# Geometry and Gauge Theory

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March 8, 2012

**Abstract**

Course at IST, spring 2012.
**Starting date of the file:** January 31, 2012
**Starting date of the course:** February 13, 2012

# Contents

# Chapter 1

# Introduction

## 1.1 Some general BlaBla

- Usually have **determinism** in physics: Given some fields (tensor fields, sections of fiber bundles, position of point-particle, etc), initial conditions and equations of motion uniquely fix the time evolution (the solution in spacetime).
- Gauge Invariance of a Physical system:
  - time evolution contains time-dependent arbitrary functions (i.e. not fixed):
    e.g. $q_1(t) = v_1 t + \varepsilon(t)$, $q_2(t) = v_2 t - \varepsilon(t)$.
    Can be seen as coming from local symmetries of the action:
    Symmetries map solutions of eom's to solutions.
    $\Rightarrow$ local symmetries (i.e. with time-dependent transformation-parameter) thus allow arbitrary time dependent parameters in the solution, i.e. solution is not fixed.
  - determinism can be reinstalled by identifying solutions that differ by this shift (**mod out gauge orbits**)
  - or by '**fixing the gauge**', i.e. choosing a representative of the orbit.
  - In most cases one can think of these symmetries as a $\boxed{\textbf{change of coordinates or change of reference frame}}$
  - Note that one can also mod out discrete symmetries (e.g. **dualities**$\cong \mathbb{Z}^2$: mirror symmetry (T-duality), S-duality: electric magnetic duality, T-folds, U-folds)
- Examples:
  - **YM:** Yang Mills theories (including Maxwell U(1), QCD SU(3), electroweak SU(2)xU(1)):
    standard class of examples. Relation to **fiber bundles** manifest. Gauge trafo=transition functions (choice of basis in the fiber)
  - classical mechanics:
    Mostly related to implementation of reparametrization invariance, e.g. relativistic point particle (geodesic equations).
    But one can easily construct plenty of artificial examples.
  - **GR:** General relativity :
    Again reparametrization invariance. Metric (instead of connection) plays the role of the gauge field.
    $\Rightarrow$In flat space, metric can be gauged away completely (by fixing an orthonormal coordinate system).
    One then says that the metric is "pure gauge".
    In curved space this is possible only for a given point of the manifold (where one can even bring the Christoffel symbols to zero).
  - Pure math example: geodesic equations (minimal volumes) from variational principle. (Nambu-Goto action with Euclidean signature):
    Worldvolume-diffeomorphism-invariance
  - Similar, but with Minkowskian signature:
    - Relativistic point particle.
    - Or even superparticle (particle propagating through superspace)$\Rightarrow$super-diffeomorphism-invariance

* **ST:** (bosonic) string theory:
  Worldsheet diffeo invariance. (with gauge field(metric): Polyakov-string, without: Nambu-Goto)
  Local Weyl-invariance (without gauge field)
  Gauge-constraints: Virasoro-Constraints -> Virasoro-algebra
* Dp-branes (world-volume diffeomorphism invariant Dirac Born Infeld action + other gauge symmetries)

– **gauged WZW:** Wess Zumino (Novikov) Witten theory (2dim sigma-models with (coset) group manifold target space):
  Fields live in the group $G$ itself. Can act on them with other group elements by either left or right multiplication.
  One of these is realized for a subgroup $H \subset G$ as local symmetry (gauge symmetry), the other just a global symmetry. Modding out the local symmetry leads effectively to a sigma-model on the coset G/H.

– higher gauge fields (B-field, TQFT):
  $\delta B^{(2)} = \mathbf{d}\Lambda^{(1)}$ (compare $\delta \boldsymbol{A}^{(1)} = \mathbf{d}\Lambda^{(0)}$)
  Fiber bundles not enough. Need **gerbes**.
  (Čech-2-Cocycles: $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$, $\delta g = g_{\beta\gamma}\delta g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta}\delta g^{-1}_{\alpha\beta\gamma} = 1$ (0?),
  $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}, \quad B_\beta - B_\alpha = dA_{\alpha\beta}$, $dB_\beta = dB_\alpha = H|_{U_\alpha}$. Hitchin-talk)

– topological models (Poisson sigma-model: no propagating degrees of freedom, →Kontsevich star product)

– can gauge any global symmetry (**Noether-method**)
  ⇒was used to derive **SUGRA** from SUSY

## 1.2 Plan

- Definition of gauge transformations
- Examples (including a first glance on Yang Mills)
- Back to general discussion: Noether theorem and Noether identities
- **Hamiltonian** description:
  – Phase space $T^*M$ with natural Poisson bracket
  – $H : \quad T^*M \to \mathbb{R}$
  – Constraints in the phase space (1st and 2nd class).
    (presymplectic Hamiltonian systems)
  – Koszul Tate differential $\delta$ reduces to the corresponding submanifold.
  – BRST differential (longitudinal exterior derivative) mods out gauge transformations (generated by the BRST charge via the Poisson bracket).
- For both ↕ need: **homological perturbation theory**: $\underbrace{\mathbf{s}}_{\text{BRST}} = \underbrace{\delta}_{\text{Koszul Tate}} + \underbrace{\mathbf{d}}_{\text{longitudinal exterior derivative}} + \ldots$
- **Lagrangian** description (**Legendre**-transformation $T^*M \to TM$)
  – Tangent space $TM$ a priori without natural Poisson bracket
  – $L : \quad TM \to \mathbb{R}$
  – Space of paths $\mathcal{P}M$ in $M$ (determined by initial conditions on $TM$ and eom's).
  – Eom's are constraints in $\mathcal{P}M$
  – extend $M$ to antifields. Functions on extended space$\cong \Gamma\left(\bigwedge^\bullet TM\right)$ (multivector-fields)
    Graded Lie algebra with natural "odd Poisson bracket": Schouten-bracket or antibracket
  – Koszul tate differential $\delta$ reduces to the surface given by the solution to the equations of motion ("on-shell")
  – BRST differential mods out gauge transformations (generated by the action $S$ via the antibracket).

- Fiber bundle geometry and relation to YM
  - characteristic classes (fiber bundle obstructions)
  - Index theorem (anomalies)
- Maybe more on:
  - Anomalies
  - Fermions
  - Solitons (magnetic monopole)
  - ...

## 1.3 Definition of a gauge transformation

- [Henneaux,p.67]: Consider a manifold (?) $M$ and the space of all paths $\mathcal{P}M$. I.e. an arbitrary path $q \in \mathcal{P}M$ is of the form

$$\begin{aligned} q: \quad \mathbb{R} \supset [a,b] &\rightarrow M \\ t &\mapsto q(t) \quad (\text{with coords } q^i(t), \quad i \in \{1, \ldots, \dim M\}) \end{aligned} \tag{1.1}$$

where we assume the paths to be smooth. Consider further a functional on this space

$$\begin{aligned} S: \quad \overbrace{\mathcal{P}M}^{\text{all paths on } M} &\rightarrow \mathbb{R} \\ q &\mapsto S[q] \equiv \int_a^b dt \quad L(q(t), \dot{q}(t), \ddot{q}(t), \ldots, t) \end{aligned} \tag{1.2}$$

which will be called an **action functional** . (It is called **local** if the integrand contains only a finite number of derivatives)

- The equations

$$0 \stackrel{!}{=} \delta S[q] = \int dt\, \delta q^i(t) \frac{\delta}{\delta q^i(t)} S[q] \quad \forall \underline{\text{variations}}\, \delta q \qquad (\tilde{q}(t) = q(t) + \delta q(t)) \tag{1.3}$$

which extremize $S$ are called the **equations of motion (eom)**

- If there exists a particular (infinitesimal) variation $\delta_\varepsilon q$ (parametrized by some **transformation parameters** $\varepsilon^a$) with

$$\delta_\varepsilon S[q] = 0 \quad \underline{\forall q} \tag{1.4}$$

it is called a **symmetry transformation** of the action.

- Or finite (more general, as there might not be an infinitesimal version, e.g. for discrete symmetries): A map

$$f: \quad \mathcal{P}M \rightarrow \mathcal{P}M \tag{1.5}$$

$$q \mapsto \tilde{q} = f(q) \quad (= q + \delta_\varepsilon q) \tag{1.6}$$

with

$$S[f(q)] = S[q] \quad \forall q \tag{1.7}$$

`finite-sym`

is called a symmetry trafo.

- Symmetry transformations of an action form a group:
  - identity is always a symmetry
  - Composition of two symmetries is always a symmetry

$$S[f(q)] = S[q], \quad S[g(q)] = S[q] \quad \forall f, g \qquad \Rightarrow S[f(g(q))] = S[g(q)] = S[q] \tag{1.8}$$

– Inverse is always a symmetry (plug $f^{-1}(q)$ into (1.7)

$$S[q] = S[f^{-1}(q)] \tag{1.9}$$

– counterexample to last: $S[q_1, q_2] = \int dt \quad L(q_1, \dot{q}_1)$ (action does not depend on $q_2$). The function $f : q_2 \mapsto 0$ is a non-invertible symmetry transformation of this action.

– In order to avoid this:

  * Restrict to trafos generated by infinitesimal ones (which are always invertible)
  * Or restrict the trafos to be elements of the diffeomorphisms on $\mathcal{P}M$. Then the symmetry group is a subgroup

- A symmetry of the action is also a symmetry of the equations of motion, in the sense that it maps solutions of the equations to other solutions:

$$\frac{\delta}{\delta q^i(t)} S[q_c + \delta_\varepsilon q] = \frac{\delta}{\delta q^i(t)} S[q_c] + \int dt' \, \delta_\varepsilon q^j(t') \frac{\delta^2}{\delta q^j(t') \delta q^i(t)} S[q] = \tag{1.10}$$

$$= \underbrace{\frac{\delta}{\delta q^i(t)} S[q_c]}_{=0} + \underbrace{\frac{\delta}{\delta q^i(t)} \, \delta_\varepsilon S[q]}_{=0} = 0 \tag{1.11}$$

- **On shell** = either on the "shell" defined by the equations of motion (that's how I will call it, while probably writing "**eom**"), or on the constraint surface (I will say "on the constraint surface"), in particular on the mass-shell $p^2 = -m^2$ (which I will call – if ever – "on mass-shell").

- **Off shell** = Not on the "shell".

**Definition 1.1.** Assume we can expand the symmetry transformation $\delta_\varepsilon q^i(t)$ in some (infinitesimal) and maybe time-dependent parameter $\varepsilon$ and its time derivatives:

$$\delta_\varepsilon q^i(t) = \varepsilon^a(t) \underbrace{\delta_a^{(0)} q^i}_{R^i_{(0)a}}(t) + \dot{\varepsilon}^a(t) \delta_a^{(1)} q^i(t) + \ldots = \tag{1.12}$$

$$= \int dt' \quad \varepsilon^a(t') \underbrace{\delta_a q^i(t', t)}_{R^i_a(t',t)} \tag{1.13}$$

If $\delta_\varepsilon q^i(t)$ is a symmetry transformation for any (infinitesimal) function $\varepsilon^a(t)$ of $t$, i.e.

$$\delta_\varepsilon S[q] = 0 \quad \forall \varepsilon(t) \tag{1.14}$$

then $\delta_\varepsilon$ is called a **local symmetry of S** or (in particular if those $q$'s related by a local transformation $\delta_\varepsilon$ are identified) a **gauge transformation**.
(some ambiguity for the latter: as soon as one uses a symmetry to identify solutions – could also be a discrete symmetry like dualities – one can call these symmetries gauge transformations, even in the discrete case. However, as a local symmetry enforces the identification, one tends to identify gauge symmetry with local symmetry)

**Remarks:**

- Allowing $\varepsilon(t)$ to be an arbitrary function of $t$ is quite a strong condition on the symmetry. It is quite common that one has some restrictions on $\varepsilon(t)$. For example, $\delta_\varepsilon q^i(t)$ might be a symmetry transformation only if $\varepsilon(t) = \varepsilon = const$, in which case all the derivative terms drop. Such a symmetry is called a **global symmetry**.

- The definition of a local symmetry requires arbitrary $\varepsilon$, but not arbitrary form of $\delta_\varepsilon q^i(t)$. In particular there can be also local symmetries that don't have derivative terms, i.e. that are of the form $\delta_\varepsilon q^i(t) = \varepsilon^a(t) \delta_a^{(0)} q^i(t)$. Still they differ from global symmetries by the fact that $\varepsilon(t)$ has an arbitrary $t$-dependence.

