}\ldots\partial_{\mu_k}\rho^a$ and then add all the terms. Interchanging μ_k and μ_1 for the k-th equation affects (because of the symmetries) only the term $A_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} \mapsto A_a^{\mu_1\,\mu_k\cdots\mu_2}$. We will sort the A_a-terms with respect to the number of indices on A_a and the E_a-terms with respect to the number of derivatives on ρ^a:

$$\begin{aligned} j^a_{(\rho)} &= \sum_{k=2}^{N}\underbrace{\left(\sum_{i=0}^{k-2} -(-)^{k-i}\partial_{\mu_2}\ldots\partial_{\mu_{2+i-1}}\rho^a\partial_{\mu_{2+i}}\ldots\partial_{\mu_k} A_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} + \partial_{\mu_2}\ldots\partial_{\mu_k}\rho^a A_a^{\mu_1\,\mu_k\cdots\mu_2}\right)}_{\equiv t^{\mu_1}_{(\rho,k)}} + \\ &\quad -\sum_{k=1}^{N}\partial_{\mu_2}\ldots\partial_{\mu_k}\rho^a\sum_{i=0}^{N-k}(-)^i\partial_{\mu_{k+1}}\ldots\partial_{\mu_{k+i}} E_a^{\mu_{k+i}\cdots\mu_{k+1}\mu_k\cdots\mu_1} \qquad (E.33)\end{aligned}$$

The second line vanishes on-shell, but it remains to show that the first line $t^{\mu_1}_{(\rho)} \equiv \sum_{k=2}^{N} t^{\mu_1}_{(\rho)}$ has trivially vanishing divergence. The second term in the first line is written separately (not in the sum over i), because in contrast to the other terms it has the μ_1 index at the first position (which is not symmetrized like the other positions). This difference in treatment disappears in the divergence with contracted μ_1. We use this fact to show the trivial vanishing (without the use of equations of motion) of the divergence of for every single $t^{\mu_1}_{(\rho,k)}$:

$$\begin{aligned} &\partial_{\mu_1} t^{\mu_1}_{(\rho,k)} = \\ &= \sum_{i=0}^{k-1}(-)^{k-i+1}\partial_{\mu_1}\ldots\partial_{\mu_{i+1}}\rho^a\partial_{\mu_{i+2}}\ldots\partial_{\mu_k} A_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} - \sum_{i=0}^{k-1}(-)^{k-i}\partial_{\mu_2}\ldots\partial_{\mu_{2+i-1}}\rho^a\partial_{\mu_{2+i}}\ldots\partial_{\mu_k}\partial_{\mu_1} A_a^{\mu_1\,\mu_k\cdots\mu_2} \\ &= \sum_{i=1}^{k-1} -(-)^{k-i+1}\partial_{\mu_1}\ldots\partial_{\mu_i}\rho^a\partial_{\mu_{i+1}}\ldots\partial_{\mu_k} A_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} - \sum_{i=1}^{k-1}(-)^{k-i}\partial_{\mu_1}\ldots\partial_{\mu_i}\rho^a\partial_{\mu_{i+1}}\ldots\partial_{\mu_k} A_a^{\mu_k\,\mu_{k-1}\cdots\mu_1} + \\ &\quad -(-)^{k-i}\partial_{\mu_1}\ldots\partial_{\mu_k}\rho^a\underbrace{A_a^{(\mu_k\,\mu_{k-1}\cdots\mu_1)}}_{=0} - (-)^k\rho^a\partial_{\mu_1}\ldots\partial_{\mu_k}\underbrace{A_a^{(\mu_k\,\mu_{k-1}\cdots\mu_1)}}_{=0} \quad = 0 \qquad (E.34)\end{aligned}$$

This completes the proof of (E.21) or of one direction of the proposition.

Now consider that we have a global transformation (constant parameter ρ_c) $\delta^0_{(\rho_c)}\phi^{\mathcal{I}}_{\text{all}} = \rho^a_c\delta_a\phi^{\mathcal{I}}_{\text{all}}$ with Noether current $j^\mu_{(\rho_c)} = \rho^a_c j^\mu_a$, which itself vanishes on-shell

$$j^\mu_a = \sum_{k=0}^{N} \lambda_a^{\mu\mathcal{I}\mu_1\ldots\mu_k} \partial_{\mu_1}\ldots\partial_{\mu_k} \frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.35}$$

$$\partial_\mu j^\mu_a = -\delta_a \phi^{\mathcal{I}}_{\text{all}} \frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.36}$$

If we plug (E.35) into (E.36) we already discover relations between the equations of motion, which look like the Noether identities for local symmetries. Indeed, if j^μ_a vanishes on-shell, also $\rho^a j^\mu_a$ vanishes on-shell, even for local ρ^a. For consistent equations of motion (some which have solutions at all) certainly also its derivative vanishes on-shell. The combination $j^0_{(\rho)} \equiv \rho^a j^\mu_a$ therefore corresponds to a symmetry transformation with a local parameter, i.e. a gauge symmetry, although this current is in general not yet in the standard form of a Noether current (where its divergence does not contain derivatives of $\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}$, but only the plain equations of motion):

$$\partial_\mu(\rho^a j^\mu_a) = \partial_\mu\rho^a \cdot j^\mu_a + \rho^a\partial_\mu j^\mu_a = \tag{E.37}$$

$$= \sum_{k=1}^{N} \partial_\mu\rho^a \lambda_a^{\mu\mathcal{I}\mu_1\ldots\mu_k} \partial_{\mu_1}\ldots\partial_{\mu_k} \frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \left(\rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}} - \partial_\mu\rho^a\lambda_a^{\mu\mathcal{I}}\right)\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.38}$$

In order to get a proper Noether current (where the righthand side does not contain any derivatives of the equations of motion) we can use our insights from the proof of Noether's theorem, i.e. equations (E.8)-(E.12). We learn that if we define the whole current to be

$$j^\mu_{(\rho)} \equiv \rho^a j^\mu_a - \sum_{k=1}^{N}\sum_{i=0}^{k-1}(-)^i \partial_{\mu_1}\ldots\partial_{\mu_i}\partial_\nu\rho^a\lambda_a^{\nu\mathcal{I}\mu\mu_1\ldots\mu_{k-1}} \cdot \partial_{\mu_{i+1}}\ldots\partial_{\mu_{k-1}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.39}$$

we get a proper Noether current with corresponding symmetry transformations

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}} - \partial_\mu\rho^a\lambda_a^{\mu\mathcal{I}} + \sum_{k=1}^{N}(-)^{k+1}\partial_{\mu_1}\ldots\partial_{\mu_k}\left(\partial_\nu\rho^a\lambda_a^{\nu\mathcal{I}\mu_1\ldots\mu_k}\right) \tag{E.40}$$

The transformation (E.40) is a local symmetry transformation which completes the proof of the proposition. □

Theorem 3 *Every on-shell vanishing symmetry transformation is a* ***trivial gauge transformation*** *as defined below:*

$$\delta\phi^{\mathcal{I}}_{all} \overset{\text{on-shell}}{=} 0, \quad \delta S = 0 \quad\Rightarrow\quad \delta\phi^{\mathcal{I}}_{all} = \int d^d\sigma \quad \mathcal{A}^{\mathcal{I}\mathcal{J}}(\sigma,\sigma')\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}(\sigma')} \quad \textit{with}\ \mathcal{A}^{\mathcal{I}\mathcal{J}}(\sigma,\sigma') = -\mathcal{A}^{\mathcal{J}\mathcal{I}}(\sigma',\sigma) \tag{E.41}$$

See in [95] (theorem 17.3 on page 414 or theorem 3.1 on page 17) for a proof of this theorem. See [95, p.69] for a discussion of trivial gauge transformations.

E.3 Shortcut to calculate the Noether current

There is a nice shortcut to calculate the current: multiply both sides of (E.7) with some local parameter $\eta(\sigma)$, integrate over the world-volume Σ and perform a partial integration to arrive at

$$\boxed{\int_\Sigma d^n\sigma\ \partial_\mu\eta \cdot j^\mu_{(\rho)} + \int_{\partial\Sigma}(\ldots) = \delta_{(\eta,\rho)}S} \tag{E.42}$$

where $\delta_{(\eta,\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \eta\cdot\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$. One thus obtains the current by multiplying the variation with an independent local parameter η and reading off the coefficient of $\partial_\mu\eta$. This trick is better known for global symmetries[4] calculating just j^μ_a.

[4]If one is just interested in j^μ_a one can consider a variation not with the full variation $\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$, but only with its derivative free part $\delta^0_{(\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}}$ (see (E.2)) and allow local ρ^a even in the case of a global symmetry. Multiplying both sides of (E.15) with ρ^a we get $\rho^a\partial_\mu j^\mu_a = -\delta^0_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}$. Integrating over Σ and partially integrating finally yields

$$\delta^0_{(\rho)}S = \int_\Sigma d^n\sigma \quad \partial_\mu\rho^a\, j^\mu_a + \int_{\partial\Sigma}(\ldots)$$

The (conserved) Noether current thus can be read off from the derivative-free variation of the action as the coefficient of $\partial_\mu\rho^a$. We could then proceed with a variation $\delta^1_{(\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \partial_\mu\rho^a\delta^\mu_a\phi^{\mathcal{I}}_{\text{all}}$ to derive $j^{\mu\mu_1}_a$ from the coefficient of $\partial_\mu\partial_{\mu_1}\rho^a$, and so on. All this is done at the same time in (E.42). ◇

E.4 Noether current for the commutator of two symmetries

Determining the Noether charge for the commutator of two symmetries is a very simple task in the Hamiltonian formalism. As the charges generate the symmetries via the Poisson bracket, we have $\delta_1\phi^{\mathcal{I}}_{\text{all}} = \left\{Q_1, \phi^{\mathcal{I}}_{\text{all}}\right\}$ and $\delta_2\phi^{\mathcal{I}}_{\text{all}} = \left\{Q_2, \phi^{\mathcal{I}}_{\text{all}}\right\}$. The Jacobi identity for the Poisson bracket then implies for the commutator of the symmetry transformations that $[\delta_1,\delta_2]\,\phi^{\mathcal{I}}_{\text{all}} = \left\{\{Q_1,Q_2\},\phi^{\mathcal{I}}_{\text{all}}\right\}$. In other words $\{Q_1,Q_2\} = \delta_1 Q_2 = -\delta_2 Q_1$ is the charge corresponding to the symmetry transformation $[\delta_1,\delta_2]$. After dropping the integration over space, this relation also holds for the currents, i.e. $\delta_1 j_2^\mu = -\delta_2 j_1^\mu$ is the divergence-free (on-shell) current corresponding to the transformation $[\delta_1,\delta_2]$.

Of course one expects to obtain the same result within the Lagrangian formalism. And on-shell this indeed has to be the case. Off-shell, however, there might be a difference to the Hamiltonian formalism. In order to capture all the subtleties, we will therefore derive in the following the off-shell Noether current corresponding to $[\delta_1,\delta_2]$ within in the Lagrangian formalism. As it turns out, the derivation is a bit more involved than one might expect.

The current corresponding to the symmetry transformation $[\delta_1,\delta_2]$ can in principle easily be computed if we know $\mathcal{K}_1^\mu$ and $\mathcal{K}_2^\mu$ with $\delta\mathcal{L} = \partial_\mu\mathcal{K}^\mu$ for the symmetries δ_1 and δ_2. By acting with the commutator symmetry on the Lagrangian, we get a simple expression for the total derivative term for this symmetry:

$$[\delta_1,\delta_2]\mathcal{L} \;=\; \delta_1\partial_\mu\mathcal{K}_2^\mu - \delta_2\partial_\mu\mathcal{K}_1^\mu = \partial_\mu\left(2\delta_{[1}\mathcal{K}^\mu_{2]}\right) \tag{E.43}$$

Knowing the total derivative term (up to trivially conserved terms), the corresponding current is simply (according to (E.6))

$$\begin{aligned} j^\mu_{[\delta_1,\delta_2]} \;&=\; [\delta_1,\delta_2]\phi^{\mathcal{I}}_{\text{all}}\frac{\partial}{\partial(\partial_\mu\phi^{\mathcal{I}}_{\text{all}})}\mathcal{L} + \\ &+\sum_{k\geq 1}\sum_{i=0}^{k}(-)^i\partial_{\nu_1}\ldots\partial_{\nu_{k-i}}[\delta_1,\delta_2]\phi^{\mathcal{I}}_{\text{all}}\cdot\partial_{\nu_{k-i+1}}\ldots\partial_{\nu_k}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\nu_1}\ldots\partial_{\nu_k}\phi^{\mathcal{I}}_{\text{all}})} - 2\delta_{[1}\mathcal{K}^\mu_{2]} \end{aligned} \tag{E.44}$$

The nontrivial part is now to show that this current is (at least on-shell) equal to $\delta_1 j_2^\mu$ or $-\delta_2 j_1^\mu$, which was suggested by the Hamiltonian formalism. We start with two currents corresponding to two symmetry transformations

$$\partial_\mu j_1^\mu \;=\; -\delta_1\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}},\quad \partial_\mu j_2^\mu = -\delta_2\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.45}$$

How not to do it. The way to derive the desired result presented in the main part of the original version of this thesis was unfortunately wrong (although luckily without bad consequences). Let me shortly sketch it and point out the trap. Acting in (E.45) with δ_1 on $\partial_\mu j_2^\mu$ and subtracting δ_2 of $\partial_\mu j_1^\mu$, one obtains

$$\partial_\mu\left(\delta_1 j_2^\mu - \delta_2 j_1^\mu\right) \;=\; -[\delta_1,\delta_2]\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} + 2\delta_{[1}\phi^{\mathcal{I}}_{\text{all}}\delta_{2]}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.46}$$

So far everything is correct, and it is tempting to argue that the last term is vanishing. The reasoning would be $\delta_{[1}\phi^{\mathcal{I}}_{\text{all}}\delta_{2]}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \stackrel{?}{=} \delta_{[1}\phi^{\mathcal{I}}_{\text{all}}\delta_{2]}\phi^{\mathcal{J}}_{\text{all}}\frac{\delta^2 S}{\delta\phi^{\mathcal{J}}_{\text{all}}\delta\phi^{\mathcal{I}}_{\text{all}}} = 0$. The last step is true for symmetry reasons, but the step before is simply wrong, because it misses an integration of the form $\delta_{[1}\phi^{\mathcal{I}}_{\text{all}}\delta_{2]}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} = \int d\tilde\sigma \quad \delta_{[1}\phi^{\mathcal{I}}_{\text{all}}\delta_{2]}\phi^{\mathcal{J}}_{\text{all}}(\tilde\sigma)\frac{\delta^2 S}{\delta\phi^{\mathcal{J}}_{\text{all}}(\tilde\sigma)\delta\phi^{\mathcal{I}}_{\text{all}}}$. This integration, however, destroys the symmetry argument. Moreover, not only the derivation is wrong, but also the result (by a factor of two). Following the wrong argument of above, $\delta_1 j_2^\mu - \delta_2 j_1^\mu$ would be the current of $[\delta_1,\delta_2]$ instead of $\delta_1 j_2 = -\delta_2 j_1 = \frac{1}{2}\left(\delta_1 j_2^\mu - \delta_2 j_1^\mu\right)$ (the result from the Hamiltonian reasoning).

Correct derivation in the Lagrangian formalism. It will be very useful in the following to use a shorthand notation in which repeated indices which are at the same vertical position are simply symmetrized, like for example in $(\partial_\nu)^2 A_\nu \equiv \partial_\nu\partial_\nu A_\nu \hat{\equiv} \partial_{(\nu_1}\partial_{\nu_2}A_{\nu_3)}$. Only if they are at opposite vertical position they are summed over. In this context one should also be aware that lower index positions in the denominator correspond to upper index positions in the nominator. This notation is similar to the one introduced on page 147 for antisymmetrized indices.

Let us now once more act in (E.45) with δ_1 on $\partial_\mu j_2^\mu$ (without subtracting $\delta_2\partial_\mu j_1^\mu$) and reformulate the righthand side such that we obtain the desired result plus some rest:

$$\partial_\mu\left(\delta_1 j_2^\mu\right) \;=\; -\delta_1\delta_2\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \delta_2\phi^{\mathcal{I}}_{\text{all}}\delta_1\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} = \tag{E.47}$$

$$\begin{aligned} &= -[\delta_1,\delta_2]\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} + \\ &\quad -\delta_2\delta_1\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \delta_2\phi^{\mathcal{I}}_{\text{all}}\delta_1\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \end{aligned} \tag{E.48}$$

This time we should be more careful about the variation δ_1 of the variational derivative $\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}$ and we assume that we study a point σ^μ which is not at the boundary of the manifold Σ (which means that the variational derivative of boundary terms with respect to $\phi^{\mathcal{I}}_{\text{all}}(\sigma)$ vanishes):

$$\delta_1 \frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)} = \frac{\delta(\overbrace{\delta_1 S}^{0})}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)} + [\delta_1, \frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}]S = \tag{E.49}$$

$$= \int d^n\tilde{\sigma} \left(\delta_1\phi^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})\frac{\delta^2 S}{\delta\phi^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)} - \frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}(\delta_1\phi^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})\frac{\delta S}{\delta\phi^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})})\right) = \tag{E.50}$$

$$= -\int d^n\tilde{\sigma} \underbrace{\left(\frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}\delta_1\phi^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})\right)}_{\delta(\sigma-\tilde{\sigma})\frac{\partial(\delta_1\phi^{\mathcal{J}}_{\text{all}})}{\partial\phi^{\mathcal{I}}_{\text{all}}}(\sigma)+\tilde{\partial}_\mu\delta(\sigma-\tilde{\sigma})\frac{\partial(\delta_1\phi^{\mathcal{J}}_{\text{all}})}{\partial(\partial_\mu\phi^{\mathcal{I}}_{\text{all}})}(\tilde{\sigma})+\tilde{\partial}_\mu\tilde{\partial}_\mu\delta(\sigma-\tilde{\sigma})\frac{\partial(\delta_1\phi^{\mathcal{J}}_{\text{all}})}{\partial(\partial_\mu\partial_\mu\phi^{\mathcal{I}}_{\text{all}})}(\tilde{\sigma})+\ldots} \frac{\delta S}{\delta\phi^{\mathcal{J}}_{\text{all}}(\tilde{\sigma})} = \tag{E.51}$$

$$= -\frac{\partial(\delta_1\phi^{\mathcal{J}}_{\text{all}})}{\partial\phi^{\mathcal{I}}_{\text{all}}}\frac{\delta S}{\delta\phi^{\mathcal{J}}_{\text{all}}} - \sum_{k\geq 1}(-)^k(\partial_\mu)^k\left(\frac{\partial(\delta_1\phi^{\mathcal{J}}_{\text{all}})}{\partial((\partial_\mu)^k\phi^{\mathcal{I}}_{\text{all}})}\frac{\delta S}{\delta\phi^{\mathcal{J}}_{\text{all}}}\right) \tag{E.52}$$

The righthand side vanishes on-shell which shows that the symmetry transformation of an equation of motion is always another valid equation of motion. Likewise we can expand $\delta_2\delta_1\phi^{\mathcal{I}}_{\text{all}}$ as

$$\delta_2\delta_1\phi^{\mathcal{I}}_{\text{all}} = \delta_2\phi^{\mathcal{K}}_{\text{all}}\frac{\partial(\delta_1\phi^{\mathcal{I}}_{\text{all}})}{\partial\phi^{\mathcal{K}}_{\text{all}}} + \sum_{k\geq 1}(\partial_\mu)^k\delta_2\phi^{\mathcal{K}}_{\text{all}}\cdot\frac{\partial(\delta_1\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_\mu)^k\phi^{\mathcal{K}}_{\text{all}}} \tag{E.53}$$

Plugging the above two expansions into the variation (E.48) of the current-divergence yields

$$\begin{aligned}\partial_\mu(\delta_1 j^\mu_2) = &-[\delta_1,\delta_2]\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} + \\ &-\sum_{k\geq 1}(\partial_\mu)^k\delta_2\phi^{\mathcal{K}}_{\text{all}}\cdot\frac{\partial(\delta_1\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_\mu)^k\phi^{\mathcal{K}}_{\text{all}}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} + \sum_{k\geq 1}(-)^k\delta_2\phi^{\mathcal{I}}_{\text{all}}(\partial_\mu)^k\left(\frac{\partial(\delta_1\phi^{\mathcal{J}}_{\text{all}})}{\partial((\partial_\mu)^k\phi^{\mathcal{I}}_{\text{all}})}\frac{\delta S}{\delta\phi^{\mathcal{J}}_{\text{all}}}\right)\end{aligned} \tag{E.54}$$

Now we can use the schematic formula $-\partial^k a\cdot b + (-)^k a\partial^k b = -\partial\left(\sum_{l=0}^{k-1}(-)^l\partial^{k-1-l}a\cdot\partial^l b\right)$ from footnote 1. The total derivative can then be added to $\delta_1 j^\mu_2$ on the lefthand side. Therefore the current defined by

$$\boxed{j^\mu_{[\delta_1,\delta_2]} \equiv \delta_1 j^\mu_2 + \sum_{k\geq 1}\sum_{l=0}^{k-1}(-)^l(\partial_\mu)^l\left(\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\frac{\partial(\delta_1\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_\mu)^k\phi^{\mathcal{K}}_{\text{all}}}\right)\cdot(\partial_\mu)^{k-1-l}\delta_2\phi^{\mathcal{K}}_{\text{all}}} \tag{E.55}$$

obeys

$$\partial_\mu j^\mu_{[\delta_1,\delta_2]} = -[\delta_1,\delta_2]\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} \tag{E.56}$$

and is thus the off-shell Noether current corresponding to the commutator symmetry $[\delta_1,\delta_2]$. Remember that this Noether current is defined only up to trivially conserved terms. The fact that the current $j^\mu_{[\delta_1,\delta_2]}$ is antisymmetric in 1 and 2 also implies that

$$\delta_1 j^\mu_2 = -\delta_2 j^\mu_1 - 2\sum_{k\geq 1}\sum_{l=0}^{k-1}(-)^l(\partial_\mu)^{k-1-l}\delta_{(1|}\phi^{\mathcal{K}}_{\text{all}}\cdot(\partial_\mu)^l\left(\frac{\partial(\delta_{|2)}\phi^{\mathcal{I}}_{\text{all}})}{\partial(\partial_\mu)^k\phi^{\mathcal{K}}_{\text{all}}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\right) \tag{E.57}$$

Only on-shell these results coincide with the ones from the Hamiltonian formalism.

Note that one could also start with equation (E.6) for the current j^μ_2 and act on it with δ_1 (instead of acting on the divergence of this equation). In order to turn the result into something resembling (E.44), one needs to make use of several commutators like $[\frac{\partial}{\partial((\partial_\nu)^k\phi^{\mathcal{I}}_{\text{all}})},\partial_\mu] = \delta^\nu_\mu\frac{\partial}{\partial((\partial_\nu)^{k-1}\phi^{\mathcal{I}}_{\text{all}})}\quad\forall k\geq 1\quad(0\text{ for }k=0)$, which imply by induction $[\frac{\partial}{\partial(\partial^k_\nu\phi^{\mathcal{I}}_{\text{all}})},\partial^l_\rho] = \sum_{c=1}^k\binom{l}{c}\delta^{\nu\ldots\nu}_{\rho\ldots\rho}\partial^{l-c}_\rho\frac{\partial}{\partial(\partial^{k-c}_\nu\phi^{\mathcal{I}}_{\text{all}})}\quad\forall k\geq 1$. The derivation of the latter commutators involves the formula $\sum_{i=0}^r\binom{i}{c} = \binom{r+1}{c+1}$. Following this path becomes extremely clumsy and I managed to follow it to the end only if the Lagrangian depends maximally on first order derivatives.

Appendix F

Torsion, Curvature H-field and their Bianchi identities

In the following we are frequently making use of the (super)vielbein and its inverse, i.e. a local frame in (co)tangent space different from the coordinate basis. We denote it via

$$
\begin{aligned}
E^A &\equiv \mathbf{d}x^M E_M{}^A && \text{(F.1)}\\
E_A{}^K E_K{}^B &\equiv \delta_A{}^B && \text{(F.2)}\\
E_A &\equiv E_A{}^K \boldsymbol{\partial}_K && \text{(F.3)}
\end{aligned}
$$

The one forms E^A are chosen in such a way that they obey nice properties, i.e. in a Riemannian space it is natural to choose an orthonormal frame, while if no metric is present, it can be replaced by other requirements like e.g. invariance under supersymmetry for flat superspace. The structure group is then the set of transformations of the vielbein which do not change these properties.

To be a useful concept, the frame should be invariant under the covariant derivative.

$$
0 \stackrel{!}{=} \nabla_M E_N{}^A \equiv \partial_M E_N{}^A + \Omega_{MB}{}^A E_N{}^B - \Gamma_{MN}{}^K E_K{}^A \tag{F.4}
$$

This relates the spacetime connection to the structure group connection.

F.1 Definition of torsion and curvature and H-field

F.1.1 Torsion

There are at least three ways to define the torsion. Let us start with the component based one and derive from this the more geometric (coordinate independent) definintion. So at first we define the (super) torsion components simply as the antisymmetric part of the connection coefficients

$$
\boxed{T_{MN}{}^K \equiv \Gamma_{[MN]}{}^K} \tag{F.5}
$$

The structure group connection $\Omega_{MA}{}^B$ is given by demanding that the covariant derivative of the vielbein vanishes

$$
0 \stackrel{!}{=} \nabla_M E_N{}^A = \partial_M E_N{}^A - \Gamma_{MN}{}^K E_K{}^A + \Omega_{MB}{}^A E_N{}^B \tag{F.6}
$$

Antisymmetrizing in (M, N) and comparing with (F.5) yields[1]

$$
\boxed{T^A = \mathbf{d}E^A - E^B \wedge \Omega_B{}^A} \tag{F.7}
$$

This can be used as an alternative definition to (F.5). Consider now the commutator of two covariant derivatives on a scalar (super) field (with $\nabla_K \varphi = \partial_K \varphi$)

$$
\begin{aligned}
[\nabla_M, \nabla_N]\,\varphi &= 2\nabla_{[M}\partial_{N]}\varphi = && \text{(F.8)}\\
&= -2\Gamma_{[MN]}{}^K \partial_K \varphi && \text{(F.9)}
\end{aligned}
$$

[1] Note that in the present text form components are defined as e.g. $T^A = T_{MN}{}^A \mathbf{d}x^M \wedge \mathbf{d}x^N$ with no (!) factor $\frac{1}{2}$ in front which corresponds to a definition of the wedge product as $\mathbf{d}x^M \mathbf{d}x^N \equiv \mathbf{d}x^M \wedge \mathbf{d}x^N \equiv \mathbf{d}x^{[M} \otimes \mathbf{d}x^{N]} \equiv \frac{1}{2}\left(\mathbf{d}x^M \otimes \mathbf{d}x^N - \mathbf{d}x^M \otimes \mathbf{d}x^N\right)$. You will thus usually find in literature a factor of 2 on the righthand side of (F.5) and a factor $\frac{1}{2}$ in (F.10). To go from one convention to the other, simply replace $T_{MN}{}^K$ by $2T_{MN}{}^K$ in all equations in component form. (For a p-form the factor is of course $p!$). Coordinate independent equations like (F.7) remain untouched because of the compensating redefinition of the wedge product and the resulting redefinition of the exterior product. $\diamond$

or simply

$$\boxed{\nabla_{[M}\nabla_{N]}\varphi = -T_{MN}{}^{K}\nabla_{K}\varphi} \tag{F.10}$$

which is yet an alternative and equivalent definition of the torsion.

F.1.2 Curvature

For the curvature, let us start with the definition via the commutator of covariant derivatives acting on vector fields

$$\boxed{\nabla_{[M}\nabla_{N]}v^{A} = -T_{MN}{}^{K}\nabla_{K}v^{A} + R_{MNB}{}^{A}v^{B}} \tag{F.11}$$

This is not only a definition, but also a proposition that the commutator takes this form. Let us check this and by doing this get a definition of the curvature in component form

$$\begin{aligned}
&\nabla_{[M}\nabla_{N]}v^{A} = \\
&= \partial_{[M}(\partial_{N]}v^{A} + \Omega_{N]B}{}^{A}v^{B}) + \Omega_{[M|C}{}^{A}(\partial_{|N]}v^{C} + \Omega_{|N]B}{}^{C}v^{B}) - \Gamma_{[MN]}{}^{K}(\partial_{K}v^{A} + \Omega_{KB}{}^{A}v^{B}) = &\text{(F.12)}\\
&= \partial_{[M}\Omega_{N]B}{}^{A}v^{B} + \Omega_{[N|B}{}^{A}\partial_{|M]}v^{B} + \Omega_{[M|C}{}^{A}(\partial_{|N]}v^{C} + \Omega_{|N]B}{}^{C}v^{B}) - T_{[MN]}{}^{K}\nabla_{K}v^{A} = &\text{(F.13)}\\
&= -T_{[MN]}{}^{K}\nabla_{K}v^{A} + \left(\partial_{[M}\Omega_{N]B}{}^{A} + \Omega_{[M|C}{}^{A}\Omega_{|N]B}{}^{C}\right)v^{B} &\text{(F.14)}
\end{aligned}$$

We can thus read off

$$R_{MNB}{}^{A} = \partial_{[M}\Omega_{N]B}{}^{A} - \Omega_{[M|B}{}^{C}\Omega_{|N]C}{}^{A} \tag{F.15}$$

which in form language reads

$$\boxed{R_{A}{}^{B} = \mathbf{d}\Omega_{A}{}^{B} - \Omega_{A}{}^{C}\wedge\Omega_{C}{}^{B}} \tag{F.16}$$

We finally can rewrite this in terms of Γ by using (F.6) in the simplified form

$$\Omega_{MB}{}^{A} = \Gamma_{MB}{}^{A} - E_{B}{}^{R}\partial_{M}E_{R}{}^{A} \tag{F.17}$$

$\Rightarrow$

$$\begin{aligned}
R_{MNB}{}^{A} &= \partial_{[M|}\left(\Gamma_{|N]B}{}^{A} - E_{B}{}^{R}\partial_{|N]}E_{R}{}^{A}\right) - \left(\Gamma_{[M|B}{}^{C} - E_{B}{}^{R}\partial_{[M|}E_{R}{}^{C}\right)\left(\Gamma_{|N]C}{}^{A} - E_{C}{}^{S}\partial_{|N]}E_{S}{}^{A}\right) &\text{(F.18)}\\
R_{MNK}{}^{L} &= \partial_{[M|}\Gamma_{|N]K}{}^{L} + E_{K}{}^{B}\partial_{[M|}E_{B}{}^{R}\Gamma_{|N]R}{}^{L} + E_{A}{}^{L}\partial_{[M|}E_{S}{}^{A}\Gamma_{|N]K}{}^{S} - E_{K}{}^{B}E_{A}{}^{L}\partial_{[M|}E_{B}{}^{R}\partial_{|N]}E_{R}{}^{A} + \\
&\quad - \left(\Gamma_{[M|K}{}^{C} - \partial_{[M|}E_{K}{}^{C}\right)\left(\Gamma_{|N]C}{}^{L} - E_{C}{}^{S}\partial_{|N]}E_{S}{}^{A}E_{A}{}^{L}\right) = &\text{(F.19)}\\
&= \partial_{[M|}\Gamma_{|N]K}{}^{L} - \Gamma_{[M|K}{}^{P}\Gamma_{|N]P}{}^{L} &\text{(F.20)}
\end{aligned}$$

$$\boxed{R_{MNK}{}^{L} = \partial_{[M|}\Gamma_{|N]K}{}^{L} - \Gamma_{[M|K}{}^{P}\Gamma_{|N]P}{}^{L}} \tag{F.21}$$

The same expression can be derived (even simpler) by acting with the commutator of covariant deriavtives on a vector v^M with a curved index instead of the flat index.