- **Remark on the notation:** Note that $M$ can not only be finite dimensional (for example for a point particle moving in space-time), but also infinite dimensional, like in field theorey where one should replace the discrete index $i$ by a continuous (perhaps combined with a discrete). Take for example a co-vector field $A$ in spacetime (section of the cotangent bundle of $\mathbb{R}^{3,1}$, where "3,1" means that the signature of the metric is $(-1,1,1,1)$). Its components are $A_\mu(x^0,x^1,\ldots,x^3) \equiv A_\mu(t,\vec{x})$. One can think of $\vec{x}$ as a continous "index", i.e. $q^i(t) \equiv A_{\mu,\vec{x}}(t)$. Many of the statements derived for finite dimensions carry over for infinite dimensions. Sometimes, however, it is nice to avoid the split of the spacetime coordinates into time $t$ and space $\vec{x}$. So some of our discussion will be made explicitly for the field-theory case. When we don't want to specify, whether we are talking about vector fields or other tensor fields or whatever, we will collectively denote all the fields of some theory by $\phi^I(x)$ (where x is now space and time). One then obtains the particle case by splitting spacetime into time and space and taking the space-dimension to be zero. So $q^i(t)$ in these notes can be both, either the point particle case (space-dimension=0) of the general field theory or a **condensed notation** of the general field theory:

$$q^i(t) \equiv \phi^{\mathcal{I},\vec{x}}_{\text{all}}(t) \equiv \phi^{\mathcal{I}}_{\text{all}}(x) \quad (\text{i is then a cont } \infty\text{-dim index} : \sum_i(\ldots) = \sum_{\mathcal{I}}\int d^{dim}\vec{x}(\ldots)) \tag{1.15}$$

$$\text{or } q^i(t) \equiv q^i(t) = \phi^{\mathcal{I}}_{\text{all}}(t) \quad (i \text{ is discrete and finite dimensional}) \tag{1.16}$$

Rarely we might use even more condensed notation, where also time is included in the index:

$$q^i \hat{=} q^{i,t}, \quad \sum_i \hat{=} \sum_i \int dt \tag{1.17}$$

## 1.4 Examples

A few examples which we might use from time to time to illustrate some general concepts / techniques.

### 1.4.1 Stupid example

- $$S[q_1,q_2] = \int_a^b dt \quad \tfrac{1}{2}(\dot{q}_1-\dot{q}_2)^2 - V(q_1-q_2) \tag{1.18}$$
- Local symmetry
$$\delta_\varepsilon q_1(t) = \varepsilon(t), \quad \delta_\varepsilon q_2(t) = \varepsilon(t) \tag{1.19}$$
- Eom's ($\delta q|_a = 0 = \delta q|_b$ general variations, but with fixed end-points)
$$0 = \int dt \quad (\delta\dot{q}_1 - \delta\dot{q}_2)(\dot{q}_1-\dot{q}_2) - (\delta q_1 - \delta q_2)V'(q_1-q_2) = \tag{1.20}$$
$$= \int dt \quad (\delta q_2 - \delta q_1)\underbrace{((\ddot{q}_1-\ddot{q}_2) + V'(q_1-q_2))}_{\frac{\delta S}{\delta q_2(t)} = -\frac{\delta S}{\delta q_1(t)}} \tag{1.21}$$
- equations of motion are dependent (Noether identities):
$$\frac{\delta S}{\delta q_1(t)} + \frac{\delta S}{\delta q_2(t)} = 0 \tag{1.22}$$
- Solutions to eom's are not unique. Only the difference is unique after fixing initial conditions
$$q_1(t) = \tfrac{1}{2}q_0 + \tfrac{1}{2}vt + f(t) \quad (\text{arbitrary } f(t)) \tag{1.23}$$
$$q_2(t) = -\tfrac{1}{2}q_0 - \tfrac{1}{2}vt + f(t) \tag{1.24}$$
$$q_1(t) - q_2(t) = q_0 + vt \tag{1.25}$$
- Physically identify solutions that differ by a gauge transformation (equivalence relation)
$$(q_1,q_2) \sim (q_1 + \delta_\varepsilon q_1, q_2 + \delta_\varepsilon q_2) \tag{1.26}$$
- Can **gauge fix** (= choose a representative of the equivalence class of paths) e.g. via
$$f(t) \stackrel{!}{=} \tfrac{1}{2}q_0 + \tfrac{1}{2}vt \quad (q_2(t) = 0, \quad q_1 \text{ as effective variable}) \tag{1.27}$$
- This is an example for a gauge symmetry without the explicit appearance of a gauge connection ($\delta\boldsymbol{A} = \mathbf{d}\varepsilon + \ldots$).

### 1.4.2 Geodesic equation from the variational principle

- Embedding $\Sigma \xrightarrow{X} M$
- $\dim \Sigma = 1$: Geodesic:

$$S[X] = -\int_\Sigma d\sigma \quad \underbrace{\sqrt{|\, \dot{X}^m g_{mn}(X) \dot{X}^n \,|}}_{\sqrt{X^* g}} \tag{1.28}$$

  Variation yields in proper-time parametrization (affine parametrization) the geodesic equation:

$$\delta S = 0 \quad \Longleftrightarrow \quad \ddot{X}^m - \dot{X}^k \dot{X}^l \Gamma_{kl}^m = 0 \tag{1.29}$$

  (e.g. point-particle in GR-background)

- $\dim \Sigma$ arbitrary:

$$S[X] = -\int_\Sigma d^d\sigma \quad \underbrace{\sqrt{|\det(\partial_\mu X^m g_{mn} \partial_\nu X^n)\,|}}_{\sqrt{|\det(X^* g)|}} \tag{1.30}$$

  (e.g. bosonic string, Dp-brane). Variation yields a generalized geodesic equation (again in a particular parametrization)

$$\partial_\mu \partial^\mu X^m - \partial^\mu X^k \partial_\mu X^l \Gamma_{kl}^m = 0 \tag{1.31}$$

- Gauge symmetry: local worldsheet-reparametrizations:

$$\delta_\varepsilon X^m = \mathcal{L}_\varepsilon X^m \quad = \quad \varepsilon^\kappa(\sigma) \partial_\kappa X^m(\sigma) \tag{1.32}$$

- E.g. for 1-dim case can gauge fix $\sigma^0 = X^0 \Rightarrow \dot{X}^0 = 1$, and assume that the remaining $\dot{X}^i \ll 1$ (where 1 can be thought of as the speed of light). Then we can expand the square root as follows:

$$\sqrt{|\, \dot{X}^m g_{mn}(X) \dot{X}^n \,|} = \sqrt{1 - \dot{X}^i g_{ij}(X) \dot{X}^j} = 1 - \tfrac{1}{2} \dot{X}^i g_{ij}(X) \dot{X}^j + \ldots \tag{1.33}$$

  The second term is just the kinetic energy of a point particle in Newtonian mechanics which serves in this case as a Lagrangian.

- Above examples are without worldvolume metric. So one has gauge invariance without a "gauge field".
- With independent worldvolume-metric (Polyakov-action, works at least in 2dim)

$$S[X] = \int_{\Sigma^{(2)}} d^2\sigma \sqrt{h} \quad \tfrac{1}{2} h^{\mu\nu} \partial_\mu X^m g_{mn} \partial_\nu X^n \tag{1.34}$$

  In 1d one has to add an extra term containing only $h$ (or the vielbein $e$).
  $0 \stackrel{!}{=} \frac{\delta S}{\delta h_{\underline{mn}}} \Rightarrow$ eom for $h_{\mu\nu}$, can be plugged back. One obtains back the old action.

**Remark on condensed notation:**

In case of the worldsheet (dimΣ=2)

$$\Sigma^{(2)} \cong \mathbb{R} \times \Sigma^{(1)} \tag{1.35}$$

$$\delta_\varepsilon X^m(\underbrace{\sigma^0}_{\tau}, \sigma^1) \quad = \quad \varepsilon^\kappa(\sigma^0, \sigma^1) \partial_\kappa X^m(\sigma^0, \sigma^1) = \tag{1.36}$$

$$\delta_\varepsilon X^{m,\sigma^1}(\tau) \quad = \quad \int d\tilde{\sigma}^1 \varepsilon^{\kappa, \tilde{\sigma}^1}(\tau) \underbrace{\delta(\sigma' - \sigma^1) \partial_\kappa X^m(\tau, \sigma^1)}_{\delta^{(0)}_{\kappa, \tilde{\sigma}^1} X^{m,\sigma^1}(\tau)} \tag{1.37}$$

Compare to

$$\delta_\varepsilon q^i(t) \quad = \quad \varepsilon^a(t) \underbrace{\delta_a^{(0)} q^i}_{R^i_{(0)a}}(t) + \dot{\varepsilon}^a(t) \delta_a^{(1)} q^i(t) + \ldots = \tag{1.38}$$

Local in $\sigma^0$ →gauge symmetry. Local in $\sigma^1$ →infinite global symmetries (conserved charges). Usually local in both.

### 1.4.3 General Relativity

Einstein-Hilbert action without matter

$$S_{EH}[g] \equiv \frac{1}{16\pi G} \int d^d x \quad \sqrt{|\det g|} \underbrace{R}_{\mathrm{Ricci-scalar\,corresp\,to}\, g_{mn}(x)} \tag{1.39}$$

It is diffeomorphism invariant (gauge invariance)

$$\delta_\xi g_{\mu\nu}(x) = \mathcal{L}_\xi g_{\mu\nu}(x) = \xi^\kappa(x)\partial_\kappa g_{\mu\nu}(x) + 2\partial_{(\mu|}\xi^\kappa(x) g_{|\kappa)\nu}(x) \tag{1.40}$$

Equations of motion give Ricci-flatness

$$0 \stackrel{!}{=} \frac{\delta}{\delta g_{\mu\nu}(x)} S \propto \quad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \tag{1.41}$$

$$\Longleftrightarrow 0 = R_{\mu\nu} \tag{1.42}$$

### 1.4.4 Yang Mills I - a first glance

The aim of this subsection is to quickly introduce the Yang Mills action in order to have it at hand as an example. This will be done without introducing in detail fiber bundle geometry (this will be done later). Some names will be dropped in order to specify what the objects are, but without yet carefully defining these.

- Conventions:

| here | [Nak] | [Kugo] | [Hitchin] | [thesis] |
|---|---|---|---|---|
| $\boldsymbol{A}, \boldsymbol{D}, F$ <br> tr | $\mathcal{A} = iA, \mathcal{D}, \mathcal{F} = iF$ | $\frac{g}{i}A, D, \frac{g}{i}F$ | $A, D, F$ <br> $\frac{1}{2}\mathrm{tr}$ | $-\boldsymbol{\Omega}, \nabla, -R$ |

- $\boldsymbol{A}$: Lie algebra valued one form, or to be more precise a local section of $T^*M \otimes \mathfrak{g}$.
  In fact it is the pullback of the so called connection 1-form (which is an element of $\Omega^1(P(M,G)) \otimes \mathfrak{g}$) of a principle fiber bundle along a local section.
  $F$: pullback of the corresponding curvature 2-form [Nak, 355]

- Action[1]

$$S[\boldsymbol{A}] \equiv \int_{\mathbb{R}^{D-1,1}} (-)^{p(D-p)} \epsilon_{(D,p)} \tfrac{1}{4} \mathrm{tr}\, F \wedge \star F = \tag{1.43}$$

$$= \int_{\mathbb{R}^{D-1,1}} d^D x \quad \tfrac{1}{4} \mathrm{tr}\, F^{mn} F_{mn} \quad (m,n \in \{0,1,\ldots,D-1\}) \tag{1.44}$$

with [Nak,p.353]

$$F = \mathbf{d}\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} \tag{1.45}$$

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] \tag{1.46}$$

[1] [thesis, p.169, (D.24)] Define the components of the $\varepsilon$-tensor (volume-form) for Minkowskian signature via

$$\varepsilon_{m_1\ldots m_D} \equiv \sqrt{|g|}\epsilon_{m_1\ldots m_D}, \quad \epsilon_{01\ldots(D-1)} \equiv 1 = i\epsilon_{D,1\ldots(D-1)} = \underbrace{(-)^{D-1} i \epsilon_{1\ldots D}}_{\equiv \epsilon^{(E)}_{1\ldots D}}$$

The last step defines the Euclidean version. The actual D-form thus reads

$$\varepsilon \equiv \tfrac{1}{D!}\varepsilon_{m_1\ldots m_D} \mathbf{d}x^{m_1} \cdots \mathbf{d}x^{m_D} = (-)^D i \varepsilon^{(E)}$$

Its components obey the following identity:

$$\tfrac{1}{p!}\epsilon_{a_1\ldots a_{D-p}c_1\ldots c_p}\epsilon^{b_1\ldots b_{D-p}c_1\ldots c_p} = -\delta^{b_1\ldots b_{D-p}}_{a_1\ldots a_{D-p}}, \quad \tfrac{1}{p!}\epsilon^{(E)}_{a_1\ldots a_{D-p}c_1\ldots c_p}\epsilon_{(E)}^{b_1\ldots b_{D-p}c_1\ldots c_p} = \delta^{b_1\ldots b_{D-p}}_{a_1\ldots a_{D-p}}$$

Using this tensor, we can define the Hodge dual as ([thesis,p.171,fn3], with redef $\omega^{(p)}_{mmm} \to \frac{1}{p!}\omega^{(p)}_{mmm}$, $\varepsilon \to d!\varepsilon$)

$$(\star\omega^{(p)})_{m_1\ldots m_{D-p}} \equiv \tfrac{\epsilon_{(D,p)}}{p!}\varepsilon_{m_1\ldots m_{D-p}}{}^{k_1\ldots k_p}\omega^{(p)}_{k_1\ldots k_p} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$

where $\epsilon_{(D,p)}$ is some $p$ and $D$ dependent sign-factor which can be chosen in any convenient way. Natural choices are $1, (-)^{p(D-p)}, (-)^{\frac{p(p-1)}{2}}$ or $(-)^{\frac{p(p-1)}{2}}(-)^{p(D-p)}$. The second is most common in literature ([Nak]), the last was used by myself in [thesis]. (The factor $(-)^{p(D-p)}$ corresponds to contracting $\omega^{(p)}$ with the first instead of the last $p$ indices of the $\varepsilon$-tensor, the factor $(-)^{\frac{p(p-1)}{2}}$ reverses the order of $\omega^{(p)}$'s indices.)