F.1.3 Summary, including H-field-strength

Let us add the field strength H of the antisymmetric tensor field B to our considerations. We then have

$$\begin{aligned}
H &\equiv \mathbf{d}B &\text{(F.22)}\\
T^{A} &\equiv \mathbf{d}E^{A} - E^{C}\wedge\Omega_{C}{}^{A} &\text{(F.23)}\\
R_{A}{}^{B} &\equiv \mathbf{d}\Omega_{A}{}^{B} - \Omega_{A}{}^{C}\wedge\Omega_{C}{}^{B} &\text{(F.24)}
\end{aligned}$$

In coordinate basis ('curved indices') we have

$$\begin{aligned}
H_{MNK} &\equiv \partial_{[M}B_{NK]} &\text{(F.25)}\\
T_{MN}{}^{K} &\equiv \Gamma_{[MN]}{}^{K} &\text{(F.26)}\\
R_{MNK}{}^{L} &\equiv \partial_{[M|}\Gamma_{|N]K}{}^{L} - \Gamma_{[M|K}{}^{C}\Gamma_{|N]C}{}^{L} &\text{(F.27)}
\end{aligned}$$

The commutator of covariant derivatives on an arbitrary rank (p,q)-tensor fields (as a generalization of (F.10) and (F.11)) reads

$$\begin{aligned}
&\nabla_{[M}\nabla_{N]}t^{A_1\ldots A_q}_{B_1\ldots B_p} = \\
&= -T_{MN}{}^{K}\nabla_{K}t^{A_1\ldots A_q}_{B_1\ldots B_p} + \sum_{i=1}^{q} R_{MNC}{}^{A_i}t^{A_1\ldots A_{i-1}CA_{i+1}\ldots A_q}_{B_1\ldots B_p} - \sum_{i=1}^{q} R_{MNB_i}{}^{C}t^{A_1\ldots A_q}_{B_1\ldots B_{i-1}CB_{i+1}\ldots B_p} &\text{(F.28)}
\end{aligned}$$

This can be generalized yet a bit more, if we want to include fields that do not transform tensorial, like e.g. the compensator field. If we denote the representation of the structure group transformation, or better the representation of an Lie algebra element, by $\mathcal{R}(L_{\cdot}{}^{\cdot})$ (where $L_A{}^B$ is the matrix of the fundamental representation), the covariant derivative can be written as

$$\nabla_M = \partial_M + \mathcal{R}(\Omega_{M\cdot}{}^{\cdot}) \tag{F.29}$$

The commutator takes the general form

$$\nabla_{[M}\nabla_{N]} = -T_{MN}{}^K\nabla_K + \mathcal{R}(R_{MN\cdot}{}^{\cdot}) \tag{F.30}$$

This is in particular interesting for the compensator field, where we have a negative shift as representations and therefore[2]

$$\nabla_M\Phi = \partial_M\Phi - \Omega_M^{(D)} \tag{F.31}$$

$$\nabla_{[M}\nabla_{N]}\Phi = -T_{MN}{}^K\nabla_K\Phi - F_{MN}^{(D)} \tag{F.32}$$

Using the definition of the torsion, exterior derivatives of p-forms $\eta^{(p)}$ can be rewritten with covariant derivatives, thus allowing to switch to flat coordinates

$$\partial_{[M_1}\eta_{M_2\ldots M_{p+1}]} = \nabla_{[M_1}\eta_{M_2\ldots M_{p+1}]} + pT_{[M_1M_2|}{}^K\eta_{K|M_3\ldots M_{p+1}]} \tag{F.33}$$

In particular

$$H = \partial_{\boldsymbol{M}}B_{\boldsymbol{MM}} = \nabla_{\boldsymbol{A}}B_{\boldsymbol{AA}} + 2T_{\boldsymbol{AA}}{}^C B_{C\boldsymbol{A}} \tag{F.34}$$

F.2 The Bianchi identities

Bianchi identities all base on the nilpotency of the exterior derivative $\mathbf{d}^2 = 0$. The objects H, T^A and $R_A{}^B$ are all defined using the exterior derivative. Acting a second time with the exterior derivative (using $\mathbf{d}^2 = 0$) yields consitency conditions (the Bianchi identities) which have to be fulfilled by any valid H, T^Aor $R_A{}^B$. While these identities are trivially fulfilled, if the original definitions for these objects are used, the imposure of constraints on them makes a check necessary.[3]

F.2.1 BI for H_{ABC}

The most simple Bianchi identity is the one of the H-field $H = \mathbf{d}B$ (F.22). It just reads

$$\mathbf{d}H \stackrel{!}{=} 0 \tag{F.35}$$

The supergravity constraints on H that we will obtain, however, are all in flat coordinates, so that it is convenient to rewrite the Bianchi identity (using (F.33)) with covariant derivatives and then contract with vielbeins in order to turn the curved indices into flat ones:

$$\boxed{\nabla_{\boldsymbol{A}}H_{\boldsymbol{AAA}} \stackrel{!}{=} -3T_{\boldsymbol{AA}}{}^C H_{C\boldsymbol{AA}}} \tag{F.36}$$

Regarding the torsion as a vector valued 2-form and using the generalized definition of the interior product, this can also be written as

$$\boldsymbol{\nabla}H \equiv \mathbf{d}H - \imath_T H \stackrel{!}{=} -\imath_T H \tag{F.37}$$

[2] It is even possible now to define a covariant derivative of a connection (see (5.62) on page 49 or footnote 2 on page 209 for the representation of the structure group and its algebra on the connection)

$$\nabla_M\tilde{\Omega}_{NA}{}^B = \partial_M\tilde{\Omega}_{NA}{}^B \underbrace{-\partial_N\Omega_{MA}{}^B - [\Omega_M,\tilde{\Omega}_N]_A{}^B}_{\mathcal{R}(\Omega_{M\cdot}{}^{\cdot})\tilde{\Omega}_{NA}{}^B}$$

If the two connections coincide, we obtain

$$\nabla_M\Omega_{NA}{}^B = \partial_M\Omega_{NA}{}^B - \partial_N\Omega_{MA}{}^B - [\Omega_M,\Omega_N]_A{}^B = 2R_{MNA}{}^B \qquad \diamond$$

[3] Let us look at an example to make this point clear: one of the supergravity constraints that we get is $H_{\boldsymbol{\alpha\beta\gamma}} = 0$. As H was defined via $H = \mathbf{d}B$ in the beginning, this is actually a differential equation for B of the form $E_{\boldsymbol{\alpha}}{}^M E_{\boldsymbol{\beta}}{}^N E_{\boldsymbol{\gamma}}{}^K\left(\partial_{[M}B_{NK]}\right) = 0$. One could try to calculate the general solution for this equation (which might be quite hard) and then calculate the H-field via $H = \mathbf{d}B$ which will of course trivially obey the Bianchi identities. However, one prefers not to solve for B, but to calculate additional constraints on H using the Bianchi identities. The idea is to get the full information about H without solving for B. The same story holds for the other objects. $\diamond$

F.2.2 BI for T^A

Remember $T^A = \mathbf{d}E^A - E^C \wedge \Omega_C{}^A$ (F.7). Acting on this equation with the exterior derivative yields

$$
\begin{aligned}
\mathbf{d}T^A &= -\mathbf{d}E^C \wedge \Omega_C{}^A + E^C \wedge \mathbf{d}\Omega_C{}^A = && \text{(F.38)}\\
&\stackrel{(F.16)}{=} -T^C \wedge \Omega_C{}^A - E^D \wedge \Omega_D{}^C \wedge \Omega_C{}^A + E^C \wedge R_C{}^A + E^C \wedge \Omega_C{}^D \wedge \Omega_D{}^A = && \text{(F.39)}\\
&= -T^C \wedge \Omega_C{}^A + E^C \wedge R_C{}^A && \text{(F.40)}
\end{aligned}
$$

The Bianchi identity for the torsion (sometimes also called the first Bianchi identity) thus reads

$$
\mathbf{d}T^A + T^C \wedge \Omega_C{}^A \stackrel{!}{=} E^C \wedge R_C{}^A \qquad \text{(F.41)}
$$

Again we want to rewrite it in terms of the covariant derivative. The "exterior" covariant derivative of T reads

$$
\begin{aligned}
\nabla_{\boldsymbol{M}} T_{\boldsymbol{MM}}{}^A &= \partial_{\boldsymbol{M}} T_{\boldsymbol{MM}}{}^A - 2T_{\boldsymbol{MM}}{}^K T_{K\boldsymbol{M}}{}^A + \Omega_{\boldsymbol{M}B}{}^A T_{\boldsymbol{MM}}{}^B && \text{(F.42)}\\
\boldsymbol{\nabla} T^A &= \mathbf{d}T^A + T^B \wedge \Omega_B{}^A - \imath_T T^A && \text{(F.43)}
\end{aligned}
$$

The above Bianchi-identity can thus be rewritten as

$$
\boxed{\nabla_{\boldsymbol{A}} T_{\boldsymbol{AA}}{}^D + 2T_{\boldsymbol{AA}}{}^C T_{C\boldsymbol{A}}{}^D \stackrel{!}{=} R_{\boldsymbol{AAA}}{}^D} \qquad \text{(F.44)}
$$

$$
\boldsymbol{\nabla} T^D + \imath_T T^D \stackrel{!}{=} R^D \equiv E^C \wedge R_C{}^D \qquad \text{(F.45)}
$$

F.2.3 BI for $R_A{}^B$

Remember $R_A{}^B = \mathbf{d}\Omega_A{}^B - \Omega_A{}^C \wedge \Omega_C{}^B$ (F.16). Acting on it with the exterior derivative yields

$$
\begin{aligned}
\mathbf{d}R_A{}^B &= -\mathbf{d}\Omega_A{}^C \wedge \Omega_C{}^B + \Omega_A{}^C \wedge \mathbf{d}\Omega_C{}^B = && \text{(F.46)}\\
&= -R_A{}^C \wedge \Omega_C{}^B - \Omega_A{}^D \wedge \Omega_D{}^C \wedge \Omega_C{}^B + \Omega_A{}^C \wedge R_C{}^B + \Omega_A{}^C \wedge \Omega_C{}^D \wedge \Omega_D{}^B = && \text{(F.47)}\\
&= -R_A{}^C \wedge \Omega_C{}^B + \Omega_A{}^C \wedge R_C{}^B && \text{(F.48)}
\end{aligned}
$$

The Bianchi identity for the curvature (also called second Bianchi identity) thus reads

$$
\mathbf{d}R_A{}^B + \underbrace{R_A{}^C \wedge \Omega_C{}^B - \Omega_A{}^C \wedge R_C{}^B}_{[R,\Omega]_A{}^C} \stackrel{!}{=} 0 \qquad \text{(F.49)}
$$

Again we want to rewrite this in terms of covariant derivatives and flat indices and therefore consider the antisymmetrized covariant derivative

$$
\begin{aligned}
\nabla_{\boldsymbol{M}} R_{\boldsymbol{MM}A}{}^B &= \partial_{\boldsymbol{M}} R_{\boldsymbol{MM}A}{}^B - 2T_{\boldsymbol{MM}}{}^K R_{K\boldsymbol{M}A}{}^B - \Omega_{\boldsymbol{M}A}{}^C R_{\boldsymbol{MM}C}{}^B + \Omega_{\boldsymbol{M}C}{}^B R_{\boldsymbol{MM}A}{}^C && \text{(F.50)}\\
\boldsymbol{\nabla} R_A{}^B &= \mathbf{d}R_A{}^B - \Omega_A{}^C \wedge R_C{}^B + R_A{}^C \wedge \Omega_C{}^B - \imath_T R_A{}^B && \text{(F.51)}
\end{aligned}
$$

We thus can rewrite the above Bianchi-identity as

$$
\boxed{\nabla_{\boldsymbol{M}} R_{\boldsymbol{MM}A}{}^B + 2T_{\boldsymbol{MM}}{}^K R_{K\boldsymbol{M}A}{}^B = 0} \qquad \text{(F.52)}
$$

$$
\boldsymbol{\nabla} R_A{}^B + \imath_T R_A{}^B = 0 \qquad \text{(F.53)}
$$

If the structure group is restricted to e.g. Lorentz plus scale transformations (see section F.4 on page 194), we get

$$
\begin{aligned}
R_{\boldsymbol{MM}a}{}^b &= F^{(D)}_{\boldsymbol{MM}} \delta^b_a + R^{(L)}_{\boldsymbol{MM}a}{}^b && \text{(F.54)}\\
\text{and } R_{\boldsymbol{MM}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \frac{1}{2} F^{(D)}_{\boldsymbol{MM}} \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4} R^{(L)}_{\boldsymbol{MM}ab} \gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} && \text{(F.55)}
\end{aligned}
$$

The above Bianchi identity then has to hold seperately for Lorentz and Dilatation part. In particular we have

$$
\boxed{\nabla_{\boldsymbol{M}} F^{(D)}_{\boldsymbol{MM}} + 2T_{\boldsymbol{MM}}{}^K F^{(D)}_{K\boldsymbol{M}} = 0} \qquad \text{(F.56)}
$$

F.2.4 Alternative derivation from the Jacobi identity

The above derivations of the Bianchi identities were based on the nilpotency $\mathbf{d}^2 = 0$ of the exterior derivative. The Bianchi identities for curvature and torsion are equivalently obtained from the Jacobi identity for commutators:

$$[A,[B,C]]+[C,[A,B]]+[B,[C,A]] \;=\; 0 \tag{F.57}$$

Applying this to covariant derivatives, using (F.30) yields

$$\begin{aligned}
0 &= [\nabla_{\boldsymbol{M}},[\nabla_{\boldsymbol{M}},\nabla_{\boldsymbol{M}}]] = && \text{(F.58)}\\
&= -2\left[\nabla_{\boldsymbol{M}},\, T_{\boldsymbol{MM}}{}^{K}\nabla_K\right] + 2\left[\nabla_{\boldsymbol{M}},\, \mathcal{R}(R_{\boldsymbol{MM}}\,\dot{\cdot})\right] = && \text{(F.59)}\\
&= -2\nabla_{\boldsymbol{M}}T_{\boldsymbol{MM}}{}^{K}\nabla_K - 2T_{\boldsymbol{MM}}{}^{K}[\nabla_{\boldsymbol{M}},\nabla_K] + 2\mathcal{R}(\nabla_{\boldsymbol{M}}R_{\boldsymbol{MM}}\,\dot{\cdot}) + 2R_{\boldsymbol{MMM}}{}^{K}\nabla_K = && \text{(F.60)}\\
&= 2\left(R_{\boldsymbol{MMM}}{}^{K} - \nabla_{\boldsymbol{M}}T_{\boldsymbol{MM}}{}^{K}\right)\nabla_K - 2T_{\boldsymbol{MM}}{}^{K}\left(-2T_{\boldsymbol{M}K}{}^{L}\nabla_L + 2\mathcal{R}(R_{\boldsymbol{M}K}\,\dot{\cdot})\right) + 2\mathcal{R}(\nabla_{\boldsymbol{M}}R_{\boldsymbol{MM}}\,\dot{\cdot}) = && \text{(F.61)}\\
&= 2\left(R_{\boldsymbol{MMM}}{}^{K} - \nabla_{\boldsymbol{M}}T_{\boldsymbol{MM}}{}^{K} - 2T_{\boldsymbol{MM}}{}^{L}T_{L\boldsymbol{M}}{}^{K}\right)\nabla_K + 2\mathcal{R}\left(\nabla_{\boldsymbol{M}}R_{\boldsymbol{MM}}\,\dot{\cdot} + 2T_{\boldsymbol{MM}}{}^{K}R_{K\boldsymbol{M}}\,\dot{\cdot}\right) && \text{(F.62)}
\end{aligned}$$

Both brackets have to vanish separately, which correctly reproduces the identities (F.44) and (F.52).

F.3 Shifting the connection

Some expressions might look simpler if one changes the connection $\Omega_{MA}{}^{B}$ to some new connection $\tilde{\Omega}_{MA}{}^{B}$. As usual, the difference

$$\Delta_{MA}{}^{B} \equiv \tilde{\Omega}_{MA}{}^{B} - \Omega_{MA}{}^{B} \tag{F.63}$$

transforms as a tensor (the inhomogenous term in the transformation cancels). The new torsion looks as follows:

$$\begin{aligned}
\tilde{T}^{A} &= \mathbf{d}E^{A} - E^{C}\wedge\tilde{\Omega}_{C}{}^{A} = && \text{(F.64)}\\
&= T^{A} - E^{C}\wedge\Delta_{C}{}^{A} = && \text{(F.65)}
\end{aligned}$$

Or simply

$$\boxed{\tilde{T}_{\boldsymbol{MM}}{}^{A} = T_{\boldsymbol{MM}}{}^{A} + \Delta_{\boldsymbol{MM}}{}^{A}} \tag{F.66}$$

The expression for the new curvature is a bit more involved and reads[4]

$$\begin{aligned}
\hat{R}_{A}{}^{B} &= \mathbf{d}\tilde{\Omega}_{A}{}^{B} - \tilde{\Omega}_{A}{}^{C}\wedge\tilde{\Omega}_{C}{}^{B} = && \text{(F.67)}\\
&= R_{A}{}^{B} + \mathbf{d}\Delta_{A}{}^{B} - \Delta_{A}{}^{C}\wedge\Omega_{C}{}^{B} - \Omega_{A}{}^{C}\wedge\Delta_{C}{}^{B} - \Delta_{A}{}^{C}\wedge\Delta_{C}{}^{B} = && \text{(F.68)}\\
&= R_{A}{}^{B} + \boldsymbol{\nabla}\Delta_{A}{}^{B} + T^{K}\Delta_{KA}{}^{B} - \Delta_{A}{}^{C}\wedge\Delta_{C}{}^{B} && \text{(F.69)}
\end{aligned}$$

$$\boxed{\tilde{R}_{\boldsymbol{MM}A}{}^{B} = R_{\boldsymbol{MM}A}{}^{B} + \nabla_{\boldsymbol{M}}\Delta_{\boldsymbol{M}A}{}^{B} + T_{\boldsymbol{MM}}{}^{K}\Delta_{KA}{}^{B} - \Delta_{\boldsymbol{M}A}{}^{C}\Delta_{\boldsymbol{M}C}{}^{B}} \tag{F.70}$$

or equivalently

$$\boxed{\tilde{R}_{\boldsymbol{MM}A}{}^{B} = R_{\boldsymbol{MM}A}{}^{B} + \tilde{\nabla}_{\boldsymbol{M}}\Delta_{\boldsymbol{M}A}{}^{B} + \tilde{T}_{\boldsymbol{MM}}{}^{K}\Delta_{KA}{}^{B} + \Delta_{\boldsymbol{M}A}{}^{C}\Delta_{\boldsymbol{M}C}{}^{B}} \tag{F.71}$$

Proposition 7 *The Bianchi identities for T^{A} and $R_{A}{}^{B}$ on the one hand and $\tilde{T}^{A}$ and $\tilde{R}_{A}{}^{B}$ on the other hand are equivalent if the objects are related via (F.66) and (F.70).*

[4] Of similar interest is a change in the definition of the vielbein. Note that local structure group transformations of the vielbein which go along with a structure group transformation of torsion and curvature also include a corresponding transformation of the connection. Instead we want to look at an independent transformation of the vielbein and consider general local $Gl(n)$ transformations.

$$\tilde{E}^{A} = E^{B}J_{B}{}^{A}$$

with $\tilde{\nabla}_{M}\tilde{E}^{A} = 0$. For the new torsion, we get

$$\begin{aligned}
\tilde{T}^{A} &= \mathbf{d}\tilde{E}^{A} - \tilde{E}^{C}\wedge\Omega_{C}{}^{A} =\\
&= \mathbf{d}E^{B}J_{B}{}^{A} - E^{B}\wedge\mathbf{d}J_{B}{}^{A} - E^{B}J_{B}{}^{C}\wedge\Omega_{C}{}^{A} =\\
&= T^{B}J_{B}{}^{A} - E^{B}\wedge\boldsymbol{\nabla}J_{B}{}^{A}
\end{aligned}$$

or

$$\boxed{\tilde{T}_{\boldsymbol{M}\boldsymbol{M}}{}^{B} = T_{\boldsymbol{M}\boldsymbol{M}}{}^{B}J_{B}{}^{A} + \nabla_{\boldsymbol{M}}J_{\boldsymbol{M}}{}^{A}}$$

The curvature remains untouched

$$\boxed{\tilde{R}_{A}{}^{B} = R_{A}{}^{B}}$$

Alternatively one might be interested in shifts of the vielbein (resulting in $\tilde{T} = T + \mathbf{d}(\Delta E)^{A} - (\Delta E)^{C}\wedge\Omega_{C}{}^{A}$) or linear transformations of the connection of the form $\tilde{\Omega} = J\Omega J^{-1}$ ◇

Proof In fact this is a rather trivial statement. The Bianchi identities do not put restrictions on the elementary objects (the connection and the vielbein), but on the derived objects (torsion and curvature). In the same way they do not put restrictions on the difference tensor. Let us make this statement more precise. If the Bianchi identity for $T^A$and $R_A{}^B$ is fulfilled, then these objects can locally be written as $T^A = \mathbf{d}E^A - E^C \wedge \Omega_C{}^A$ and $R_A{}^B = \mathbf{d}\Omega_A{}^B - \Omega_A{}^C \wedge \Omega_C{}^B$ for some E^A and some $\Omega_A{}^B$. If we revert the derivation of (F.66) and (F.70), these equations then simply imply that $\tilde{T}^A$ and $\tilde{R}_A{}^B$ can locally be written as $\tilde{T}^A = \mathbf{d}E^A - E^C \wedge \tilde{\Omega}_C{}^A$ and $\tilde{R}_A{}^B = \mathbf{d}\tilde{\Omega}_A{}^B - \tilde{\Omega}_A{}^C \wedge \tilde{\Omega}_C{}^B$ with $\tilde{\Omega}_{MA}{}^B = \Omega_{MA}{}^B + \Delta_{MA}{}^B$ and therefore necessarily obey the Bianchi identities. This proves the proposition. □

For the first Bianchi identity, we will also provide a brute force proof: Remember the first Bianchi identity (F.44) for which we temporarily introduce the symbol J:

$$J_{\boldsymbol{AAA}}{}^D \equiv \nabla_{\boldsymbol{A}} T_{\boldsymbol{AA}}{}^D + 2T_{\boldsymbol{AA}}{}^C T_{C\boldsymbol{A}}{}^D - R_{\boldsymbol{AAA}}{}^D \stackrel{!}{=} 0 \tag{F.72}$$

The transformed J reads

$$\begin{aligned}
\tilde{J}_{\boldsymbol{AAA}}{}^D \stackrel{(F.66)(F.70)(F.72)}{=} & J_{\boldsymbol{AAA}}{}^D + \nabla_{\boldsymbol{A}}\Delta_{\boldsymbol{AA}}{}^D + \Delta_{\boldsymbol{A}C}{}^D(T_{\boldsymbol{AA}}{}^C + \Delta_{\boldsymbol{AA}}{}^C) - 2\Delta_{\boldsymbol{AA}}{}^C(T_{C\boldsymbol{A}}{}^D + \Delta_{[C\boldsymbol{A}]}{}^D) + \\
& + 2\Delta_{\boldsymbol{AA}}{}^C(T_{C\boldsymbol{A}}{}^D + \Delta_{[C\boldsymbol{A}]}{}^D) + 2T_{\boldsymbol{AA}}{}^C\Delta_{[C\boldsymbol{A}]}{}^D + \\
& - \nabla_{\boldsymbol{A}}\Delta_{\boldsymbol{AA}}{}^D - T_{\boldsymbol{AA}}{}^C\Delta_{C\boldsymbol{A}}{}^D + \Delta_{\boldsymbol{AA}}{}^C\Delta_{\boldsymbol{A}C}{}^D = \qquad \text{(F.73)} \\
= \; & J_{\boldsymbol{AAA}}{}^D \qquad \text{(F.74)}
\end{aligned}$$

This proves the proposition again for the first Bianchi identity. The brute force proof for the second is left to the reader as an exercise ;-)

F.4 Restricted structure group

As we discussed earlier, the (infinitesimal) local structure group transformations in the type II supergravity context are block-diagonal $\Lambda_A{}^B = \mathrm{diag}\,(\Lambda_a{}^b, \Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \Lambda_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}})$ and are in addition restricted to Lorentz transformations and scale transformations in order to leave invariant the supersymmetry structure constants $\gamma^c_{\boldsymbol{\alpha\beta}}$:

$$\begin{aligned}
\Lambda_a{}^b &= \Lambda^{(D)}\delta_a^b + \Lambda^{(L)}_{a_1}{}^{a_2} && \text{(F.75)} \\
\Lambda_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \frac{1}{2}\Lambda^{(D)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Lambda^{(L)}_{a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} && \text{(F.76)} \\
\Lambda_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} &= \frac{1}{2}\Lambda^{(D)}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\Lambda^{(L)}_{a_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && \text{(F.77)}
\end{aligned}$$

Also the connection is a sum of a scaling connection and a Lorentz connection which makes perfect sense as it is supposed to be a Lie algebra valued one form:

$$\begin{aligned}
\Omega_{Ma}{}^b &= \Omega^{(D)}_M\delta_a^b + \Omega^{(L)}_{Ma_1}{}^{a_2} && \text{(F.78)} \\
\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \frac{1}{2}\Omega^{(D)}_M\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega^{(L)}_{M\,a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} && \text{(F.79)} \\
\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} &= \frac{1}{2}\Omega^{(D)}_M\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\Omega^{(L)}_{M\,a_1a_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} && \text{(F.80)}
\end{aligned}$$

with

$$\Omega^{(L)}_{M\,a_1a_2} \equiv \Omega^{(L)}_{Ma_1}{}^c\eta_{ca_2} = -\Omega^{(L)}_{M\,a_2a_1} \tag{F.81}$$

F.4.1 Curvature

It is well known that the curvature is a Lie algebra valued two form. Let us quickly recall the reason. The curvature is defined to be

$$R_A{}^B = \mathbf{d}\Omega_A{}^B - \Omega_A{}^C \wedge \Omega_C{}^B \tag{F.82}$$

If $\Omega_A{}^B$ is Lie algebra valued, $\mathbf{d}\Omega_A{}^B$ is still Lie algebra valued, as the exterior derivative acts only on the coefficient functions and not on the Lie algebra generator. In addition, the term $\Omega_A{}^C \wedge \Omega_C{}^B$ can be written as $\frac{1}{2}[\Omega, \Omega]_A{}^B$, and the commutator of two Lie algebra elements is again a Lie algebra element.

Let us now see how the structure group reduces into irreducible parts or in particular how the curvature decays into the Lorentz part and the scaling part (if the latter is present). First of all, the result is clearly block diagonal if the connection is of this type

$$R_A{}^B = \mathrm{diag}\,(R_a{}^b, R_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, R_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}) \tag{F.83}$$

such that the curvature definition (F.82) decays into the three blocks

$$R_a{}^b = \mathbf{d}\Omega_a{}^b - \Omega_a{}^c \wedge \Omega_c{}^b \tag{F.84}$$
$$R_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \mathbf{d}\Omega_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} - \Omega_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}} \wedge \Omega_{\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}} \tag{F.85}$$
$$R_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \mathbf{d}\Omega_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} - \Omega_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\gamma}}} \wedge \Omega_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\beta}}} \tag{F.86}$$

For the bosonic part of the curvature the seperation of scaling part and Lorentz part is quite obvious

$$R_a{}^b = \mathbf{d}\Big(\Omega^{(D)}\delta_a^b + \Omega_a^{(L)b}\Big) - \Big(\Omega^{(D)}\delta_a^c + \Omega_a^{(L)c}\Big) \wedge \Big(\Omega^{(D)}\delta_c^b + \Omega_c^{(L)b}\Big) = \tag{F.87}$$
$$= \underbrace{\mathbf{d}\Omega^{(D)}}_{\equiv F^{(D)}}\delta_a^b + \underbrace{\Big(\mathbf{d}\Omega_a^{(L)b} - \Omega_a^{(L)c}\wedge\Omega_c^{(L)b}\Big)}_{R_a^{(L)b}} \tag{F.88}$$

Where the Lorentz curvature $R_a^{(L)b}$ is antisymmetric if we pull down the index b with the Minkowski metric. We can thus extract from the complete curvature the scale part and the Lorentz part (here for 10 spacetime dimensions)

$$F^{(D)} = \frac{1}{10}R_a{}^a \tag{F.89}$$

For the fermionic parts we get similarly ($\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}} = -16$ in our conventions)[5]

$$R_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}F^{(D)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}R^{(L)}{}_{a_1}{}^b\eta_{ba_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{F.90}$$
$$F^{(D)} = -\frac{1}{8}R_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}} \tag{F.91}$$

and

$$R_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{2}F^{(D)}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}R^{(L)}{}_{a_1}{}^b\eta_{ba_2}\gamma^{a_1a_2}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{F.92}$$
$$F^{(D)} = -\frac{1}{8}R_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\alpha}}} \tag{F.93}$$

F.4.2 Alternative version of the first Bianchi identity

The ordinary Riemannian curvature (without torsion) obeys $R_{abcd} = -R_{bacd} = -R_{abdc}$, $R_{[abc]d} = 0$ and $R_{abcd} = R_{cdab}$ (The last is a consequence of the others). For the bosonic components of our curvature we have (using $G_{ab} = e^{2\Phi}\eta_{ab}$ with $\nabla_M G_{ab} = 2(\partial_M\Phi - \Omega_M^{(Dil)})G_{ab}$ to pull down bosonic indices)

$$R_{abcd} = -R_{bacd}, \qquad R_{(ab)cd} = 0 \tag{F.94}$$
$$R_{abcd} = -R_{abdc} + 2F_{ab}^{(Dil)}G_{cd}, \qquad R_{ab(cd)} = F_{ab}^{(Dil)}G_{cd} \tag{F.95}$$
$$R_{[abc]d} = \nabla_{[a}T_{bc]|d} - 2(\partial_{[a}\Phi - \Omega_{[a}^{(Dil)})T_{bc|d} + 2T_{[ab|}{}^E T_{E|c]|d} \tag{F.96}$$

[5]In order to see how the curvature decays into Lorentz and scale part, let us first consider the building blocks seperately:

$$\partial_{\boldsymbol{M}}\Omega_{\boldsymbol{M}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}\partial_{\boldsymbol{M}}\Omega_{\boldsymbol{M}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\partial_{\boldsymbol{M}}\Omega_{\boldsymbol{M}a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$$

$$\begin{aligned}
\Omega_{\boldsymbol{M}\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\Omega_{\boldsymbol{M}\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}} &= \left(\frac{1}{2}\Omega_{\boldsymbol{M}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}} + \frac{1}{4}\Omega_{\boldsymbol{M}a_1a_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\right)\left(\frac{1}{2}\Omega_{\boldsymbol{M}}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega_{\boldsymbol{M}b_1b_2}\gamma^{b_1b_2}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}}\right) = \\
&= \frac{1}{16}\underbrace{\Omega_{\boldsymbol{M}a_1a_2}\Omega_{\boldsymbol{M}b_1b_2}}_{\text{antisym in } (a_1a_2)\leftrightarrow(b_1b_2)}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\gamma^{b_1b_2}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}} = \\
&\overset{(D.117)}{=} \frac{1}{4}\Omega_{\boldsymbol{M}a_1c}\eta^{cd}\Omega_{\boldsymbol{M}da_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}
\end{aligned}$$

The curvature thus takes the form

$$\begin{aligned}
\Rightarrow R_{\boldsymbol{M}\boldsymbol{M}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} &= \frac{1}{2}\partial_{\boldsymbol{M}}\Omega_{\boldsymbol{M}}^{(Dil)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\left(\partial_{\boldsymbol{M}}\Omega_{\boldsymbol{M}a_1a_2}^{(Lor)} - \Omega_{\boldsymbol{M}a_1c}^{(Lor)}\eta^{cd}\Omega_{\boldsymbol{M}da_2}^{(Lor)}\right)\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \equiv \\
&\equiv \frac{1}{2}F^{(Dil)}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}R^{(Lor)}{}_{a_1}{}^b\eta_{ba_2}\gamma^{a_1a_2}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \qquad \diamond
\end{aligned}$$

Let us write down the antisymmetrization of the indices in $R_{[abc]d}$ explicitely and several times, with permuted indices:

$$R_{[abc]d} = R_{abcd} + R_{cabd} + R_{bcad} \tag{F.97}$$
$$R_{[dab]c} = R_{dabc} + R_{bdac} + R_{abdc} \tag{F.98}$$
$$R_{[cda]b} = R_{cdab} + R_{acdb} + R_{dacb} \tag{F.99}$$
$$R_{[bcd]a} = R_{bcda} + R_{dbca} + R_{cdba} \tag{F.100}$$

From this we learn, how we can express the difference $R_{abcd} - R_{cdab}$ (which vanishes in the Riemannian case), in terms of antisymmetrized and symmetrized terms. Consider the sum (F.97)-(F.98)-(F.99)+(F.100):

$$\begin{aligned} & R_{[abc]d} - R_{[dab]c} - R_{[cda]b} + R_{[bcd]a} = \\ &= 2R_{abcd} - 2R_{ab(cd)} - 2R_{cdab} + 2R_{cd(ab)} + 2R_{(ca)bd} - 2R_{ac(db)} + 2R_{bc(da)} - 2R_{da(bc)} - 2R_{bd(ac)} + 2R_{(db)ca} = \\ &= 2\left(R_{abcd} - R_{cdab}\right) + 2\left(-F_{ab}G_{cd} + F_{cd}G_{ab} - F_{ac}G_{db} + F_{bc}G_{da} - F_{da}G_{bc} - F_{bd}G_{ac}\right) \end{aligned} \tag{F.101}$$

The identity corresponding to $R_{abcd} = R_{cdab}$ in the Riemannian case thus reads

$$\begin{aligned} & 2\left(R_{abcd} - R_{cdab}\right) = \\ &= 2\left(F_{ab}G_{cd} - F_{cd}G_{ab} + F_{ac}G_{db} - F_{bc}G_{da} + F_{da}G_{bc} + F_{bd}G_{ac}\right) + R_{[abc]d} - R_{[dab]c} - R_{[cda]b} + R_{[bcd]a} \end{aligned} \tag{F.102}$$

with $R_{[abc]d} = \nabla_{[a}T_{bc]|d} - 2(\partial_{[a}\Phi - \Omega^{(Dil)}_{[a})T_{bc|d} + 2T_{[ab|}{}^{E}T_{E|c]|d}$.