or expressed in terms of the **covariant derivative**

$$\begin{aligned} F &= [\boldsymbol{D},\boldsymbol{D}] && (1.47)\\ \boldsymbol{D} &= \mathbf{d} + [\boldsymbol{A}, .] && (1.48)\end{aligned}$$

- Bianchi identity for general (nonabelian) group

$$\begin{aligned} \mathbf{d}F &= [\mathbf{d}\boldsymbol{A},\boldsymbol{A}]_\wedge && (1.49)\\ &= [F - \boldsymbol{A}\wedge\boldsymbol{A},\boldsymbol{A}]_\wedge && (1.50)\end{aligned}$$

$$\boldsymbol{D}F \equiv \mathbf{d}F - [\boldsymbol{A},F]_\wedge = 0 \quad (\iff [\boldsymbol{D},[\boldsymbol{D},\boldsymbol{D}]] = 0) \qquad (1.51)$$

- Gauge transformation (bundle transition maps)

$$\tilde{\boldsymbol{A}} = g\boldsymbol{A}g^{-1} + \underbrace{g\mathbf{d}g^{-1}}_{-\mathbf{d}g\, g^{-1}} \quad , \quad \tilde{F} = gFg^{-1} \qquad (1.52)$$

Or for an infinitesimal gauge transformation $g = 1 + \alpha$:

$$\delta\boldsymbol{A} = [\alpha,\boldsymbol{A}] - \mathbf{d}\alpha = -\boldsymbol{D}\alpha \quad , \quad \delta F = [\alpha, F] \qquad (g = 1+\alpha) \qquad (1.53)$$

- Equations of motion

$$\delta F = \mathbf{d}\delta\boldsymbol{A} + [\boldsymbol{A},\delta\boldsymbol{A}] = \boldsymbol{D}\delta\boldsymbol{A} \qquad (1.54)$$

$$\begin{aligned} \delta S &= \int -\tfrac{1}{4}\mathrm{tr}\,\delta F\wedge\star F - \tfrac{1}{4}\underbrace{\mathrm{tr}\,F\wedge\star\delta F}_{\mathrm{tr}\,\delta F\wedge\star F} = && (1.55)\\ &= \int -\tfrac{1}{2}\mathrm{tr}\,\boldsymbol{D}\delta\boldsymbol{A}\wedge\star F && (1.56)\\ &= \int -\tfrac{1}{2}\mathrm{tr}\,\delta\boldsymbol{A}\wedge\boldsymbol{D}\star F + \int \mathbf{d}\big(\tfrac{1}{2}\mathrm{tr}\,\delta\boldsymbol{A}\wedge\star F\big) && (1.57)\end{aligned}$$

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The Hodge-dual of 1 is simply

$$(\star 1)_{m_1\dots m_D} \equiv \epsilon_{(D,0)}\varepsilon_{m_1\dots m_D} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})$$

$$\star 1 = \epsilon_{(D,0)}\varepsilon = \varepsilon \quad (\text{for all above versions of } \epsilon_{(D,p)})$$

The square of the Hodge-star operation is either +1 or -1, depending on dimension $D$ and (convention-dependent) also on the form-degree $p$.

$$\star^2 = -(-)^{p(D-p)}\epsilon_{(D,p)}\epsilon_{(D,D-p)} \quad (\star^2_{(E)} = (-)^{p(D-p)}\epsilon_{(D,p)}\epsilon_{(D,D-p)})$$

$$\star^2 1 = -\underbrace{\epsilon_{(D,0)}\epsilon_{(D,D)}}_{\in\{1,\epsilon_{(D)}\}}, \quad \star\varepsilon = -\underbrace{\epsilon_{(D,D)}}_{\in\{1,\epsilon_{(D)}\}}$$

The Hodge dual of a wedge product of two forms leads to the contraction of its indices. If we denote by $\tilde{\omega}$ the p-vector obtained by raising all $p$ indices of the p-form $\omega^{(p)}$, then we can write

$$\star(\omega^{(p)}\wedge\eta^{(q)}) = (-)^{pq+p(D-p)}\epsilon_{(D,p+q)}\epsilon_{(D,q)}(-)^{p(p-1)/2}\imath_{\tilde{\omega}^{(p)}}\star\eta^{(q)} = (-)^{q(D-q)}\epsilon_{(D,p+q)}\epsilon_{(D,p)}(-)^{q(q-1)/2}\imath_{\tilde{\eta}^{(q)}}\omega^{(p)}$$

Replacing $\eta^{(q)}$ by $\star\eta^{(p)}$ (of degree $q = D-p$) and acting with another star, yields a **symmetric inner product**

$$(\omega^{(p)}\wedge\star\eta^{(p)}) = (-)^{p(D-p)}\epsilon_{(D,p)}\underbrace{(\imath_{(-)^{p(p-1)/2}\tilde{\omega}^{(p)}}\eta^{(p)})}_{\text{posdef for Euklid}}\varepsilon = (\eta^{(p)}\wedge\star\omega^{(p)})$$

Above we actually have a volume form, so that the proper inner product is its integral over the manifold.

For $\epsilon_{(D,p)} = (-)^{p(D-p)}(-)^{p(p-1)/2}$ we have

$$\begin{aligned} (\star\omega^{(p)})_{m_1\dots m_{D-p}} &\equiv \tfrac{1}{p!}\omega^{(p)}_{k_p\dots k_1}\varepsilon^{k_1\dots k_p}{}_{m_1\dots m_{D-p}} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})\\ (\star 1)_{m_1\dots m_D} &\equiv \varepsilon_{m_1\dots m_D} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})\\ \star^2 &= -(-)^{D(D-1)/2}\\ (\omega^{(p)}\wedge\star\eta^{(p)}) &= (\imath_{\tilde{\omega}^{(p)}}\eta^{(p)})\varepsilon = (\eta^{(p)}\wedge\star\omega^{(p)})\end{aligned}$$

For $\epsilon_{(D,p)} = (-)^{p(D-p)}$ (e.g. [Nak])

$$\begin{aligned} (\star\omega^{(p)})_{m_1\dots m_{D-p}} &\equiv \tfrac{1}{p!}\omega^{(p)}_{k_1\dots k_p}\varepsilon^{k_1\dots k_p}{}_{m_1\dots m_{D-p}} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})\\ (\star 1)_{m_1\dots m_D} &\equiv \varepsilon_{m_1\dots m_D} \quad (\star_{(E)} \quad \text{with } \varepsilon_{(E)})\\ \star^2 &= -(-)^{p(D-p)}, \quad \star^2_{(E)} = (-)^{p(D-p)}\\ (\omega^{(p)}\wedge\star\eta^{(p)}) &= (\imath_{(-)^{p(p-1)/2}\tilde{\omega}^{(p)}}\eta^{(p)})\varepsilon = (\eta^{(p)}\wedge\star\omega^{(p)}) \qquad \diamond\end{aligned}$$

$$\delta S[\boldsymbol{A}] = 0 \Rightarrow \quad \boldsymbol{D} \star F = 0 \quad (\boldsymbol{D}_m F^{mn} = 0) \quad \& \quad F_{mn} n^n|_{\partial M} = 0 \tag{1.58}$$

- Gauge group U(1) (commutative, linear eom's): Maxwell electromagnetic field $\boldsymbol{E}$ :1-form on $\mathbb{R}^3$,

$$E_i = F_{i0} \tag{1.59}$$

$\boldsymbol{B}$ :Hodge dual of a 2-form $B^{(2)}$ (in 3dim: 1-form $\cong$1-vector),

$$B^i = \tfrac{1}{2}\epsilon^{ijk} \underbrace{F_{jk}}_{\equiv B_{jk}} \tag{1.60}$$

So $F$ written in terms of electric and magnetic field reads

$$F = B + c\boldsymbol{E} \wedge \mathbf{d}t \tag{1.61}$$

This is a closed 2-form in $\mathbb{R}^4$ upon some of the Maxwell-equations (the Bianchi-part of Maxwell eqs)

$$\mathbf{d}F = \mathbf{d}B + \mathbf{d}t \wedge \mathbf{d}\boldsymbol{E} = \tag{1.62}$$

$$= \mathbf{d}t \wedge \mathbf{d}x^i \wedge \mathbf{d}x^j \left(\tfrac{1}{2}\partial_t B_{ij} + c\partial_i E_j\right) + \mathbf{d}x^k \wedge \mathbf{d}x^i \wedge \mathbf{d}x^j \left(\tfrac{1}{2}\partial_k B_{ij}\right) \tag{1.63}$$

$$\mathbf{d}F = 0 \quad \Longleftrightarrow \quad \partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla}\vec{B} = 0 \tag{1.64}$$

Similarly

$$\mathbf{d}\star F = 0 \iff \partial_t \vec{E} - \vec{\nabla} \times \vec{B} = 0, \quad \vec{\nabla}\vec{E} = 0 \tag{1.65}$$

- Noether identities: (see later)

$$0 = D_\mu \left(D_\nu F^{\mu\nu}\right) = [F_{\mu\nu}, F^{\mu\nu}] = 0 \tag{1.66}$$

- Remark on "gauging": fermion action with global invariance -> gauge field action,

$$S[\psi, \bar{\psi}] = \int d^4x \quad \bar{\psi}(\Gamma^\mu \partial_\mu - m)\psi \tag{1.67}$$

$$\text{or } S[\phi, \phi*] = \int d^4x \quad \partial_\mu \phi^* \partial^\mu \phi - V(\phi^*\phi) \tag{1.68}$$

Invariant under global U(1) transformations

$$\tilde{\psi} = e^{i\theta}\psi, \quad \theta = \text{const}, \qquad \delta\psi = i\theta\psi \tag{1.69}$$

$$\text{or } \tilde{\phi} = e^{i\theta}\phi \tag{1.70}$$

Not invariant under local transformation, because of derivative. Introducing gauge field $A_\mu$ ("minimal coupling") leads to locally invariant theory:

$$S[\psi, \bar{\psi}, A] = \int d^4x \quad \bar{\psi}(\Gamma^\mu \underbrace{(\partial_\mu - A_\mu)}_{D_\mu} - m)\psi + \mathcal{L}_{MW} \tag{1.71}$$

$$S[\phi, \phi^*, A] = \int d^4x \quad D_\mu \phi^* D^\mu \phi - V(\phi^*\phi) + \mathcal{L}_{MW} \tag{1.72}$$

Similarly: $S$ is invariant under global Poincaré-transformations (translation+$SO(1,3)$). Can make the local by introducing metric and vielbeins (local diffeo-invariance+local Lorentz invariance). Adding some kinetic term for metric ⇒general relativity coupled to fermions+Maxwell;
Note that $A_\mu$ is coupled to the current $j^\mu = \bar{\psi}\Gamma^\mu\psi$. This is the starting point of the "Noether procedure" to "gauge" a global symmetry.

## 1.5 Noether Theorem and Noether identities

Consider a quite general action functional of the form

$$S[\phi^{\mathcal{I}}_{\text{all}}] \equiv \overbrace{\int_\Sigma \underbrace{d^d\sigma}_{\mu}}^{\int_{\mathbb{R}} d\tau \int_{\Sigma^{(d-1)}} d^{(d-1)}\sigma} \quad \mathcal{L}(\phi^{\mathcal{I}}_{\text{all}}, \partial_\mu \phi^{\mathcal{I}}_{\text{all}}, \partial_{\mu_1}\partial_{\mu_2}\phi^{\mathcal{I}}_{\text{all}}, \ldots) \tag{1.73}$$

**Theorem 1.1** (Noether)**.** *To every transformation $\delta_{(\rho)}\phi^{\mathcal{I}}_{all}$ which leaves the action $S$ invariant, i.e. transforms the Lagrangian as*

$$\delta_{(\rho)}\mathcal{L} \stackrel{!}{=} \partial_\mu \mathcal{K}^\mu_{(\rho)} \qquad \textit{with} \quad n_\mu \mathcal{K}^\mu_{(\rho)}\Big|_{\partial\Sigma} = 0 \tag{1.74}$$

`symmetry-cond-Lagr`

*there is an on-shell divergence-free current $j^\mu_{(\rho)}$ whose explicit form can be chosen to be*

$$j^\mu_{(\rho)} \equiv \delta_{(\rho)}\phi^{\mathcal{I}}_{all} \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{\mathcal{I}}_{all})} + \sum_{k\geq 1}\sum_{i=0}^{k}(-)^i \partial_{\nu_1}\ldots\partial_{\nu_{k-i}}\delta\phi^{\mathcal{I}}_{all}\cdot\partial_{\nu_{k-i+1}}\ldots\partial_{\nu_k}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\nu_1}\ldots\partial_{\nu_k}\phi^{\mathcal{I}}_{all})} - \mathcal{K}^\mu_{(\rho)} \tag{1.75}$$

*Its off-shell divergence is given by*

$$\boxed{\partial_\mu j^\mu_{(\rho)} = -\delta_{(\rho)}\phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}}} \tag{1.76}$$