F.4.3 Scaling-curvature

A covariant way to calculate the scaling field strength $F^{(D)}_{MN}$ is as follows: Consider the covariant derivative $\nabla_M\Phi = \partial_M\Phi - \Omega^{(D)}_M$ of a compensator field Φ (a field transforming with a shift under scaling transformations $\delta\Phi = -\Lambda^{(D)}$). We can calculate $F^{(D)}_{MN}$ via the ususal commutator of covariant derivatives[6]

$$\nabla_{[M}\nabla_{N]}\Phi = -T_{MN}{}^{K}\nabla_K\Phi \underbrace{-F^{(D)}_{MN}}_{\mathcal{R}\left(F^{(D)}_{MN}\right)\Phi} \tag{F.103}$$

Note that the curvature (or field strength) appears "naked" in difference to any action on tensor fields. The above equation will be particularly useful when we have constraints on $\nabla_M\Phi$ which then determine the scaling curvature via

$$\boxed{F^{(D)}_{MN} = -\nabla_{[M}\nabla_{N]}\Phi - T_{MN}{}^{K}\nabla_K\Phi} \tag{F.104}$$

F.5 Dragon's theorem

In the following we will need the commutator of two covariant derivatives acting on the torsion with afterwards all lower indices antisymmetrized. Due to (F.28), it is given by[7]

$$\nabla_{\boldsymbol{M}}\nabla_{\boldsymbol{M}}T_{\boldsymbol{MM}}{}^{A} = -T_{\boldsymbol{MM}}{}^{K}\nabla_K T_{\boldsymbol{MM}}{}^{A} - 2R_{\boldsymbol{MMM}}{}^{K}T_{K\boldsymbol{M}}{}^{A} + R_{\boldsymbol{MM}B}{}^{A}T_{\boldsymbol{MM}}{}^{B} \tag{F.105}$$

and can, using the first Bianchi identity (F.44), be rewritten as

$$\begin{aligned} & R_{\boldsymbol{MM}B}{}^{A}T_{\boldsymbol{MM}}{}^{B} = \\ &= \nabla_{\boldsymbol{M}}\nabla_{\boldsymbol{M}}T_{\boldsymbol{MM}}{}^{A} + T_{\boldsymbol{MM}}{}^{K}\nabla_K T_{\boldsymbol{MM}}{}^{A} + 2\left(\nabla_{\boldsymbol{M}}T_{\boldsymbol{MM}}{}^{K} + 2T_{\boldsymbol{MM}}{}^{L}T_{L\boldsymbol{M}}{}^{K}\right)T_{K\boldsymbol{M}}{}^{A} \end{aligned} \tag{F.106}$$

It is convenient to introduce a new symbol for the terms of the curvature Bianchi identity

$$I_A{}^B \equiv I_{\boldsymbol{CCC}A}{}^{B} \equiv \nabla_{\boldsymbol{C}}R_{\boldsymbol{CC}A}{}^{B} + 2T_{\boldsymbol{CC}}{}^{D}R_{D\boldsymbol{C}A}{}^{B} \tag{F.107}$$

so that the Bianchi identity (F.52) simply reads $I_A{}^B \stackrel{!}{=} 0$. Then the following theorem holds (originally due to Dragon in [15]; slightly modified in order to include dilatations):

[6]Let us check explicitely the validity of (F.103):

$$\begin{aligned} \nabla_{[M}\nabla_{N]}\Phi &= \partial_{[M}\nabla_{N]}\Phi - \Gamma_{[MN]}{}^{K}\nabla_K\Phi = \\ &= \partial_{[M}(\partial_{N]}\Phi - \Omega^{(D)}_{N]}) - T_{[MN]}{}^{K}\nabla_K\Phi = \\ &= -F^{(D)}_{MN} - T_{[MN]}{}^{K}\nabla_K\Phi \qquad \diamond \end{aligned}$$

[7]Of course (F.28) implies a more general relation than (F.105), namely one of the form $[\nabla_M, \nabla_N]T_{KL}{}^{A} = \ldots$. However, the lower indices are intentionally antisymmetrized in (F.105), in order to get the weakest possible condition that we need to proof the theorem later on. You'll see... $\diamond$

Theorem 4 (Dragon) *Given a block diagonal structure group consisting of Lorentz transformation and dilatation in a type II superspace, the torsion Bianchi identity (F.44) together with the algebra (F.105) or equivalently (F.106) imply the curvature Bianchi identities (F.52) $I_A{}^B = 0$ up to one remaining equation for the scale part, namely $I^{(D)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}c} \stackrel{!}{=} 0$ or equivalently*

$$\nabla_{[\boldsymbol{\gamma}} F^{(D)}_{\hat{\boldsymbol{\gamma}}c]} + 2T_{[\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}|}{}^D F^{(D)}_{D|c]} \stackrel{!}{=} 0 \tag{F.108}$$

where $F^{(D)}_{MN}$ is the field strength of the scale connection $\Omega^{(D)}_M$.

It is natural to proof this theorem in two steps, the first being useful enough to write it as a seperate proposition. Let us include one more index into the antisymmetrization of $I_A{}^B$ and define

$$I^B \equiv I_{\boldsymbol{CCCC}}{}^B \equiv \nabla_{\boldsymbol{C}} R_{\boldsymbol{CCC}}{}^B + 2T_{\boldsymbol{CC}}{}^D R_{D\boldsymbol{CC}}{}^B \tag{F.109}$$

so that we can make direct use of the torsion-Bianchi-identity (F.44) due to the appearance of $R_{\boldsymbol{CCC}}{}^B$. Clearly $I^B \stackrel{!}{=} 0$ is a consequence of $I_A{}^B \stackrel{!}{=} 0$ and is in general a weaker condition. The following proposition treats this weaker condition:

Proposition 8 *In any dimension and for any structure group, the equation $I^B \stackrel{!}{=} 0$ (with I^B given by (F.109)) is implied by the first Bianchi identity (F.44) and the algebra (F.105) or equivalently (F.106).*

Proof of the proposition:

$$\begin{aligned}
I^B &= \nabla_{\boldsymbol{M}} R_{\boldsymbol{MMM}}{}^B + 2T_{\boldsymbol{MM}}{}^K R_{K\boldsymbol{MM}}{}^B = && \text{(F.110)}\\
&\stackrel{(F.44)}{=} \nabla_{\boldsymbol{M}} \left(\nabla_{\boldsymbol{M}} T_{\boldsymbol{MM}}{}^B + 2T_{\boldsymbol{MM}}{}^C T_{C\boldsymbol{M}}{}^B\right) + 2T_{\boldsymbol{MM}}{}^K R_{K\boldsymbol{MM}}{}^B = && \text{(F.111)}\\
&\stackrel{(F.105)}{=} -T_{\boldsymbol{MM}}{}^C \nabla_C T_{\boldsymbol{MM}}{}^B - 2R_{\boldsymbol{MMM}}{}^C T_{C\boldsymbol{M}}{}^B + R_{\boldsymbol{MM}C}{}^B T_{\boldsymbol{MM}}{}^C + && \text{(F.112)}\\
&\quad +2\nabla_{\boldsymbol{M}} T_{\boldsymbol{MM}}{}^C T_{C\boldsymbol{M}}{}^B + 2T_{\boldsymbol{MM}}{}^C \nabla_{\boldsymbol{M}} T_{C\boldsymbol{M}}{}^B + 2T_{\boldsymbol{MM}}{}^K R_{K\boldsymbol{MM}}{}^B = && \text{(F.113)}\\
&= 3T_{\boldsymbol{MM}}{}^C \left(R_{[C\boldsymbol{MM}]}{}^B - \nabla_{[C} T_{\boldsymbol{MM}]}{}^B\right) - 2\left(R_{\boldsymbol{MMM}}{}^C - \nabla_{\boldsymbol{M}} T_{\boldsymbol{MM}}{}^C\right) T_{C\boldsymbol{M}}{}^B = && \text{(F.114)}\\
&\stackrel{(F.44)}{=} 6T_{\boldsymbol{MM}}{}^C T_{[C\boldsymbol{M}|}{}^D T_{D|\boldsymbol{M}]}{}^B - 4T_{\boldsymbol{MM}}{}^D T_{D\boldsymbol{M}}{}^C T_{C\boldsymbol{M}}{}^B = && \text{(F.115)}\\
&= 2T_{\boldsymbol{MM}}{}^C T_{\boldsymbol{MM}}{}^D T_{DC}{}^B = 0 && \text{(F.116)}
\end{aligned}$$

Indeed $I^B = 0$ is a consequence of the torsion Bianchi identity (F.44) $R_{\boldsymbol{MMM}}{}^B = \nabla_{\boldsymbol{M}} T_{\boldsymbol{MM}}{}^B + 2T_{\boldsymbol{MM}}{}^C T_{C\boldsymbol{M}}{}^B$ and (F.105). □

Proof of the theorem: Let us now show that in the case of the type II superspace the antisymmetrized version already implies (up to one term) the complete one. Remember the object $I_{\boldsymbol{CCC}A}{}^B \equiv \nabla_{\boldsymbol{C}} R_{\boldsymbol{CC}A}{}^B + 2T_{\boldsymbol{CC}}{}^D R_{DMA}{}^B$ introduced in (F.107). It is Lie algebra valued and thus has (for our block diagonal structure group) no mixed components in A, B:

$$I_{\boldsymbol{CCC}A}{}^B = \mathrm{diag}\,(I_{\boldsymbol{CCC}a}{}^b, I_{\boldsymbol{CCC\alpha}}{}^{\boldsymbol{\beta}}, I_{\boldsymbol{CCC}\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}) \tag{F.117}$$

In addition it splits into dilatation and Lorentz part

$$I_{\boldsymbol{CCC}A}{}^B = I^{(D)}_{\boldsymbol{CCC}} \delta_A{}^B + I^{(L)}_{\boldsymbol{CCC}A}{}^B \tag{F.118}$$

with the latter term being antisymmetric in A, B for bosonic a, b. The complete object is fixed by determing[8] $I_{\boldsymbol{CCC}a}{}^b$. Given the equation $I_{\boldsymbol{CCCC}}{}^B = 0$, we want to show that $I_{\boldsymbol{CCC}A}{}^B = 0$. Consider first $B = b$:

$$0 = 4I_{[\boldsymbol{ccc}a]}{}^b = I_{\boldsymbol{ccc}a}{}^b \tag{F.119}$$

Similarly, for $B = \boldsymbol{\beta}$:

$$\begin{aligned}
0 &= 4I_{[\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} = I_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0 && \text{(F.120)}\\
0 &= 4I_{[c\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} = I_{c\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0 && \text{(F.121)}\\
0 &= 4I_{cc\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} = I_{cc\hat{\boldsymbol{\gamma}}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0 && \text{(F.122)}\\
0 &= 4I_{ccc\boldsymbol{\alpha}]}{}^{\boldsymbol{\beta}} = I_{ccc\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = 0 && \text{(F.123)}
\end{aligned}$$

[8]The following proof is based on a block-diagonal connection of the form $\Omega_{MA}{}^B = \mathrm{diag}\,(\Omega_{Ma}{}^b, \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}})$ where the three entries are related by $\nabla_M \gamma^a_{\boldsymbol{\alpha\beta}} = \nabla_M \gamma^a_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0$ which in turn is equivalent to $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{4}\Omega_{Ma}{}^b \gamma^a{}_{b\,\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{4}\Omega_{Ma}{}^b \gamma^a{}_{b\,\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$. The Bianchi identity for its torsion $T^A = (T^a, T^{\boldsymbol{\alpha}}, T^{\hat{\boldsymbol{\alpha}}})$ is equivalent to the one for the Torsion $\underline{T}^A = (\check{T}^a, T^{\boldsymbol{\alpha}}, \hat{T}^{\hat{\boldsymbol{\alpha}}})$ when information about the connection-difference $\Delta_{MA}{}^B$ is available. ◇

This implies

$$\begin{aligned} I_{c\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\gamma}}a}{}^{b} &= 0 && \text{(F.124)} \\ I_{cc\hat{\boldsymbol{\gamma}}a}{}^{b} &= 0 && \text{(F.125)} \\ I_{ccca}{}^{b} &= 0 && \text{(F.126)} \end{aligned}$$

Equivalently we get from the equations for $B = \hat{\boldsymbol{\beta}}$:

$$\begin{aligned} I_{c\boldsymbol{\gamma}\boldsymbol{\gamma}a}{}^{b} &= 0 && \text{(F.127)} \\ I_{cc\boldsymbol{\gamma}a}{}^{b} &= 0 && \text{(F.128)} \end{aligned}$$

There is thus only one component of $I_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}ca}{}^{b}$ left to determine. For this we get

$$\begin{aligned} 0 &= I_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}[ca]}{}^{b} = && \text{(F.129)} \\ &= I^{(D)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}[c}\delta^{b}_{a]} + I^{(L)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}[ca]}{}^{b} && \text{(F.130)} \end{aligned}$$

Taking the trace in (a,b) yields

$$0 = 9I^{(D)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}c} + I^{(L)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}ac}{}^{a} \quad \text{(F.131)}$$

In order that they vanish independently, it is thus enough to check only one equation, namely $I^{(D)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}c} \stackrel{!}{=} 0$ which reads explicitely

$$\boxed{\nabla_{[\boldsymbol{\gamma}}F^{(D)}_{\hat{\boldsymbol{\gamma}}c]} + 2T_{[\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}|}{}^{D}F^{(D)}_{D|c]} \stackrel{!}{=} 0} \quad \text{(F.132)}$$

Appendix G

About the Connection

Let us refer to both, spacetime and structure group connection, simply as "the connection". Properties of the one are translated to the other via the condition of covariantly constant vielbeins $\nabla_M E_N{}^A = 0$:

$$\Gamma_{MN}{}^A = \partial_M E_N{}^A + \Omega_{MN}{}^A \tag{G.1}$$

We will use symbols without any decoration (like hats or whatever) to describe a general connection and objects derived from it. In our application to the Berkovits string, however, we use the undecorated symbol $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ for the leftmoving connection only, which hopefully does not lead to confusions. To be more explicit, in the application we work with several different connections which are all blockdiagonal. In the action there appear only $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$. The spinorial $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ induces via $\nabla_M \gamma^c_{\boldsymbol{\alpha\beta}}$ a connection $\Omega_{Ma}{}^b$ for the bosonic subspace which in turn induces a connection $\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ via $\nabla_M \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0$. The collection of those will be denoted by $\Omega_{MA}{}^B$ (**left-mover connection**). The same can be done for $\hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ leading to a connection $\hat{\Omega}_{MA}{}^B$ which we call the **right-mover connection**.

$$\Omega_{MA}{}^B = \begin{pmatrix} \Omega_{Ma}{}^b & 0 & 0 \\ 0 & \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix}, \qquad \hat{\Omega}_{MA}{}^B = \begin{pmatrix} \hat{\Omega}_{Ma}{}^b & 0 & 0 \\ 0 & \hat{\Omega}_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{G.2}$$

The supergravity constraints are derived from the Berkovits string using a **mixed connection**

$$\underline{\Omega}_{MA}{}^B \equiv \begin{pmatrix} \check{\Omega}_{Ma}{}^b & 0 & 0 \\ 0 & \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} \tag{G.3}$$

where $\check{\Omega}_{Ma}{}^b$ is an a priori independent connection for the bosonic part which is only at some parts of the calculation set to either the right or the left mover connection. In order to have covariantly constant structure constants $(\gamma^c_{\boldsymbol{\alpha\beta}}, \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}})$ the latter connection is inadequate and we need to use either one of the first two or s.th. inbetween, an **average connection**, which we denote by

$$\underset{\longleftrightarrow}{\Omega}_{MA}{}^B \equiv \frac{1}{2}\left(\Omega_{MA}{}^B + \hat{\Omega}_{MA}{}^B\right) \tag{G.4}$$

By definition the connections $\Omega_{MA}{}^B$, $\hat{\Omega}_{MA}{}^B$ and $\underset{\longleftrightarrow}{\Omega}_{MA}{}^B$ (but not $\underline{\Omega}_{MA}{}^B$) obey

$$\nabla_M \gamma^c_{\boldsymbol{\alpha\beta}} = \hat{\nabla}_M \gamma^c_{\boldsymbol{\alpha\beta}} = \underset{\longleftrightarrow}{\nabla}_M \gamma^c_{\boldsymbol{\alpha\beta}} = 0 \tag{G.5}$$

$$\nabla_M \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = \hat{\nabla}_M \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = \underset{\longleftrightarrow}{\nabla}_M \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0 \tag{G.6}$$

This relates the three matrix-blocks of the connection components. E.g. for the left-mover connection the spinorial connection $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$(being a sum of scale and Lorentz connection) determines the remaining two blocks (see footnote 7 on page 49 for a derivation):

$$\Omega_{Ma}{}^b = \Omega^{(D)}_M \delta^b_a + \Omega^{(L)}_{Ma}{}^b, \qquad \text{with } \Omega^{(L)}_{Mab} = -\Omega^{(L)}_{Mba} \tag{G.7}$$

$$\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{2}\Omega^{(D)}_M \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{G.8}$$

$$\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{2}\Omega^{(D)}_M \delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{G.9}$$

Please note again that the considerations in the following sections are for a general connection $\Omega_{MA}{}^B$ and not specific to the leftmoving one. In particular the block diagonality and also $\nabla_M \gamma^c_{\boldsymbol{\alpha\beta}} = \nabla_M \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0$ are only used if this is explicitely mentioned.

G.1 Connection in terms of torsion and vielbein (or metric)

A given torsion and vielbein do not determine yet the connection completely. It can be determined by having additional structures (like metric or some group structure constants) that one wants to be covariantly constant. In the case where a metric is present, the connection is uniquely determined by the torsion and the (non)metricity of the metric. Remember the form of the torsion:

$$T^A = \mathbf{d}E^A - E^C \wedge \Omega_C{}^A \tag{G.10}$$
$$T_{[MN]}{}^A = \partial_{[M}E_{N]}{}^A + \Omega_{[MN]}{}^A \tag{G.11}$$

Assume that there is some given symmetric tensor field G_{AB} (call it metric, although it might be degenerate). In flat indices, (non)metricity (metricity for $M_{ABC} = 0$) reads

$$M_{ABC} \equiv \nabla_A G_{BC} = \tag{G.12}$$
$$= E_A{}^M \left(\partial_M G_{BC} - 2\Omega_{M(B|}{}^D G_{D|C)}\right) = \tag{G.13}$$
$$\equiv E_A{}^M \left(\partial_M G_{BC} - 2\Omega_{M(B|C)}\right) \tag{G.14}$$

Here we used G_{AB} to pull down indices, although there might be no inverse to pull indices up. It is quite common that the metric in the comoving frame (i.e. in flat indices) is constant, like the Minkowski metric, and then the derivative part above vanishes. This is, however, not obligatory. In any case, nonmetricity is part of the symmetric part (in the last two indices) of $\Omega_{MB|C}$ only. Let us directly compare (G.14) (solved for the connection term) with (G.11) (rewritten in terms of flat indices and with one index pulled down via G_{AB}

$$\Omega_{A(B|C)} = \frac{1}{2}\left(E_A{}^M \partial_M G_{BC} - M_{ABC}\right) \tag{G.15}$$
$$\Omega_{[AB]|C} = T_{AB|C} - \underbrace{E_A{}^M E_B{}^N \partial_{[M}E_{N]}{}^D G_{DC}}_{(\mathbf{d}E^D)_{AB}G_{DC}} \tag{G.16}$$

From those two equations we can derive the $\Omega_{AB|C}$ without any symmetrization. To this end, write down the antisymmetrized connection three times with permuted indices

$$\Omega_{AB|C} - \Omega_{BA|C} = 2\Omega_{[AB]|C} \tag{G.17}$$
$$\Omega_{BC|A} - \Omega_{CB|A} = 2\Omega_{[BC]|A} \tag{G.18}$$
$$\Omega_{CA|B} - \Omega_{AC|B} = 2\Omega_{[CA]|B} \tag{G.19}$$

Note that

$$\Omega_{AB|C} = -\Omega_{AC|B} + 2\Omega_{A(B|C)} \tag{G.20}$$

and consider $\frac{1}{2}((G.17) + (G.19) - (G.18))$:

$$\Omega_{AB|C} - \Omega_{A(C|B)} + \Omega_{C(B|A)} - \Omega_{B(C|A)} = \Omega_{[AB]|C} + \Omega_{[CA]|B} - \Omega_{[BC]|A} \tag{G.21}$$

or

$$\boxed{\Omega_{AB|C} = \Omega_{[AB]|C} + \Omega_{[CA]|B} - \Omega_{[BC]|A} + \Omega_{A(C|B)} + \Omega_{B(C|A)} - \Omega_{C(B|A)}} \tag{G.22}$$

with $\Omega_{AB|C} \equiv E_A{}^M \Omega_{MB}{}^D G_{DC}$. Now one can plug in (G.15) and (G.16), in order to get the relation to non-metricity and torsion. For our purpose it is, however, more convenient to use only the torsion (G.16) and leave $\Omega_{A(B|C)}$ instead of replacing it by nonmetricity.

$$\boxed{\begin{aligned}\Omega_{AB|C} = {} & T_{AB|C} + T_{CA|B} - T_{BC|A} - (\mathbf{d}E^D)_{AB}G_{DC} - (\mathbf{d}E^D)_{CA}G_{DB} + (\mathbf{d}E^D)_{BC}G_{DA} + \\ & + \Omega_{A(C|B)} + \Omega_{B(C|A)} - \Omega_{C(B|A)}\end{aligned}} \tag{G.23}$$

Some readers might be more familiar with the derivation in curved indices (defining $\Gamma_{MN|K} \equiv \Gamma_{MN}{}^L G_{LK}$):

$$\Gamma_{[MN]|K} = T_{MN|K} \tag{G.24}$$
$$\Gamma_{K(M|N)} = \frac{1}{2}(\partial_K G_{MN} - \underbrace{\nabla_K G_{MN}}_{\equiv M_{KMN}}) \tag{G.25}$$

Equation (G.22) of course holds likewise for the spacetime connection

$$\boxed{\Gamma_{MN|K} = \Gamma_{[MN]|K} + \Gamma_{[KM]|N} - \Gamma_{[NK]|M} + \Gamma_{M(N|K)} + \Gamma_{N(K|M)} - \Gamma_{K(M|N)}} \tag{G.26}$$

This time we replace not only the terms antisymmetrized in the first two indices with the torsion (G.24) but also the terms symmetrized in the last two indices with the (non)metricity (G.25):

$$\boxed{\Gamma_{MN|K} = \frac{1}{2}\left(\partial_M G_{NK} + \partial_N G_{KM} - \partial_K G_{MN}\right) + T_{MN|K} + T_{KM|N} - T_{NK|M} - \frac{1}{2}\left(M_{MNK} + M_{NKM} - M_{KMN}\right)} \tag{G.27}$$

If the metric G_{MN} is nondegenerate, one can raise the index and the connection is completely determined. In ten-dimensional superspace, however, the situation is different as we have a nondegenerate metric only in the bosonic subspace.

Consider finally a second connection

$$\tilde{\Omega}_{MA}{}^B \equiv \Omega_{MA}{}^B + \Delta_{MA}{}^B \tag{G.28}$$

Due to (G.1), we also have

$$\tilde{\Gamma}_{MK}{}^L = \Gamma_{MK}{}^L + \Delta_{MK}{}^L \tag{G.29}$$

$$\Rightarrow \tilde{T}_{MK}{}^L = T_{MK}{}^L + \Delta_{[MK]}{}^L \tag{G.30}$$

The equations (G.22) and (G.26) certainly also hold for Δ:

$$\boxed{\Delta_{AB|C} = \Delta_{[AB]|C} + \Delta_{[CA]|B} - \Delta_{[BC]|A} + \Delta_{A(C|B)} + \Delta_{B(C|A)} - \Delta_{C(B|A)}} \tag{G.31}$$

The vielbein part of (G.23) drops out in the difference of two connections and we get with (G.30)[1]

$$\boxed{\Delta_{AB|C} = (\tilde{T}-T)_{AB|C} + (\tilde{T}-T)_{CA|B} - (\tilde{T}-T)_{BC|A} + \Delta_{A(C|B)} + \Delta_{B(C|A)} - \Delta_{C(B|A)}} \tag{G.32}$$

G.2 Connection in Superspace

At least in the ten dimensional type II superspace, there is no natural nondegenerate superspace metric. Only the bosonic part G_{MN} can be inverted and the remaining undetermined connection coefficients have to be fixed by additional conditions. The expression (G.23) for the structure group connection in flat indices is more appropriate than (G.27), because in flat indeces we have a clear split of the bosonic and fermionic subspace of the tangent space and the only nonvanishing components of the metric G_{AB} is the bosonic (and invertible) metric G_{ab}. The connection is from now on block diagonal of the form $\Omega_{MA}{}^B = \mathrm{diag}\,(\Omega_{Ma}{}^b, \Omega_{m\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \Omega_{m\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}})$. Due to the degeneracy of G_{AB}, equation (G.23) determines only the components $\Omega_{Ab}{}^c$ or equivalently $\Omega_{Mb}{}^c$ of the structure group connection, i.e. those with bosonic Lie algebra indices.

In order to determine the remaining components $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$, we have to give additional information on what properties we want our connection to have. In supergravity it is a reasonable demand that the structure constants of the supersymmetry algebra, i.e. the gamma matrices, are covariantly constant:

$$\nabla_M \gamma^a_{\boldsymbol{\alpha}\boldsymbol{\beta}} \overset{!}{=} 0 \tag{G.33}$$

$$\nabla_M \gamma^a_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} \overset{!}{=} 0 \tag{G.34}$$

This does not only fix uniquely the form of $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ in terms of $\Omega_{Ma}{}^b$, but it also restricts the latter to be the sum of a Lorentz connection and a scale (or dilatation) connection:[2]

$$\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = \frac{1}{4}\Omega_{Ma}{}^b \gamma^a{}_{b\,\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} + \frac{1}{2}\Omega^{(D)}_M \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \tag{G.35}$$

$$\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} = \frac{1}{4}\Omega_{Ma}{}^b \gamma^a{}_{b\,\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} + \frac{1}{2}\Omega^{(D)}_M \delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \tag{G.36}$$

[1]Some of our supergravity constraints will determine $\Delta_{[ab]|c} = -3H_{abc}$, $\Delta_{[\boldsymbol{\alpha}b]|c} = -T_{\boldsymbol{\alpha}b|c}$, $\Delta_{[\hat{\boldsymbol{\alpha}}b]|c} = \hat{T}_{\hat{\boldsymbol{\alpha}}b|c}$, $\Delta_{a(b|c)} = 0$, $\Delta_{\boldsymbol{\alpha}(b|c)} = \nabla_{\boldsymbol{\alpha}}\Phi G_{bc}$ and $\Delta_{\hat{\boldsymbol{\alpha}}(b|c)} = -\hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi G_{bc}$, so that the difference tensor reads

$$\Delta_{ab|c} = -3H_{abc} \quad (= -2T_{ab|c} = 2\hat{T}_{ab|c})$$

$$\Delta_{\boldsymbol{\alpha}b|c} = -2T_{\boldsymbol{\alpha}[b|c]} + \nabla_{\boldsymbol{\alpha}}\Phi G_{bc} = -2T_{\boldsymbol{\alpha}b|c}$$

$$\Delta_{\hat{\boldsymbol{\alpha}}b|c} = 2\hat{T}_{\hat{\boldsymbol{\alpha}}[b|c]} - \hat{\nabla}_{\hat{\boldsymbol{\alpha}}}\Phi G_{bc} = 2\hat{T}_{\hat{\boldsymbol{\alpha}}b|c} \qquad \diamond$$

[2]Let us give at this point only a short argument for this. According to (D.2)-(D.4) we have schematically $\Gamma^{[k]}\Gamma^{[1]} \propto \Gamma^{[|k-1|]} + \Gamma^{[k+1]}\quad \forall k$, if $\Gamma^{[k]}$ denotes a term proportional to a completely antisymmetrized product of k gamma matrices. Let us restrict now to ten dimensions. The same schematic equation then holds for the chiral submatrices $\gamma^{[k]}$. The connection can due to its index structure be expanded in even antisymmetrized products:

$$\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} \propto \gamma^{[0]} + \gamma^{[2]} + \gamma^{[4]}$$

When this connection acts on another gamma matrix, we get schematically

$$\Omega_{M[\boldsymbol{\alpha}|}{}^{\gamma}\gamma^c_{\gamma|\boldsymbol{\beta}]} \propto (\gamma^{[0]} + \gamma^{[2]} + \gamma^{[4]})\gamma^{[1]} \propto \gamma^{[1]} + (\gamma^{[1]} + \underbrace{\gamma^{[3]}) + (\gamma^{[3]}}_{0} + \underbrace{\gamma^{[5]}}_{0})$$

with

$$\Omega_{Ma}{}^{b} \equiv \underbrace{\Omega_{M[ac]}G^{cb}}_{\equiv\Omega^{(L)}_{Ma}{}^{b}} + \Omega^{(D)}_{M}\delta^{b}_{a} \tag{G.37}$$

Because of the split in Lorentz and scale connection, the block-diagonality of the structure group and the degeneracy of the superspace metric, equation (G.23) can be rewritten as

$$\Omega_{Ab|c} = T_{Ab|c} + T_{cA|b} - T_{bc|A} - (\mathbf{d}E^{d})_{Ab}G_{dc} - (\mathbf{d}E^{d})_{cA}G_{db} + (\mathbf{d}E^{d})_{bc}G_{dA} + \Omega^{(D)}_{A}G_{cb} + \Omega^{(D)}_{b}G_{cA} - \Omega^{(D)}_{c}G_{bA} \tag{G.38}$$

or

$$\begin{aligned}
\Omega_{ab|c} &= T_{ab|c} + T_{ca|b} - T_{bc|a} - (\mathbf{d}E^{d})_{ab}G_{dc} - (\mathbf{d}E^{d})_{ca}G_{db} + (\mathbf{d}E^{d})_{bc}G_{da} + \Omega^{(D)}_{a}G_{cb} + \Omega^{(D)}_{b}G_{ca} - \Omega^{(D)}_{c}G_{ba} && \text{(G.39)}\\
\Omega_{\boldsymbol{\alpha}b|c} &= T_{\boldsymbol{\alpha}b|c} + T_{c\boldsymbol{\alpha}|b} - (\mathbf{d}E^{d})_{\boldsymbol{\alpha}b}G_{dc} - (\mathbf{d}E^{d})_{c\boldsymbol{\alpha}}G_{db} + \Omega^{(D)}_{\boldsymbol{\alpha}}G_{cb} && \text{(G.40)}\\
\Omega_{\hat{\boldsymbol{\alpha}}b|c} &= T_{\hat{\boldsymbol{\alpha}}b|c} + T_{c\hat{\boldsymbol{\alpha}}|b} - (\mathbf{d}E^{d})_{\hat{\boldsymbol{\alpha}}b}G_{dc} - (\mathbf{d}E^{d})_{c\hat{\boldsymbol{\alpha}}}G_{db} + \Omega^{(D)}_{\hat{\boldsymbol{\alpha}}}G_{cb} && \text{(G.41)}
\end{aligned}$$

which determines $\Omega_{Ma}{}^{b}$ via

$$\Omega_{Ma}{}^{b} = E_{M}{}^{C}\Omega_{Ca|d}G^{db} \quad \text{with } G_{ac}G^{cb} \equiv \delta^{b}_{a} \tag{G.42}$$

The remaining components $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$ and $\Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}}$ are then fixed via (G.35) and (G.36).

Let us in the following calculate $\Omega_{Ma}{}^{b}$ more explicitely in the WZ gauge in order to extract the Levi Civita connection of the bosonic subspace.

G.3 Extracting Levi Civita from whole superspace connection (in WZ-gauge)

Remember our definition $G_{MN} = E_{M}{}^{a}\underbrace{e^{2\Phi}\eta_{ab}}_{G_{ab}}E_{N}{}^{b}$ in the application to the Berkovits string and the Wess Zumino gauge (H.76,H.77,H.92):

$$E_{M}{}^{A}\big|_{\vec{\boldsymbol{\theta}}=0} = \begin{pmatrix} e_{m}{}^{a} & \psi_{m}{}^{\boldsymbol{\alpha}} & \hat{\psi}_{m}{}^{\hat{\boldsymbol{\alpha}}} \\ 0 & \delta_{\boldsymbol{\mu}}{}^{\boldsymbol{\alpha}} & 0 \\ 0 & 0 & \delta_{\hat{\boldsymbol{\mu}}}{}^{\hat{\boldsymbol{\alpha}}} \end{pmatrix}, \quad E_{A}{}^{M}\big| = \begin{pmatrix} e_{a}{}^{m} & -\psi_{a}{}^{\boldsymbol{\mu}} & -\hat{\psi}_{a}{}^{\hat{\boldsymbol{\mu}}} \\ 0 & \delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\mu}} & 0 \\ 0 & 0 & \delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\mu}}} \end{pmatrix}, \quad \Omega_{\boldsymbol{\mathcal{M}}A}{}^{B}\big| = 0 \tag{G.43}$$

$$\text{with} \quad e_{m}{}^{a}e_{a}{}^{n} = \delta^{n}_{m}, \quad \psi_{a}{}^{\boldsymbol{\mu}} \equiv e_{a}{}^{m}\psi_{m}{}^{\boldsymbol{\alpha}}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\mu}}, \quad \hat{\psi}_{a}{}^{\hat{\boldsymbol{\mu}}} \equiv e_{a}{}^{m}\psi_{m}{}^{\hat{\boldsymbol{\alpha}}}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\mu}}}$$

As bosonic metric, we could either take just the leading component in the $\vec{\boldsymbol{\theta}}$-expansion of G_{mn}, or the one given by the bosonic vielbein $e_{m}{}^{a}$ and the Minkowski metric:

$$\tilde{g}_{mn} \equiv G_{mn}\big| = e_{m}{}^{a}\underbrace{e^{2\phi}\eta_{ab}}_{\tilde{g}_{ab}}e_{n}{}^{b}, \qquad g_{mn} \equiv e_{m}{}^{a}\eta_{ab}e_{n}{}^{b} = e^{-2\phi}\tilde{g}_{mn} \tag{G.44}$$

The first is naturally induced by the superspace 'metric', while the second is by construction covariantly conserved with respect to the connection $\omega_{ma}{}^{b} \equiv \Omega_{ma}{}^{b}\big|$ (in contrast to $\tilde{g}_{mn}$ because of the scaling compensator field ϕ). We want to write the superspace connection at $\vec{\boldsymbol{\theta}} = 0$ as the Levi Civita connection w.r.t. $\tilde{g}_{mn}$ or g_{mn} plus additional terms.