`eq:noet:currentdivergence`

*The such defined Noether current is unique up to trivially conserved terms of the form $\partial_\nu S^{[\nu\mu]}$.*
*In turn, for any given on-shell divergence-free current $\tilde{j}^\mu$ with*

$$\partial_\mu \tilde{j}^\mu = -y^{\mathcal{I}}_{(0)}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} - y^{\mathcal{I}\mu_1}_{(1)}\partial_{\mu_1}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} - \ldots - y^{\mathcal{I}\mu_N\ldots\mu_1}_{(N)}\partial_{\mu_1}\ldots\partial_{\mu_N}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} \tag{1.77}$$

*which is furthermore itself on-shell neither vanishing nor trivial, there is a corresponding nonzero symmetry transformation $\delta\phi^{\mathcal{I}}_{all}$ of the form*

$$\delta\phi^{\mathcal{I}}_{all} \equiv \sum_{k=0}^{N}(-)^k\partial_{\mu_1}\ldots\partial_{\mu_k}y^{\mathcal{I}\mu_1\ldots\mu_k}_{(k)} \tag{1.78}$$

*The simple form of the off-shell divergence given in (1.76) can be recovered upon redefining*

$$j^\mu \equiv \tilde{j}^\mu + \sum_{k=1}^{N}\sum_{i=0}^{k-1}(-)^i\partial_{\mu_1}\ldots\partial_{\mu_i}y^{\mathcal{I}\mu\,\mu_1\ldots\mu_{k-1}}_{(k)}\cdot\partial_{\mu_{i+1}}\ldots\partial_{\mu_{k-1}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} \tag{1.79}$$

*Proof.* See [thesis, p.182]. Let's prove only the mechanics-case $\phi^{\mathcal{I}}_{\text{all}}(\sigma) \to q^i(t)$ and even there further restrict to Lagrangians of the form $L(q,\dot{q})$ without higher derivatives. We start from the symmetry requirement (1.74)

$$\begin{aligned}
\dot{K}_{(\rho)} &\stackrel{!}{=} \delta_{(\rho)}L = &(1.80)\\
&= \delta_{(\rho)}q^i\frac{\partial}{\partial q^i}L + \delta_{(\rho)}\dot{q}^i\frac{\partial}{\partial\dot{q}^i}L = &(1.81)\\
&= \delta_{(\rho)}q^i\underbrace{\left(\frac{\partial}{\partial q^i}L - \frac{d}{dt}\frac{\partial}{\partial\dot{q}^i}L\right)}_{\frac{\delta}{\delta q^i(t)}S} + \frac{d}{dt}\left(\delta_{(\rho)}q^i\frac{\partial}{\partial\dot{q}^i}L\right) &(1.82)
\end{aligned}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\delta_{(\rho)}q^i\frac{\partial}{\partial\dot{q}^i}L - K_{(\rho)}\right) = -\delta_{(\rho)}q^i\frac{\delta}{\delta q^i(t)}S \tag{1.83}$$

The main difference for higher derivatives is that one needs to use a generalized formula for partial integration which is of the form

$$\partial^k a\cdot b = \partial\left[\sum_{i=0}^{k-1}(-)^i\partial^{k-1-i}a\cdot\partial^i b\right] + (-)^k a\cdot\partial^k b \tag{1.84}$$

Instead for $L(q,\dot{q})$ we just needed simple partial integration. The inverse direction is left as an exercise... □

**Fact.** *Every Noether current with on-shell vanishing divergence leads to a conserved* ***Noether charge*** *:*

$$\begin{aligned}
Q &\equiv \int_{\Sigma^{(d-1)}} d^{d-1}\sigma\, j^0 &(1.85)\\
\frac{\partial}{\partial t}Q &= \int_{\Sigma^{(d-1)}} d^{d-1}\sigma\,\partial_0 j^0 = -\int_{\Sigma^{(d-1)}} d^{d-1}\sigma\,\partial_i j^i \stackrel{\text{no bdry}}{=} 0 &(1.86)
\end{aligned}$$

**Theorem 1.2** (2nd Noether theorem)**.** *If $\delta_\rho\phi^{\mathcal{I}}_{all}$ is a symmetry transformation for arbitrary local (gauge)parameters $\rho^a(\sigma)$ of the form*

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{all} \equiv \underbrace{\rho^a\delta_a\phi^{\mathcal{I}}_{all}}_{\delta^0_{(\rho)}\phi^{\mathcal{I}}_{all}} + \underbrace{\partial_\mu\rho^a\delta^\mu_a\phi^{\mathcal{I}}_{all}}_{\delta^1_{(\rho)}\phi^{\mathcal{I}}_{all}} + \underbrace{\partial_{\mu_1}\partial_{\mu_2}\rho^a\delta^{\mu_1\mu_2}_a\phi^{\mathcal{I}}_{all}}_{\delta^2_{(\rho)}\phi^{\mathcal{I}}_{all}} + \ldots \tag{1.87}$$

*then the following **Noether identities** hold and are equivalent to (1.76)*

$$\delta_a\phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} - \partial_{\mu_1}\left(\delta^{\mu_1}_a\phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}}\right) + \ldots + (-)^{N+1}\partial_{\mu_1}\ldots\partial_{\mu_{N+1}}\left(\delta^{\mu_{N+1}\cdots\mu_1}_a\phi^{\mathcal{I}}_{all}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}}\right) = 0 \tag{1.88}$$

*Proof.* Plugging (1.87) into

$$0 \stackrel{!}{=} \int d^d\sigma \quad \delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}(\sigma)\frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}S[\phi_{\text{all}}] = \tag{1.89}$$

$$= \int d^d\sigma \quad \left(\rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}} + \partial_\mu\rho^a\delta^\mu_a\phi^{\mathcal{I}}_{\text{all}} + \partial_{\mu_1}\partial_{\mu_2}\rho^a\delta^{\mu_1\mu_2}_a\phi^{\mathcal{I}}_{\text{all}} + \ldots\right)\frac{\delta}{\delta\phi^{\mathcal{I}}_{\text{all}}(\sigma)}S[\phi_{\text{all}}] \quad \forall\rho \tag{1.90}$$

This shows the Noether identities.

Similar to $\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}$, we can expand also $j^\mu_{(\rho)}$:

$$j^\mu_{(\rho)} \equiv \rho^a j^\mu_a + \partial_{\mu_1}\rho^a j^{\mu\,\mu_1}_a + \ldots + \partial_{\mu_1}\ldots\partial_{\mu_{N-1}}\rho^a j^{\mu\,\mu_1\ldots\mu_{N-1}}_a \tag{1.91}$$

Plugging (1.87) and the expension of the current (1.91) into (1.76), one can show the equivalence of the Noether identities to (1.76). □

**Example:** Noether identity for Yang Mills: The equations of motion where

$$\frac{\delta S}{\delta A^c_\mu} \propto D_\nu F^{\nu\mu}_c \tag{1.92}$$

The variation of the gauge field was:

$$\delta A_\mu = [\alpha, A_\mu] - \partial_\mu\alpha = \tag{1.93}$$

$$= \alpha^a[T_a, A_\mu] - \partial_\nu a^a\delta^\nu_\mu T_a \tag{1.94}$$

$$\delta A^c_\mu = \alpha^a f_{ab}{}^c A^b_\mu - \partial_\nu a^a\delta^\nu_\mu\delta^c_a \tag{1.95}$$

Compare to

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \underbrace{\rho^a\delta_a\phi^{\mathcal{I}}_{\text{all}}}_{\delta^0_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}} + \underbrace{\partial_\mu\rho^a\delta^\mu_a\phi^{\mathcal{I}}_{\text{all}}}_{\delta^1_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}} + \underbrace{\partial_{\mu_1}\partial_{\mu_2}\rho^a\delta^{\mu_1\mu_2}_a\phi^{\mathcal{I}}_{\text{all}}}_{\delta^2_{(\rho)}\phi^{\mathcal{I}}_{\text{all}}} + \ldots \tag{1.96}$$

The Noether identities are in general

$$\delta_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}} - \partial_{\mu_1}\left(\delta^{\mu_1}_a\phi^{\mathcal{I}}_{\text{all}}\frac{\delta S}{\delta\phi^{\mathcal{I}}_{\text{all}}}\right) + \ldots = 0 \tag{1.97}$$

So this translates for Yang Mills into

$$0 \stackrel{?}{=} f_{ab}{}^c A^b_\mu\frac{\delta S}{\delta A^c_\mu} + \partial_\nu\left(\delta^\nu_\mu\delta^c_a\frac{\delta S}{\delta A^c_\mu}\right) = \tag{1.98}$$

$$\propto f_{ab}{}^c A^b_\mu D_\nu F^{\nu\mu}_c + \partial_\mu\left(D_\nu F^{\nu\mu}_a\right) = \tag{1.99}$$

$$= f_{ab}{}^c A^b_\mu\left(\partial_\nu F^{\nu\mu}_c + A^d_\nu f_{dec}F^{\nu\mu e}\right) + \partial_\mu\left(\partial_\nu F^{\nu\mu}_a + A^c_\nu f_{cba}F^{\nu\mu b}\right) = \tag{1.100}$$

$$= f_{ab}{}^c A^b_\mu\partial_\nu F^{\nu\mu}_c + f_{ab}{}^c A^b_\mu A^d_\nu f_{dec}F^{\nu\mu e} + \partial_\mu A^c_\nu f_{cba}F^{\nu\mu}_b + A^c_\nu f_{cba}\partial_\mu F^{\nu\mu b} = \tag{1.101}$$

$$= \underbrace{f_{ab}{}^c A^b_\mu\partial_\nu F^{\nu\mu}_c - A^b_\mu f_{bca}\partial_\nu F^{\nu\mu c}}_{=0} + \left(\partial_\mu A^c_\nu f_{cea} - A^b_\mu A^d_\nu\underbrace{f_{ab}{}^c f_{ced}}_{-\frac{1}{2}f_{ea}{}^c f_{cbd}}\right)F^{\nu\mu e} = \tag{1.102}$$

$$= F^c_{\mu\nu}f_{cea}F^{\nu\mu e} = 0 \quad \surd \tag{1.103}$$

Probably boils down to (exercise: check)

$$0 = D_\mu\left(D_\nu F^{\mu\nu}\right) = [F_{\mu\nu}, F^{\mu\nu}] = 0 \tag{1.104}$$

prop:currentOnshellZero **Proposition 1.1.** *: The Noether current of a gauge symmetry vanishes on-shell up to trivially conserved terms*

$$j^{\mu}_{(\rho)} = \sum_{k=0}^{N} \lambda^{\mu\mathcal{I}\mu_1\ldots\mu_k}_{(\rho)} \partial_{\mu_1}\ldots\partial_{\mu_k} \frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} + \partial_\nu S^{[\mu\nu]}_{(\rho)} \tag{1.105}$$

*In turn, if a given global symmetry transformation has an on-shell vanishing current (compare [Henneaux, p.95])*

$$j^{\mu}_{a} = \sum_{k=0}^{N} \lambda^{\mu\mathcal{I}\mu_1\ldots\mu_k}_{a} \partial_{\mu_1}\ldots\partial_{\mu_k} \frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}} \tag{1.106}$$

*then one can extend the transformation to a local one*

$$\delta_{(\rho)}\phi^{\mathcal{I}}_{all} \equiv \rho^a\delta_a\phi^{\mathcal{I}}_{all} - \partial_\mu\rho^a\lambda^{\mu\mathcal{I}}_a + \sum_{k=1}^{N}(-)^{k+1}\partial_{\mu_1}\ldots\partial_{\mu_k}\left(\partial_\nu\rho^a\lambda^{\nu\mathcal{I}\,\mu_1\ldots\mu_k}_a\right) \tag{1.107}$$

*Proof.* Quite technical to prove. We'll skip that as well as the proof of the next statement. □

thm:trafoOnshellZero **Theorem 1.3.** *Every on-shell vanishing symmetry transformation is a* ***trivial gauge transformation*** *as defined below:*

$$\delta\phi^{\mathcal{I}}_{all} \overset{\text{on-shell}}{=} 0, \quad \delta S = 0 \quad \Rightarrow \quad \delta\phi^{\mathcal{I}}_{all} = \int d^d\sigma \quad \mathcal{A}^{\mathcal{I}\mathcal{J}}(\sigma,\sigma')\frac{\delta S}{\delta\phi^{\mathcal{I}}_{all}(\sigma')} \quad \textit{with}\, \mathcal{A}^{\mathcal{I}\mathcal{J}}(\sigma,\sigma') = -\mathcal{A}^{\mathcal{J}\mathcal{I}}(\sigma',\sigma) \tag{1.108}$$

*Proof.* See in [Henneaux] (theorem 17.3 on page 414 or theorem 3.1 on page 70 - see also proof(p.229) of thm 10.1(p.209)) for a proof of this theorem. See [Henneaux] p.69 for a discussion of trivial gauge transformations. □

## Remark

- Shortcut to calculate Noether current:

$$\delta_{(\rho,\eta)}\phi^{\mathcal{I}}_{\text{all}} \equiv \eta(\sigma)\cdot\delta_\rho\phi^{\mathcal{I}}_{\text{all}} \tag{1.109}$$

$$\delta_{\rho,\sigma} = \int d^d\sigma \quad \partial_\mu\eta\cdot j^{\mu}_{(\rho)} \tag{1.110}$$

  Where $j^{\mu}_{(\rho)}$ vanishes on-shell. Can instead also use $\delta^{(0)}_{(\rho)}\phi^{\mathcal{I}}_{\text{all}} \equiv \rho^a\delta^{(0)}_a\phi^{\mathcal{I}}_{\text{all}}$.