The superspace connection was derived above starting from (G.22) or (G.23), arriving at the equations (G.39-G.41) for $\Omega_{ab|c}, \Omega_{\boldsymbol{\alpha}b|c}$ and $\Omega_{\hat{\boldsymbol{\alpha}}b|c}$ in terms of the torsion and the exterior derivative of the supervielbein $\mathbf{d}E^{d}$. We can also use the general equation (G.23), in order to determine the form of the Levi Civita connection for g_{mn} in terms of the bosonic vielbein. We just have to set the torsion and the symmetric part to zero. However, as we already use the supervielbein in order to switch from flat to curved indices and vice versa, we better should write the bosonic vielbeins explicitely in the resulting equation:

$$\boxed{e_{a}{}^{m}\omega^{LC\,d}_{mb}[g]\cdot\eta_{dc} = -e_{a}{}^{m}e_{b}{}^{n}(\mathbf{d}e^{d})_{mn}\eta_{dc} - e_{c}{}^{m}e_{a}{}^{n}(\mathbf{d}e^{d})_{mn}\eta_{db} + e_{b}{}^{m}e_{c}{}^{n}(\mathbf{d}e^{d})_{mn}\eta_{da}} \tag{G.45}$$

The $\gamma^{[3]}$-parts vanish due to the graded antisymmetrization of the indices. The $\gamma^{[1]}$ parts are fine because they can be absorbed by acting with the bosonic connection on the bosonic index. Only the $\gamma^{[5]}$ part remains and cannot be removed. As it stems from the $\gamma^{[4]}$-part in $\Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}$, we conclude that the corresponding coefficient has to vanish and only scale and Lorentz connection remain. The sketched argumentation can be done rigorously which leads to the stated results for the relation between bosonic and fermionic connection. ◇

For the metric $\tilde{g}_{mn}$ instead, the symmetric part of the Levi Civita connection is no longer zero. We still have torsionlessness and metric compatibility as characterizing properties. The latter condition implies via (G.14) that

$$\omega^{(LC)}_{m(b|c)}[\tilde{g}] = \frac{1}{2}\partial_m \tilde{g}_{bc} = \partial_m \phi \cdot \tilde{g}_{bc} \tag{G.46}$$

Using again (G.23) with vanishing torsion, we arrive at

$$\boxed{\begin{aligned} e_a{}^m \omega_{mb}^{LC\,d}[\tilde{g}] \cdot \tilde{g}_{dc} \quad = \quad & -e_a{}^m e_b{}^n (\mathbf{d}e^d)_{mn} \tilde{g}_{dc} - e_c{}^m e_a{}^n (\mathbf{d}e^d)_{mn} \tilde{g}_{db} + e_b{}^m e_c{}^n (\mathbf{d}e^d)_{mn} \tilde{g}_{da} + \\ & + e_a{}^m \partial_m \phi \cdot \tilde{g}_{bc} + e_b{}^m \partial_m \phi \cdot \tilde{g}_{ca} - e_c{}^m \partial_m \phi \cdot \tilde{g}_{ab} \end{aligned}} \tag{G.47}$$

In both cases (for $\tilde{g}$ and g) the corresponding Levi Civita connection is certainly sitting in the superspace connection in the terms with $\mathbf{d}E^d$ in (G.39-G.41) at $\vec{\boldsymbol{\theta}} = 0$. Indeed one can write[3]

$$(\mathbf{d}E^a)_{mn}| \quad = \quad (\mathbf{d}e^a)_{mn} \tag{G.48}$$

$$(\mathbf{d}E^a)_{\boldsymbol{\mathcal{M}}N}| \quad = \quad T_{\boldsymbol{\mathcal{M}}N}{}^a| \tag{G.49}$$

This is consistent with the fact that $\Omega_{\boldsymbol{\alpha}b|c}$ and $\Omega_{\hat{\boldsymbol{\alpha}}b|c}$ as given in (G.40) and (G.41) vanish at $\vec{\boldsymbol{\theta}} = 0$ in the WZ-gauge (where $E_{\boldsymbol{\alpha}}{}^M\big| = \delta_{\boldsymbol{\alpha}}{}^M$ and $\Omega_{\boldsymbol{\mu}A}{}^B\big| = 0$. In order to calculate $\Omega_{ab|c}\big|$ as given in (G.39), we need the the exterior derivative of the vielbein (as given above) with flat bosonic indices. As the constraints on the torsion components will also be given in flat indices, we will express everything in terms of torsion components with flat indices:

$$\begin{aligned} (\mathbf{d}E^d)_{ab}\big| \quad &= \quad e_a{}^m e_b{}^n (\mathbf{d}e^d)_{mn} - 2\psi_{[a}{}^{\boldsymbol{\mathcal{M}}} e_{b]}{}^n\, T_{\boldsymbol{\mathcal{M}}n}{}^d\big| + \psi_a{}^{\boldsymbol{\mathcal{M}}} \psi_b{}^{\boldsymbol{\mathcal{N}}}\, T_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}}{}^d\big| = & \text{(G.50)} \\ &= \quad e_a{}^m e_b{}^n \left((\mathbf{d}e^d)_{mn} + \psi_m{}^{\boldsymbol{\mathcal{A}}} \psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^d\big| \right) - 2e_{[a|}{}^m \psi_m{}^{\boldsymbol{\mathcal{A}}} e_{b]}{}^n \left(e_n{}^c\, T_{\boldsymbol{\mathcal{A}}c}{}^d\big| + \psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^d\big| \right) = & \text{(G.51)} \\ &= \quad e_a{}^m e_b{}^n \left((\mathbf{d}e^d)_{mn} - \psi_m{}^{\boldsymbol{\mathcal{A}}} \psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^d\big| \right) - 2e_{[a|}{}^m \psi_m{}^{\boldsymbol{\mathcal{A}}}\, T_{\boldsymbol{\mathcal{A}}|b]}{}^d\big| & \text{(G.52)} \end{aligned}$$

Plugging this result into (G.39) yields $\Omega_{ab|c}$ at $\vec{\boldsymbol{\theta}} = 0$ in terms of torsion components with flat indices and derivatives of the bosonic vielbein only:

$$\begin{aligned} \Omega_{ab|c}\big| \quad = \quad & T_{ab|c}\big| + T_{ca|b}\big| - T_{bc|a}\big| - \left(e_a{}^m e_b{}^n \left((\mathbf{d}e^d)_{mn} \tilde{g}_{dc} - \psi_m{}^{\boldsymbol{\mathcal{A}}} \psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}|c}\big| \right) - 2e_{[a|}{}^m \psi_m{}^{\boldsymbol{\mathcal{A}}}\, T_{\boldsymbol{\mathcal{A}}|b]c}\big| \right) + \\ & - \left(e_c{}^m e_a{}^n \left((\mathbf{d}e^d)_{mn} \tilde{g}_{db} - \psi_m{}^{\boldsymbol{\mathcal{A}}} \psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}|b}\big| \right) - 2e_{[c|}{}^m \psi_m{}^{\boldsymbol{\mathcal{A}}}\, T_{\boldsymbol{\mathcal{A}}|a]b}\big| \right) + \\ & + \left(e_b{}^m e_c{}^n \left((\mathbf{d}e^d)_{mn} \tilde{g}_{da} - \psi_m{}^{\boldsymbol{\mathcal{A}}} \psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}|a}\big| \right) - 2e_{[b|}{}^m \psi_m{}^{\boldsymbol{\mathcal{A}}}\, T_{\boldsymbol{\mathcal{A}}|c]a}\big| \right) + \\ & + \Omega_a\big|\, \tilde{g}_{cb} + \Omega_b\big|\, \tilde{g}_{ca} - \Omega_c\big|\, \tilde{g}_{ba} \end{aligned} \tag{G.53}$$

Now we can express everything in terms of the Levi Civita connection w.r.t. $\tilde{g}$ (G.47), torsion terms with flat indices and covariant derivatives of the compensator field:

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[3]In the Wess Zumino gauge we can express $\mathbf{d}E^a|$ by $\mathbf{d}e^a$ plus torsion terms as we will demonstrate now. First we have

$$\mathbf{d}E^a| \quad = \quad \overbrace{\partial_{[m} E_{n]}{}^a\big|\, \mathbf{d}x^m \mathbf{d}x^n}^{\mathbf{d}e^a} + 2\, \partial_{[m} E_{\boldsymbol{\mathcal{N}}]}{}^a\big|\, \mathbf{d}x^m \mathbf{d}x^{\boldsymbol{\mathcal{N}}} + \partial_{[\boldsymbol{\mathcal{M}}} E_{\boldsymbol{\mathcal{N}}]}\big|\, \mathbf{d}x^{\boldsymbol{\mathcal{M}}} \mathbf{d}x^{\boldsymbol{\mathcal{N}}}$$

As $E_m{}^a| = e_m{}^a$, we have

$$(\mathbf{d}E^a)_{mn}| = (\mathbf{d}e^a)_{mn}$$

Now remember the definition of the torsion $T^A = \mathbf{d}E^A - E^B \wedge \Omega_B{}^A$ which reads for fermionic form indices at $\vec{\boldsymbol{\theta}} = 0$ in the Wess-Zumino gauge (H.95,H.96):

$$\partial_{[\boldsymbol{\mathcal{M}}} E_{\boldsymbol{\mathcal{N}}]}{}^A\Big| \quad = \quad T_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}}{}^A\Big| - \Omega_{[\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}]}{}^A\Big| \overset{(H.96)}{=} T_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}}{}^A\Big|$$

Similarly we have

$$\partial_{[\boldsymbol{\mathcal{M}}} E_{n]}{}^A\Big| \quad = \quad T_{\boldsymbol{\mathcal{M}}n}{}^A\Big| - \Omega_{[\boldsymbol{\mathcal{M}}n]}{}^A\Big| \overset{(H.96)}{=} T_{\boldsymbol{\mathcal{M}}n}{}^A\Big| + \frac{1}{2}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{B}}}\, \Omega_{n\boldsymbol{\mathcal{B}}}{}^A\Big|$$

For $A = a$, we can thus write in summary

$$(\mathbf{d}E^a)_{\boldsymbol{\mathcal{M}}N}| \quad = \quad T_{\boldsymbol{\mathcal{M}}N}{}^a| \qquad \diamond$$

$$\begin{aligned}
\Omega_{ab|c}\big| &= e_a{}^m \omega^{LC\,d}_{mb}[\tilde g]\tilde g_{dc} + T_{ab|c}\big| + T_{ca|b}\big| - T_{bc|a}\big| + \\
&- \underbrace{(e_a{}^m\partial_m\phi - \Omega_a|)}_{\nabla_a\Phi|+\psi_a{}^{\mathcal{M}}\ \underbrace{(\partial_{\mathcal{M}}\Phi)|}_{\delta_{\mathcal{M}}{}^{\mathcal{A}}(\nabla_{\mathcal{A}}\Phi)|}}\ \tilde g_{cb} - (e_b{}^m\partial_m\phi - \Omega_b|)\,\tilde g_{ca} + (e_c{}^m\partial_m\phi - \Omega_c|)\,\tilde g_{ba} + \\
&+ (e_a{}^m e_b{}^n \tilde g_{cd} + e_c{}^m e_a{}^n \tilde g_{bd} - e_b{}^m e_c{}^n \tilde g_{ad})\,\psi_m{}^{\mathcal{A}}\psi_n{}^{\mathcal{B}}\, T_{\mathcal{AB}}{}^d\big| \\
&+ 2e_{[a|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}|b]c}\big| + 2e_{[c|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}|a]b}\big| - 2e_{[b|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}|c]a}\big|
\end{aligned} \tag{G.54}$$

While for the use of $\omega^{LC\,d}_{mb}[\tilde g]$ above the partial derivatives of the compensator ϕ combine with the scale connections to covariant derivatives, either the scale connections or the partial derivatives remain explicitely for the use of $\omega^{LC\,d}_{mb}[g]$ (G.45). In summary we have for the two cases

$$\boxed{\begin{aligned}
\underbrace{\Omega_{ab|c}\big|}_{e_a{}^m\omega_{mb}{}^d e^{2\phi}\eta_{dc}} &= e_a{}^m \omega^{LC\,d}_{mb}[\tilde g]\tilde g_{dc} + 2\,T_{a[b|c]}\big| - T_{bc|a}\big| + \\
&-2\,\nabla_{[b}\Phi\big|\,\tilde g_{c]a} - \nabla_a\Phi\big|\,\tilde g_{bc} - \left(2e_{[b}{}^m\tilde g_{c]a} + e_a{}^m\tilde g_{bc}\right)\psi_m{}^{\mathcal{A}}\,(\nabla_{\mathcal{A}}\Phi)| + \\
&+ \left(2e_a{}^m e_{[b}{}^n\tilde g_{c]d} - e_{[b}{}^m e_{c]}{}^n\tilde g_{ad}\right)\psi_m{}^{\mathcal{A}}\psi_n{}^{\mathcal{B}}\, T_{\mathcal{AB}}{}^d\big| \\
&+2e_a{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}[b|c]}\big| - 2e_{[b|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}a|c]}\big| - 2e_{[b|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}|c]a}\big| = \qquad (G.55)\\
&= e_a{}^m\omega^{LC\,d}_{mb}[g]\eta_{dc}e^{2\phi} + 2\,T_{a[b|c]}\big| - T_{bc|a}\big| + \\
&-2(\nabla_{[b|}\Phi\big| - e_{[b|}{}^n\partial_n\phi)\eta_{c]a} - (\nabla_a\Phi\big| - e_a{}^n\partial_n\phi)\eta_{bc} - \left(2e_{[b}{}^m\tilde g_{c]a} + e_a{}^m\tilde g_{bc}\right)\psi_m{}^{\mathcal{A}}\,(\nabla_{\mathcal{A}}\Phi)| + \\
&+e^{2\phi}\left(2e_a{}^m e_{[b}{}^n\eta_{c]d} - e_{[b}{}^m e_{c]}{}^n\eta_{ad}\right)\psi_m{}^{\mathcal{A}}\psi_n{}^{\mathcal{B}}\, T_{\mathcal{AB}}{}^d\big| + \\
&+2e_a{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}[b|c]}\big| - 2e_{[b|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}a|c]}\big| - 2e_{[b|}{}^m\psi_m{}^{\mathcal{A}}\, T_{\mathcal{A}|c]a}\big| \qquad (G.56)
\end{aligned}}$$

We have written the terms in a way that one can clearly distinguish between terms anti-symmetric in b,c (Lorentz-part) and terms symmetric in b,c (scale-part). In the second version (G.56), the whole second line could be written as $+2\,\Omega^{(D)}_{[b}\Big|\,e^{2\phi}\eta_{c]a} + \Omega^{(D)}_a\Big|\,e^{2\phi}\eta_{bc}$ which is, however, less convenient for plugging the constraints into it. The Levi Civita connection $\omega^{LC\,d}_{mb}[g]$ does not transform under scale transformations in the way it should, which is repaired by the non-covariantly transforming partial derivatives $\partial_k\phi$. They are thus the minimal extension of the Levi-Civita connection to make it transforming properly under the whole structure group. Combining these terms with $\omega^{LC\,d}_{mb}[g]$ just leads back to $\omega^{LC\,d}_{mb}[\tilde g]$ which apparently has a scale part. This seems strange for a Levi Civita connection, but is only true in the frame $e_m{}^a$ where the flat metric is not Minkowski.

Assuming that $\nabla_M\gamma^a_{\boldsymbol{\alpha\beta}} = \nabla_M\gamma^a_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}} = 0$, we can finally (according to (G.35) and (G.36)) write down the connection when acting on fermionic indices. We restrict to the version with the Levi Civita action for $g_{mn} = e_m{}^a\eta_{ab}e_n{}^b$:

$$\begin{aligned}
\Omega_{m\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}| &= \frac14\omega_{m[b|c]}\tilde\gamma^{bc}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}} + \frac12\omega^{(D)}_m\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}} = \\
&= \frac14 e_m{}^a\Big\{e_a{}^n\omega^{LC\,d}_{n[b|}[g]\eta_{d|c]} + 2e^{-2\phi}\,T_{a[b|c]}\big| - e^{-2\phi}\,T_{bc|a}\big| + \\
&-2(\nabla_{[b|}\Phi\big| - e_{[b|}{}^n\partial_n\phi)\eta_{c]a} - 2e_{[b}{}^m\eta_{c]a}\psi_m{}^{\mathcal{A}}\,(\nabla_{\mathcal{A}}\Phi)| + \\
&+ \left(2e_a{}^k e_{[b}{}^n\eta_{c]d} - e_b{}^k e_c{}^n\eta_{ad}\right)\psi_k{}^{\mathcal{A}}\psi_n{}^{\mathcal{B}}\, T_{\mathcal{AB}}{}^d\big| + \\
&+e^{-2\phi}\Big(2e_a{}^n\psi_n{}^{\mathcal{A}}\, T_{\mathcal{A}[b|c]}\big| \underbrace{-2e_b{}^n\psi_n{}^{\mathcal{A}}\, T_{\mathcal{A}(a|c)}\big| + 2e_c{}^n\psi_n{}^{\mathcal{A}}\, T_{\mathcal{A}(a|b)}\big|}_{-2e_{[b|}{}^n\psi_n{}^{\mathcal{A}}T_{\mathcal{A}a|c]}\big|-2e_{[b|}{}^n\psi_n{}^{\mathcal{A}}T_{\mathcal{A}|c]a}\big|}\Big)\Big\}\gamma^{bc}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}} \\
&-\frac12\underbrace{\left(\nabla_a\Phi\big| - e_a{}^n\partial_n\phi + e_a{}^m\psi_m{}^{\mathcal{A}}\,(\nabla_{\mathcal{A}}\Phi)|\right)}_{\equiv-\omega^{(D)}_m=-\,\Omega^{(D)}_m|}\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}
\end{aligned} \tag{G.57}$$

An equivalent expression with $\gamma^{bc}{}_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}$ and $\delta_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}$ replaced by $\gamma^{bc}{}_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}$ and $\delta_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}$ is obtained for $\Omega_{m\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\alpha}}}$.

A second useful way to write the connection $\Omega_{ab|c}\big|$ is to bring it to a form which is the bosonic version of (G.23) and from which we can read off the bosonic torsion and nonmetricity. To this end, we rewrite (G.50) as

$$(\mathbf{d}E^d)_{ab}\big| = e_a{}^m e_b{}^n\left((\mathbf{d}e^d)_{mn} - T_{mn}{}^d\right) + T_{ab}{}^d\big| \tag{G.58}$$

Plugging this into (G.39) yields

$$\boxed{\begin{aligned}\underbrace{\Omega_{ab|c}\big|}_{e_a{}^m\omega_{mb|c}} &= -e_a{}^m e_b{}^n(\mathbf{d}e^d)_{mn}\tilde{g}_{dc} - e_c{}^m e_a{}^n(\mathbf{d}e^d)_{mn}\tilde{g}_{db} + e_b{}^m e_c{}^n(\mathbf{d}e^d)_{mn}\tilde{g}_{da} + \\ &\quad + e_a{}^m e_b{}^n\, T_{mn}{}^d\big|\,\tilde{g}_{dc} + e_c{}^m e_a{}^n\, T_{mn}{}^d\big|\,\tilde{g}_{db} - e_b{}^m e_c{}^n\, T_{mn}{}^d\big|\,\tilde{g}_{da} \\ &\quad + \Omega_a^{(D)}\Big|\,\tilde{g}_{cb} + \Omega_b^{(D)}\Big|\,\tilde{g}_{ca} - \Omega_c^{(D)}\Big|\,\tilde{g}_{ba}\end{aligned}} \tag{G.59}$$

As we have in the Wess-Zumino gauge $\Omega_{mb}{}^e\big| = e_m{}^a\omega_{ab}{}^e$, the obtained equation is simply the bosonic version of (G.23) with $\omega_{ma}^{(D)b} = \Omega_m^{(D)}\Big|\,\delta_a^b$. The bosonic torsion coincides with $T_{mn}{}^d\big|$.

$$T_{mn}{}^d\big| \quad = \quad e_m{}^a e_n{}^b\, T_{ab}{}^d\big| + 2e_{[m}{}^a\psi_{n]}{}^{\boldsymbol{\mathcal{B}}}\, T_{a\boldsymbol{\mathcal{B}}}{}^d\big| + \psi_m{}^{\boldsymbol{\mathcal{A}}}\psi_n{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{B}}}{}^d\big| \tag{G.60}$$

Appendix H

Supergauge Transformations, their Algebra and the Wess Zumino Gauge

This appendix contains, like most of the others, considerations which are valid not only for our application to the Berkovits string in ten dimensions, but as well for other dimensions and for different supergravity theories. The curved indices M as well as the flat indices A contain bosonic indices m or a as well as fermionic indices $\boldsymbol{\mathcal{M}}$ or $\boldsymbol{\mathcal{A}}$. For extended supersymmetry the latter are further split into several irreducible fermionic indices. E.g. for type II in ten dimensions (our application) we have $\boldsymbol{\mathcal{M}}=(\boldsymbol{\mu},\hat{\boldsymbol{\mu}})$ and $\boldsymbol{\mathcal{A}}=(\boldsymbol{\alpha},\hat{\boldsymbol{\alpha}})$ where $\hat{\boldsymbol{\alpha}}$ is either of the same or of opposite chirality as $\boldsymbol{\alpha}$. We only assume the presence of a (super)vielbein $E_M{}^A$ and of a (super)connection $\Omega_{MA}{}^B$ in the supergravity theory. Discussions of other fields (like the B-field) are of course only relevant for theories containing these fields.

The supergravity transformation (local supersymmetry) is in some sense a special class of superdiffeomorphism transformations. If the general superdiffeomorphisms are parametrized by a vector field $\xi^A(\vec{x}\,) \equiv \xi^A(x,\vec{\boldsymbol{\theta}})$, the local supersymmetry will be parametrized by only $\xi^{\boldsymbol{\mathcal{A}}}(x,0)$. Likewise, general coordinate transformations in the bosonic submanifold are parametrized by $\xi^a(x,0)$, while all the higher $\vec{\boldsymbol{\theta}}$-components of ξ^A correspond to additional auxiliary gauge degrees of freedom. Similarly, the local structure group transformations $L_{ab}(\vec{x}\,)$ (e.g. Lorentz-transformations or in our application also scale transformations) have auxiliary gauge degrees in the higher $\vec{\boldsymbol{\theta}}$-parts. Following roughly [17, p.127-144], we want to bring e.g. the vielbein into a particular form, using (and thereby fixing) some of those shift symmetries, and to identify the bosonic spacetime diffeomorphisms and the local supersymmetry transformations with the bosonic and fermionic stabilizers of this (Wess-Zumino-like) gauge respectively. But let us at first have a look at the general transformation properties of the superfields.

H.1 Supergauge transformations of the superfields

H.1.1 Infinitesimal form

In the following, we make frequent use of some structure group connection $\Omega_{MA}{}^B$ and the corresponding covariant derivative ∇_M. As long as nothing else is announced, the equations are valid for any connection (in particular, it is not meant to be the left-moving connection only).

Transformation of a general tensor field We are interested in a combination of an infinitesimal superdiffeomorphism transformation (or better the corresponding Lie derivative) and a local structure group transformation. For an object with only curved indices, the transformation reduces to the Lie derivative. The Lie derivative of a vector field $\vec{v} \equiv v^M \boldsymbol{\partial}_M$ e.g. reads as usual

$$\boldsymbol{\mathcal{L}}_{\vec{\xi}} v^M \equiv (\boldsymbol{\mathcal{L}}_{\vec{\xi}} \vec{v}\,)^M = \tag{H.1}$$

$$= \xi^K \partial_K v^M - \partial_K \xi^M v^K \tag{H.2}$$

It can be rewritten in terms of covariant derivatives as

$$\boldsymbol{\mathcal{L}}_{\vec{\xi}} v^M = \xi^K \nabla_K v^M - \nabla_K \xi^M v^K - 2\xi^K T_{KL}{}^M v^L \tag{H.3}$$

For one-forms the covariant expression of the Lie derivative contains a torsion term with opposite sign:

$$\boldsymbol{\mathcal{L}}_{\vec{\xi}} \omega_M \equiv \left(\boldsymbol{\mathcal{L}}_{\vec{\xi}}(\omega_N \mathbf{d}x^N)\right)_M \tag{H.4}$$

$$= \xi^K \partial_K \omega_M + \partial_M \xi^K \omega_K = \tag{H.5}$$

$$= \xi^K \nabla_K \omega_M + \nabla_M \xi^K \omega_K + 2\xi^K T_{KM}{}^L \omega_L \tag{H.6}$$

In contrast to the above, it is convenient for objects with flat indices, not to consider them as being contracted with basis elements, when acting with the Lie derivative, but to really only act on the component functions, which transform like scalars under diffeomorphisms[1].

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\vec{\xi}} v^A &= \xi^K \partial_K v^A = && \text{(H.7)} \\ &= \xi^K \nabla_K v^A - \xi^K \Omega_{KB}{}^A v^B && \text{(H.8)} \end{aligned}$$

This is a covariant object from the diffeomorphism point of view, but the connection transforms inhomogenously under the structure group transformations. The entire gauge transformation of v^A, however, contains also a local structure group transformation:

$$\delta v^A = \boldsymbol{\mathcal{L}}_{\vec{\xi}} v^A + \tilde{L}_B{}^A v^B \tag{H.9}$$

As the structure group connection itself is Lie algebra valued, the second term in (H.8) can be absorbed in the structure group transformation:

$$L_B{}^A \equiv \tilde{L}_B{}^A - \xi^K \Omega_{KB}{}^A \tag{H.10}$$

The combined diffeomorphism and local structure group transformation can thus be written as

$$\delta v^A = \xi^K \nabla_K v^A + L_B{}^A v^B \tag{H.11}$$

The first term is a covariantized (w.r.t. the structure group) version of the Lie derivative (H.7), and we will therefore denote it by

$$\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} v^A \equiv \xi^K \nabla_K v^A \tag{H.12}$$

In general $\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}}$ will be defined as the $L_A{}^B = 0$ part of the complete transformation, i.e. a Lie derivative w.r.t. $\vec{\xi}$, accompanied by a structure group transformation with $\tilde{L}_A{}^B = \xi^K \Omega_{KA}{}^B$ whose representation we denote with $\mathcal{R}\left(\tilde{L}_{\cdot}{}^{\cdot}\right)$ (see also before (F.29) on page 191):

$$\boxed{\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} \equiv \boldsymbol{\mathcal{L}}_{\vec{\xi}} + \mathcal{R}\left(\xi^K \Omega_{K\cdot}{}^{\cdot}\right)} \tag{H.13}$$

[1]Note the (common) convention used in (H.1) to define $\boldsymbol{\mathcal{L}}_{\vec{\xi}} v^M$ as the M-th component of the Lie derivative of $\vec{v}$ and not the Lie derivative of the M-th component function! This convention is extended to objects with an arbitrary number of curved indices, i.e.

$$\boldsymbol{\mathcal{L}}_{\vec{\xi}} t_{M_1 \ldots M_p}^{N_1 \ldots N_q} \equiv \left(\boldsymbol{\mathcal{L}}_{\vec{\xi}} (t_{K_1 \ldots K_p}^{L_1 \ldots L_q} \mathbf{d}x^{K_1} \otimes \ldots \otimes \mathbf{d}x^{K_p} \otimes \boldsymbol{\partial}_{L_1} \otimes \ldots \otimes \boldsymbol{\partial}_{L_q})\right)_{M_1 \ldots M_p}^{N_1 \ldots N_q}$$

In cases where we want to act explicitely on e.g. the component functions, we can denote it with e.g. $\boldsymbol{\mathcal{L}}_{\vec{\xi}}(v^M) = \xi^K \partial_K v^M$. This is of course not the component of a tensor, but it makes sense in calculations like $\boldsymbol{\mathcal{L}}_{\vec{\xi}}(v^M \boldsymbol{\partial}_M) = \boldsymbol{\mathcal{L}}_{\vec{\xi}}(v^M) \cdot \boldsymbol{\partial}_M + v^M \boldsymbol{\mathcal{L}}_{\vec{\xi}}(\boldsymbol{\partial}_M)$. From the Lie derivatives for general vectors (H.2) and one forms (H.5) we can in turn read off the transformation of the basis elements

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\vec{\xi}}(\boldsymbol{\partial}_M) &= -\partial_M \xi^N \, \boldsymbol{\partial}_N \\ \boldsymbol{\mathcal{L}}_{\vec{\xi}}(\mathbf{d}x^M) &= \partial_N \xi^M \, \mathbf{d}x^N \end{aligned}$$

For flat indices, however, we use just the opposite convention, i.e. we do not regard the flat index to be contracted with any basis element when acting with the Lie derivative. The action on an object with both, flat and curved indices will thus be defined as follows

$$\boldsymbol{\mathcal{L}}_{\vec{\xi}} t_{MA}^{NB} \equiv \left(\boldsymbol{\mathcal{L}}_{\vec{\xi}} (t_{KA}^{LB} \mathbf{d}x^K \otimes \boldsymbol{\partial}_L)\right)_M^N$$

In cases where we want to calculate something different we will use a more explicit notation like on the righthand side in the above equation. The reason for this convention is the following. Starting in a coordinate basis, it is natural to express the transformed tensor in the coordinate basis again, while if one starts in a non-coordinate frame e_A, it is more natural to express the result in the transformed basis:

$$\tilde{v} \equiv v + \boldsymbol{\mathcal{L}}_{\vec{\xi}} v = v + \boldsymbol{\mathcal{L}}_{\vec{\xi}} v^A \cdot e_A + v^A \boldsymbol{\mathcal{L}}_{\vec{\xi}} e_A = \left(v^A + \boldsymbol{\mathcal{L}}_{\vec{\xi}} v^A\right) \underbrace{\left(e_A + \boldsymbol{\mathcal{L}}_{\vec{\xi}} e_A\right)}_{\tilde{e}_A}$$

Let us finally give the Lie derivative of the local vielbein and its inverse (using (H.3) and (H.6)) which will also be discussed in the equations (H.16) and following:

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{\vec{\xi}}(E_A) &= \left(\xi^K \Omega_{KA}{}^B - \nabla_A \xi^B - 2\xi^K T_{KA}{}^B\right) E_B \\ \boldsymbol{\mathcal{L}}_{\vec{\xi}}(E^A) &= \left(-\xi^K \Omega_{KB}{}^A + \nabla_B \xi^A + 2\xi^K T_{KB}{}^A\right) E^B \qquad \diamond \end{aligned}$$

On one-forms we thus have $\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}}\omega_A \equiv \xi^K \nabla_K \omega_A$, while on objects with curved index the structure group transformation has no effect and the covariantized Lie derivative reduces to the ordinary Lie derivative. When acting on a more general tensor with curved and flat indices, $\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}}$ thus takes the following form:

$$\boxed{\begin{aligned} \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{NB} &= \xi^K \partial_K t_{MA}^{NB} - \partial_K \xi^N t_{MA}^{KB} + \partial_M \xi^K t_{MA}^{NB} + \xi^K \Omega_{KC}{}^B t_{MA}^{NC} - \xi^K \Omega_{KA}{}^C t_{MC}^{NB} = \quad &\text{(H.14)}\\ &= \xi^K \nabla_K t_{MA}^{NB} - \left(\nabla_L \xi^N + 2\xi^K T_{KL}{}^N\right) t_{MA}^{LB} + \left(\nabla_M \xi^L + 2\xi^K T_{KM}{}^L\right) t_{LA}^{NB} \quad &\text{(H.15)}\end{aligned}}$$

This transformation is usually called a **supergauge transformation** [17, chapter XVI]. As it reduces for curved indices to the ordinary Lie derivative, its action on tensor components (given above) is determined by the Lie derivative, the Leibniz rule and the transformation of the supervielbein. In addition the transformation of the structure group connection will be of interest, as it transforms inhomogenously under the structure group transformation. For completeness (even if the given information will be a bit redundant), let us write down explicitely the transformations (supergauge + structure group) for all the type II supergravity superfields of our interest:

Supervielbein A general infinitesimal gauge transformation (a Lie derivative corresponding to a superdiffeomorphism plus a local structure group transformation) of the supervielbein $E_M{}^A$ looks as follows:

$$\delta E_M{}^A = \xi^K \partial_K E_M{}^A + \partial_M \xi^K E_K{}^A + E_M{}^B \tilde{L}_B{}^A \tag{H.16}$$

Redefining the local structure group transformation parameter, this can be written in terms of covariant derivatives

$$\begin{aligned} \delta E_M{}^A &= \xi^K \underbrace{\nabla_K E_M{}^A}_{0} + \nabla_M \xi^K E_K{}^A + \xi^K \underbrace{\left(\Gamma_{KM}{}^L - \Gamma_{MK}{}^L\right) E_L{}^A}_{2T_{KM}{}^A} + E_M{}^B \underbrace{\left(\tilde{L}_B{}^A - \xi^K \Omega_{KB}{}^A\right)}_{L_B{}^A} = &\text{(H.17)}\\ &= \underbrace{\nabla_M \xi^A + 2\xi^C T_{CM}{}^A}_{\equiv \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} E_M{}^A} + L_B{}^A E_M{}^B &\text{(H.18)}\end{aligned}$$

For some purposes, also the explicit form with partial derivatives (but in the new parametrization) will be useful:

$$\boxed{\delta E_M{}^A = \underbrace{\overbrace{\partial_M \xi^A + \Omega_{MC}{}^A \xi^C}^{\nabla_M \xi^A} + 2\xi^C T_{CM}{}^A}_{\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} E_M{}^A} + \underbrace{L_B{}^A E_M{}^B}_{\mathcal{R}(L) E_M{}^A}} \tag{H.19}$$

For the inverse vielbein we get likewise (or via $\delta E^{-1} = -E^{-1}\delta E \cdot E^{-1}$)

$$\begin{aligned} \delta E_A{}^M &= \xi^K \partial_K E_A{}^M - \partial_K \xi^M E_A{}^K - \tilde{L}_A{}^B E_B{}^M &\text{(H.20)}\\ \text{or } \delta E_A{}^M &= \underbrace{-\nabla_A \xi^M - 2\xi^C T_{CA}{}^M}_{\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} E_A{}^M} - L_B{}^A E_A{}^N &\text{(H.21)}\end{aligned}$$

The structure group connection transforms tensorial with respect to the superdiffeomorphisms but of course not like a tensor (but inhomogenous) with respect to the structure group transformation.[2]

$$\begin{aligned}
\delta\Omega_{MA}{}^{B} &= \xi^K\partial_K\Omega_{MA}{}^B + \partial_M\xi^K\Omega_{KA}{}^B - \partial_M \underbrace{\tilde{L}_A{}^B}_{L_A{}^B+\xi^K\Omega_{KA}{}^B} - [\tilde{L},\Omega_M]_A{}^B = && \text{(H.22)}\\
&= \xi^K\partial_K\Omega_{MA}{}^B + \partial_M\xi^K\Omega_{KA}{}^B - \partial_M L_A{}^B - \partial_M\xi^K\Omega_{KA}{}^B - \xi^K\partial_M\Omega_{KA}{}^B + \\
&\quad -[L+\xi^K\Omega_K,\Omega_M]_A{}^B = && \text{(H.23)}\\
&= 2\xi^K\partial_{[K}\Omega_{M]A}{}^B - \xi^K[\Omega_K,\Omega_M]_A{}^B - \partial_M L_A{}^B - [L,\Omega_M]_A{}^B && \text{(H.24)}
\end{aligned}$$

$\Rightarrow$

$$\boxed{\delta\Omega_{MA}{}^B = \underbrace{2\xi^K R_{KMA}{}^B}_{\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\Omega_{MA}{}^B} \underbrace{-\partial_M L_A{}^B - [L,\Omega_M]_A{}^B}_{-\nabla_M L_A{}^B = \mathcal{R}(L)\Omega_{MA}{}^B}} \qquad \text{(H.25)}$$

The scale connection The above transformation of the connection is valid for a general one. In our application to the Berkovits-string, however, the structure group on the supermanifold is restricted as follows. Firstly, the connection is block-diagonal. Secondly, each block decays into Lorentz- plus scale transformation. Finally, the blocks are not independent in the end, but let us assume for the moment, that they are. Then we have three scale connections, namely the trace of each block respectively. In detail we have for the "mixed connection" (see appendix G)

$$\begin{aligned}
\underline{\Omega}_{MA}{}^B &= \begin{pmatrix} \check{\Omega}_{Ma}{}^b & 0 & 0 \\ 0 & \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \hat{\Omega}_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} = && \text{(H.26)}\\
&= \begin{pmatrix} \check{\Omega}^{(D)}_M\delta^b_a & 0 & 0 \\ 0 & \frac{1}{2}\Omega^{(D)}_M\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \frac{1}{2}\hat{\Omega}^{(D)}_M\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} + \begin{pmatrix} \check{\Omega}^{(L)}_{Ma}{}^{b} & 0 & 0 \\ 0 & \frac{1}{4}\Omega^{(L)}_{Mab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \frac{1}{4}\hat{\Omega}^{(L)}_{Mab}\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} && \text{(H.27)}\\
\underline{R}_{MNA}{}^B &= \begin{pmatrix} \check{F}^{(D)}_{MN}\delta^b_a & 0 & 0 \\ 0 & \frac{1}{2}F^{(D)}_{MN}\delta_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \frac{1}{2}\hat{F}^{(D)}_{MN}\delta_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} + \begin{pmatrix} \check{R}^{(L)}_{MNa}{}^{b} & 0 & 0 \\ 0 & \frac{1}{4}R^{(L)}_{MNab}\gamma^{ab}{}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} & 0 \\ 0 & 0 & \frac{1}{4}\hat{R}^{(L)}_{MNab}\gamma^{ab}{}_{\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}} \end{pmatrix} && \text{(H.28)}
\end{aligned}$$

The scale connection (or dilatation connection) simply transforms as

$$\begin{aligned}
\delta\Omega^{(D)}_M &= \xi^K\partial_K\Omega^{(D)}_M + \partial_M\xi^K\Omega^{(D)}_K - \partial_M\tilde{L}^{(D)}, & \delta\hat{\Omega}^{(D)}_M &= \xi^K\partial_K\hat{\Omega}^{(D)}_M + \partial_M\xi^K\hat{\Omega}^{(D)}_K - \partial_M\tilde{\hat{L}}^{(D)} && \text{(H.29)}\\
\boxed{\delta\Omega^{(D)}_M = 2\xi^K F^{(D)}_{KM} - \partial_M L^{(D)}}&, & \delta\hat{\Omega}^{(D)}_M &= 2\xi^K\hat{F}^{(D)}_{KM} - \partial_M\hat{L}^{(D)} && \text{(H.30)}\\
\text{with } F^{(D)}_{KM} &= \partial_{[K}\Omega_{M]}, & \hat{F}^{(D)}_{KM} &= \partial_{[K}\hat{\Omega}_{M]} && \text{(H.31)}
\end{aligned}$$

We also could have started with the pure left-mover connection $\Omega_{MA}{}^B = \mathrm{diag}\,(\Omega_{Ma}{}^b, \Omega_{M\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}, \Omega_{M\hat{\boldsymbol{\alpha}}}{}^{\hat{\boldsymbol{\beta}}})$ to derive $\delta\Omega^{(D)}_M$ or the pure right-mover connection $\hat{\Omega}_{MA}{}^B$ to derive $\delta\hat{\Omega}^{(D)}_M$. We will now return to the notation of this appendix, where $\Omega_{MA}{}^B$ is just a general connection, and not necessarily the left-mover one.