- Conserved Noether Charges = Integrals of Motion

# Chapter 2

# Constrained Hamiltonian/Lagrangian Systems – Classical

## 2.1 Formulations with brackets (without gauge symmetry)

### 2.1.1 Legendre Transform without constraints

- The Lagrangian is a function on $TM$ (assuming that it depends maximally on first time derivatives of $q$)[1]:

$$\begin{aligned} L: \quad TM &\rightarrow \mathbb{R} \qquad (2.1)\\ (q^m, v^m) &\mapsto L(q,v) \qquad (2.2)\end{aligned}$$

- The Hamiltonian instead is a function on $T^*M$:

$$\begin{aligned} H: \quad T^*M &\rightarrow \mathbb{R} \qquad (2.3)\\ (q^m, p_m) &\mapsto H(q,p) \qquad (2.4)\end{aligned}$$

- The Legendre transformation brings one from one to the other by assigning a momentum (cotangent vector) $p$ for a given $q$ and $v$ as follows

$$p_m \equiv \frac{\partial L(q,v)}{\partial v^m} \qquad (2.5)$$

  Assuming that the resulting relation between $p_m$ and $v^m$ is invertible, i.e.

$$\det\left(\frac{\partial^2 L(q,v)}{\partial v^m \partial v^n}\right) \neq 0 \qquad (2.6)$$

  one can build the following function on the cotangent bundle as follows (**Legendre transform**)

$$H(q,p) = p_m v^m(q,p) - L(q, v(q,p)) \qquad (2.7)$$

- One can calculate the partial derivatives of $H$ without knowing the explicit form of the inverse transformation $v^m(q,p)$, because $\delta v^m(q,p)$ drops from the variation of $H$:

$$\begin{aligned} \delta H(q,p) &= \delta p_m v^m(q,p) + p_m \delta v^m(q,p) - \delta q^m \frac{\partial}{\partial q^m} L(q, v(q,p)) - \delta v^m \underbrace{\frac{\partial}{\partial v^m} L(q, v(q,p))}_{p_m} = \qquad (2.8)\\ &= \delta p_m v^m(q,p) - \delta q^m \frac{\partial}{\partial q^m} L(q, v(q,p)) \qquad (2.9)\end{aligned}$$

[1] For Lagrangians that depend on higher derivatives of $q^m$, see either the comment at the end of the subsequent subsection on first order Lagrangians (suggesting to iteratively replace $\dot{q}$ by momenta and build a "first order Lagrangian" which might not be first order yet, but can be used again to define momenta and so on) , or see [Henneaux, p.47, exercise 1.26]: simply introduce one new variable for each of $\dot{q}, \ddot{q}$ and so on:

$$\begin{aligned} p^{(1)} &\equiv \frac{\partial L}{\partial \dot{q}}, \quad p^{(2)} \equiv \frac{\partial L}{\partial \ddot{q}}, \ldots\\ H &\equiv p^{(1)}\dot{q} + p^{(2)}\ddot{q} - L(q, \dot{q}(q,p), \ddot{q}(q,p), \ldots) \qquad \diamond\end{aligned}$$

The partial derivatives of the two functions thus read:

$$\frac{\partial H}{\partial q^m} = -\frac{\partial L}{\partial q^m} \tag{2.10}$$

$$\frac{\partial H}{\partial p_m} = v^m \quad , \quad p_m = \frac{\partial L}{\partial v^m} \tag{2.11}$$

### 2.1.2 Hamiltonian eom's with Poisson bracket

- From the Lagrangian eom's

$$\frac{\partial L}{\partial q^m} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^m} = 0 \tag{2.12}$$

one can easily obtain the Hamiltonian ones (using the above relations):

$$\dot{q}^m = \frac{\partial H}{\partial p_m} = \{H, q\} \tag{2.13}$$

$$\dot{p}_m = -\frac{\partial H}{\partial q^m} = \{H, p\} \tag{2.14}$$

The Hamiltonian thus generates the time evolution via the Poisson bracket.

$$\{F, G\} \equiv \partial F/\partial p_m \frac{\partial}{\partial q^m} G - (-)^{FG} \partial G/\partial p_m \frac{\partial}{\partial q^m} F = \tag{2.15}$$

$$= \partial F/\partial p_m \frac{\partial}{\partial q^m} G - \partial F/\partial q^m \frac{\partial}{\partial p_m} G \tag{2.16}$$

$$\{F, G\} = -(-)^{FG}\{G, F\} \tag{2.17}$$

$$\{p_m, q^n\} = \delta_m{}^n \tag{2.18}$$

- Hamiltonian vector field

$$X_H \equiv \{H, -\} = \frac{\partial H}{\partial p_m}\frac{\partial}{\partial q^m} - \frac{\partial H}{\partial q^m}\frac{\partial}{\partial p_m} = \frac{\partial H}{\partial y^M} P^{MN} \frac{\partial}{\partial y^N} \tag{2.19}$$

or equivalently

$$\mathbf{d}H = \imath_{X_H}\omega \tag{2.20}$$

$$\partial_M H = X_H^N \omega_{NM} \tag{2.21}$$

So

$$\dot{y}^M = X_H y \tag{2.22}$$

- The Hamiltonian equations of motion can also be obtained from an action principle (first order action)

$$\tilde{L}: \quad T(T^*M) \to \mathbb{R} \tag{2.23}$$

$$(q, p, \dot{q}, \dot{p}) \mapsto \tilde{L}(q, p, \dot{q}, \dot{p}) \equiv \dot{q}^m p_m - H(q, p) \tag{2.24}$$

$$\frac{\partial \tilde{L}}{\partial q^m} = \frac{\partial L}{\partial q^m} \tag{2.25}$$

$$\frac{\partial \tilde{L}}{\partial \dot{q}^m} = p_m = \frac{\partial L}{\partial \dot{q}^m} \tag{2.26}$$

$$\frac{\partial \tilde{L}}{\partial p_m} = \dot{q}^m - \frac{\partial H}{\partial p_m} \tag{2.27}$$

- If the original Lagrangian depends on *higher derivatives* of $q$, say $\ddot{q}$, then $\tilde{L}$ will also depend on $\dot{p}$. But not on $\ddot{q}$ any more. Than one can do the procedure of defining momenta again. If one has even higher derivatives, one does this iteratively.

### 2.1.3 Schouten-Nijenhuis bracket on $\Gamma(\Lambda^{\bullet}TM)$

- Generalization of Lie-bracket of vector fields and of Lie-derivative
- Remember for vector fields $\boldsymbol{v} = v^m \boldsymbol{\partial}_m$ the **Cartan formulae**:

$$\begin{aligned}
[\imath_{\boldsymbol{v}}, \imath_{\boldsymbol{w}}] &= 0 \quad (\equiv \imath_{[v,w]_{alg}}) && (2.28)\\
[\mathbf{d}, \mathbf{d}] &= 0 && (2.29)\\
\boldsymbol{\mathcal{L}}_{\boldsymbol{v}} &\equiv [\imath_{\boldsymbol{v}}, \boldsymbol{d}] && (2.30)\\
[\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}, \mathbf{d}] &= 0 && (2.31)\\
[\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{w}}] &= \boldsymbol{\mathcal{L}}_{[\boldsymbol{v},\boldsymbol{w}]} && (2.32)\\
[\underbrace{[\imath_{\boldsymbol{v}}, \mathbf{d}]}_{\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}}, \imath_{\boldsymbol{w}}] &= \imath_{[\boldsymbol{v},\boldsymbol{w}]} && (2.33)
\end{aligned}$$

- Last line shows that the Lie bracket is a derived bracket. Inherits the Jacobi-property.
- Take now multivectors

$$\begin{aligned}
v^{(p)} &\equiv \tfrac{1}{p!} v^{m_1\ldots m_p} \boldsymbol{\partial}_{m_1} \wedge \ldots \wedge \boldsymbol{\partial}_{m_p} && (2.34)\\
\imath_{v^{(p)}} \rho^{(r)} &\equiv \tfrac{1}{p!(r-p)!} v^{k_1\ldots k_p} \rho_{k_p\ldots k_1 m_1\ldots m_{r-p}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p}} && (2.35)
\end{aligned}$$

- Define Lie derivative in the same way

$$\begin{aligned}
\boldsymbol{\mathcal{L}}_{v^{(p)}} \rho^{(r)} &\equiv [\imath_{v^{(p)}}, \boldsymbol{d}] \rho^{(r)} = && (2.36)\\
&= \frac{1}{(p-1)!(r-p+1)!} v^{k_1\ldots k_p} \partial_{k_p} \rho_{k_{p-1}\ldots k_1 m_1 \ldots m_{r-p+1}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p+1}} + \\
&\quad -(-)^p \frac{1}{p!(r-p)!} \partial_{m_1} v^{k_1\ldots k_p} \rho_{k_p\ldots k_1 m_1 \ldots m_{r-p}} \mathbf{d}x^{m_1} \wedge \ldots \wedge \mathbf{d}x^{m_{r-p}} && (2.37)
\end{aligned}$$

- The Cartan formulae then hold exactly in the same way and define a generalization of the Lie-bracket of vector fields, namely the **Schouten-Nijenhuis-bracket** of multivector fields

$$\begin{aligned}
[\underbrace{[\imath_{v^{(p)}}, \mathbf{d}]}_{\boldsymbol{\mathcal{L}}_{v^{(p)}}}, \imath_{w^{(p)}}] &\equiv \imath_{[v^{(p)}, w^{(p)}]} && (2.38)\\
[v^{(p)}, w^{(q)}] &= \Big( \tfrac{1}{(p-1)!q!} v^{[m_1\ldots m_{p-1}|k} \partial_k w^{|m_p \ldots m_{p+q-1}]} + \\
&\quad -(-)^{(p-1)(q-1)} \tfrac{1}{(q-1)!p!} w^{[m_1\ldots m_{q-1}|k} \partial_k v^{|m_q\ldots m_{p+q-1}]} \Big) \boldsymbol{\partial}_{m_1} \ldots \boldsymbol{\partial}_{m_{p+q-1}} && (2.39)
\end{aligned}$$

- It is a **Lie bracket of degree -1** :

$$\begin{aligned}
\deg[v^{(p)}, w^{(q)}] &= p+q-1 && (2.40)\\
[v^{(p)}, w^{(q)}] &= -(-)^{(p-1)(q-1)} (w^{(q)}, v^{(p)}) && (2.41)\\
[v,[w,u]] &= ((v,w),u) + (-)^{(v-1)(w-1)} (w,(v,u)) && (2.42)\\
[v, w \wedge u] &= [v,w] \wedge u + (-)^{(v-1)w} w \wedge [v,u] && (2.43)
\end{aligned}$$

- Alternative approach: Start by identifying the bracket of a vector $v$ with a general tensor $T$ with the Lie-derivative of the tensor $T$ with respect to the vector:

$$[v, T] \equiv \mathcal{L}_v T \tag{2.44}$$

In particular for a scalar $\phi$ or a vector $w$ we have

$$[v,\phi] \equiv \mathcal{L}_v \phi \;=\; v^m \partial_m \phi, \quad [v,w] \equiv \mathcal{L}_v w = (v^k \partial_k w^m - w^k \partial_k v^m) \boldsymbol{\partial}_m \tag{2.45}$$

Note that this definition implies a Leibniz rule for tensor products in the right argument of the bracket

$$[v,(w \otimes y)] = [v,w] \otimes y + w \otimes [v,y] \tag{2.46}$$

Next one can try to generalize also the lefthand side of the bracket $[v, T]$ to tensors, by demanding some Leibniz rule for tensor products $[v \otimes w, T]$. For some reason (which I cannot reproduce at the moment) this turns out not to be possible for general tensors and for the general tensor product $v \otimes w$. However, if one restricts $T$ to multivectors $y^{(p)}$ and demands the graded Leibniz rule

$$[v \wedge w, y] = v \wedge [w, y] + (-)^{(y-1)v}[v, y] \wedge w \tag{2.47}$$

when acting on the wedge product of two other multivectors, then this works and one obtains precisely the SN-bracket. The above Leibniz rule is of course equivalent to (2.43) when taking into account the graded antisymmetry of the bracket.