[2] Let us quickly rederive the correct structure group transformation of the connection via the transformation property of the covariant derivative:

$$\begin{aligned}
\delta_{(L)}v^A &= v^B L_B{}^A\\
\delta_{(L)}\nabla_M v^A &= \delta_{(L)}\left(\partial_M v^A + \Omega_{MB}{}^A v^B\right) =\\
&= \partial_M\left(v^B L_B{}^A\right) + \delta_L\Omega_{MB}{}^A v^B + \Omega_{MB}{}^A\delta_L v^B =\\
&= \partial_M v^B\cdot L_B{}^A + v^B\partial_M L_B{}^A + \delta_L\Omega_{MB}{}^A v^B + \Omega_{MB}{}^A v^C L_C{}^B =\\
&= \left(\partial_M v^B + \Omega_{MC}{}^B v^C\right)\cdot L_B{}^A + v^C\left(\partial_M L_C{}^A + \delta_L\Omega_{MC}{}^A + L_C{}^B\Omega_{MB}{}^A - \Omega_{MC}{}^B L_B{}^A\right)
\end{aligned}$$

For $\nabla_M v^A$ to transform covariantly, we need to have

$$\begin{aligned}
\delta_{(L)}\Omega_{MC}{}^A &= -\partial_M L_C{}^A \underbrace{-L_C{}^B\Omega_{MB}{}^A + \Omega_{MC}{}^B L_B{}^A}_{\equiv -[L,\Omega_M]_C{}^A} =\\
&= -\nabla_M L_C{}^A \qquad \diamond
\end{aligned}$$

The superspace connection We will not need the superspace connection $\Gamma_{MN}{}^{K}$ as frequently as the structure group connection, but let us discuss its transformation for completeness. As it is inert under structure group transformations, the supergauge transformation reduces to the Lie derivative. Remember the relation

$$\Gamma_{MN}{}^{K} = \Omega_{MN}{}^{K} + \partial_M E_N{}^{A} \cdot E_A{}^{K} \tag{H.32}$$

which is a direct consequence of $\nabla_M E_M{}^{A} = 0$. The Lie derivative of $\Gamma_{MN}{}^{K}$ can thus be derived from the Lie derivative (or alternatively from the supergauge transformation) of the structure group transformation and the vielbein. Both, vielbein and structure group transformation are tensorial with respect to diffeomorphisms and thus the inhomogenity in the transformation of $\Gamma_{MN}{}^{K}$ can only result from the inhomogenity of the Lie derivative of $\partial_M E_N{}^{A}$, which is (using commutativity of partial and Lie derivative[3]) $\partial_M\partial_N\xi^L E_L{}^{A}$. The Lie derivative of the connection thus reads

$$\mathcal{L}_{\vec{\xi}}\Gamma_{MN}{}^{K} = \xi^L\partial_L\Gamma_{MN}{}^{K} + \partial_M\xi^L\Gamma_{LN}{}^{K} + \underbrace{\partial_N\xi^L\Gamma_{ML}{}^{K} - \partial_L\xi^K\Gamma_{MN}{}^{L} + \partial_M\partial_N\xi^K}_{[\partial\xi,\Gamma_M]_N{}^L + \partial_M(\partial\xi)_N{}^K} \tag{H.33}$$

The first two terms are just the Lie derivative of a matrix valued one form $\mathbf{d}x^M\Gamma_{MN}{}^{K}$, while the last three terms are the usual inhomogenous transformation of a structure group connection (compare (H.25)), here with the Gl(n)-matrix $\tilde{M}_N{}^{K} \equiv -\partial_N\xi^K$. The same transformation can be derived by comparing e.g. the tensorial transformation of $\mathcal{L}_{\vec{\xi}}\nabla_M v^K$ on the one side with $\partial_M(\mathcal{L}_{\vec{\xi}}v^K) + \mathcal{L}_{\vec{\xi}}\Gamma_{MN}{}^{K}\cdot v^N + \Gamma_{MN}{}^{K}\mathcal{L}_{\vec{\xi}}v^N$ on the other side (using again that Lie and partial derivative commute). The Lie derivative of the connection is in some sense the

[3]For a scalar field $\Phi_{(ph)}$, whose partial derivative becomes the component of a vector field, it is quite obvious that partial and Lie derivative commute:

$$\mathcal{L}_{\vec{\xi}}\partial_M\Phi_{(ph)} = \xi^K\partial_K\partial_M\Phi_{(ph)} + \partial_M\xi^K\partial_K\Phi_{(ph)} = \partial_M(\xi^K\partial_K\Phi_{(ph)}) = \partial_M\mathcal{L}_{\vec{\xi}}\Phi_{(ph)}$$

For a nontensorial object like $\partial_M t_{M_1\ldots M_p}^{N_1\ldots N_q}$ (or also the connection) it is less clear whether it makes sense to define a Lie derivative on it. However, it will be very convenient to do so, and we will simply take the definition coming from infinitesimal diffeomorphisms (with $x' = x+\xi$). Note that $\partial'_M t'^{N_1\ldots N_q}_{M_1\ldots M_p}(x')\Big|_{x'=x} = \partial_M t'^{N_1\ldots N_q}_{M_1\ldots M_p}(x)$, which leads to

$$\mathcal{L}_{\vec{\xi}}\partial_M t_{M_1\ldots M_p}^{N_1\ldots N_q}(x) \equiv \partial_M t_{M_1\ldots M_p}^{N_1\ldots N_q}(x) - \partial'_M t'^{N_1\ldots N_q}_{M_1\ldots M_p}(x')\Big|_{x'=x} = \partial_M(\mathcal{L}_{\vec{\xi}} t_{M_1\ldots M_p}^{N_1\ldots N_q}(x))$$

We can likewise extend the definition of $\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} = \mathcal{L}_{\vec{\xi}} + \mathcal{R}(\xi^K\Omega_K{}_{\cdot}{}^{\cdot})$ to nontensorial objects by defining e.g.

$$\mathcal{R}(L)\ \partial_P t_{MA}^{NB} \equiv \partial_P\left(\mathcal{R}(L)\, t_{MA}^{NB}\right)$$

The structure group transformation $\mathcal{R}(L)$ thus commutes with the partial derivative by definition and we thus have the same property for the covariantized Lie derivative

$$\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\partial_P t_{MA}^{NB} = \partial_P(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{NB})$$

Note that this is also consistent with a proper transformation property of the covariant derivative:

$$\begin{aligned}
\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\nabla_P t_{MA}^{NB} &= \mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\left(\partial_P t_{MA}^{NB} + \Gamma_{PK}{}^{N} t_{MA}^{KB} - \Gamma_{PM}{}^{K} t_{KA}^{NB} + \mathcal{R}(\Omega_P{}_{\cdot}{}^{\cdot})\, t_{MA}^{NB}\right) = \\
&= \partial_P\left(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{NB}\right) + \left(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\Gamma_{PK}{}^{N}\right) t_{MA}^{KB} + \Gamma_{PK}{}^{N}\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{KB} - \left(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\Gamma_{PM}{}^{K}\right) t_{KA}^{NB} - \Gamma_{PM}{}^{K}\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} t_{KA}^{NB} + \\
&\quad + \mathcal{R}\left(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\Omega_P{}_{\cdot}{}^{\cdot}\right) t_{MA}^{NB} + \mathcal{R}(\Omega_P{}_{\cdot}{}^{\cdot})\,\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{NB} = \\
&= \nabla_P\left(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{NB}\right) + \left(\mathcal{L}_{\vec{\xi}}\Gamma_{PK}{}^{N}\right) t_{MA}^{KB} - \left(\mathcal{L}_{\vec{\xi}}\Gamma_{PM}{}^{K}\right) t_{KA}^{NB} + \mathcal{R}\left(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}\Omega_P{}_{\cdot}{}^{\cdot}\right) t_{MA}^{NB} = \\
&= \nabla_P\left(\xi^K\nabla_K t_{MA}^{NB} + \left(\nabla_M\xi^K + 2\xi^L T_{LM}{}^{K}\right) t_{KA}^{NB} - \left(\nabla_K\xi^N + 2\xi^L T_{LK}{}^{N}\right) t_{MA}^{KB}\right) + \\
&\quad + \left(2\xi^L R_{LPK}{}^{N} + \nabla_P(\nabla_K\xi^N + 2\xi^L T_{LK}{}^{N})\right) t_{MA}^{KB} - \left(2\xi^L R_{LPM}{}^{K} + \nabla_P(\nabla_M\xi^K + 2\xi^L T_{LM}{}^{K})\right) t_{KA}^{NB} + \\
&\quad + \mathcal{R}\left(2\xi^L R_{LP}{}_{\cdot}{}^{\cdot}\right) t_{MA}^{NB} = \\
&= \xi^K \underbrace{\nabla_P\nabla_K t_{MA}^{NB}}_{\nabla_K\nabla_P t_{MA}^{NB} - 2T_{PK}{}^{L}\nabla_L t_{MA}^{NB} + 2R_{PKL}{}^{N} t_{MA}^{LB} - 2R_{PKM}{}^{L} t_{LA}^{NB} + \mathcal{R}(2R_{PK}{}_{\cdot}{}^{\cdot})\, t_{MA}^{NB}} + \left(\nabla_M\xi^K + 2\xi^L T_{LM}{}^{K}\right)\nabla_P t_{KA}^{NB} - \left(\nabla_K\xi^N + 2\xi^L T_{LK}{}^{N}\right)\nabla_P t_{MA}^{KB} + \\
&\quad + \nabla_P\xi^K\nabla_K t_{MA}^{NB} + \nabla_P\left(\nabla_M\xi^K + 2\xi^L T_{LM}{}^{K}\right) t_{KA}^{NB} - \nabla_P\left(\nabla_K\xi^N + 2\xi^L T_{LK}{}^{N}\right) t_{MA}^{KB} \\
&\quad + \left(2\xi^L R_{LPK}{}^{N} + \nabla_P(\nabla_K\xi^N + 2\xi^L T_{LK}{}^{N})\right) t_{MA}^{KB} - \left(2\xi^L R_{LPM}{}^{K} + \nabla_P(\nabla_M\xi^K + 2\xi^L T_{LM}{}^{K})\right) t_{KA}^{NB} + \\
&\quad + \mathcal{R}\left(2\xi^L R_{LP}{}_{\cdot}{}^{\cdot}\right) t_{MA}^{NB} = \\
&= \xi^K\nabla_K\nabla_P t_{MA}^{NB} + \left(\nabla_P\xi^K + 2\xi^L T_{LP}{}^{K}\right)\nabla_K t_{MA}^{NB} + \left(\nabla_M\xi^K + 2\xi^L T_{LM}{}^{K}\right)\nabla_P t_{KA}^{NB} - \left(\nabla_K\xi^N + 2\xi^L T_{LK}{}^{N}\right)\nabla_P t_{MA}^{KB} \quad \diamond
\end{aligned}$$

difference of two connections and is therefore a tensor. This can be seen by expressing the partial derivatives on ξ^M in terms of covariant ones and discover that the remaining connection terms combine to curvature and torsion.[4]

$$\boxed{\mathcal{L}_{\vec{\xi}}\Gamma_{MN}{}^K = 2\xi^L R_{LMN}{}^K + \nabla_M \underbrace{\left(\nabla_N \xi^K + 2\xi^L T_{LN}{}^K\right)}_{\equiv -M_N{}^K}} \tag{H.34}$$

Remember that above we have seen the Lie derivative of the superspace connection as a combination of a Lie derivative on its form index (the first lower index) plus a Gl(n) structure group transformation with transformation matrix $\tilde{M}_N{}^K \equiv -\partial_N \xi^K$. Equivalently it can be seen as a combination of a supergauge transformation (regarding only the first index as curved one) plus a modified Gl(n) transformation with the matrix (compare (H.10))

$$\begin{aligned} M_N{}^K &\equiv -\partial_N \xi^K - \xi^P \Gamma_{PN}{}^K = && \text{(H.35)} \\ &= -\nabla_N \xi^K - 2\xi^P T_{PN}{}^K && \text{(H.36)} \end{aligned}$$

Indeed the above Lie transformation can be written as

$$\mathcal{L}_{\vec{\xi}}\Gamma_{MN}{}^K = 2\xi^L R_{LMN}{}^K \underbrace{-\partial_M M_N{}^K - [M, \Gamma_M]_N{}^K}_{= -\nabla_M M_N{}^K} \tag{H.37}$$

which perfectly agrees with the form of a gauge transformation of a structure group connection given in (H.25).

Let us finally note that

$$[\mathcal{L}_{\vec{\xi}}, \nabla_M] v^K = (\mathcal{L}_{\vec{\xi}}\Gamma_{MN}{}^K) v^N \tag{H.38}$$

which provides another way to calculate the Lie derivative of the connection. For the Levi Civita connection this equation implies that the Lie derivative commutes with the covariant derivative, if $\vec{\xi}$ is a killing vector.

Tensorial superfields Usually, all additional fields present in a supergravity theory (like B-field, RR-fields or dilaton) are contained in superfields that transform homogenously (tensorial) under supergauge transformations and structure group transformations. The gauge transformation of a tensor field with index structure t_{MA}^{NB} transforms as

$$\delta t_{MA}^{NB} = \mathcal{L}^{(\text{cov})}_{\vec{\xi}} t_{MA}^{NB} + \underbrace{L_C{}^B t_{MA}^{NC} - L_A{}^C t_{MC}^{NB}}_{\mathcal{R}(L\,\cdot\,)t_{MA}^{NB}} \tag{H.39}$$

where $\mathcal{L}^{(\text{cov})}_{\vec{\xi}}$ was given in (H.15). The above transformation is of course also valid for scalar fields where simply the structure group transformation vanishes. If a B-field (a two form, i.e. an antisymmetric rank two tensor) is present, its general gauge transformation contains in addition the one-form gauge transformation $B \to B + \mathbf{d}\Lambda$ which will briefly be discussed in a separate section at a later point. Another example of a tensorial superfield in our application to the Berkovits string is the bispinor-superfield $\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}$ which contains the RR-fields in the leading component in the $\vec{\boldsymbol{\theta}}$-expansion. In order to act with the structure group transformation $L_A{}^B$ (appearing in the general transformation (H.39)) on the bispinor indices, we need $L_A{}^B$ to be block diagonal. This is described in the main part (see (5.65)). A final remark about our application in the main part is about the appearance of a compensator field Φ which does not transform homogenously under the structure group, but via a shift (see discussion below (5.159)).

H.1.2 Algebra of Lie derivatives and supergauge transformations

H.1.2.1 Commutator of Lie derivatives

The SUSY algebra on scalar fields and tensors with curved indices should be entirely implemented in the superdiffeomorphisms (independent from any accompanying local structure group transformation which appeared above). The commutator of two diffeomorphisms yields the vector Lie bracket of the transformation parameters

$$\left[\mathcal{L}_{\vec{\xi}_1}, \mathcal{L}_{\vec{\xi}_2}\right] = \mathcal{L}_{[\vec{\xi}_1, \vec{\xi}_2]} \tag{H.40}$$

[4]Alternatively we can derive the same result, starting from (H.32)

$$\mathcal{L}_{\vec{\xi}}\Gamma_{MN}{}^K = \mathcal{L}^{(\text{cov})}_{\vec{\xi}}(\Omega_{MA}{}^B E_N{}^A E_B{}^K) + \partial_M(\mathcal{L}^{(\text{cov})}_{\vec{\xi}} E_N{}^A) \cdot E_A{}^K + \partial_M E_N{}^A \cdot \mathcal{L}^{(\text{cov})}_{\vec{\xi}} E_A{}^K$$

Using the covariant expressions of the supergauge transformation of $\Omega_{MA}{}^B$ and $E_M{}^A$ then leads to (H.34). ⋄

where the vector Lie bracket reads

$$\left[\vec{\xi}_1, \vec{\xi}_2\right]^M = \xi_1^K \partial_K \xi_2^M - \xi_2^K \partial_K \xi_1^M = \tag{H.41}$$

$$= \xi_1^K \nabla_K \xi_2^M - \xi_2^K \nabla_K \xi_1^M - 2\xi_1^K T_{KL}{}^M \xi_2^L \tag{H.42}$$

If we plug in the local basis elements $\vec{E}_A \equiv E_A{}^M \boldsymbol{\partial}_M$ in place of $\xi_{1/2}$, the above equation only holds, if the covariant derivative acts only on the curved index. The covariant derivatives do not vanish when we act on the curved index of $E_A{}^M$ only. We thus do not only get the torsion term, as one would naively expect, but instead

$$[\vec{E}_A, \vec{E}_B] = \left(2\Omega_{[AB]}{}^C - 2T_{AB}{}^C\right) \vec{E}_C = \tag{H.43}$$

$$= -2(\mathbf{d}E^C)_{AB}\, \vec{E}_C \tag{H.44}$$

For objects with flat indices it is thus convenient to extend the Lie derivative to the supergauge transformation, which is covariantized with respect to the structure group.

H.1.2.2 Algebra of covariant Lie derivative and structure group action

Let us restrict our considerations for a moment to a structure group vector v^A. We first want to study the commutator of two covariantized Lie derivatives.

$$[\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}}, \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\eta}}]v^A = \xi^L \nabla_L \left(\eta^K \nabla_K v^A\right) - (\xi \leftrightarrow \eta) = \tag{H.45}$$

$$= \left(\xi^L \nabla_L \eta^K - \eta^L \nabla_L \xi^K\right) \nabla_K v^A + \xi^L \eta^K \left[\nabla_L, \nabla_K\right] v^A = \tag{H.46}$$

$$= \left(\xi^L \nabla_L \eta^K - \eta^L \nabla_L \xi^K - 2\xi^L T_{LP}{}^K \eta^P\right) \nabla_K v^A + 2\xi^L \eta^K R_{LKB}{}^A v^B = \tag{H.47}$$

$$= \boldsymbol{\mathcal{L}}^{(\text{cov})}_{[\vec{\xi}, \vec{\eta}]} v^A + 2\xi^L \eta^K R_{LKB}{}^A v^B \tag{H.48}$$

For a one form we arrive likewise at

$$[\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}}, \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\eta}}]\omega_A = \boldsymbol{\mathcal{L}}^{(\text{cov})}_{[\vec{\xi}, \vec{\eta}]} \omega_A - 2\xi^L \eta^K R_{LKA}{}^B \omega_B \tag{H.49}$$

On curved indices, however, the super gauge transformation reduces to the Lie derivative

$$\left[\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}}, \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\eta}}\right] v^M = \left[\boldsymbol{\mathcal{L}}_{\vec{\xi}}, \boldsymbol{\mathcal{L}}_{\vec{\eta}}\right] v^M = \boldsymbol{\mathcal{L}}_{[\vec{\xi}, \vec{\eta}]} v^M = \boldsymbol{\mathcal{L}}^{(\text{cov})}_{[\vec{\xi}, \vec{\eta}]} v^M \tag{H.50}$$

$$\left[\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}}, \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\eta}}\right] \omega_M = \boldsymbol{\mathcal{L}}^{(\text{cov})}_{[\vec{\xi}, \vec{\eta}]} \omega_M \tag{H.51}$$

On a more general tensor t_{MA}^{NB} we therefore have the following commutator of supergauge transformations (remember footnote 1)

$$\boxed{\left[\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}}, \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\eta}}\right] t_{MA}^{NB} = \boldsymbol{\mathcal{L}}^{(\text{cov})}_{[\vec{\xi}, \vec{\eta}]} t_{MA}^{NB} + \underbrace{2\xi^K \eta^L R_{KLC}{}^B t_{MA}^{NC} - 2\xi^K \eta^L R_{KLA}{}^C t_{MC}^{NB}}_{\mathcal{R}\left(-\imath_{\vec{\xi}}\imath_{\vec{\eta}}(R.\cdot)\right) t_{MA}^{NB}}} \tag{H.52}$$

In particular we have for supergauge transformations along the coordinate basis

$$[\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\boldsymbol{\partial}_K}, \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\boldsymbol{\partial}_L}] t_{MA}^{NB} = 2R_{KLC}{}^B t_{MA}^{NC} - 2R_{KLA}{}^C t_{MC}^{NB} = \mathcal{R}\left(-\imath_{\boldsymbol{\partial}_K} \imath_{\boldsymbol{\partial}_L}(R_C{}^D)\right) t_{MA}^{NB} \tag{H.53}$$

The algebra of two infinitesimal structure group transformations is rather simple[5]

$$\boxed{[\mathcal{R}(L_1), \mathcal{R}(L_2)] = -\mathcal{R}([L_1, L_2])} \tag{H.54}$$

[5]The minus sign comes from our definition how the structure group matrix acts on vectors and forms. E.g. on a vector we have $\mathcal{R}(L_1)\,\mathcal{R}(L_2)\,v^A = \mathcal{R}(L_1)\,(L_2{}_B{}^A v^B) = L_1{}_C{}^A L_2{}_B{}^C v^B = (L_2 L_1)_B{}^A v^B = \mathcal{R}(L_2 L_1)\, v^A \Rightarrow [\mathcal{R}(L_1), \mathcal{R}(L_2)]v^A = -\mathcal{R}([L_1, L_2])\, v^A$. Similarly for one forms $\mathcal{R}(L_1)\,\mathcal{R}(L_2)\,\omega_A = \mathcal{R}(L_1)\,(-L_2{}_A{}^B \omega_B) = L_1{}_A{}^C L_2{}_C{}^B \omega_B = (L_1 L_2)_A{}^B \omega_B = -\mathcal{R}(L_1 L_2)\,\omega_A \Rightarrow [\mathcal{R}(L_1), \mathcal{R}(L_2)]\omega_A = -\mathcal{R}([L_1, L_2])\,\omega_A$. If one prefers, one can get rid of the minus sign by either redefining the action of $\mathcal{R}(L)$ with a minus sign or with a transposed L (not only for antisymmetric L). This is because $[L_1^T, L_2^T]^T = -[L_1, L_2]$ and $-[-L_1, -L_2] = -[L_1, L_2]$. ◇

The commutator between supergauge transformation and structure group transformation finally reads

$$\boxed{\left[\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}, \mathcal{R}(L)\right] = \mathcal{R}\Big((\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} L)\Big)} \tag{H.55}$$

which is easily checked by acting e.g. on a vector v^A. The complete algebra can be written in one single equation as

$$\boxed{\left[\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} + \mathcal{R}(L_1)\ ,\ \mathcal{L}^{(\mathrm{cov})}_{\vec{\eta}} + \mathcal{R}(L_2)\right] = \mathcal{L}^{(\mathrm{cov})}_{[\vec{\xi},\vec{\eta}]} + \mathcal{R}\Big(2\xi^K\eta^L R_{KL}{}^{\cdot}_{\cdot} + \mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} L_2{}^{\cdot}_{\cdot} - \mathcal{L}^{(\mathrm{cov})}_{\vec{\eta}} L_1{}^{\cdot}_{\cdot} - [L_1, L_2]{}^{\cdot}_{\cdot}\Big)} \tag{H.56}$$

H.1.2.3 Commutator of covariantized Lie derivative (supergauge) and covariant derivative

In Riemannian geometry the commutator of Lie derivative and covariant derivative vanishes, if the vector along which the Lie derivative is taken is a killing vector. We want to see what relation there is for a more general connection. Let us first consider the commutator of the Lie derivative and the covariant derivative with curved index on a superspace vector

$$\left[\mathcal{L}_{\vec{\xi}}, \nabla_M\right] v^K = \underbrace{[\mathcal{L}_{\vec{\xi}}, \partial_M]}_{=0} v^K + \mathcal{L}_{\vec{\xi}} \Gamma_{MN}{}^K \cdot v^N \tag{H.57}$$

According to footnote 3, the first term vanishes and we have

$$\left[\mathcal{L}_{\vec{\xi}}, \nabla_M\right] = 0 \quad \Longleftrightarrow \quad 0 = \mathcal{L}_{\vec{\xi}} \Gamma_{MN}{}^K \quad \left(\overset{(H.34)}{=} 2\xi^L R_{LMN}{}^K + \nabla_M \left(\nabla_N \xi^K + 2\xi^L T_{LN}{}^K\right)\right) \tag{H.58}$$

In the case of a Levi Civita connection, the Lie derivative of the connection vanishes, if the Lie derivative of the metric vanishes, i.e. if $\vec{\xi}$ is a killing vector[6]. In general, however, we have the condition that the Lie derivative of the connection has to vanish.

Let us introduce just for the moment the symbol $\tilde{\mathcal{R}}$ to denote the action of a Gl(n) matrix (like the superspace connection $\Gamma_M{}^{\cdot}_{\cdot}$) on the curved indices. Acting on an arbitrary tensor, the commutator of above becomes

$$\left[\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}, \nabla_M\right] = \tilde{\mathcal{R}}\left(\mathcal{L}_{\vec{\xi}} \Gamma_M{}^{\cdot}_{\cdot}\right) + \mathcal{R}\Big(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} \Omega_M{}^{\cdot}_{\cdot}\Big) \tag{H.59}$$

How does this commutator modify, if we choose the covariant derivative with flat index?

$$\left[\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}, \nabla_A\right] = \left[\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}}, E_A{}^M \nabla_M\right] = \tag{H.60}$$

$$\overset{(H.21)}{=} -\left(\nabla_A \xi^M + 2\xi^C T_{CA}{}^M\right) \nabla_M + E_A{}^M \tilde{\mathcal{R}}\left(\mathcal{L}_{\vec{\xi}} \Gamma_M{}^{\cdot}_{\cdot}\right) + E_A{}^M \mathcal{R}\Big(\mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}} \Omega_M{}^{\cdot}_{\cdot}\Big) \tag{H.61}$$

[6] This is quite natural, as the Levi Civita connection is built only out of the metric. Nevertheless, let us check this statement explicitly with the derived formula, in order to see whether it is consistent. In the Riemannian case we have

$$\mathcal{L}_{\vec{\xi}} \Gamma_{mn}{}^k = 2\xi^l R_{lmn}{}^k + \nabla_m \nabla_n \xi^k$$

and the killing vector condition reads (pulling down the indices with the covariantly conserved metric g_{mn})

$$\nabla_{(m}\xi_{n)} = 0$$

We can rewrite the above Lie derivative as

$$\begin{aligned}
\mathcal{L}_{\vec{\xi}} \Gamma_{mn|k} &= 2\xi^l R_{lmnk} + \nabla_m \nabla_n \xi_k = \\
&= 2\xi^l R_{lmnk} + \frac{1}{2}\nabla_m \nabla_n \xi_k + \frac{1}{2}\nabla_n \nabla_m \xi_k - R_{mnk}{}^l \xi_l = \\
&\overset{\text{killing}}{=} 2\xi^l R_{lmnk} - \frac{1}{2}\nabla_m \nabla_k \xi_n - \frac{1}{2}\nabla_n \nabla_k \xi_m - R_{mnk}{}^l \xi_l = \\
&= 2\xi^l R_{lmnk} - \frac{1}{2}\nabla_k \nabla_m \xi_n + R_{mkn}{}^l \xi_l - \frac{1}{2}\nabla_k \nabla_n \xi_m + R_{nkm}{}^l \xi_l - R_{mnk}{}^l \xi_l \\
&= 2\xi^l \underbrace{R_{lmnk}}_{-R_{nkml}} - R_{kmn}{}^l \xi_l + R_{nkm}{}^l \xi_l - R_{mnk}{}^l \xi_l = \\
&= -\left(R_{nkm}{}^l + R_{kmn}{}^l + R_{mnk}{}^l\right) \xi_l = 0 \qquad \diamond
\end{aligned}$$

Finally we allow for an additional structure group transformation, in order to see the commutator of a general gauge transformation with the covariant derivative:

$$\boxed{\begin{aligned}\left[\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} + \mathcal{R}(L.^{\cdot})\ ,\ \nabla_A\right] \quad = \quad & \Big(\underbrace{-(\nabla_A\xi^D + 2\xi^C T_{CA}{}^D)}_{(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} E_A{}^M) E_M{}^D} - L_A{}^D\Big)\nabla_D + \\ & +\tilde{\mathcal{R}}\underbrace{\left(2\xi^L R_{LA}.^{\cdot} + \nabla_A\left(\nabla.\xi^{\cdot} + 2\xi^L T_L.^{\cdot}\right)\right)}_{E_A{}^M \boldsymbol{\mathcal{L}}_{\vec{\xi}}\Gamma_M.^{\cdot}} + \ \mathcal{R}\big(\underbrace{2\xi^C R_{CA}.^{\cdot}}_{E_A{}^M \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}}\Omega_M.^{\cdot}} - \nabla_A L.^{\cdot}\big)\end{aligned}} \tag{H.62}$$

When acting on scalar fields, only the first term remains.

The idea of the above considerations was of course that part of the gauge transformations become just the local supersymmetry transformations, while the fermionic components of the covariant derivative should contain the supersymmetric covariant derivative. We therefore expect, at least for the flat case, a vanishing result for the fermionic components of this commutator. We will come back to this question after having established the WZ-gauge.