- Application: the SN-bracket appears in the criterion for integrability of a Poisson-structure:

$$0 = [P, P] = P^{[m_1|k}\partial_k P^{|m_2 m_3]}\boldsymbol{\partial}_{m_1}\boldsymbol{\partial}_{m_2}\boldsymbol{\partial}_{m_3} \iff \text{Jacobi of Poisson-bracket} \tag{2.48}$$

If the Poisson bivector $P$ is non-degenerate, then this condition is equivalent to closure of its inverse $\omega$ which is a symplectic structure

$$\overset{\text{if P invertible}}{\iff} \mathbf{d}x^m \mathbf{d}x^n \mathbf{d}x^k \partial_m \omega_{nk} = \mathbf{d}\omega = 0 \tag{2.49}$$

### 2.1.4 Lagrangian eom's with antibracket

- Note first the formal isomorphism

$$\Gamma(\Lambda^\bullet TM) \cong \mathcal{F}(\Pi T^* M) \tag{2.50}$$

where

$$\Pi T^* M: \quad T^* M \text{ with "parity-inversed fiber": } \mathbb{R}^N \to \Lambda^{N(odd)}, \quad \mathcal{F}: \text{into } \Lambda^N \tag{2.51}$$

Take coordinates $(q^m, \boldsymbol{q}^+_m)$ on $\Pi T^* M$ and $f \in \mathcal{F}(\Pi T^* M)$. It can be expanded as

$$f(q, \boldsymbol{q}^+) = \sum_{k=0}^{D} \tfrac{1}{k!} f^{m_1 \dots m_k}(q) \boldsymbol{q}^+_{m_1} \dots \boldsymbol{q}^+_{m_k} \tag{2.52}$$

Each of the expansion coefficients corresponds to a section of $\Lambda^k TM$.

- The Schouten-Nijenhuis bracket of above now corresponds to a bracket $[\boldsymbol{q}^+_m, q^n] = \delta^n_m$. However, as we are actually interested in paths on $M$ $(\in \mathcal{P}M)$ we will actually need to extend this bracket from $\mathcal{F}(\Pi T^* M)$ to $\mathcal{F}(\Pi T^* \mathcal{P}M)$ (i.e. functionals) which will be denoted by round brackets and defined by simply setting

$$(\boldsymbol{q}^+_m(\tau), q^n(\tau')) = \delta^n_m \delta(\tau - \tau') \tag{2.53}$$

This bracket is called the **antibracket**. The $\boldsymbol{q}^+_m$ are known as **antifields**.

- For general functionals on $\Pi T^* \mathcal{P}M$ the antibracket thus reads

$$\begin{aligned}
(F, G) &\equiv \int d\tau\, \delta F / \delta \boldsymbol{q}^+_m(\tau) \frac{\delta}{\delta q^m(\tau)} G - (-)^{(F-1)(G-1)} \delta G / \delta \boldsymbol{q}^+_m(\tau) \frac{\delta}{\delta q^m(\tau)} F && (2.54)\\
(\boldsymbol{q}^+_m(\tau), G) &= \frac{\delta}{\delta q^m(\tau)} G && (2.55)\\
(q^m(\tau), G) &= -\frac{\delta}{\delta \boldsymbol{q}^+_m(\tau)} G && (2.56)\\
(F, G) &= -(-)^{(F-1)(G-1)} (G, F) && (2.57)\\
(F, (G, H)) &= ((F, G), H) + (-)^{(F-1)(G-1)} (G, (F, H)) \quad \text{(graded Jacobi)} && (2.58)
\end{aligned}$$

It is a graded Lie-bracket of degree -1.

- Using this bracket, the equations of motion appear in the bracket of the action with the antifields:

$$(\boldsymbol{q}^+_m(\tau), S) = \frac{\delta}{\delta q_m(\tau)} S \tag{2.59}$$

This can be either seen as $\boldsymbol{q}^+_m$ acting on the action functional $S$ or as $S$ acting on $\boldsymbol{q}^+_m$. The latter interpretation is interesting, because

$$\mathbf{s} \equiv (S, -) \tag{2.60}$$

defines a differential (of degree $-1$) which (as a differential should) squares to zero

$$\begin{aligned}
\mathbf{s}^2 &= (S, (S, -)) \overset{\text{Jac}}{=} (\underbrace{(S, S)}_{=0}, -) - (S, (S, -)) && (2.61)\\
\Rightarrow \mathbf{s}^2 &= 0 && (2.62)
\end{aligned}$$

$\mathbf{s}$ can thus be used to build a homology (as it reduces the multivector degree). In this homology the equations of motion $\frac{\delta}{\delta q_m(\tau)} S$ are apparently $\mathbf{s}$-exact and are thus implemented homologically

$$\frac{\delta}{\delta q_m(\tau)} S = -\mathbf{s} \boldsymbol{q}^+_m(\tau) \tag{2.63}$$

In other words the functionals on the physical subspace of $\Pi T^* \mathcal{P}M$ (consisting of those paths $\mathcal{P}_{phys} M$ on $M$ that obey the equations of motion) is given by the zero-degree homology

$$\mathcal{F}(\mathcal{P}_{phys} M) = H_0(\mathbf{s} | \mathcal{F}(\Pi T^* \mathcal{P}M)) \equiv \left( \frac{\mathrm{Ker}(\mathbf{s})}{\mathrm{Im}(\mathbf{s})} \right) \Bigg|_{\deg 0} \tag{2.64}$$

- At degree 1 we have[2] we have vector fields which we can identify with symmetry transformations along these vectors

$$U[q,\boldsymbol{q}^+] \equiv \int d\tau' \quad V^m[q](\tau')\boldsymbol{q}_m^+(\tau') \quad (\delta_{(\varepsilon)}q^m(\tau) \equiv V^m_{(\varepsilon)}[q](\tau)) \tag{2.65}$$

And indeed the requirement of $\mathbf{s}$-invariance (BRST invariance) of the vector field is equivalent to invariance of the action under the corresponding transformation

$$\mathbf{s}U[q,\boldsymbol{q}^+] \quad = \quad -\int d\tau' \quad V^m[q](\tau')\frac{\delta S}{\delta q^m(\tau')} \stackrel{!}{=} 0 \quad (\iff \delta_{(\varepsilon)}S = 0) \tag{2.66}$$

So it's the symmetry transformations which are BRST invariant. Instead the exact ones are obtained from

$$\mathbf{s}\int d\tau' \int d\tau \tfrac{1}{2}\Omega^{mn}(\tau,\tau')\boldsymbol{q}_m^+\boldsymbol{q}_n^+(\tau') \quad = \quad -\int d\tau' \underbrace{\int d\tau \Omega^{[mn]}[q](\tau,\tau')\frac{\delta S}{\delta q^m(\tau)}}_{-\delta V^m[q](\tau')}{}^{+}\boldsymbol{q}_n^+(\tau') \tag{2.67}$$

So the exact functionals contain just the trivial gauge transformations.

- If we have a local symmetry, then the Noether identities imply that there are also non-integrated BRST-invariant (and non-exact) vertices

$$\begin{aligned} 0 \quad &\stackrel{NI}{=} \quad \delta_a^{(0)}q^m\frac{\delta}{\delta q^m}S - \frac{d}{dt}\left(\delta_a^{(1)}q^m\frac{\delta}{\delta q^m}S\right) + \ldots = &(2.68)\\ &= \quad -\mathbf{s}\left(\delta_a^{(0)}q^m\boldsymbol{q}_m^+ - \frac{d}{dt}\left(\delta_a^{(1)}q^m\boldsymbol{q}_m^+\right) + \ldots\right) &(2.69)\end{aligned}$$

Trivial gauge transformations can also be written in an unintegrated form

$$\begin{aligned} \mathbf{s}\left(\sum_{r,s\geq 0}\tfrac{1}{2}\Omega^{mn}_{(r)(s)}[q]\boldsymbol{q}_m^{(r)+}\boldsymbol{q}_n^{(s)+}\right) \quad &= \quad -\sum_{r,s\geq 0}\tfrac{1}{2}\Omega^{mn}_{(r)(s)}[q]\left(\left(\frac{d}{dt}\right)^r\frac{\delta S}{\delta q^m}\boldsymbol{q}_n^{(s)+} - \left(\frac{d}{dt}\right)^s\frac{\delta S}{\delta q^n}\boldsymbol{q}_m^{(r)+}\right) = &(2.70)\\ &= \quad -\sum_{r,s\geq 0}\tfrac{1}{2}\left(\Omega^{mn}_{(r)(s)}[q] - \Omega^{nm}_{(s)(r)}[q]\right)\left(\frac{d}{dt}\right)^r\frac{\delta S}{\delta q^m}\boldsymbol{q}_n^{(s)+} &(2.71)\end{aligned}$$

- Noether current

$$\partial_\mu j^\mu(\tau) = -\delta_{(\varepsilon)}q^m(\tau)\frac{\delta S}{\delta q^m(\tau)} = \mathbf{s}\big(V^m[q](\tau)\boldsymbol{q}_m^+(\tau)\big) \tag{2.72}$$

For a typical global symmetry, only the integrated vertex $U = \int V$ is $\mathbf{s}$-invariant. The integrand $V$ instead induces descent equations

$$\begin{aligned} \mathbf{s}V \quad &= \quad \mathbf{d}j &(2.73)\\ \mathbf{s}j \quad &= \quad 0 &(2.74)\end{aligned}$$

[2] If one sticks to the finite dimensional manifold $M$ (instead of the space of paths $\mathcal{P}M$) together with the Schouten-Nijenhuis-bracket on $\Gamma(\Lambda^\bullet TM) \cong \mathcal{F}(\Pi T^*\mathcal{P}M)$, then we have

$$[v\boldsymbol{,}w] \quad = \quad \partial v/\partial(\boldsymbol{\partial}_m)\frac{\partial}{\partial q^m}w - (-)^{(v-1)(w-1)}\partial w/\partial(\boldsymbol{\partial}_m)\frac{\partial}{\partial q^m}v, \quad [\boldsymbol{\partial}_m\boldsymbol{,}q^n] = \delta_m^n$$

In the same way as in the infinite dimensional case let us define a functional $\mathbf{s}$ generated by a 0-vector (a function on $M$) $S(q)$ via

$$\mathbf{s} \equiv [S\boldsymbol{,}-]$$

Then at 0-grade all 0-vectors (functions) are in the kernel. They are $\mathbf{s}$-exact (the image of some 1-vector) if they are the Lie-derivative (directional derivative) of the function $S$ along that vector:

$$\mathbf{s}f = 0 \quad \forall f \in \mathcal{F}(M), \quad f(q) \sim f(q) + \mathbf{s}(\xi^m\boldsymbol{\partial}_m) = f(q) - \underbrace{\xi^m\partial_m S}_{\imath_{\boldsymbol{\xi}}\mathbf{d}S}(q)$$

So

$$H_0(\mathbf{s}|\Gamma(\Lambda^\bullet TM)) \equiv \left(\frac{\mathrm{Ker}(\mathbf{s})}{\mathrm{Im}(\mathbf{s})}\right)\Big|_{\deg 0} = \mathcal{F}(\{q \in M | \partial_m S(q) = 0\})$$

At grade 1, for a vector to be in the kernel it has to be a symmetry-direction (flat direction) of the function $S$

$$\begin{aligned} v \quad &\equiv \quad v^m(q)\boldsymbol{\partial}_m\\ \mathbf{s}v \quad &= \quad -v^m(q)\partial_m S = -\mathcal{L}_v S = [S\boldsymbol{,}v]\\ v \quad &\sim \quad v + \mathbf{s}(\tfrac{1}{2}\xi^{mn}(q)\boldsymbol{\partial}_m\boldsymbol{\partial}_n) = (v^n - \xi^{mn}(q)\partial_m S(q))\,\boldsymbol{\partial}_n = v + \xi(\mathbf{d}S,-) \quad \diamond\end{aligned}$$

The current is thus the non-integrated $\mathbf{s}$-invariant vertex corresponding to the symmetry transformation $V^m$ in the integrated vertex.

- We can also generate the transformation itself with the antibracket

$$\delta q^m[q](\tau) \quad = \quad (\int d\tau' \delta q^k[q](\tau')\boldsymbol{q}_k^+(\tau'),q^m(\tau)) = (\boldsymbol{U},q^m(\tau)) \tag{2.75}$$

What do the non-integrated vertices induce? E.g. $(j, F[q]) = 0$.

**Remarks**

- Future: BRST-differential

$$\mathbf{s} = \boldsymbol{\delta} + \mathbf{d}_l + \ldots \tag{2.76}$$

  - $\delta$:Koszul-Tate-differential: homology puts you on the constraint surface (Hamiltonian formalism=BRST-formalism) or on the equations of motion (Antifield-formalism=BV=BFV). (Different $\delta$'s!)
  - $\mathbf{d}_l$: longitudinal exterior derivative: cohomology restricts to gauge invariant objects
  - combining them to $\mathbf{s}$ is known as **homological perturbation theory**

$$H^\bullet(\mathbf{s}| \ldots) = H^\bullet(\mathbf{d}_l|H_\bullet(\boldsymbol{\delta}| \ldots)) \tag{2.77}$$

  - in the above antifield-discussion the differential $\mathbf{s} = (S,\ldots)$ did not yet take care of any gauge symmetry. So we had just $\mathbf{s} = \boldsymbol{\delta}$. Indeed we observed that $\mathbf{s}$ just put us on the equations of motion.

- Manifold: $M^D$ looks locally like $\mathbb{R}^D$ (charts map to $\mathbb{R}^D$)
**Supermanifold**: $M^{D_c|D_a}$ looks locally like $\mathbb{R}_c^{D_c} \times \mathbb{R}_a^{D_a}$ where $\mathbb{R}_c$ and $\mathbb{R}_a$ are commuting/anticommuting supernumbers respectively, where the **supernumbers** $\Lambda_\infty \equiv \mathbb{R}_c \oplus \mathbb{R}_a$ are the formal limit of a Grassmann algebra $\Lambda_N$ with $\infty$ generators: see next item. It can be expanded as follows:

$$z \quad = \quad \sum_{k=0}^{\infty} \tfrac{1}{k!} z_{i_1\ldots i_k}\boldsymbol{\eta}^{i_1}\cdots\boldsymbol{\eta}^{i_k} \tag{2.78}$$

$$\equiv \quad \underbrace{z_B}_{\text{body}} + \underbrace{\sum_{k=1}^{\infty} \tfrac{1}{k!} z_{i_1\ldots i_k}\boldsymbol{\eta}^{i_1}\cdots\boldsymbol{\eta}^{i_k}}_{\equiv z_S\text{ (soul)}} \qquad (z_B, z_{i_1\ldots i_k} \in \mathbb{R}) \tag{2.79}$$

$$\equiv \quad \underbrace{z_{even}}_{\in\mathbb{R}_c} + \underbrace{z_{odd}}_{\in\mathbb{R}_a} \tag{2.80}$$

Transition functions for the supermanifold have to be **superanalytic functions** $\Lambda_\infty \to \Lambda_\infty$.

- **Grassmann algebra:** $\Lambda_N$ generated by $\boldsymbol{\eta}^i$, $\quad i \in \{1,\ldots,N\}$ with

$$\boldsymbol{\eta}^i\boldsymbol{\eta}^j = -\boldsymbol{\eta}^j\boldsymbol{\eta}^i, \quad \not\sum (\boldsymbol{\eta}^i)^2 = 0 \quad \forall i \tag{2.81}$$

$\sum_{k=0}^{N}\binom{N}{k} = 2^N$ dimensional (e.g. real or complex) vector space. (think of exterior algebra of differential forms $\boldsymbol{\eta}^i = \mathbf{d}x^i$).

- Cotangent bundle with parity reversed fiber $\Pi T^*M$

$$T^*M \quad \overset{local}{=} \quad M^D \times \mathbb{R}^D, \quad \&\text{transition functions} \tag{2.82}$$

$$\Pi T^*M \quad \overset{local}{=} \quad M^D \times \mathbb{R}_a^D, \quad \&\text{same transition functions} \tag{2.83}$$

- Schouten-Nijenhuis-bracket (partial derivative)

$$[-,-]: \quad \Gamma(\Lambda^\bullet TM) \times \Gamma(\Lambda^\bullet TM) \quad \to \quad \Gamma(\Lambda^\bullet TM) \tag{2.84}$$

$$\mathcal{F}(\Pi T^*M) \times \mathcal{F}(\Pi T^*M) \quad \to \quad \mathcal{F}(\Pi T^*M) \tag{2.85}$$

$$[\boldsymbol{q}_m^+,q^n] \quad = \quad \delta_m^n \tag{2.86}$$

$$q^m \in M, \quad (q^m, \boldsymbol{q}_m^+) \in \Pi T^*M \tag{2.87}$$

<-> antibracket (variational derivative):

$$
\begin{aligned}
\Gamma(\Lambda^\bullet T\mathcal{P}M) \times \Gamma(\Lambda^\bullet T\mathcal{P}M) &\rightarrow \mathcal{P}\Gamma(\Lambda^\bullet T\mathcal{P}M) && (2.88)\\
\mathcal{F}(\Pi T^*\mathcal{P}M) \times \mathcal{F}(\Pi T^*\mathcal{P}M) &\rightarrow \mathcal{F}(\Pi T^*\mathcal{P}M) && (2.89)
\end{aligned}
$$

$$
\begin{aligned}
q^m \in \mathcal{P}M, \quad (q^m, \boldsymbol{q}^+_m) \in \mathcal{P}\Pi T^*M && (2.90)\\
q^m(\tau) \in M, \quad (q^m(\tau), \boldsymbol{q}^+_m(\tau)) \in \Pi T^*M && (2.91)
\end{aligned}
$$

- graded commutator

$$
\begin{aligned}
[\imath_v, \imath_w] &\equiv \imath_v\imath_w + (-)^{vw}\imath_w\imath_v \quad |\, \imath_v \,|=|\, v \,| && (2.92)\\
\boldsymbol{\mathcal{L}}_v &\equiv [\mathbf{d}, \imath_v] \quad |\, \boldsymbol{\mathcal{L}}_v \,|=|\, v \,| +1 && (2.93)
\end{aligned}
$$

### 2.1.5 More about symmetries

- A symmetry $\delta q(q, \dot{q}, \ddot{q}, \dddot{q}, ..., t)$ of a Langrangian $L(q, \dot{q})$ can always be split into a symmetry $\delta q(q, \dot{q}, t)$ plus a trivial symmetry! (Henneaux, Exercise 3.8, page 96). This allows to write symmetries in the Hamiltonian formalism as functions of $q$ and $p$ (and maybe t) only.
- Similarly for the Noether charge, see exercise 3.28, page 100

#### Moment map

- Noether theorem $\leftrightarrow$ moment map? [Silva,p.131]
  - **Action** $\psi$ of a Lie group $G$ on a (symplectic) manifold $M$

    $$
    \begin{aligned}
    \psi: \quad G &\rightarrow Diff(M)\\
    g &\mapsto \psi_g && (2.94)
    \end{aligned}
    $$

    Similar to embedding (group homomorphism, so a representation? right-action: $\psi$ is an antihomomorphism). The associated **evaluation map** is

    $$
    \begin{aligned}
    \mathrm{ev}_\psi : M \times G &\rightarrow M\\
    (p, g) &\mapsto \psi_g(p) && (2.95)
    \end{aligned}
    $$

    Action is called smooth, if $\mathrm{ev}_\psi$ is smooth.
  - **symplectic action**

    $$
    \psi: \quad G \rightarrow Sympl(M, \omega) \subset Diff(M) \qquad (2.96)
    $$

  - a symplectic action $\psi$ of $S^1$ (or $\mathbb{R}$) on $(M, \omega)$ is **hamiltonian** if the vector field generated by $\psi$ is hamiltonian ($\exists H$ s.t. $X = \{H, -\}$ or $\mathbf{d}H = \imath_\omega X$)
  - a symplectic action $\psi$ of $G$ on $(M, \omega)$ is a **hamiltonian action** if there exists a map

    $$
    \mu : M \rightarrow \mathfrak{g}^* \qquad (2.97)
    $$

    satisfying:
    * $\forall X \in \mathfrak{g}$ let
      $\mu^X : M \rightarrow \mathbb{R}, \mu^X(p) := \langle\mu(p), X\rangle$ be the component of $\mu$ along $X$.
      $X^\#$ be the vector field on $M$ generated by the one-parameter subgroup $\{\exp tX | t \in \mathbb{R}\} \subseteq G$
      Then

      $$
      \mathbf{d}\mu^X = \imath_{X^\#}\omega \qquad (2.98)
      $$

      i.e. $\mu^X$ is a hamiltonian function for the vector field $X^\#$.
    * $\mu$ is equivariant with respect to the given action $\psi$ of $G$ on $M$ and the coadjoint action $Ad^*$ of $G$ on $\mathfrak{g}^*$:

      $$
      \mu \circ \psi_g = Ad^*_g \circ \mu \quad \forall g \in G \qquad (2.99)
      $$

      The tuple $(M, \omega, G, \mu)$ is then called a **hamiltonian G-space** and $\mu$ is a **moment map**.
  - It would be nice to translate the last general definition of a hamiltonian action of a nonabelian group into the Poisson-bracket language. For me it would be natural to call such an action hamiltonian, if there exist generators $G_a$ such that $\delta_\varepsilon = \varepsilon^a\{G_a, -\}$.

## 2.2 Hamiltonian system with constraints

### 2.2.1 Primary phase space constraints

- Remember that the Legendre transformation

$$H(q,p) = p_m v^m(q,p) - L(q, v(q,p)) \tag{2.100}$$

with the assignment of a momentum (cotangent vector) $p$ for a given $q$ and $v$ via

$$p_m \equiv \frac{\partial L(q,v)}{\partial v^m} \tag{2.101}$$

needs a nondegenerate Hessian (for the relation between $p_m$ and $v^m$ to be invertible), i.e.

$$\det\left(\frac{\partial^2 L(q,v)}{\partial v^m \partial v^n}\right) \neq 0 \tag{2.102}$$

- If instead $\det\left(\frac{\partial^2 L(q,v)}{\partial v^m \partial v^n}\right) = 0$, it means that the relation between $p_m$ and $v^m$ is not invertible and therefore that the $p_m$ are not all independent. So the map from $TM$ to $T^*M$ is not surjective but will instead map only onto a constraint surface $\Sigma_{\text{prim}}$ within $T^*M$ (phase space). They will be defined by some functions

$$\phi_a(q,p) = 0 \tag{2.103}$$

- Examples:

  –

$$\begin{aligned} L(q_1, q_2, \dot{q}_1, \dot{q}_2) &= \tfrac{1}{2}\dot{q}_1^2 - V(q_1,q_2) && (2.104)\\ p_1 &= \dot{q}_1, \quad p_2 = 0 && (2.105)\\ \Rightarrow \phi(q_1,q_2,p_1,p_2) &= p_2 && (2.106) \end{aligned}$$

  –

$$\begin{aligned} L &= q_1 \dot{q}_2 && (2.107)\\ p_1 &= 0, \quad p_2 = q_1 && (2.108)\\ \phi_1 = p_1 & \qquad \phi_2 = p_2 - q_1 && (2.109) \end{aligned}$$

- **Def:** A function $F : T^*M \to \mathbb{R}$ is called **first class** if

$$\{F, \phi_a\} = 0 \quad \forall a \tag{2.110}$$

otherwise it is called **second class**.

-

$$\{\phi_a, \phi_b\} = C_{ab} \begin{array}{l} \overset{?}{\approx} 0 \\ \overset{?}{\approx} \det \neq 0 \end{array} \tag{2.111}$$

- All constraint functions are second class $\iff$

$$\{\phi_a, \phi_b\} = C_{ab} \quad \text{is nondegen} \tag{2.112}$$

- All constraint functions are first class

$$\{G_a, G_b\} = f_{ab}{}^c G_c \approx 0 \tag{2.113}$$

Constraints generate gauge transformation. Each constraint removes 2dof's!

- The constraints so far are called **primary** as they were obtained just from the definition $p_m = \frac{\partial L}{\partial \dot{q}^m}$ without taking into account the equations of motion which can imply further (**secondary**) constraints. For their treatment, this distinction is not at all essential.

- Constraints are called **reducible**, if they are not (linearly) independent

### 2.2.2 Total Hamiltonian

In the presence of constraints, the map $v^m \mapsto p_m = \frac{\partial L(q,v)}{\partial v^m}$ is not invertible. In order to better understand what happens, let us split the *Legendre transformation* into
- building the difference $vp - L(q,v)$
- and only then identify (parts of) $p$ and $v$ via $p = \frac{\partial L(q,v)}{\partial v}$.

So a priori we will obtain a function $H$ which depends on $q^m, v^m$ and $p_m$. Geometrically one could say it is defined on the fiberwise direct product of $TM$ with $T^*M$, which we will denote simply by $TM \times T^*M$).

$$H(q,p,v) \equiv (v^m p_m - L(q,v)) \tag{2.114}$$

As noted already earlier, the variation of $v^m$ drops completely if we restrict to the subspace of $\{(q,p,v)\}$ where $p = \frac{\partial L(q,v)}{\partial v}$. This is true even if $p(q,v)$ is not invertible:

$$\begin{aligned} \delta\, H(q,p,v)\big|_{p=\frac{\partial L(q,v)}{\partial v}} &= \left(v^m \delta p_m + \delta v^m \left(p_m - \frac{\partial L(q,v)}{\partial v^m}\right) - \delta q^m \frac{\partial L(q,v)}{\partial q^m}\right)\Big|_{p=\frac{\partial L(q,v)}{\partial v}} = && (2.115)\\ &= v^m \delta p_m - \delta q^m \frac{\partial L(q,v)}{\partial q^m} && (2.116) \end{aligned}$$

This means that $H(q,p,v)$ does not depend on $v$ when we restrict to a subspace $\Xi \subset TM \times T^*M$ defined via $p = \frac{\partial L(q,v)}{\partial v}$. Remember that the same equation $p = \frac{\partial L(q,v)}{\partial v}$ implies also the primary constraints $\Sigma_{\rm prim} \subset T^*M$ in phase space. This means that the we can define a well-defined $q,p$-dependent Hamiltonian on $\Sigma_{\rm prim} \subset T^*M$ by restricting[3] $H(q,p,v)$ to $\Xi$:

$$H(q,p)|_{\Sigma_{\rm prim}} \equiv H(q,p,v)\big|_{p=\frac{\partial L(q,v)}{\partial v}} \tag{2.117}$$

As $H(q,p)$ coincides only on $\Sigma_{\rm prim}$ with $H(q,p,v)$, also their $p$ and $q$-derivatives coincide only when they are along the surface. So we can say something about the variation of $H(q,p)$ if we restrict in (2.116) not only the variables to the surface, but also the variations to be along $\Sigma$. Any such constrained variation $\delta_\Sigma$ has to obey $\delta_{\Sigma_{\rm prim}}\phi_a = 0$ or any linear combination thereof::

$$\begin{aligned} 0 &\overset{!}{=} u^a \delta_{\Sigma_{\rm prim}} \phi_a(q,p) = && (2.118)\\ &= \delta_{\Sigma_{\rm prim}} q^m \cdot u^a \frac{\partial \phi_a(q,p)}{\partial q^m} + \delta_{\Sigma_{\rm prim}} p_m \cdot u^a \frac{\partial \phi_a(q,p)}{\partial p_m} && (2.119) \end{aligned}$$