H.1.2.4 Algebra of the gauge transformations

The algebra in subsection H.1.2.2 was assuming that the variation acts on all objects, including the transformation parameter of the first transformation. This is not true for field-independent transformation parameters. If $\vec{\xi}$ is just the transformation parameter of the symmetry, then this parameter does not transform itself. On the other hand, there is no need for the transformation parameter to coincide with $\vec{\xi}$. Instead, $\vec{\xi}$ can be a functional of transformation parameter and of the the fields. We thus have to treat its variation seperately. A general gauge variation has the form $\delta t_{MA}^{NB} = \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}} t_{MA}^{NB} + \mathcal{R}(L\cdot{}^{\cdot})\, t_{MA}^{NB}$, where $\vec{\xi}$ and the structure group matrix L are local and may or may not depend on the fields of the theory. Acting a second time with such a variation yields

$$\begin{aligned}
&\delta_1\delta_2(\ldots) = \\
&= \delta_1\left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_2} + \mathcal{R}(L_2.^{\cdot})\right) = && \text{(H.63)}\\
&= \delta_1\left(\boldsymbol{\mathcal{L}}_{\vec{\xi}_2} + \mathcal{R}\big(\xi_2^K\Omega_K.^{\cdot} + L_2.^{\cdot}\big)\right)(\ldots) = && \text{(H.64)}\\
&= \left(\boldsymbol{\mathcal{L}}_{\delta_1\vec{\xi}_2} + \mathcal{R}\big(\delta_1\xi_2^K\Omega_K.^{\cdot} + \xi_2^K\delta_1\Omega_K.^{\cdot} + \delta_1 L_2.^{\cdot}\big)\right)(\ldots) + \left(\boldsymbol{\mathcal{L}}_{\vec{\xi}_2} + \mathcal{R}\big(\xi_2^K\Omega_K.^{\cdot} + L_2.^{\cdot}\big)\right)\delta_1(\ldots) = && \text{(H.65)}\\
&= \left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\delta_1\vec{\xi}_2} + \mathcal{R}\left(\xi_2^K\left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_1}\Omega_K.^{\cdot} - \nabla_K L_1.^{\cdot}\right) + \delta_1 L_2.^{\cdot}\right)\right)(\ldots) + \\
&\quad + \left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_2} + \mathcal{R}(L_2.^{\cdot})\right)\left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_1} + \mathcal{R}(L_1)\right)(\ldots) = && \text{(H.66)}\\
&= \left[\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\delta_1\vec{\xi}_2} + \mathcal{R}\big(2\xi_2^K\xi_1^L R_{LK}.^{\cdot} - \xi_2^K\nabla_K L_1 + \delta_1 L_2.^{\cdot}\big) + \left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_2} + \mathcal{R}(L_2.^{\cdot})\right)\left(\boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_1} + \mathcal{R}(L_1.^{\cdot})\right)\right](\ldots) && \text{(H.67)}
\end{aligned}$$

Finally we take the commutator and use the commutation relation (H.56) of above

$$\begin{aligned}
[\delta_1, \delta_2] \quad = \quad & \mathcal{R}\big(4\xi_2^K\xi_1^L R_{LK}.^{\cdot} + \xi_1^K\nabla_K L_2 - \xi_2^K\nabla_K L_1 + \delta_1 L_2\cdot{}^{\cdot} - \delta_2 L_1\big) + \\
& + \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{[\vec{\xi}_2,\vec{\xi}_1] + \delta_1\vec{\xi}_2 - \delta_2\vec{\xi}_1} + \mathcal{R}\left(2\xi_2^K\xi_1^L R_{KL}.^{\cdot} + \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_2} L_1 - \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_1} L_2 - [L_2, L_1]\right)
\end{aligned} \tag{H.68}$$

$$\boxed{[\delta_1, \delta_2] = \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{[\vec{\xi}_2,\vec{\xi}_1] + \delta_1\vec{\xi}_2 - \delta_2\vec{\xi}_1} + \mathcal{R}\big(2\xi_1^K\xi_2^L R_{KL}.^{\cdot} + [L_1, L_2].^{\cdot} + \delta_1 L_2.^{\cdot} - \delta_2 L_1.^{\cdot}\big)} \tag{H.69}$$

If $\vec{\xi}$ and L are field dependent and transform like all the other fields, we have $\delta_1\vec{\xi}_2 = [\vec{\xi}_1, \vec{\xi}_2]$ and $\delta_1 L_2 = \boldsymbol{\mathcal{L}}^{(\mathrm{cov})}_{\vec{\xi}_1} L_2 - [L_1, L_2]$ and the above equation is the same as (H.56), while if both parameters do not transform at all, we have a similar, but still different algebra with some different signs and some terms missing. The above important equation will help us to find the SUSY-algebra in this huge algebra. By going to the WZ-gauge, we will fix part of the superdiffeomorphisms and local structure group transformations. The remaining transformations, which stabilize this gauge will then have a field-dependent $\vec{\xi}$, which we can plug into the above equation.

H.1.3 Finite gauge transformations

In order to choose an explicit gauge it is useful to know the finite form of the gauge transformations (only then you can decide whether a particular gauge is accessible or not). For superdiffeomorphisms and local structure group transformations (i.e. Lorentz transformations and perhaps dilatations), we know the finite form anyway. Let us denote the transformed fields by a prime (for superdiffeomorphisms) and by a tilde (for structure group transformations). The vielbein transforms homogenously under both transformations, i.e. $E'_M{}^A(\vec{x}\,') = \frac{\partial x^N}{\partial x'^M} E_N{}^A(\vec{x}\,')$ under superdiffeomorphisms and $\tilde{E}_M{}^A(\vec{x}\,) = E_M{}^B(\vec{x}\,)\Lambda_B{}^A(\vec{x}\,)$ under structure group transformations. Altogether this reads

$$\tilde{E}'_M{}^A(\vec{x}\,') = \frac{\partial x^N}{\partial x'^M}\left(E_N{}^B(\vec{x}\,)\Lambda_B{}^A(\vec{x}\,)\right) = \left(\frac{\partial x^N}{\partial x'^M}E_N{}^B(\vec{x}\,)\right)\Lambda'_B{}^A(\vec{x}\,') \tag{H.70}$$

Likewise a more general tensor field with index structure t_{MA}^{NB} transfoms as

$$\tilde{t}'^{NB}_{MA}(\vec{x}\,') = \frac{\partial x^K}{\partial x'^M}\frac{\partial x'^N}{\partial x^L} t_{KC}^{LD}(\vec{x}\,)\Lambda_A{}^C(\vec{x}\,)(\Lambda^{-1})_D{}^B(\vec{x}\,) \tag{H.71}$$

Other examples for such homogenous transformations (apart from the vielbein) are a RR-superfield with $\tilde{\mathcal{P}}'^{\boldsymbol{\delta}\hat{\boldsymbol{\delta}}}(\vec{x}\,') = \mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\gamma}}}(\vec{x}\,)\Lambda_{\boldsymbol{\gamma}}{}^{\boldsymbol{\delta}}\Lambda_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\delta}}}$ (where the structure group transformation $\Lambda_A{}^B$ is supposed to be a blockdiagonal one), or a dilaton scalar superfield with simply $\widetilde{\Phi_{(ph)}}'(\vec{x}\,') = \Phi_{(ph)}(\vec{x}\,)$.

The finite inhomogenous transformation of the connection superfield reads[7]

$$\tilde{\Omega}'_{MA}{}^B(\vec{x}\,') = \frac{\partial x^N}{\partial x'^M}\left(-\partial_N\Lambda_A{}^B + (\Lambda^{-1})_A{}^D\Omega_{ND}{}^C(\vec{x}\,)\Lambda_C{}^B\right) \tag{H.72}$$

$$\tilde{\Omega}^{(D)\prime}{}_M(\vec{x}\,') = \frac{\partial x^N}{\partial x'^M}\left(\Omega_N^{(D)}(\vec{x}\,) - \partial_N\Lambda^{(D)}(\vec{x}\,)\right) \tag{H.73}$$

In the main part of this thesis we have also introduced a compensator field Φ, which transforms by a shift under scale transformations, i.e. $\tilde{\Phi}'(\vec{x}\,') = \Phi(\vec{x}\,) - \check{\Lambda}^{(D)}(\vec{x}\,)$ (where $\check{\Lambda}^{(D)}$ denotes the dilatation or scale part of the bosonic block).

H.2 Wess-Zumino gauge

H.2.1 WZ gauge for the vielbein

Superdiffeomorphisms $x'^M = F^M(\vec{x}\,) \overset{inf}{=} x^M + \xi^M(\vec{x}\,)$ with $\vec{x} = (\vec{x}\,, \vec{\boldsymbol{\theta}})$ parametrise many more gauge degrees of freedom than just the bosonic diffeomorphisms $x'^m = f^m(\vec{x}\,) \overset{inf}{=} x^m + \xi^m(\vec{x}\,, \vec{\boldsymbol{\theta}} = 0)$. Let us write $\vec{x}\,'$ as

$$x'^M = x'^M_0(\vec{x}\,) + \underbrace{x^{\boldsymbol{\mathcal{N}}}}_{\boldsymbol{\theta}^{\mathcal{N}}} x'^M_{\boldsymbol{\mathcal{N}}}(\vec{x}\,) + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) \tag{H.74}$$

We have

$$\frac{\partial x'^M}{\partial x^N} = \begin{pmatrix} \frac{\partial x'^m}{\partial x^n} & \frac{\partial x'^m}{\partial x^{\boldsymbol{\mathcal{N}}}} \\ \frac{\partial x'^{\boldsymbol{\mathcal{M}}}}{\partial x^n} & \frac{\partial x'^{\boldsymbol{\mathcal{M}}}}{\partial x^{\boldsymbol{\mathcal{N}}}} \end{pmatrix} \overset{\vec{\boldsymbol{\theta}}=0}{=} \begin{pmatrix} \frac{\partial x'^m_0}{\partial x^n} & x'^m_{\boldsymbol{\mathcal{N}}} \\ \frac{\partial x'^{\boldsymbol{\mathcal{M}}}_0}{\partial x^n} & x'^{\boldsymbol{\mathcal{M}}}_{\boldsymbol{\mathcal{N}}} \end{pmatrix} \tag{H.75}$$

In the following we will see that it is possible to fix the vielbein for vanishing $\vec{\boldsymbol{\theta}}$ to

$$\boxed{E_M{}^A\big| = \begin{pmatrix} e_m{}^a & \psi_m{}^{\boldsymbol{\mathcal{A}}} \\ 0 & \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}} \end{pmatrix}} \tag{H.76}$$

[7]Defining $\Omega_M^{(D)} \equiv \frac{1}{\dim}\Omega_{Ma}{}^a$ and $\Lambda^{(D)} \equiv \frac{1}{\dim}\Lambda_a{}^a$ yields the transformation (H.73) in the second line. However, having in mind the definition of the mixed connection (H.27) yields the same transformation for each of the scale connections $\Omega_M^{(D)}$ (with $\Lambda^{(D)}$), $\hat{\Omega}_M^{(D)}$ (with $\hat{\Lambda}^{(D)}$) and $\check{\Omega}_M^{(D)}$ (with $\check{\Lambda}^{(D)}$) respectively.

In our application to the Berkovits string, we have introduced a compensator field Φ via $G_{ab} = e^{2\Phi}\eta_{ab}$ which transforms under the bosonic scale transformations $\check{\Lambda}$. The distinction, however, is not important, as Λ, $\hat{\Lambda}$ and $\check{\Lambda}$ get coupled by the gauge fixing of $T_{\boldsymbol{\alpha\beta}}{}^c = \gamma^c_{\boldsymbol{\alpha\beta}}$ and $T_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}{}^c = \gamma^c_{\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\beta}}}$ anyway. ◇

with inverse

$$\boxed{E_A{}^M| = \begin{pmatrix} e_a{}^m & -\psi_a{}^{\boldsymbol{\mathcal{M}}} \\ 0 & \delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}} \end{pmatrix}} \tag{H.77}$$

$$\text{where } \psi_a{}^{\boldsymbol{\mathcal{M}}} \equiv e_a{}^m \psi_m{}^{\boldsymbol{\mathcal{A}}} \delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}} \tag{H.78}$$

$$e_m{}^a e_a{}^n = \delta_m^n \tag{H.79}$$

We want to show that the above gauge can always be reached if the original supervielbein had full rank. To this end, let us call the supervielbein in the above gauge $E'_M{}^A(x')$ and only the original general one $E_M{}^A$. We should have the relation $\frac{\partial x'^M}{\partial x^N} E'_M{}^A(x') = E_N{}^A(x)$. Indeed, multiplying $E'_M{}^A(x')$ from the left with the transposed ($\vec{\boldsymbol{\theta}} = 0$)-Jacobian without ordinary diffeos ($\frac{\partial x'^m_0}{\partial x^n} = \delta_n^m$) yields

$$\begin{pmatrix} \delta_n^m & \frac{\partial x'^{\boldsymbol{\mathcal{M}}}_0}{\partial x^n} \\ x'^m_{\boldsymbol{\mathcal{N}}} & x'^{\boldsymbol{\mathcal{M}}}_{\boldsymbol{\mathcal{N}}} \end{pmatrix} \begin{pmatrix} e_m{}^a & \psi_m{}^{\boldsymbol{\mathcal{A}}} \\ 0 & \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}} \end{pmatrix} = \begin{pmatrix} e_n{}^a & \left(\psi_n{}^{\boldsymbol{\mathcal{A}}} + \frac{\partial x'^{\boldsymbol{\mathcal{M}}}_0}{\partial x^n} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}}\right) \\ x'^m_{\boldsymbol{\mathcal{N}}} e_m{}^a & \left(x'^m_{\boldsymbol{\mathcal{N}}} \psi_m{}^{\boldsymbol{\mathcal{A}}} + x'^{\boldsymbol{\mathcal{M}}}_{\boldsymbol{\mathcal{N}}} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}}\right) \end{pmatrix} \stackrel{!}{=} E_N{}^A| \tag{H.80}$$

This fixes some of the auxiliary gauge parameters:

$$x'^m_{\boldsymbol{\mathcal{N}}} = e_a{}^m \, E_{\boldsymbol{\mathcal{N}}}{}^a| , \qquad x'^{\boldsymbol{\mathcal{M}}}_{\boldsymbol{\mathcal{N}}} = \left(E_{\boldsymbol{\mathcal{N}}}{}^{\boldsymbol{\mathcal{A}}} - x'^m_{\boldsymbol{\mathcal{N}}} \psi_m{}^{\boldsymbol{\mathcal{A}}}\right) \delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}} \tag{H.81}$$

So all the $x'^M_{\boldsymbol{\mathcal{N}}}$ are fixed. In contrast, $x'^M_0(\vec{x})$ are still free and they parametrize bosonic diffeomorphisms and local supersymmmetry. We still have many more unfixed auxiliary gauge parameters (the higher $\boldsymbol{\theta}$-derivatives of x') whose fixing we will discuss in subsection H.2.4.

H.2.2 Calculus with the gauge fixed vielbein

Before we proceed with the gauge fixing of the connection, let us have a look at some consequences of the special vielbein gauge. The new bosonic vielbein $e_m{}^a(\vec{x}) = E_m{}^a(\vec{x}, 0)$ offers a second possibility to switch from curved to flat indices and one has to be careful, in order not to mix up things. The inverse of the supervielbein behaves differently than the inverse of the bosonic vielbein. While in superspace the inverse is with respect to a sum over all superspace indices, the sum for the bosonic inverse runs only over the bosonic indices

$$E_M{}^A E_B{}^M = \delta^A{}_B \quad \Rightarrow E_M{}^a| \, E_b{}^M| = \delta_b^a \tag{H.82}$$

$$E_m{}^a| \, e_b{}^m = \delta_b^a \tag{H.83}$$

It therefore makes a difference which vielbein is used to change from flat to curved indices and vice verse. Consider an arbitrary supervector V_M:

$$V_m| \, e_a{}^m = V_C E_m{}^C| \, e_a{}^m = \tag{H.84}$$

$$= V_c E_m{}^c| \, e_a{}^m + V_{\boldsymbol{\mathcal{C}}} E_m{}^{\boldsymbol{\mathcal{C}}}| \, e_a{}^m \tag{H.85}$$

or in summary

$$\boxed{V_m| \, e_a{}^m = V_a| + V_{\boldsymbol{\mathcal{C}}}| \, \psi_m{}^{\boldsymbol{\mathcal{C}}} e_a{}^m} \tag{H.86}$$

For upper bosonic indices the situation is better because the WZ-gauge removes the disturbing additional term:

$$V^a| \, e_a{}^m = V^N E_N{}^a| \, e_a{}^m = \tag{H.87}$$

$$= V^n E_n{}^a| \, e_a{}^m + V^{\boldsymbol{\mathcal{N}}} \underbrace{E_{\boldsymbol{\mathcal{N}}}{}^a|}_{=0} \, e_a{}^m \tag{H.88}$$

so that we get the nice relation

$$\boxed{V^a| \, e_a{}^m = V^m|} \tag{H.89}$$

We can do the same considerations for fermionic indices and arrive at the opposite situation

$$\Psi_{\boldsymbol{\mathcal{M}}}| \, \delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}} = \Psi_{\boldsymbol{\mathcal{A}}}| \tag{H.90}$$

$$\Xi^{\boldsymbol{\mathcal{M}}}| \, \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}} = \Xi^{\boldsymbol{\mathcal{A}}}| - \Xi^b| \, \psi_b{}^{\boldsymbol{\mathcal{M}}} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}} \tag{H.91}$$

H.2.3 WZ gauge for the connection

Similar to the supervielbein-case it is likewise possible to reach a special gauge at $\vec{\boldsymbol{\theta}} = 0$ for the connection componets with fermionic form-index:

$$\boxed{\Omega_{\boldsymbol{\mathcal{M}}\,A}{}^B\big| = 0} \tag{H.92}$$

Let us show that this gauge fixing is really accessible. We would like to reach the gauge (H.92) using the local structure group transformations of higher order in $\vec{\boldsymbol{\theta}}$ (i.e. with $\Lambda_A{}^B\big| = \delta_A{}^B$). Remember the structure group transformation of the connection

$$\tilde{\Omega}_{MA}{}^B(x) \quad = \quad -\partial_M \Lambda_A{}^B + (\Lambda^{-1})_A{}^D \Omega_{MD}{}^C(x)\Lambda_C{}^B \tag{H.93}$$

Reaching the gauge fixing condition (H.92) is thus possible by simply choosing

$$\Lambda_{\boldsymbol{\mathcal{M}}A}{}^B \quad \equiv \quad \partial_{\boldsymbol{\mathcal{M}}}\Lambda_A{}^B\big| \stackrel{!}{=} \Omega_{\boldsymbol{\mathcal{M}}A}{}^B(x)\big| \tag{H.94}$$

H.2.4 Gauge fixing the remaining auxiliary gauge freedom

In addition to the ordinary Wess Zumino gauge

$$E_{\boldsymbol{\mathcal{M}}}{}^A\big| \quad = \quad \delta_{\boldsymbol{\mathcal{M}}}{}^A \tag{H.95}$$

$$\Omega_{\boldsymbol{\mathcal{M}}A}{}^B\big| \quad = \quad 0 \tag{H.96}$$

we can demand the gauge fixing condition $\partial_{(\boldsymbol{\mathcal{M}}}E_{\boldsymbol{\mathcal{N}})}{}^A\big| \stackrel{!}{=} 0$ using the gauge parameter $\partial_{\boldsymbol{\mathcal{M}}}\partial_{\boldsymbol{\mathcal{N}}}\xi^A\big|$. Indeed all the other higher components of ξ^A and $L_A{}^B$ can be fixed by imposing[8] (see e.g. [119])

$$\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}E_{\boldsymbol{\mathcal{M}}_{n+1})}{}^A\big| \quad \stackrel{!}{=} \quad 0 \tag{H.97}$$

$$\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1})A}{}^B\big| \quad \stackrel{!}{=} \quad 0 \qquad \forall n \in \{1,\ldots,\dim(\boldsymbol{\mathcal{M}})-1\} \tag{H.98}$$

where $\dim(\boldsymbol{\mathcal{M}})$ shall denote the number of fermionic dimensions, e.g. 32 for type II in ten dimensions. Actually the above equations even hold for $n = \dim(\boldsymbol{\mathcal{M}})$ (the highest components of E and Ω), but then trivially, as the total graded symmetrization of $n+1$ fermionic indices (which is an antisymmetrization in fact) in n dimensions always vanishes. For $n > \dim(\boldsymbol{\mathcal{M}})$ even the derivative without graded symmetrization vanishes trivially as usual. The second equation is even true for $n = 0$ (due to (H.96)) while the first is modified for $n = 0$ to $E_{\boldsymbol{\mathcal{M}}}{}^A\big| = \delta_{\boldsymbol{\mathcal{M}}}{}^A$ (H.95).

This gauge is useful to calculate explicitely higher orders in the $\vec{\boldsymbol{\theta}}$-expansion of the vielbein or the connection in terms of torsion and curvature. Let us consider at first the connection. For the n-th partial derivative of the component with fermionic form index we can write

$$\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1}A}{}^B\big| =$$

$$= \underbrace{\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1})A}{}^B\big|}_{=0\ (H.98)} + \frac{2}{n+1}\sum_{i=1}^n \partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{[\boldsymbol{\mathcal{M}}_i|}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{|\boldsymbol{\mathcal{M}}_{n+1}]A}{}^B\big| = \tag{H.99}$$

$$= \frac{2}{n+1}\sum_{i=1}^n \partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_{i-1}}\partial_{\boldsymbol{\mathcal{M}}_{i+1}}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\left(R_{\boldsymbol{\mathcal{M}}_i\boldsymbol{\mathcal{M}}_{n+1}A}{}^B + \Omega_{[\boldsymbol{\mathcal{M}}_i|A}{}^C\Omega_{|\boldsymbol{\mathcal{M}}_{n+1}]C}{}^B\right)\Big| = \tag{H.100}$$

$$= \frac{n}{n+1}\,\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_{n-1}|}\left(2R_{|\boldsymbol{\mathcal{M}}_n)\boldsymbol{\mathcal{M}}_{n+1}A}{}^B + \Omega_{|\boldsymbol{\mathcal{M}}_n)A}{}^C\cdot\Omega_{\boldsymbol{\mathcal{M}}_{n+1}C}{}^B - \Omega_{\boldsymbol{\mathcal{M}}_{n+1}A}{}^C\cdot\Omega_{|\boldsymbol{\mathcal{M}}_n)C}{}^B\right)\Big| \tag{H.101}$$

$$\stackrel{(H.98)}{\Rightarrow} \boxed{\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1}A}{}^B\big| = \frac{2n}{n+1}\,\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_{n-1}}R_{\boldsymbol{\mathcal{M}}_n)\boldsymbol{\mathcal{M}}_{n+1}A}{}^B\big|} \quad \forall n \geq 1 \tag{H.102}$$

Unfortunately, due to the n-dependent factor $\frac{2n}{n+1}$, this relation cannot easily be integrated. In particular, although the above equation implies $\partial_{\boldsymbol{\mathcal{M}}}\Omega_{\boldsymbol{\mathcal{N}}A}{}^B\big| = R_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}A}{}^B\big|$, we have in general $\partial_{\boldsymbol{\mathcal{M}}}\Omega_{\boldsymbol{\mathcal{N}}A}{}^B \neq R_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}A}{}^B$. Also $\Omega_{\boldsymbol{\mathcal{N}}A}{}^B \neq x^{\boldsymbol{\mathcal{M}}}R_{\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}A}{}^B$.

---

[8] Looking at the infinitesimal transformations

$$\delta\left(\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}E_{\boldsymbol{\mathcal{M}}_{n+1}}{}^A\right)\Big| \quad = \quad \partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\left(\partial_{\boldsymbol{\mathcal{M}}_{n+1}}\xi^A + \Omega_{\boldsymbol{\mathcal{M}}_{n+1}B}{}^A\xi^B + 2\xi^C T_{CM}{}^A\right)\Big| =$$

$$\delta\left(\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1}A}{}^B\right)\Big| \quad = \quad -\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\left(\partial_{\boldsymbol{\mathcal{M}}_{n+1}}L_A{}^B + [L,\Omega_{\boldsymbol{\mathcal{M}}_{n+1}}]\right)\Big|$$

it seems quite obvious that the parameters $\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_{n+1}}\xi^A\big|$ and $\partial_{\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_{n+1}}L_A{}^B\big|$ can be used to shift $\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}E_{\boldsymbol{\mathcal{M}}_{n+1})}{}^A\big|$ and $\partial_{(\boldsymbol{\mathcal{M}}_1}\cdots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1})A}{}^B\big|$ to whatever value one likes. A rigorous proof that (H.97) and (H.98) are accessible, however, should consider the finite transformations. ◇

The calculation for the components of the vielbein is very similar

$$\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}E_{\mathcal{M}_{n+1}}{}^A\big| =$$

$$= \underbrace{\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}E_{\mathcal{M}_{n+1})}{}^A\big|}_{=0\;(H.97)} + \frac{2}{n+1}\sum_{i=1}^{n}\partial_{\mathcal{M}_1}\dots\partial_{[\mathcal{M}_i|}\dots\partial_{\mathcal{M}_n}E_{|\mathcal{M}_{n+1}]}{}^A\big| = \tag{H.103}$$

$$= \frac{2}{n+1}\sum_{i=1}^{n}\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_{i-1}}\partial_{\mathcal{M}_{i+1}}\dots\partial_{\mathcal{M}_n}\left(T_{\mathcal{M}_i\mathcal{M}_{n+1}}{}^A + E_{[\mathcal{M}_i}{}^B\Omega_{\mathcal{M}_{n+1}]B}{}^A\right)\big| = \tag{H.104}$$

$$= \frac{n}{n+1}\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}|}\left(2T_{|\mathcal{M}_n)\mathcal{M}_{n+1}}{}^A + E_{|\mathcal{M}_n)}{}^B\Omega_{\mathcal{M}_{n+1}B}{}^A - E_{\mathcal{M}_{n+1}}{}^B\Omega_{|\mathcal{M}_n)B}{}^A\right)\big| \tag{H.105}$$

For the second and third term in the bracket we can use (H.97) and (H.98) again, so that the third term will vanish while from the second term we get a contribution only when all derivatives act on the connection, because $E_{\mathcal{M}_n}{}^B\big| = \delta_{\mathcal{M}_n}{}^B$. Using (H.102), we arrive at

$$\boxed{\begin{aligned}&\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}E_{\mathcal{M}_{n+1}}{}^A\big| = \qquad\qquad\qquad\qquad \forall n\geq 1\\ &= \frac{2n}{n+1}\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}}T_{\mathcal{M}_n)\mathcal{M}_{n+1}}{}^A\big| + \frac{2(n-1)}{n+1}\delta_{(\mathcal{M}_1}{}^{\mathcal{B}}\,\partial_{\mathcal{M}_2}\dots\partial_{\mathcal{M}_{n-1}}R_{\mathcal{M}_n)\mathcal{M}_{n+1}\mathcal{B}}{}^A\big|\end{aligned}} \tag{H.106}$$

In particular we get for $n = 1$

$$\partial_{\mathcal{M}}E_{\mathcal{N}}{}^A\big| = T_{\mathcal{M}\mathcal{N}}{}^A\big|, \qquad \partial_{\mathcal{M}}\Omega_{\mathcal{N}A}{}^B\big| = R_{\mathcal{M}\mathcal{N}A}{}^B\big| \tag{H.107}$$

The higher $\vec{\boldsymbol{\theta}}$-components of the vielbein and connection parts with bosonic form index ($E_m{}^A$ and $\Omega_{mA}{}^B$) can likewise be expressed in terms of torsion and curvature:

$$\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}\Omega_{mA}{}^B\big| = \frac{2}{n}\sum_{i=1}^{n}\partial_{\mathcal{M}_1}\dots\partial_{[\mathcal{M}_i|}\dots\partial_{\mathcal{M}_n}\Omega_{|m]A}{}^B\big| + \partial_m\underbrace{\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}}\Omega_{\mathcal{M}_n)A}{}^B\big|}_{=0\;(H.98)} = \tag{H.108}$$

$$= 2\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}|}\left(R_{|\mathcal{M}_n)mA}{}^B + \frac{1}{2}\Omega_{|\mathcal{M}_n)A}{}^C\Omega_{mC}{}^B - \frac{1}{2}\Omega_{mA}{}^C\Omega_{|\mathcal{M}_n)C}{}^B\right)\Bigg| \tag{H.109}$$

$$\overset{(H.98)}{\Rightarrow}\boxed{\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}\Omega_{mA}{}^B\big| = 2\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}|}R_{|\mathcal{M}_n)mA}{}^B\big|}\;\forall n\geq 1 \tag{H.110}$$

Although in contrast to (H.102) we do not have an n-dependent factor, we have in general $\partial_{\mathcal{M}}\Omega_{mA}{}^B \neq 2R_{\mathcal{M}mA}{}^B$ away from $\vec{\boldsymbol{\theta}} = 0$. The reason for this fact is the symmetrization on the righthand side. Also we have $\Omega_{mA}{}^B \neq 2x^{\mathcal{M}}R_{\mathcal{M}mA}{}^B$ for $\vec{\boldsymbol{\theta}} \neq 0$.

For the vielbein the situation is again similar:

$$\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}E_n{}^A\big| = \frac{2}{n}\sum_{i=1}^{n}\partial_{\mathcal{M}_1}\dots\partial_{[\mathcal{M}_i|}\dots\partial_{\mathcal{M}_n}E_{|m]}{}^A\big| + \underbrace{\partial_m\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}}E_{\mathcal{M}_n)}{}^A\big|}_{=0\;(H.97),(H.95)} = \tag{H.111}$$

$$= 2\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}|}\left(T_{|\mathcal{M}_n)m}{}^A + \frac{1}{2}E_{|\mathcal{M}_n)}{}^B\Omega_{mB}{}^A - \frac{1}{2}E_m{}^B\Omega_{|\mathcal{M}_n)B}{}^A\right)\Bigg| = \tag{H.112}$$

$$\underset{(H.95)}{\overset{(H.97),(H.98)}{=}} 2\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}}T_{\mathcal{M}_n)m}{}^A\big| + \delta_{(\mathcal{M}_n}{}^B\,\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1})}\Omega_{mB}{}^A\big| \tag{H.113}$$

In particular for $n = 1$ we get

$$\boxed{\partial_{\mathcal{M}}\,E_m{}^A\big| = 2\,T_{\mathcal{M}m}{}^A\big| + \delta_{\mathcal{M}}{}^B\,\Omega_{mB}{}^A\big|} \tag{H.114}$$

while for $n > 1$ we can use (H.110) to arrive at

$$\boxed{\partial_{\mathcal{M}_1}\dots\partial_{\mathcal{M}_n}E_n{}^A\big| = 2\,\partial_{(\mathcal{M}_1}\dots\partial_{\mathcal{M}_{n-1}}T_{\mathcal{M}_n)m}{}^A\big| + 2\delta_{(\mathcal{M}_1}{}^{\mathcal{B}}\,\partial_{\mathcal{M}_2}\dots\partial_{\mathcal{M}_{n-1}}R_{\mathcal{M}_n)m\mathcal{B}}{}^A\big|}\;\forall n\geq 2 \tag{H.115}$$

In practice we are given constraints on torsion and curvature components with only flat indices. Rewriting the equations (H.102),(H.106),(H.110),(H.114) and (H.115) with flat components yields the following rekursion realtions

$$
\begin{aligned}
\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_n}\Omega_{\mathcal{M}_{n+1}A}{}^B\big| &= \frac{2n}{n+1}\,\delta_{(\mathcal{M}_n}{}^{\mathcal{C}}\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_{n-1})}(E_{\mathcal{M}_{n+1}}{}^D R_{\mathcal{C}DA}{}^B)\big| \quad \forall n\geq 1 && \text{(H.116)}\\
\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_n}E_{\mathcal{M}_{n+1}}{}^A\big| &= \frac{2n}{n+1}\,\delta_{(\mathcal{M}_n}{}^{\mathcal{C}}\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_{n-1})}(E_{\mathcal{M}_{n+1}}{}^D T_{\mathcal{C}D}{}^A)\big| + \qquad (\forall n\geq 1)\\
&\quad +\frac{2(n-1)}{n+1}\delta_{(\mathcal{M}_{n-1}}{}^{\mathcal{C}}\delta_{\mathcal{M}_n}{}^{\mathcal{B}}\,\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_{n-2})}(E_{\mathcal{M}_{n+1}}{}^D R_{\mathcal{C}D\mathcal{B}}{}^A)\big| && \text{(H.117)}\\
\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_n}\Omega_{mA}{}^B\big| &= 2\delta_{(\mathcal{M}_n}{}^{\mathcal{C}}\,\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_{n-1})}(E_m{}^D R_{\mathcal{C}DA}{}^B)\big| \;\forall n\geq 1 && \text{(H.118)}\\
\partial_{\mathcal{M}}\,E_m{}^A\big| &= 2\delta_{\mathcal{M}}{}^{\mathcal{C}}\,E_m{}^D T_{\mathcal{C}D}{}^A\big| + \delta_{\mathcal{M}}{}^{\mathcal{B}}\,\Omega_{m\mathcal{B}}{}^A\big| && \text{(H.119)}\\
\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_n}E_n{}^A\big| &= 2\delta_{(\mathcal{M}_n}{}^{\mathcal{C}}\,\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_{n-1})}(E_m{}^D T_{\mathcal{C}D}{}^A)\big| +\\
&\quad +2\delta_{(\mathcal{M}_n}{}^{\mathcal{B}}\delta_{\mathcal{M}_{n-1}}{}^{\mathcal{C}}\,\partial_{\mathcal{M}_1}\ldots\partial_{\mathcal{M}_{n-2})}(E_m{}^D R_{\mathcal{C}D\mathcal{B}}{}^A)\big| \;\forall n\geq 2 && \text{(H.120)}
\end{aligned}
$$

Let us do the first steps of the iteration, in order to see what is happening:

$$
\begin{aligned}
n=0:\quad \Omega_{\mathcal{M}A}{}^B\big| &= 0,\quad \Omega_{mA}{}^B\big| \equiv \omega_{mA}{}^B && \text{(H.121)}\\
E_{\mathcal{M}}{}^A\big| &= \delta_{\mathcal{M}}{}^A,\quad E_m{}^a\big| \equiv e_m{}^a,\quad E_m{}^{\mathcal{A}}\big| \equiv \psi_m{}^{\mathcal{A}} && \text{(H.122)}\\
n=1:\quad \partial_{\mathcal{M}_1}\Omega_{\mathcal{M}_2A}{}^B\big| &= \delta_{\mathcal{M}_1}{}^{\mathcal{C}}\delta_{\mathcal{M}_2}{}^{\mathcal{D}}\,R_{\mathcal{C}\mathcal{D}A}{}^B\big|,\quad \partial_{\mathcal{M}}\Omega_{nA}{}^B\big| = 2\delta_{\mathcal{M}}{}^{\mathcal{C}}e_n{}^d\,R_{\mathcal{C}dA}{}^B\big| + 2\delta_{\mathcal{M}}{}^{\mathcal{C}}\psi_n{}^{\mathcal{D}}\,R_{\mathcal{C}\mathcal{D}A}{}^B\big| && \text{(H.123)}\\
\partial_{\mathcal{M}_1}E_{\mathcal{M}_2}{}^A\big| &= \delta_{\mathcal{M}_1}{}^{\mathcal{C}}\delta_{\mathcal{M}_2}{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^A\big|,\quad \partial_{\mathcal{M}}E_n{}^a\big| = 2\delta_{\mathcal{M}}{}^{\mathcal{C}}e_n{}^d\,T_{\mathcal{C}d}{}^a\big| + 2\delta_{\mathcal{M}}{}^{\mathcal{C}}\psi_n{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^a\big|\\
&\quad \partial_{\mathcal{M}}E_n{}^{\mathcal{A}}\big| = 2\delta_{\mathcal{M}}{}^{\mathcal{C}}e_n{}^d\,T_{\mathcal{C}d}{}^{\mathcal{A}}\big| + 2\delta_{\mathcal{M}}{}^{\mathcal{C}}\psi_n{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^{\mathcal{A}}\big| + \delta_{\mathcal{M}}{}^{\mathcal{B}}\omega_{n\mathcal{B}}{}^{\mathcal{A}} && \text{(H.124)}\\
n=2:\ \partial_{\mathcal{M}_1}\partial_{\mathcal{M}_2}\Omega_{\mathcal{M}_3A}{}^B\big| &= \frac{4}{3}\delta_{(\mathcal{M}_2|}{}^{\mathcal{C}}\delta_{|\mathcal{M}_1)}{}^{\mathcal{E}}\delta_{\mathcal{M}_3}{}^{\mathcal{F}}\,T_{\mathcal{E}\mathcal{F}}{}^D\big|\,R_{\mathcal{C}DA}{}^B\big| + \frac{4}{3}\delta_{(\mathcal{M}_2|}{}^{\mathcal{C}}\delta_{\mathcal{M}_3}{}^{\mathcal{D}}\partial_{|\mathcal{M}_1)}R_{\mathcal{C}\mathcal{D}A}{}^B && \text{(H.125)}\\
\partial_{\mathcal{M}_1}\partial_{\mathcal{M}_2}\Omega_{mA}{}^B\big| &= 2\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}\left(2\delta_{\mathcal{M}_1)}{}^{\mathcal{E}}e_n{}^f\,T_{\mathcal{E}f}{}^D\big| + 2\delta_{\mathcal{M}_1)}{}^{\mathcal{E}}\psi_m{}^{\mathcal{F}}\,T_{\mathcal{E}\mathcal{F}}{}^D\big| + \delta_{\mathcal{M}_1)}{}^{\mathcal{E}}\omega_{m\mathcal{E}}{}^D\right)R_{\mathcal{C}DA}{}^B\big| +\\
&\quad +2\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}e_m{}^d\,\partial_{\mathcal{M}_1)}R_{\mathcal{C}dA}{}^B\big| + 2\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}\psi_m{}^{\mathcal{D}}\,\partial_{\mathcal{M}_1)}R_{\mathcal{C}\mathcal{D}A}{}^B\big| && \text{(H.126)}\\
\partial_{\mathcal{M}_1}\partial_{\mathcal{M}_2}E_{\mathcal{M}_3}{}^A\big| &= \frac{4}{3}\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}\delta_{\mathcal{M}_1)}{}^{\mathcal{E}}\delta_{\mathcal{M}_3}{}^{\mathcal{F}}\,T_{\mathcal{E}\mathcal{F}}{}^D\big|\,T_{\mathcal{C}D}{}^A\big| + \frac{4}{3}\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}\delta_{\mathcal{M}_3}{}^{\mathcal{D}}\,\partial_{\mathcal{M}_1)}T_{\mathcal{C}\mathcal{D}}{}^A\big| +\\
&\quad +\frac{2}{3}\delta_{(\mathcal{M}_1}{}^{\mathcal{C}}\delta_{\mathcal{M}_2)}{}^{\mathcal{B}}\delta_{\mathcal{M}_3}{}^{\mathcal{D}}\,R_{\mathcal{C}\mathcal{D}\mathcal{B}}{}^A\big| && \text{(H.127)}\\
\partial_{\mathcal{M}_1}\partial_{\mathcal{M}_2}E_n{}^A\big| &= 2\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}\left(2\delta_{\mathcal{M}_1)}{}^{\mathcal{E}}e_m{}^f\,T_{\mathcal{E}f}{}^D\big| + 2\delta_{\mathcal{M}_1)}{}^{\mathcal{E}}\psi_m{}^{\mathcal{F}}\,T_{\mathcal{E}\mathcal{F}}{}^D\big| + \delta_{\mathcal{M}_1)}{}^{\mathcal{E}}\omega_{m\mathcal{E}}{}^D\right)T_{\mathcal{C}D}{}^A\big| +\\
&\quad +2\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}e_m{}^d\,\partial_{\mathcal{M}_1)}T_{\mathcal{C}d}{}^A\big| + 2\delta_{(\mathcal{M}_2}{}^{\mathcal{C}}\psi_m{}^{\mathcal{D}}\,\partial_{\mathcal{M}_1)}T_{\mathcal{C}\mathcal{D}}{}^A\big| +\\
&\quad +2\delta_{(\mathcal{M}_2}{}^{\mathcal{B}}\delta_{\mathcal{M}_1)}{}^{\mathcal{C}}e_m{}^d\,R_{\mathcal{C}d\mathcal{B}}{}^A\big| + 2\delta_{(\mathcal{M}_2}{}^{\mathcal{B}}\delta_{\mathcal{M}_1)}{}^{\mathcal{C}}\psi_m{}^{\mathcal{D}}\,R_{\mathcal{C}\mathcal{D}\mathcal{B}}{}^A\big| && \text{(H.128)}\\
&\ddots
\end{aligned}
$$

Apparently this iteration gets very involved for higher orders, but in principle we can express every supervielbein component and superconnection component in terms of the bosonic vielbein, the gravitinos, the bosonic connection and the torsion and curvature components. Note finally that the components $\Gamma_{\mathcal{M}N}{}^K$ of the superspace connection do not vanish at leading order like the structure group connection. Instead we find because of $\Gamma_{MN}{}^K = \left(\partial_M E_N{}^C + E_N{}^B\Omega_{MB}{}^C\right)E_C{}^K$ for the leading order that

$$
\Gamma_{\mathcal{M}N}{}^K\big| \overset{(H.96)}{=} \partial_{\mathcal{M}}E_N{}^C\big|\,E_C{}^K\big| \qquad \text{(H.129)}
$$

Using some of the equations above, this implies in particular

$$
\begin{aligned}
\Gamma_{\mathcal{M}n}{}^K\big| &= 2\,T_{\mathcal{M}n}{}^a\big|\,E_a{}^K\big| + 2\,T_{\mathcal{M}n}{}^{\mathcal{A}}\big|\,\delta_{\mathcal{A}}{}^K + \delta_{\mathcal{M}}{}^{\mathcal{B}}\omega_{n\mathcal{B}}{}^{\mathcal{A}}\delta_{\mathcal{A}}{}^K = && \text{(H.130)}\\
&= 2\delta_{\mathcal{M}}{}^{\mathcal{C}}e_n{}^d\,T_{\mathcal{C}d}{}^a\big|\,E_a{}^K\big| + 2\delta_{\mathcal{M}}{}^{\mathcal{C}}\psi_n{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^a\big|\,E_a{}^K\big| +\\
&\quad +2\delta_{\mathcal{M}}{}^{\mathcal{C}}e_n{}^d\,T_{\mathcal{C}d}{}^{\mathcal{A}}\big|\,\delta_{\mathcal{A}}{}^K + 2\delta_{\mathcal{M}}{}^{\mathcal{C}}\psi_n{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^{\mathcal{A}}\big|\,\delta_{\mathcal{A}}{}^K + \delta_{\mathcal{M}}{}^{\mathcal{B}}\omega_{n\mathcal{B}}{}^{\mathcal{A}}\delta_{\mathcal{A}}{}^K && \text{(H.131)}\\
\Gamma_{\mathcal{M}\mathcal{N}}{}^K\big| &= \delta_{\mathcal{M}}{}^{\mathcal{C}}\delta_{\mathcal{N}}{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^a\big|\,E_a{}^K\big| + \delta_{\mathcal{M}}{}^{\mathcal{C}}\delta_{\mathcal{N}}{}^{\mathcal{D}}\,T_{\mathcal{C}\mathcal{D}}{}^{\mathcal{A}}\big|\,\delta_{\mathcal{A}}{}^K && \text{(H.132)}
\end{aligned}
$$

H.3 Partial Gauge Fixing of the B-superfield

Although the gauge fixing of the B-field is not necessary in order to obtain the supergravity transformations, we will discuss it at this place, as it is very similar to the gauge fixings of connection and vielbein. Again we want to fix only the auxiliary gauge degrees but leave the gauge freedom of the bosonic two-form. The B-field gauge symmetry is of the form $B \to B + \mathbf{d}\Lambda$, with some one-form Λ. Let us split the gauge transformation into three cases with different index structures:

$$B_{\mathcal{MN}} \to B_{\mathcal{MN}} + \partial_{[\mathcal{M}}\Lambda_{\mathcal{N}]} \tag{H.133}$$
$$B_{\mathcal{M}n} \to B_{\mathcal{M}n} + \partial_{[\mathcal{M}}\Lambda_{n]} \tag{H.134}$$
$$B_{mn} \to B_{mn} + \partial_{[m}\Lambda_{n]} \tag{H.135}$$

In the $\vec{\boldsymbol{\theta}}$-expansion, we thus have

$$\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{MN}}\big| \to \partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{MN}}\big| + \frac{1}{2}\,\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\partial_{\mathcal{M}}\Lambda_{\mathcal{N}}\big| - \frac{1}{2}\,\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\partial_{\mathcal{N}}\Lambda_{\mathcal{M}}\big| \tag{H.136}$$
$$\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M}n}\big| \to \partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M}n}\big| + \frac{1}{2}\,\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\partial_{\mathcal{M}}\Lambda_n\big| - \frac{1}{2}\partial_n\,\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\Lambda_{\mathcal{M}}\big| \tag{H.137}$$
$$\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{mn}\big| \to \partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{mn}\big| + \frac{1}{2}\partial_m\,\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\Lambda_n\big| - \frac{1}{2}\partial_n\,\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\Lambda_m\big| \tag{H.138}$$

The gauge symmetries of the first two lines can be used to set $\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})\mathcal{N}}\big| - \partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{N})\mathcal{M}}\big|$ and $\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})n}\big|$ to any value one likes. This fixes Λ_M up to a de-Rham closed term (as usual) and up to the bosonic gauge parameter $\Lambda_m|$. We want to choose a gauge in such a way that for $p \geq 1$, the higher orders in the $\vec{\boldsymbol{\theta}}$-expansion can be expressed in a simple way in terms of the H-flux $H_{MNK} \equiv \partial_{[M}B_{NK]}$. To this end consider

$$\begin{aligned}
&3p\cdot\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}H_{\mathcal{K}_p)\mathcal{MN}} = \\
&= 3\sum_{i=1}^{p}\partial_{\mathcal{K}_1}\dots\partial_{[\mathcal{K}_i|}\dots\partial_{\mathcal{K}_p}B_{|\mathcal{MN}]} = && \text{(H.139)}\\
&= p\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{MN}} - \sum_{i=1}^{p}\left(\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{M}}\dots\partial_{\mathcal{K}_p}B_{\mathcal{K}_i\mathcal{N}} - \partial_{\mathcal{K}_1}\dots\partial_{\mathcal{N}}\dots\partial_{\mathcal{K}_p}B_{\mathcal{K}_i\mathcal{M}}\right) = && \text{(H.140)}\\
&= (p+2)\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{MN}} - (p+1)\left(\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})\mathcal{N}} - \partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{N})\mathcal{M}}\right) && \text{(H.141)}
\end{aligned}$$

This suggests to choose the gauge

$$\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})\mathcal{N}}\big| - \partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{N})\mathcal{M}}\big| \overset{!}{=} 0 \quad \forall p \tag{H.142}$$

which fixes $\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\partial_{[\mathcal{M}}\Lambda_{\mathcal{N}]}\big|$. The above equation is a trivial statement for p equal or bigger as the fermionic dimensions (i.e. 32 for a ten-dimensional spacetime and type II), because the graded symmetrization of fermionic indices (i.e. their antisymmetrization) vanishes when the number of indices exceeds the dimension. On the other hand the statement is a very strong one for $p = 0$, where we simply get $B_{\mathcal{MN}}| = 0$.

The choice for the gauge in the case with mixed index structure is not as obvious as above:

$$\begin{aligned}
&3p\cdot\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}H_{\mathcal{K}_p)\mathcal{M}n} = \\
&= 3\sum_{i=1}^{p}\partial_{\mathcal{K}_1}\dots\partial_{[\mathcal{K}_i|}\dots\partial_{\mathcal{K}_p}B_{|\mathcal{M}n]} = && \text{(H.143)}\\
&= p\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M}n} - \sum_{i=1}^{p}\left(\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{M}}\dots\partial_{\mathcal{K}_p}B_{\mathcal{K}_i n} - \partial_{\mathcal{K}_1}\dots\partial_n\dots\partial_{\mathcal{K}_p}B_{\mathcal{K}_i\mathcal{M}}\right) = && \text{(H.144)}\\
&= (p+1)\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M}n} - \left((p+1)\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})n} - p\partial_n\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}B_{\mathcal{K}_p)\mathcal{M}}\right) && \text{(H.145)}
\end{aligned}$$

Instead of setting $\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})n}\big|$ to zero (which is of course a valid choice, too), it seems more convenient here to choose

$$\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{\mathcal{M})n}\big| = \frac{p}{p+1}\partial_n\,\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}B_{\mathcal{K}_p)\mathcal{M}}\big| \quad \forall p \tag{H.146}$$

which fixes $\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}\partial_{\mathcal{M}}\Lambda_n\big|$. Now we have fixed as much as we can and hope that the remaining components behave in a nice way:

$$\begin{aligned}
&3p\cdot\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}H_{\mathcal{K}_p)mn} = \\
&= 3\sum_{i=1}^{p}\partial_{\mathcal{K}_1}\dots\partial_{[\mathcal{K}_i|}\dots\partial_{\mathcal{K}_p}B_{|mn]} = && \text{(H.147)}\\
&= p\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{mn} - \sum_{i=1}^{p}\left(\partial_{\mathcal{K}_1}\dots\partial_m\dots\partial_{\mathcal{K}_p}B_{\mathcal{K}_i n} - \partial_{\mathcal{K}_1}\dots\partial_n\dots\partial_{\mathcal{K}_p}B_{\mathcal{K}_i m}\right) = && \text{(H.148)}\\
&= p\partial_{\mathcal{K}_1}\dots\partial_{\mathcal{K}_p}B_{mn} - p\left(\partial_m\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}B_{\mathcal{K}_p)n} - \partial_n\partial_{(\mathcal{K}_1}\dots\partial_{\mathcal{K}_{p-1}}B_{\mathcal{K}_p)m}\right) && \text{(H.149)}
\end{aligned}$$

Indeed, the gauge fixing condition (H.146) is fine to remove the last terms for $\vec{\boldsymbol{\theta}} = 0$. Plugging (H.142) in (H.141) and (H.146) in (H.145) and (H.149), we can express all auxiliary components of B in terms of some H-field components:

$$\partial_{\mathcal{K}_1}\ldots\partial_{\mathcal{K}_p}B_{\mathcal{M}\mathcal{N}}\big| = \frac{3p}{p+2}\cdot\partial_{(\mathcal{K}_1}\ldots\partial_{\mathcal{K}_{p-1}}H_{\mathcal{K}_p)\mathcal{M}\mathcal{N}}\big| \quad \forall p\geq 1, \qquad B_{\mathcal{M}\mathcal{N}}\big| = 0 \quad (p=0) \tag{H.150}$$

$$\partial_{\mathcal{K}_1}\ldots\partial_{\mathcal{K}_p}B_{\mathcal{M}n}\big| = \frac{3p}{p+1}\cdot\partial_{(\mathcal{K}_1}\ldots\partial_{\mathcal{K}_{p-1}}H_{\mathcal{K}_p)\mathcal{M}n}\big| \quad \forall p\geq 1, \qquad B_{\mathcal{M}n}\big| = 0 \quad (p=0) \tag{H.151}$$

$$\partial_{\mathcal{K}_1}\ldots\partial_{\mathcal{K}_p}B_{mn}\big| = \underbrace{3}_{\frac{3p}{p}}\,\partial_{(\mathcal{K}_1}\ldots\partial_{\mathcal{K}_{p-1}}H_{\mathcal{K}_p)mn}\big| \quad \forall p\geq 1 \tag{H.152}$$

Again, the constraints on the components of H wil be given in flat coordinates. Rewriting the above set of equations correspondingly, produces derivatives acting on the vielbein. We thus get again a recursion relation which is coupled to the recursion relation for the vielbein.

H.4 Stabilizer

In order to recover the supergravity transformations, we need to determine those supergauge transformations which leave the Wess-Zumino-gauge and the additional gauge fixing conditions untouched.

H.4.1 Stabilizer of the Wess Zumino gauge

Let us start with the vielbein which was fixed to $E_{\mathcal{M}}{}^{A}\big| = \delta_{\mathcal{M}}{}^{A}$ (H.76), and remember the general transformation (H.19)

$$\delta E_M{}^A = \underbrace{\partial_M\xi^A + \Omega_{MC}{}^A\xi^C}_{\nabla_M\xi^A} + 2\xi^C T_{CM}{}^A + L_B{}^A E_M{}^B \tag{H.153}$$

Let us denote the first components in the $\vec{\boldsymbol{\theta}}$-expansion of the transformation parameters as follows

$$\xi^A \equiv \xi_0^A + x^{\mathcal{M}}\xi^A_{\mathcal{M}} + \ldots \tag{H.154}$$

$$L_A{}^B \equiv L_{0\,A}{}^B + x^{\mathcal{M}}L_{\mathcal{M}\,A}{}^B + \ldots \tag{H.155}$$

The $\vec{\boldsymbol{\theta}} = 0$ component of $E_{\mathcal{M}}{}^A$ in the WZ gauge then transforms as

$$\delta\, E_{\mathcal{M}}{}^A\big| = \xi^A_{\mathcal{M}} + \underbrace{\Omega_{\mathcal{M}C}{}^A\big|}_{=0\ (H.92)}\xi_0^C + 2\xi_0^C\ T_{C\mathcal{M}}{}^A\big| + L_{0\,B}{}^A\ \underbrace{E_{\mathcal{M}}{}^B\big|}_{\delta_{\mathcal{M}}{}^B\ (H.76)} = \tag{H.156}$$

$$= \xi^A_{\mathcal{M}} + 2\xi_0^C\ T_{C\mathcal{M}}{}^A\big| + L_{0\,\mathcal{B}}{}^A\delta_{\mathcal{M}}{}^{\mathcal{B}} \tag{H.157}$$

In order to preserve the gauge of the vielbein, we thus need that the above variation vanishes

$$\boxed{\xi^A_{\mathcal{M}} = -\delta_{\mathcal{M}}{}^{\mathcal{B}}\left(2\xi_0^C\ T_{C\mathcal{B}}{}^A\big| + L_{0\,\mathcal{B}}{}^A\right)} \tag{H.158}$$

This result is very general, without any restriction on the structure group. In order to become more explicit, let us now assume that the structure group is **block-diagonal** and split the index A into $(a, \boldsymbol{\mathcal{A}})$. (Remember, the fermionic index might further decay, e.g. for type II in ten dimensions into $\boldsymbol{\mathcal{A}} = (\boldsymbol{\alpha}, \hat{\boldsymbol{\alpha}})$.) The vector ξ^A can then be written as

$$\xi^a = \xi_0^a - 2x^{\mathcal{M}}\delta_{\mathcal{M}}{}^{\mathcal{B}}\xi_0^C\ T_{C\mathcal{B}}{}^a\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) \tag{H.159}$$

$$\xi^{\boldsymbol{\mathcal{A}}} = \xi_0^{\boldsymbol{\mathcal{A}}} - x^{\mathcal{M}}\delta_{\mathcal{M}}{}^{\mathcal{B}}\left(2\xi_0^C\ T_{C\mathcal{B}}{}^{\boldsymbol{\mathcal{A}}}\big| + L_{0\,\mathcal{B}}{}^{\boldsymbol{\mathcal{A}}}\right) + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) \tag{H.160}$$

In this appendix, we will not make use of any torsion constraints. This will be done in the main part.

The gauge fixing condition of the connection was $\Omega_{\mathcal{M}A}{}^B\big| = 0$, while its general gauge transformation reads (H.25)

$$\delta\Omega_{MA}{}^B = 2\xi^K R_{KMA}{}^B - \partial_M L_A{}^B - [L, \Omega_M]_A{}^B \tag{H.161}$$

The gauge is thus preserved if

$$\boxed{L_{\mathcal{M}A}{}^B \stackrel{!}{=} 2\delta_{\mathcal{M}}{}^{\mathcal{D}}\xi_0^C\ R_{C\mathcal{D}A}{}^B\big|} \tag{H.162}$$

or

$$L_A{}^B(\vec{x}, \vec{\boldsymbol{\theta}}) = L_{0\,A}{}^B(\vec{x}) + 2x^{\mathcal{M}}\delta_{\mathcal{M}}{}^{\mathcal{D}}\xi_0^C\ R_{C\mathcal{D}A}{}^B\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) \tag{H.163}$$

H.4.2 Stabilizer of the additional gauge fixing conditions

Remember the additional gauge fixing conditions (H.97) and (H.98)

$$\partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n}E_{\boldsymbol{\mathcal{M}}_{n+1})}{}^A\big| \stackrel{!}{=} 0, \qquad \partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1})A}{}^B\big| \stackrel{!}{=} 0 \qquad \forall n\geq 1 \tag{H.164}$$

Stabilizing the first condition

$$\begin{aligned}
&\delta\,\partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n}E_{\boldsymbol{\mathcal{M}}_{n+1})}{}^A\big| = \\
&= \partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n|}\left(\partial_{|\boldsymbol{\mathcal{M}}_{n+1})}\xi^A + \Omega_{|\boldsymbol{\mathcal{M}}_{n+1})C}{}^A\xi^C + 2\xi^C T_{C|\boldsymbol{\mathcal{M}}_{n+1})}{}^A + L_{\boldsymbol{\mathcal{B}}}{}^A E_{|\boldsymbol{\mathcal{M}}_{n+1})}{}^{\boldsymbol{\mathcal{B}}}\right)\big| = &\text{(H.165)}\\
&= \partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n|}\left(\partial_{|\boldsymbol{\mathcal{M}}_{n+1})}\xi^A + \delta_{|\boldsymbol{\mathcal{M}}_{n+1})}{}^{\boldsymbol{\mathcal{B}}}\left(2\xi^C T_{C\boldsymbol{\mathcal{B}}}{}^A + L_{\boldsymbol{\mathcal{B}}}{}^A\right)\right)\big| &\text{(H.166)}
\end{aligned}$$

implies

$$\boxed{\partial_{\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_{n+1}}\xi^A\big| = -\partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n|}\left(2\xi^C T_{C\boldsymbol{\mathcal{B}}}{}^A + L_{\boldsymbol{\mathcal{B}}}{}^A\right)\big|\,\delta_{|\boldsymbol{\mathcal{M}}_{n+1})}{}^{\boldsymbol{\mathcal{B}}}} \qquad \forall n\geq 1 \tag{H.167}$$

This is actually recursion relation again. For the second fermionic derivative of the transformation parameter e.g., we get

$$\begin{aligned}
\partial_{\boldsymbol{\mathcal{M}}_1}\partial_{\boldsymbol{\mathcal{M}}_2}\xi^A\big| &= -2\xi^C_{(\boldsymbol{\mathcal{M}}_1|}\,T_{C|\boldsymbol{\mathcal{M}}_2)}{}^A\big| - 2\xi^C_0\,\partial_{(\boldsymbol{\mathcal{M}}_1|}T_{C|\boldsymbol{\mathcal{M}}_2)}{}^A\big| - L_{(\boldsymbol{\mathcal{M}}_1\boldsymbol{\mathcal{M}}_2)}{}^A = &\text{(H.168)}\\
&= 2\xi^C_0\left(2\,T_{C(\boldsymbol{\mathcal{M}}_1|}{}^E\big|\,T_{E|\boldsymbol{\mathcal{M}}_2)}{}^A\big| - \partial_{(\boldsymbol{\mathcal{M}}_1|}T_{C|\boldsymbol{\mathcal{M}}_2)}{}^A\big| - R_{C(\boldsymbol{\mathcal{M}}_1\boldsymbol{\mathcal{M}}_2)}{}^A\big|\right) + \\
&\quad +2L_{0\,(\boldsymbol{\mathcal{M}}_1|}{}^C\,T_{C|\boldsymbol{\mathcal{M}}_2)}{}^A\big| &\text{(H.169)}
\end{aligned}$$

Stabilizing finally the second additional condition (the one on the connection)

$$\begin{aligned}
&\delta\,\partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n}\Omega_{\boldsymbol{\mathcal{M}}_{n+1})A}{}^B\big| = \\
&= \partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n|}\left(2\xi^K R_{K|\boldsymbol{\mathcal{M}}_{n+1})A}{}^B - \partial_{|\boldsymbol{\mathcal{M}}_{n+1})}L_A{}^B - [L,\Omega_{|\boldsymbol{\mathcal{M}}_{n+1})}]_A{}^B\right)\big| = &\text{(H.170)}\\
&= \partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n|}\left(2\xi^K R_{K|\boldsymbol{\mathcal{M}}_{n+1})A}{}^B - \partial_{|\boldsymbol{\mathcal{M}}_{n+1})}L_A{}^B\right)\big| &\text{(H.171)}
\end{aligned}$$

implies

$$\boxed{\partial_{\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_{n+1}}L_A{}^B\big| = 2\,\partial_{(\boldsymbol{\mathcal{M}}_1}\ldots\partial_{\boldsymbol{\mathcal{M}}_n|}\left(\xi^C R_{C\boldsymbol{\mathcal{D}}A}{}^B\right)\big|\,\delta_{|\boldsymbol{\mathcal{M}}_{n+1})}{}^{\boldsymbol{\mathcal{D}}}} \qquad \forall n\geq 1 \tag{H.172}$$

Like above, this is a recursion relation, starting with the second fermionic derivative

$$\begin{aligned}
\partial_{\boldsymbol{\mathcal{M}}_1}\partial_{\boldsymbol{\mathcal{M}}_2}L_A{}^B\big| &= 2\xi^C_{(\boldsymbol{\mathcal{M}}_1|}\,R_{C|\boldsymbol{\mathcal{M}}_2)A}{}^B\big| + 2\xi^C_0\,\partial_{(\boldsymbol{\mathcal{M}}_1|}R_{C|\boldsymbol{\mathcal{M}}_2)A}{}^B\big| = \\
&= 2\xi^C_0\left(-2\,T_{C(\boldsymbol{\mathcal{M}}_1|}{}^E\big|\,R_{E|\boldsymbol{\mathcal{M}}_2)A}{}^B\big| + \partial_{(\boldsymbol{\mathcal{M}}_1|}R_{C|\boldsymbol{\mathcal{M}}_2)A}{}^B\big|\right) - 2L_{0\,(\boldsymbol{\mathcal{M}}_1|}{}^C\,R_{C|\boldsymbol{\mathcal{M}}_2)A}{}^B\big|
\end{aligned}$$

The two conditions (H.167) and (H.172) are restricting only terms of order 2 and higher in $\vec{\boldsymbol{\theta}}$ of the transformation parameters ξ^A and $L_A{}^B$ and therefore do not affect our earlier result (H.159)-(H.160) and (H.163) for the stabilizer of the WZ gauge.

H.4.3 Local Lorentz transformations as part of the stabilizer

For a reasonable gauge fixing we should still have local Lorentz invariance and the bosonic diffeomorphism as part of the stabilizer group. We recover the local structure group transformations, if we set

$$\xi^C_0 = 0 \tag{H.173}$$

which leads to

$$\begin{aligned}
L_A{}^B(\vec{x},\vec{\boldsymbol{\theta}}) &= L_{0\,A}{}^B(\vec{x}) + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) &\text{(H.174)}\\
\xi^a &= \mathcal{O}(\vec{\boldsymbol{\theta}}^2) &\text{(H.175)}\\
\xi^{\boldsymbol{\mathcal{A}}} &= -x^{\boldsymbol{\mathcal{M}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{B}}}L_{0\,\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}} + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) &\text{(H.176)}
\end{aligned}$$

The leading components of all superfields with flat indices obviously then transform only under the local structure group transformation $L_{0\,A}{}^B$, because the coupled superdiffeomorphism affects only higher orders in $\vec{\boldsymbol{\theta}}$. When acting on a more general tensor of e.g. the form t^{NB}_{MA}, the coupled diffeomorphism contributes via the matrix $(\nabla_L\xi^K + 2\xi^C T_{CL}{}^K)$ acting on the curved indices (compare (H.15)). For the leading component, i.e. $\vec{\boldsymbol{\theta}} = 0$, the nonvanishing part of this matrix is just

$$-(\nabla_{\boldsymbol{\mathcal{K}}}\xi^P + 2\xi^D T_{CK}{}^P)\big| = \delta_{\boldsymbol{\mathcal{K}}}{}^{\boldsymbol{\mathcal{B}}}L_{0\,\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}\delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{P}}} \tag{H.177}$$

In other words, the bosonic curved indices $m, n, \ldots$ do not transform, while the fermionic curved indices $\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{N}}, \ldots$ transform under the structure group.

For the behaviour on first order in $\vec{\boldsymbol{\theta}}$, it is already instructive to consider the action of the above transformation on a scalar superfield like a dilaton superfield $\Phi_{(ph)}$:

$$\delta\Phi_{(ph)} = \xi^C \nabla_C \Phi_{(ph)} = -x^{\boldsymbol{\mathcal{M}}} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{B}}} L_{0\,\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} \nabla_{\boldsymbol{\mathcal{C}}} \Phi_{(ph)} + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) \tag{H.178}$$

That means for the $\vec{\boldsymbol{\theta}}$-component $\lambda_{\boldsymbol{\mathcal{M}}} \equiv \nabla_{\boldsymbol{\mathcal{M}}} \Phi_{(ph)}\big|$, that it transforms, as if $\boldsymbol{\mathcal{M}}$ was a spinor index.

$$\begin{aligned} \delta\lambda_{\boldsymbol{\mathcal{M}}} &= \partial_{\boldsymbol{\mathcal{M}}} \delta(\Phi_{(ph)})\big| = & \text{(H.179)}\\ &= -\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{B}}} L_{0\,\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}}\, \nabla_{\boldsymbol{\mathcal{C}}} \Phi_{(ph)}\big| = & \text{(H.180)}\\ &= -\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{B}}} L_{0\,\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{C}}} \delta_{\boldsymbol{\mathcal{C}}}{}^{\boldsymbol{\mathcal{N}}} \lambda_{\boldsymbol{\mathcal{N}}} & \text{(H.181)} \end{aligned}$$

Although it might seem intuitive that (curved) fermionic indices transform under the structure group, it is important to note that this is only due to the WZ-gauge, which couples part of the superdiffeomorphisms to the local structure group transformations. Originally, the curved fermionic index $\boldsymbol{\mathcal{M}}$ does not transform under structure group transformations.

H.4.4 Bosonic diffeomorphisms as part of the stabilizer

The equations for the stabilizer are given in flat indices ξ^A. We will need this to extract the local supersymmetry transformations. But in order to see whether the transformation with parameters $\xi^M(\vec{x}) = (\xi_0^m(\vec{x}), 0, 0)$ and $\tilde{L}_A{}^B = 0$ (not $L_A{}^B$, which has absorbed part of the diffeomorphism), corresponding to bosonic diffeomorphisms, is contained in the stabilizer, a change to curved indices is preferable. Instead of using the vielbein to switch from flat to curved index, we check this directly. The transformation of the vielbein components with this parameter is

$$\delta\, E_{\boldsymbol{\mathcal{M}}}{}^A\big| = \xi_0^k \partial_k \underbrace{E_{\boldsymbol{\mathcal{M}}}{}^A\big|}_{\delta_{\boldsymbol{\mathcal{M}}}{}^A} + \underbrace{\partial_{\boldsymbol{\mathcal{M}}} \xi^K\big|}_{=0}\, E_K{}^A\big| = 0 \tag{H.182}$$

$$\begin{aligned} \delta\partial_{(\boldsymbol{\mathcal{M}}_1} \ldots \partial_{\boldsymbol{\mathcal{M}}_n} E_{\boldsymbol{\mathcal{M}}_{n+1})}{}^A\big| &= \partial_{(\boldsymbol{\mathcal{M}}_1} \ldots \partial_{\boldsymbol{\mathcal{M}}_n|} \left(\xi^k \partial_k E_{|\boldsymbol{\mathcal{M}}_{n+1})}{}^A + \partial_{|\boldsymbol{\mathcal{M}}_{n+1})} \xi^k E_k{}^A\right)\big| = & \text{(H.183)}\\ &= \xi^k \partial_k \partial_{(\boldsymbol{\mathcal{M}}_1} \ldots \partial_{\boldsymbol{\mathcal{M}}_n|} E_{|\boldsymbol{\mathcal{M}}_{n+1})}{}^A\big| = 0 & \text{(H.184)} \end{aligned}$$

The same is true for the connection

$$\delta\, \Omega_{\boldsymbol{\mathcal{M}}A}{}^B\big| = \xi_0^k \partial_k\, \Omega_{\boldsymbol{\mathcal{M}}A}{}^B\big| + \underbrace{\partial_{\boldsymbol{\mathcal{M}}} \xi^K\big|}_{=0}\, \Omega_{KA}{}^B\big| = 0 \tag{H.185}$$

$$\delta\partial_{(\boldsymbol{\mathcal{M}}_1} \ldots \partial_{\boldsymbol{\mathcal{M}}_n} \Omega_{\boldsymbol{\mathcal{M}}_{n+1})A}{}^B\big| = \ldots = 0 \tag{H.186}$$

H.5 Local SUSY-transformation

This section could actually be another subsection of the "stabilizer" section. But as we have special interest in the local SUSY transformations, we make it a seperate section.

H.5.1 The transformation parameter

The supersymmetry transformations are defined to be the set of transformations within the stabilizer with

$$\text{SUSY:} \quad \xi_0^c = L_{0\,A}{}^B = 0, \qquad 0 \neq \xi_0^{\boldsymbol{\mathcal{C}}} \equiv \varepsilon^{\boldsymbol{\mathcal{C}}} \tag{H.187}$$

From (H.158) and (H.162) we thus get

$$\xi_{\boldsymbol{\mathcal{M}}}{}^A = -2\varepsilon^{\boldsymbol{\mathcal{C}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{M}}}{}^A\big|, \qquad L_{\boldsymbol{\mathcal{M}}A}{}^B = 2\varepsilon^{\boldsymbol{\mathcal{C}}}\, R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{M}}A}{}^B\big| \tag{H.188}$$

Or more explicitely (compare (H.159),(H.160) and (H.163)):

$$\begin{aligned} \xi^a(\varepsilon) &= -2x^{\boldsymbol{\mathcal{M}}} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{D}}} \varepsilon^{\boldsymbol{\mathcal{C}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^a| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) & \text{(H.189)}\\ \xi^{\boldsymbol{\mathcal{A}}}(\varepsilon) &= \varepsilon^{\boldsymbol{\mathcal{A}}} - 2x^{\boldsymbol{\mathcal{M}}} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{D}}} \varepsilon^{\boldsymbol{\mathcal{C}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^{\boldsymbol{\mathcal{A}}}\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) & \text{(H.190)}\\ L_A{}^B(\varepsilon) &= 2x^{\boldsymbol{\mathcal{M}}} \delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{D}}} \varepsilon^{\boldsymbol{\mathcal{C}}}\, R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}A}{}^B\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) & \text{(H.191)} \end{aligned}$$

Remember that the gauge transformation corresponding to these parameters is of the form

$$\delta_\varepsilon = \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}\,(\varepsilon)} + \mathcal{R}(L(\varepsilon)\,.^{\cdot}) \tag{H.192}$$

We should finally note that the separation of the gauge transformations into local structure group transformations, local bosonic diffeomorphisms and local supersymmetry contains some arbitraryness. In particular when the structure group contains an abelian subgroup (e.g. dilatations), a redefinition of local supersymmetry with such an abelian structure group transformation does not change the supersymmetry algebra. In fact the choice $L_{0\,A}{}^B = 0$ as part of the stabilizer of the gauge fixing is not possible any longer if such a subgroup (e.g. the local scale transformation) is fixed. In the case where we fix for example (in our application in the main part) the leading component of the (bosonic) compensator field Φ to $\Phi| \stackrel{!}{=} 0$ or $\Phi| \stackrel{!}{=} \Phi_{(ph)}|$, we get the additional stabilizer condition $\left(\xi^{C}"\nabla_C\Phi" - L^{(D)}\right)\big| \stackrel{!}{=} 0$ or $\left(\xi^{C}"\nabla_C\Phi" - L^{(D)}\right)\big| \stackrel{!}{=} \xi^C\nabla_C\,\Phi_{(ph)}\big|$ or equivalently

$$L_0^{(D)}(\varepsilon) \stackrel{!}{=} \varepsilon^{\boldsymbol{\mathcal{C}}}\,\nabla_{\boldsymbol{\mathcal{C}}}\Phi| \quad \& \quad \xi^{\boldsymbol{\mathcal{A}}}(\varepsilon) \to \xi^{\boldsymbol{\mathcal{A}}}(\varepsilon) - \frac{1}{2}x^{\boldsymbol{\mathcal{M}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}}\varepsilon^{\boldsymbol{\mathcal{C}}}\,\nabla_{\boldsymbol{\mathcal{C}}}\Phi| \tag{H.193}$$

$$\text{or } L_0^{(D)}(\varepsilon) \stackrel{!}{=} \varepsilon^{\boldsymbol{\mathcal{C}}}\left("\nabla_{\boldsymbol{\mathcal{C}}}\Phi"| - \nabla_{\boldsymbol{\mathcal{C}}}\,\Phi_{(ph)}|\right) \quad \& \quad \xi^{\boldsymbol{\mathcal{A}}}(\varepsilon) \to \xi^{\boldsymbol{\mathcal{A}}}(\varepsilon) - \frac{1}{2}x^{\boldsymbol{\mathcal{M}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{A}}}\varepsilon^{\boldsymbol{\mathcal{C}}}\left("\nabla_{\boldsymbol{\mathcal{C}}}\Phi"| - \nabla_{\boldsymbol{\mathcal{C}}}\,\Phi_{(ph)}|\right) \tag{H.194}$$

Alternatively, we could have fixed the complete superfield Φ to zero (before going to WZ-gauge). Then the scale part of the connection is not structure group valued and therefore has to be treated as a difference tensor. Only the Lorentz part can then be used for the implementation of the WZ-gauge.

H.5.2 The supersymmetry algebra

In order to read off the algebra of the local supersymmetry transformations from (H.69), we need the transformation of $\vec{\xi}$ itself under a second supersymmetry transformation

$$\begin{aligned}
\delta_{\varepsilon_1}\xi^A(\varepsilon_2) &= -2x^{\boldsymbol{\mathcal{M}}}\varepsilon_2^{\boldsymbol{\mathcal{C}}}\,\delta_{\varepsilon_1}T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{M}}}{}^A\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) = &\text{(H.195)}\\
&= -2x^{\boldsymbol{\mathcal{M}}}\varepsilon_2^{\boldsymbol{\mathcal{C}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{B}}}\,\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}\,(\varepsilon_1)}\,T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^A\Big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) = &\text{(H.196)}\\
&= -2x^{\boldsymbol{\mathcal{M}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{D}}}\varepsilon_2^{\boldsymbol{\mathcal{C}}}\varepsilon_1^{\boldsymbol{\mathcal{B}}}\,\nabla_{\boldsymbol{\mathcal{B}}}T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^A\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) &\text{(H.197)}
\end{aligned}$$

and also the transformation of $L_A{}^B$ under supersymmetry:

$$\begin{aligned}
\delta_{\varepsilon_1}L_A{}^B(\varepsilon_2) &= 2x^{\boldsymbol{\mathcal{M}}}\varepsilon_2^{\boldsymbol{\mathcal{C}}}\,\delta_{\varepsilon_1}R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{M}}A}{}^B\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) = &\text{(H.198)}\\
&= 2x^{\boldsymbol{\mathcal{M}}}\varepsilon_2^{\boldsymbol{\mathcal{C}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{D}}}\,\boldsymbol{\mathcal{L}}^{(\text{cov})}_{\vec{\xi}\,(\varepsilon_1)}\,R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}A}{}^B\Big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) = &\text{(H.199)}\\
&= 2x^{\boldsymbol{\mathcal{M}}}\varepsilon_2^{\boldsymbol{\mathcal{C}}}\delta_{\boldsymbol{\mathcal{M}}}{}^{\boldsymbol{\mathcal{D}}}\varepsilon_1^{\boldsymbol{\mathcal{E}}}\,\nabla_{\boldsymbol{\mathcal{E}}}R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}A}{}^B\big| + \mathcal{O}(\vec{\boldsymbol{\theta}}^2) &\text{(H.200)}
\end{aligned}$$

For the algebra (H.69), we still need the Lie bracket of the vector field:

$$[\vec{\xi}_1, \vec{\xi}_2]^A = \xi_1^C\nabla_C\xi_2^A - \xi_2^C\nabla_C\xi_1^A - 2\xi_1^C T_{CB}{}^A\xi_2^B \tag{H.201}$$

For simplicity, let us restrict to the leading component, although we would have enough information to calculate higher orders as well:

$$\begin{aligned}
[\vec{\xi}\,(\varepsilon_1), \vec{\xi}\,(\varepsilon_2)]^A\Big| &= \varepsilon_1^{\boldsymbol{\mathcal{C}}}\delta_{\boldsymbol{\mathcal{C}}}{}^{\boldsymbol{\mathcal{M}}}\xi_{\boldsymbol{\mathcal{M}}}{}^A(\varepsilon_2) - \varepsilon_2^{\boldsymbol{\mathcal{B}}}\delta_{\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{M}}}\xi_{\boldsymbol{\mathcal{M}}}{}^A(\varepsilon_1) - 2\varepsilon_1^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^A\big|\,\varepsilon_2^{\boldsymbol{\mathcal{B}}} = &\text{(H.202)}\\
&= -2\varepsilon_1^{\boldsymbol{\mathcal{C}}}\varepsilon_2^{\boldsymbol{\mathcal{B}}}\,T_{\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}}}{}^A\big| + 2\varepsilon_2^{\boldsymbol{\mathcal{B}}}\varepsilon_1^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^A\big| - 2\varepsilon_1^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^A\big|\,\varepsilon_2^{\boldsymbol{\mathcal{B}}} = &\text{(H.203)}\\
&= 2\varepsilon_1^{\boldsymbol{\mathcal{C}}}\,T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^A\big|\,\varepsilon_2^{\boldsymbol{\mathcal{B}}} &\text{(H.204)}
\end{aligned}$$

Having derived only the leading component of the vector-Lie bracket, we should restrict to the leading component for the rest as well. The algebra (H.69) then becomes

$$\boxed{[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \boldsymbol{\mathcal{L}}^{(\text{cov})}_{\left(-2\varepsilon_1^{\boldsymbol{\mathcal{C}}}T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}{}^A|\varepsilon_2^{\boldsymbol{\mathcal{D}}} + \mathcal{O}(\vec{\boldsymbol{\theta}})\right)\vec{E}_A} + \mathcal{R}\Big(2\varepsilon_1^{\boldsymbol{\mathcal{C}}}\varepsilon_2^{\boldsymbol{\mathcal{D}}}R_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{D}}}\,.^{\cdot} + \mathcal{O}(\vec{\boldsymbol{\theta}})\Big)} \tag{H.205}$$

H.5.3 Transformation of the fields

The supersymmetry transformation of the fields is simply given by

$$\delta_\varepsilon = \mathcal{L}^{(\mathrm{cov})}_{\vec{\xi}\,(\varepsilon)} + \mathcal{R}(L(\varepsilon).\dot{}) \tag{H.206}$$

where $\xi^A(\varepsilon)$ and $L_A{}^B(\varepsilon)$ are of the special form given in (H.187)-(H.191). Let us derive the transformations of all the fields that we will need. In order to extract the transformation of the (leading) components, we will again make frequent use of the Wess Zumino gauge (H.76) and (H.92) (using $E_m{}^a| \equiv e_m{}^a$, $E_m{}^{\boldsymbol{\mathcal{A}}}| \equiv \psi_m{}^{\boldsymbol{\mathcal{A}}}$). In any supergravity theory we have a vielbein and a structure group connection which we will consider first.

H.5.3.1 Vielbein (bosonic vielbein and gravitino)

Remember, the vielbein transforms according to (H.19) as

$$\delta E_M{}^A = \underbrace{\partial_M \xi^A + \Omega_{MC}{}^A \xi^C}_{\nabla_M \xi^A} + 2\xi^C T_{CM}{}^A + L_B{}^A E_M{}^B \tag{H.207}$$

In practice, we will be given constraints on torsion components with flat indices, s.t. it is useful to rewrite the equations in those components. In addition, we plug in the explicit form of $\xi^A(\varepsilon)$ and $L_B{}^A(\varepsilon)$ given in (H.189)-(H.191) to obtain the local supersymmetry transformation of the nonvanishing leading vielbein components (the bosonic vielbein and the gravitino(s))::

$$\boxed{\begin{aligned} \delta_\varepsilon e_m{}^a &= 2\varepsilon^{\boldsymbol{\mathcal{C}}} e_m{}^b\, T_{\boldsymbol{\mathcal{C}}b}{}^a| + 2\varepsilon^{\boldsymbol{\mathcal{C}}} \psi_m{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^a| && \text{(H.208)}\\ \delta_\varepsilon \psi_m{}^{\boldsymbol{\mathcal{A}}} &= \underbrace{\partial_m \varepsilon^{\boldsymbol{\mathcal{A}}} + \omega_{m\boldsymbol{\mathcal{C}}}{}^{\boldsymbol{\mathcal{A}}} \varepsilon^{\boldsymbol{\mathcal{C}}}}_{\nabla_m \varepsilon^{\boldsymbol{\mathcal{A}}}} + 2\varepsilon^{\boldsymbol{\mathcal{C}}} e_m{}^b\, T_{\boldsymbol{\mathcal{C}}b}{}^{\boldsymbol{\mathcal{A}}}\Big| + 2\varepsilon^{\boldsymbol{\mathcal{C}}} \psi_m{}^{\boldsymbol{\mathcal{B}}}\, T_{\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}}{}^{\boldsymbol{\mathcal{A}}}\Big| && \text{(H.209)} \end{aligned}}$$

H.5.3.2 Connection

Remember the general gauge transformation of the structure group connection (H.25)

$$\delta \Omega_{MA}{}^B = 2\xi^K R_{KMA}{}^B - \partial_M L_A{}^B - [L, \Omega_M]_A{}^B \tag{H.210}$$

In the case where a scale part of the connection is present, this transforms accordingly as (see (H.30))

$$\delta \Omega^{(D)}_M = 2\xi^C F^{(D)}_{CM} - \partial_M L^{(D)} \tag{H.211}$$

For the stabilizer of WZ-gauge with $\Omega_{\boldsymbol{\mathcal{M}}A}{}^B\big| = 0$ and $\delta\, \Omega_{\boldsymbol{\mathcal{M}}A}{}^B\big| = 0$ and for the choice $\xi^c_0 = L_0{}_A{}^B$ (corresponding to local supersymmetry (H.187) and (H.188)) the nontrivial part of the above equations becomes (for $\vec{\boldsymbol{\theta}} = 0$):

$$\delta\, \Omega_{mA}{}^B\big| = 2\xi_0^{\boldsymbol{\mathcal{C}}}\, R_{\boldsymbol{\mathcal{C}}mA}{}^B\big| \tag{H.212}$$

$$\delta\, \Omega^{(D)}_m\Big| = 2\xi_0^{\boldsymbol{\mathcal{C}}}\, F^{(D)}_{\boldsymbol{\mathcal{C}}m}\Big| \tag{H.213}$$

More explicitely (replacing $\varepsilon^{\boldsymbol{\gamma}} \equiv \xi_0^{\boldsymbol{\gamma}}$, $\hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}} \equiv \xi_0^{\hat{\boldsymbol{\gamma}}}$) this reads

$$\boxed{\begin{aligned} \delta\, \Omega_{ma}{}^b\big| &= 2\varepsilon^{\boldsymbol{\gamma}} \left(e_m{}^d\, R_{\boldsymbol{\gamma}da}{}^b\big| + \psi_m{}^{\boldsymbol{\delta}}\, R_{\boldsymbol{\gamma\delta}a}{}^b\big| + \hat{\psi}_m{}^{\hat{\boldsymbol{\delta}}}\, R_{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}a}{}^b\Big| \right) + \\ &\quad + 2\varepsilon^{\hat{\boldsymbol{\gamma}}} \left(e_m{}^d\, R_{\hat{\boldsymbol{\gamma}}da}{}^b\big| + \psi_m{}^{\boldsymbol{\delta}}\, R_{\hat{\boldsymbol{\gamma}}\boldsymbol{\delta}a}{}^b\big| + \hat{\psi}_m{}^{\hat{\boldsymbol{\delta}}}\, R_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}a}{}^b\Big| \right) && \text{(H.214)}\\ \delta\, \Omega^{(D)}_m\Big| &= 2\varepsilon^{\boldsymbol{\gamma}} \left(e_m{}^d\, F^{(D)}_{\boldsymbol{\gamma}d}\Big| + \psi_m{}^{\boldsymbol{\delta}}\, F^{(D)}_{\boldsymbol{\gamma\delta}}\Big| + \hat{\psi}_m{}^{\hat{\boldsymbol{\delta}}}\, F^{(D)}_{\boldsymbol{\gamma}\hat{\boldsymbol{\delta}}}\Big| \right) + \\ &\quad + 2\varepsilon^{\hat{\boldsymbol{\gamma}}} \left(e_m{}^d\, F^{(D)}_{\hat{\boldsymbol{\gamma}}d}\Big| + \psi_m{}^{\boldsymbol{\delta}}\, F^{(D)}_{\hat{\boldsymbol{\gamma}}\boldsymbol{\delta}}\Big| + \hat{\psi}_m{}^{\hat{\boldsymbol{\delta}}}\, F^{(D)}_{\hat{\boldsymbol{\gamma}}\hat{\boldsymbol{\delta}}}\Big| \right) && \text{(H.215)} \end{aligned}}$$

H.5.3.3 Compensator field

A compensator field is not necessarily present in a supergravity theory. In our context such a field Φ is used to allow a scale transformation of the metric in flat indices:

$$G_{AB} \equiv e^{2\Phi} \eta_{AB} \tag{H.216}$$

Where η_{AB} is some constant metric which is invariant under the orthogonal transformations. In our case, its bosonic part is just the Minkowski metric and the rest is zero. There is no way, how a constant metric can scale. Therefore the compensator field Φ takes over the scaling of G_{AB} under scale transformation by simply getting shifted with the scale parameter

$$\mathcal{R}(L)\,\Phi = \Phi - L^{(D)} \tag{H.217}$$

Similarly, the covariant derivative will be defined to act only on Φ (and not on η_{AB}) in such a way that the covariant derivative of G_{AB} has the form that is indicated by its indices.

$$\begin{aligned} \nabla_M G_{AB} &= 2(\partial_M \Phi - \Omega_M^{(D)})G_{AB} && \text{(H.218)} \\ \Rightarrow \text{"}\nabla_M \Phi\text{"} &= 2(\partial_M \Phi - \Omega_M^{(D)}) && \text{(H.219)} \end{aligned}$$

The general gauge transformation of the compensator field thus reads

$$\delta\Phi = \xi^K \underbrace{\left(\partial_K \Phi - \Omega_K^{(D)}\right)}_{\text{"}\nabla_K\Phi\text{"}} - L^{(D)} \tag{H.220}$$

Define

$$\begin{aligned} \phi &\equiv \Phi| && \text{(H.221)} \\ \phi_{\boldsymbol{\mathcal{M}}} &\equiv \partial_{\boldsymbol{\mathcal{M}}}\Phi| && \text{(H.222)} \end{aligned}$$

For the lowest component , this implies the following local SUSY transformation in the WZ gauge

$$\boxed{\delta_\varepsilon \phi = \varepsilon^{\boldsymbol{\gamma}}\phi_{\boldsymbol{\gamma}} + \hat{\varepsilon}^{\hat{\boldsymbol{\gamma}}}\phi_{\hat{\boldsymbol{\gamma}}}} \tag{H.223}$$

The transformation is zero, if we combine it with an additional scale stabilizer transformation (H.193)

$$L^{(D)} = \xi_0^{\boldsymbol{\mathcal{C}}}\phi_{\boldsymbol{\mathcal{C}}} \tag{H.224}$$

Note that the transformation of the connection is such that the covariant derivative of the compensator field transforms like a vector

$$\delta\nabla_A\Phi = \xi^B\nabla_B\nabla_A\Phi - L_A{}^B\nabla_B\Phi \tag{H.225}$$

In particular we have for the SUSY transformation of the first theta-components

$$\boxed{\delta_\varepsilon\ \nabla_{\boldsymbol{\mathcal{A}}}\Phi| = \varepsilon^{\boldsymbol{\mathcal{B}}}\ \nabla_{\boldsymbol{\mathcal{B}}}\nabla_{\boldsymbol{\mathcal{A}}}\Phi|} \tag{H.226}$$

H.5.3.4 Scalar super field (e.g. dilaton and dilatino)

The Dilaton field is a scalar and thus has the simple transformation

$$\delta\Phi_{(ph)} = \xi^C \underbrace{\nabla_C\Phi_{(ph)}}_{E_C{}^M\partial_M\Phi_{(ph)}} = \boldsymbol{\mathcal{L}}_{\vec{\xi}}\Phi_{(ph)} \tag{H.227}$$

Define now the **dilatino** to be

$$\begin{aligned} \lambda_{\boldsymbol{\mathcal{A}}} &\equiv \nabla_{\boldsymbol{\mathcal{A}}}\Phi_{(ph)}\big| = \delta_{\boldsymbol{\mathcal{A}}}{}^{\boldsymbol{\mathcal{M}}}\ \partial_{\boldsymbol{\mathcal{M}}}\Phi_{(ph)}\big| && \text{(H.228)} \\ \lambda_{\boldsymbol{\mathcal{M}}} &= \partial_{\boldsymbol{\mathcal{M}}}\Phi_{(ph)}\big| && \text{(H.229)} \\ \Rightarrow \Phi_{(ph)} &= \phi_{(ph)} + x^{\boldsymbol{\mu}}\lambda_{\boldsymbol{\mu}} + x^{\hat{\boldsymbol{\mu}}}\hat{\lambda}_{\hat{\boldsymbol{\mu}}} + \frac{1}{2}x^{\boldsymbol{\mathcal{M}}}x^{\boldsymbol{\mathcal{N}}}\ \partial_{\boldsymbol{\mathcal{M}}}\partial_{\boldsymbol{\mathcal{N}}}\Phi| + \ldots && \text{(H.230)} \end{aligned}$$

This definition of the dilatino implies according to (H.227) for the dilaton $\phi_{(ph)}$ the transformation

$$\boxed{\delta\Phi_{(ph)} = \varepsilon^{\boldsymbol{\mathcal{C}}}\lambda_{\boldsymbol{\mathcal{C}}}} \tag{H.231}$$

For the transformation of the dilatino we use the fact that the variation of a covariant derivative is simply the covariantized Lie derivative (supergauge transformation) plus the structure group transformation of the new tensor according to the new index structure (see footnote 3 on page 210 and (H.15)). We thus have

$$\delta(\nabla_A\Phi_{(ph)}) = \xi^C\nabla_C\nabla_A\Phi_{(ph)} - L_A{}^B\nabla_B\Phi_{(ph)} \tag{H.232}$$

with ξ^C and $L_A{}^B$ given in (H.187)-(H.191). For the fermionic components at $\vec{\boldsymbol{\theta}}=0$, this reads simply

$$\boxed{\delta\lambda_{\mathcal{A}} = \varepsilon^{\mathcal{C}}\, \nabla_{\mathcal{C}}\nabla_{\mathcal{A}}\Phi_{(ph)}\big|} \tag{H.233}$$

Apparently, we need some equations of motion at this point, in order to say more. We can, however, relate this expression explicitely to the $\vec{\boldsymbol{\theta}}^2$ component $\partial_{\mathcal{M}}\partial_{\mathcal{N}}\Phi_{(ph)}\big|$ of the dilaton:

$$\begin{aligned} \delta\lambda_{\mathcal{A}} &= \varepsilon^{\mathcal{C}}\delta_{\mathcal{C}}{}^{\mathcal{M}}\ \partial_{\mathcal{M}}(E_{\mathcal{A}}{}^K\partial_K\Phi_{(ph)})\big| = && \text{(H.234)}\\ &= \varepsilon^{\mathcal{C}}\delta_{\mathcal{C}}{}^{\mathcal{M}}\left(\partial_{\mathcal{M}}E_{\mathcal{A}}{}^K\big|\ \partial_K\Phi_{(ph)}\big| + \delta_{\mathcal{A}}{}^{\mathcal{K}}\ \partial_{\mathcal{M}}\partial_{\mathcal{K}}\Phi_{(ph)}\big|\right) && \text{(H.235)} \end{aligned}$$

Now we can use that

$$\begin{aligned} \partial_{\mathcal{M}}E_{\mathcal{A}}{}^K\big| &= -\,E_{\mathcal{A}}{}^L\big|\ \partial_{\mathcal{M}}E_L{}^B\big|\ E_B{}^K\big| = && \text{(H.236)}\\ &= -\,E_{\mathcal{A}}{}^{\mathcal{L}}\big|\ \underbrace{\partial_{\mathcal{M}}E_{\mathcal{L}}{}^B\big|}_{\partial_{[\mathcal{M}}E_{\mathcal{L}]}{}^B|}\ E_B{}^K\big| = && \text{(H.237)}\\ &= -\delta_{\mathcal{A}}{}^{\mathcal{L}}\ T_{\mathcal{M}\mathcal{L}}{}^B\big|\ E_B{}^K\big| && \text{(H.238)} \end{aligned}$$

The transformation of before can then be rewritten as

$$\boxed{\begin{aligned} \delta\lambda_{\mathcal{A}} \;=\;& -\varepsilon^{\mathcal{C}}\ T_{\mathcal{C}\mathcal{A}}{}^b\big|\ e_b{}^k\partial_k\phi_{(ph)} + \varepsilon^{\mathcal{C}}\ T_{\mathcal{C}\mathcal{A}}{}^b\big|\ \psi_b{}^{\mathcal{K}}\lambda_{\mathcal{K}} - \varepsilon^{\mathcal{C}}\ T_{\mathcal{C}\mathcal{A}}{}^{\mathcal{B}}\big|\ \lambda_{\mathcal{B}} + \\ & +\varepsilon^{\mathcal{C}}\delta_{\mathcal{C}}{}^{\mathcal{M}}\delta_{\mathcal{A}}{}^{\mathcal{K}}\ \partial_{\mathcal{M}}\partial_{\mathcal{K}}\Phi_{(ph)}\big| \end{aligned}} \tag{H.239}$$

H.5.3.5 Bispinor fields (RR-fields)

Apart from that we will be interested in the transformation of RR-fields

$$\delta\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = \xi^C\nabla_C\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} + L_{\boldsymbol{\gamma}}{}^{\boldsymbol{\alpha}}\mathcal{P}^{\boldsymbol{\gamma}\hat{\boldsymbol{\beta}}} + L_{\hat{\boldsymbol{\gamma}}}{}^{\hat{\boldsymbol{\beta}}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\gamma}}} \tag{H.240}$$

The leading component, that we defined in the main text as $\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} = e^{-8\phi_{(ph)}}\ \mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}\Big|$, then transforms as

$$\boxed{\delta\mathfrak{p}^{\alpha\hat{\boldsymbol{\beta}}} = -8\varepsilon^{\mathcal{C}}\lambda_{\mathcal{C}}\mathfrak{p}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}} + e^{-8\phi_{(ph)}}\varepsilon^{\mathcal{C}}\ \nabla_{\mathcal{C}}\mathcal{P}^{\boldsymbol{\alpha}\hat{\boldsymbol{\beta}}}\Big|} \tag{H.241}$$

H.5.3.6 Two or three form (e.g. B-field and H-field)

Finally we consider the transformation of a two form (e.g. the B-field) and of a three form (e.g the H-field):

$$\begin{aligned} \delta B_{AB} &= \xi^D\nabla_D B_{AB} - 2L_{[A|}{}^D B_{D|B]} && \text{(H.242)}\\ \delta B_{MN} &= \xi^D\nabla_D B_{MN} + 2(\nabla_{[M|}\xi^L + 2\xi^P T_{P[M|}{}^L)B_{L|N]} = \xi^K\partial_K B_{MN} + 2\partial_{[M|}\xi^L B_{L|N]} && \text{(H.243)}\\ \delta H_{ABC} &= \xi^D\nabla_D H_{ABC} - 3L_{[A|}{}^D H_{D|BC]} && \text{(H.244)}\\ \delta H_{MNK} &= \xi^D\nabla_D H_{MNK} + 3(\nabla_{[M|}\xi^L + 2\xi^P T_{P[M|}{}^L)H_{L|NK]} = \xi^L\partial_L H_{MNK} + 3\partial_{[M|}\xi^L H_{L|NK]} && \text{(H.245)} \end{aligned}$$

It makes some difference whether we consider the fields with flat or with curved coordinates. The difference lies in the transformation of the vielbeins. Physically, we are interested in the transformation of the bosonic B-field $B_{mn}|$ and H-field $H_{mnk}|$ with curved indices. If we assume that $H=\mathbf{d}B$ and B thus is a gauge field, we can make use of the WZ-like gauge $B_{\mathcal{M}\mathcal{N}}| = B_{m\mathcal{N}}| = 0$ and $\partial_{\mathcal{K}}B_{mn}| = 3\ H_{\mathcal{K}mn}|$, in order to become more explicit for the transformation of $B_{mn}|$. For the B-field transformation it thus makes sense to take the version in terms of partial derivatives instead of covariant ones.

$$\begin{aligned} \delta\ B_{mn}| &= \varepsilon^{\mathcal{D}}\delta_{\mathcal{D}}{}^{\mathcal{K}}\ \partial_{\mathcal{K}}B_{mn}| + 2\partial_{[m|}\varepsilon^{\mathcal{D}}\ B_{\mathcal{D}|n]}\big| = && \text{(H.246)}\\ &= 3\varepsilon^{\mathcal{D}}\ H_{\mathcal{D}mn}| && \text{(H.247)} \end{aligned}$$

Rewritten in flat coordinates, the result becomes

$$\boxed{\delta_\varepsilon\ B_{mn}| \;=\; 3\varepsilon^{\mathcal{D}}e_m{}^a e_n{}^b\ H_{\mathcal{D}ab}| + 6\varepsilon^{\mathcal{D}}\psi_{[m}{}^{\mathcal{A}}e_{n]}{}^b\ H_{\mathcal{D}\mathcal{A}b}| + 3\varepsilon^{\mathcal{D}}\psi_{[m}{}^{\mathcal{A}}\psi_{n]}{}^{\mathcal{B}}\ H_{\mathcal{D}\mathcal{A}\mathcal{B}}|} \tag{H.248}$$

So far we have only used simplifications coming from the WZ-like gauge but no supergravity constraints yet.

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Curriculum Vitae

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Education / research experience

- 1996 Abitur (final exams) at the "Gymnasium Miesbach" (highschool) in Germany
- 10/1997 - 10/2002: physics studies at the Munich University of Technology (TUM)
- 1999/2000: three and a half months studies as a guest at the "State University of New York, Stony Brook"
- 10/2001 - 10/2002 extramural diploma thesis with the title "effektive Wirkungen in der Stringtheorie" (effective actions in string theory) at the "Ludwig Maximilians Universität" in Munich with advisor Ivo Sachs. Part of this work was done from May '02 until July '02 at the Trinity College in Dublin.
- 10/1998 - fall 1999: additional studies of mathematics at the TUM, terminated after successfully having passed the first part ("Vordiplom")
- 03/2003-09/2007 PhD studies at the Vienna University of Technology (TU Wien) with supervisor Maximilian Kreuzer.
- 11/2005 - 09/2006: ten months visit in Paris, Saclay (CEA/SPhT), working in the string group with Ruben Minasian, Mariana Graña, Pierre Vanhove et al.
- 10/2007 planned start of a postdoctoral position at the 'Demokritos Nuclear Research Centre' in Athens/Greece in the group of George Savvidy

Awards and Fellowships

- 1995 Second prize in the first round of the "Bundeswettbewerb Mathematik '95" (German high school math competition)
- 1995 reaching the third round of the German selection procedure for the International Physics Olympiade '96 (One of top 50 German high school students)
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Lebenslauf

Persönliche Daten

Name:	Sebastian Guttenberg
Geburtsdatum:	13. Mai 1977
Geburtsort:	Rosenheim, Deutschland
Nationalität:	deutsch
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Ausbildung / Forschung

- 1996 Abitur am Gymnasium Miesbach
- 10/1997 - 10/2002: Physikstudium an der Technischen Universität München (TUM)
- 1999/2000: dreieinhalb-monatiges Gaststudium an der "State University of New York, Stony Brook"
- 10/2001 - 10/2002 Externe Diplomarbeit mit dem Titel "effektive Wirkungen in der Stringtheorie" an der Ludwig Maximilians Universität in München unter der Betreuung von Ivo Sachs. Ein Teil dieser Arbeit wurde zwischen Mai '02 und Juli '02 am 'Trinity College' in Dublin angefertigt.
- 10/1998 - Herbst 1999: zusätzliches Studium der Mathematik an der TUM, beendet nach erfolgreichem Ablegen der Vordiplomsprüfungen.
- 03/2003-09/2007 Doktoratsstudium an der Technischen Universität Wien unter der Leitung von Maximilian Kreuzer.
- 11/2005 - 09/2006: zehnmonatiger Aufenthalt in Paris, Saclay (CEA/SPhT); Gast der dortigen Stringtheorie-Gruppe bestehend aus Ruben Minasian, Mariana Graña, Pierre Vanhove et al.
- 10/2007 geplanter Beginn einer Postdoc-Stelle am 'Demokritos Nuclear Research Centre' in Athen/Griechenland bei George Savvidy

Auszeichnungen und Stipendien

- 1995 Zweiter Preis in der ersten Runde des Bundeswettbewerbs Mathematik 1995
- 1995 Erreichen der dritten Runde des deutschen Auswahlverfahrens zur Internationalen Physik Olympiade '96
- 1996 Erster Preis in der ersten Runde und zweiter Preis in der zweiten Runde des Bundeswettbewerbs Mathematik 1996
- Von Nov.'98 bis zum Ende des Physikstudiums: Stipendium der "Studienstiftung des deutschen Volkes".
- Mai '05 bis Juli '05 "Junior Research Fellowship in Mathematics and Mathematical Physics" am Erwin Schrödinger Institut in Wien
- "Mobilitätsstipendium der Akademisch-sozialen Arbeitsgemeinschaft Österreichs (ASAG)" und "Auslands-Stipendium der TU Wien" zur Ermöglichung eines dreimonatigen Aufenthaltes in Saclay, der dann (finanziert durch EGIDE von französischer Seite) auf zehn Monate verlängert wurde.