Therefore to a variation of $H(q,p)|_\Sigma$ which stays on the surface one can always add (or for later convenience subtract) these vanishing terms and obtains in general a variation

$$\begin{aligned} \delta_{\Sigma_{\rm prim}} H(q,p)\big|_{\Sigma_{\rm prim}} &\overset{(2.116)}{=} \delta_{\Sigma_{\rm prim}} p_m v^m - \delta_{\Sigma_{\rm prim}} q^m \frac{\partial L(q,v)}{\partial q^m} = && (2.120)\\ &\overset{(2.119)}{=} \delta_{\Sigma_{\rm prim}} p_m \left(v^m - u^a \frac{\partial \phi_a(q,p)}{\partial p_m}\right) + \delta_{\Sigma_{\rm prim}} q^m \left(-u^a \frac{\partial \phi_a(q,p)}{\partial q^m} - \frac{\partial L(q,v)}{\partial q^m}\right) && (2.121) \end{aligned}$$

This tells us that independent from how we extend $H$ off the surface $\Sigma_{\rm prim}$, the partial derivatives on the surface

[3] A different approach might be to work immediately with $H(q,p,v)$, instead of first restricting to the constraint surface and then extending again with the help of Lagrange multipliers. The function $H(q,p,v)$ already provides the partial derivatives that one wants, in order to reformulate the Lagrangian equations of motion in terms of Poisson brackets:

$$\frac{\partial H}{\partial q^m} = -\frac{\partial L(q,v)}{\partial q^m}, \quad \frac{\partial H}{\partial p_m} = v^m, \quad \frac{\partial H}{\partial v^m} = p_m - \frac{\partial L(q,v)}{\partial v^m}$$

I.e. Lagrangian equations of motion $\frac{\partial L}{\partial q^m} - \frac{d}{dt}\frac{\partial L}{\partial v} = 0$ (with $v^m \equiv \dot q^m$) are equivalent to

$$\begin{aligned} p_m &= \frac{\partial L}{\partial v^m} \iff \frac{\partial H}{\partial v^m} = 0\\ \dot p_m &= \frac{\partial L}{\partial q^m} = -\frac{\partial H}{\partial q^m} \iff \dot p_m = \{H, p_m\}\\ v^m &= \frac{\partial H}{\partial p_m} \iff \dot q^m = \{H, q^m\} \quad \diamond \end{aligned}$$

will be of the above form[4]

$$\left.\frac{\partial H(q,p)}{\partial q^m}\right|_{\Sigma_{\rm prim}} = -u^a\frac{\partial\phi_a(q,p)}{\partial q^m} - \frac{\partial L(q,v)}{\partial q^m} \tag{2.122}$$

$$\left.\frac{\partial H(q,p)}{\partial p_m}\right|_{\Sigma_{\rm prim}} = v^m - u^a\frac{\partial\phi_a(q,p)}{\partial p_m} \quad \text{for some } u^a \tag{2.123}$$

It should be clear that the first naive extension $H(q,p)$ can be redefined by arbitrary linear combinations of the constraints $\phi_a(q,p)$ as this does not change $H(q,p)|_\Sigma$. In particular we can define an extension for which the partial derivatives reduce to the ones we are used to from the non-degenerate case. Remember that the coefficients $u^a$ can be completely arbitrary. So if we act on them with derivatives, we might allow also a $q$ or $p$ dependence, but we might also treat them as new independent variables. The two different points of view are[5]

$$H_{tot}(q,p) \equiv H(q,p) + u^a(q,p)\phi_a(q,p) \quad (i) \tag{2.124}$$

$$\text{or } H_{tot}(q,p,u) \equiv H(q,p) + u^a\phi_a(q,p) \quad (ii) \tag{2.125}$$

The variation of this so-called **total Hamiltonian** becomes for both approaches (remember $\approx$ means on the constraint surface, so for $\phi_a = 0$)

$$\delta H_{tot}(q,p\underbrace{,u}_{(ii)}) = \Big(\frac{\partial H}{\partial p_m} + u^a\frac{\partial\phi_a}{\partial p_m}\underbrace{+\frac{\partial}{\partial p_m}u^a\phi_a}_{(i)}\Big)\delta p_m - \delta q^m\Big(-\frac{\partial H}{\partial q^m} - u^a\frac{\partial\phi_a}{\partial q^m}\underbrace{-\frac{\partial u^a}{\partial q^m}\phi_a}_{(i)}\Big)\underbrace{+\delta u^a\phi_a}_{(ii)} \tag{2.126}$$

$$\overset{(2.122)(2.123)}{\approx} v^m\delta p_m - \delta q^m\frac{\partial L}{\partial q^m} \tag{2.127}$$

So the definition of the total Hamiltonian is such that on the constraint surface it has precisely the above simple partial derivatives $\frac{\partial}{\partial p_m}H_{tot} = v^m$ and $\frac{\partial}{\partial q^m}H_{tot} = -\frac{\partial}{\partial q^m}L$. Now we can translate the original Lagrangian equations into Hamiltonian language:

$$0 = \frac{\partial L(q,v)}{\partial q^m} - \frac{d}{dt}\underbrace{\frac{\partial L(q,v)}{\partial \dot q^m}}_{p_m} = -\left.\frac{\partial H_{tot}(q,p)}{\partial q^m}\right|_{\Sigma_{\rm prim}} - \dot p_m \tag{2.128}$$

$$(\dot q^m \equiv) v^m = \left.\frac{\partial H_{tot}(q,p)}{\partial p_m}\right|_{\Sigma_{\rm prim}} \tag{2.129}$$

Or in terms of brackets and explicitly demanding the constraints:

$$\dot p_m = \{H_{tot}, p_m\}, \quad \dot q^m = \{H_{tot}, q^m\} \tag{2.130}$$

$$\phi_a(q,p) = 0 \tag{2.131}$$

This can be obtained from a first order action, containing $u^a$ a priori as independent Lagrange multipliers which force the constraints $\phi_a = 0$.

$$\tilde S[q,p,u] \equiv \int dt \quad \dot q^m p_m - H(q,p) - u^a\phi_a(q,p) \tag{2.132}$$

Indeed the variation yields

$$\frac{\delta S}{\delta q^m} = -\dot p_m - \frac{\partial H}{\partial q^m} - u^a\frac{\partial\phi_a}{\partial q^m} \tag{2.133}$$

$$\frac{\delta S}{\delta p_m} = \dot q^m - \frac{\partial H}{\partial p_m} - u^a\frac{\partial\phi_a}{\partial p_m} \tag{2.134}$$

$$\frac{\delta S}{\delta u^a} = \phi_a(q,p) \tag{2.135}$$

[4]The appearance of $v$ on the righthand side of ((2.122))-((2.123)) seems to contradict the statement that $H(q,p)$ is a function of $q$ and $p$ only. $v$ on the righthand side has to be understood as inverting $p = \frac{\partial L}{\partial v}$ as much as possible and removing the remaining $v$'s by choosing $u^a(q,p)$ appropriately. So in fact one might want to give the coefficients $u^a$ an explicit $v$-dependence. Take for example the extreme case where $L(q,v) = L(q)$ and thus $p_m = \frac{\partial L}{\partial v^m} = 0$. In this case the $v$-dependence in the first line drops explicitly, while in the second we have $\left.\frac{\partial H(q,p)}{\partial p_m}\right|_\Sigma = v^m + u^n\frac{\partial p_n}{\partial p_m} = v^m + u^m$ for some $u^m$ which in fact is just $u^m = -v^m$. ⋄

[5]Note that the two ansatzs i) and ii) for $H_{tot}$ both don't mean any loss of generality. If we have $u^a(q,p)$, it contains also the possibility of taking $u^a$ to be a constant. Then the definition of $H_{tot}$ depends on the choice of this constant, and we are basically at ii). Instead if we start from ii), we can always plug a q,p-dependent function u(q,p) as third argument. This would mean restricting the $(q,p,u)$ space to a subspace where $u = u(q,p)$. Then we are back at i). The difference is thus more a notational one and the decision if a total derivative (or variation) should contain the derivative (variation) with respect to $u$ or not. ⋄

The equation $\dot{q}^m = \frac{\partial H}{\partial p_m} + u^a \frac{\partial \phi_a}{\partial p_m}$ should be the "inverse" of $p_m = \frac{\partial L}{\partial \dot{q}^m}$. If the latter is not invertible, the $u^a$ have to make up for it.

This indicates already what we will see in the discussion of the secondary constraints: some of the Lagrange multipliers can be integrated out via their equations of motions (those corresponding to 2nd class constraints), while others cannot and correspond to the gauge symmetries induced by 1st class constraints.

Altogether we have from the Lagrangian to the Hamiltonian formalism a coordinate transformation $(q^m, v_m) \mapsto (q^m, p_m, u^a)$ with

$$q^m = q^m \tag{2.136}$$

$$p_m = \frac{\partial L(q,v)}{\partial v^m} \tag{2.137}$$

$$\left.\frac{\partial H(q,p)}{\partial p_m}\right|_{\Sigma_{\text{prim}}} + u^a \frac{\partial \phi_a(q,p)}{\partial p_m} = v^m \tag{2.138}$$

### 2.2.3 Secondary Constraints

- Secondary constraints are obtained as consistency conditions on the equations of motion. Namely the time evolution should stay on the constraint surface:

$$0 \stackrel{!}{=} \dot{\phi}_b = \tag{2.139}$$

$$= \{H_{tot}, \phi_b\} = \tag{2.140}$$

$$\approx \{H, \phi_b\} + u^a C_{ab} \tag{2.141}$$

  Whenever possible, we use the freedom in the functions $u^a(q,p)$ to impose this equality.

- Sometimes it is not possible and leads to new **secondary constraints** on the phase space which we add to the set $\{\phi_a\}$ of primary constraints (while in the total Hamiltonian we keep a priori only the primary constraints). These new constraints again must obey the same consistency condition which can again lead to new constraints (still called secondary). And so on.

- Example for secondary constraints

$$L = L_0 + \lambda f(q) \tag{2.142}$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 \quad \phi_1 = p_\lambda \tag{2.143}$$

$$H_{tot} = p\dot{q} - L_0 - \lambda f(q) + u p_\lambda \tag{2.144}$$

$$\dot{p}_\lambda = \{H_{tot}, p_\lambda\} \approx -f(q) \stackrel{!}{\approx} 0 \Rightarrow \boxed{\phi_2 \equiv f(q)} \quad \text{(secondary)} \tag{2.145}$$

- Assume now that $\{\phi_A\}$ are all constraints (not just the primary ones} that we obtain in this way and define a submanifold $\Sigma$. Let us now focus on the constraints that we get on the functions $u^a(q,p)$:

$$0 \stackrel{!}{\approx} \{H, \phi_B\} + u^a C_{aB} \tag{2.146}$$

- If all $\{\phi_A\} = \{G_A\}$ are all first class, then $C_{aB} \approx 0$, so $u^a$ drops from the equation and we would get a new constraint on $q, p$, but by assumption all these constraints are already part of $\{G_A\}$. This means in the first class case there is <u>no condition on the $u^a$'s</u>! They are thus completely free parameters in the time evolution and thus correspond to <u>gauge symmetries</u> of the system. The time evolution using the total Hamiltonian makes only part of these gauge symmetries manifest. If one wants all of them manifest one can add also the remaining ones to the the total Hamiltonian and obtain the so-called **extended Hamiltonian**

$$H_{ext} \equiv H + u^A G_A \tag{2.147}$$

- Instead if all $\{\phi_A\} = \{\chi_A\}$ are second class we can solve explicitly (at least on the constraint surface) for $u^a$:

$$u^a \approx -\{H, \chi_B\}(C^{-1})^{Ba} \tag{2.148}$$

  However, for the remaining indicex-values of $A$, where we don't have a corresponding $u^a$, we obtain also consistency relations of the form

$$(u^\alpha \equiv)0 \approx -\{H, \chi_B\}(C^{-1})^{B\alpha} \tag{2.149}$$

  These should be automatically fulfilled, if the original Lagrangian was well-defined (as the two systems of equations were equivalent).

### 2.2.4 Second class constraints and Dirac bracket

- For purely second class constraints

$$\{\chi_a, \chi_b\} = C_{ab} \quad \det C \neq 0 \tag{2.150}$$

one can define a bracket which is compatible with the constraints. It is called the **Dirac bracket**

$$\{f, g\}_D = \{f, g\}_{PB} - \{f, \chi_a\} C^{-1\,ab} \{\chi_b, g\} \tag{2.151}$$

Indeed the bracket of a constraint with anything yields a vanishing result:

$$\{\chi_c, g\}_D = \{\chi_c, g\}_{PB} - \{\chi_c, \chi_a\} C^{-1\,ab} \{\chi_b, g\} = 0 \tag{2.152}$$

- The Dirac bracket is defined also off the constraint surface $\Sigma$, but what actually matters is only what happens on the surface. The claims are that
  - second class constraints define a surface on which the symplectic form $\omega$ of $T^*M$ induces a symplectic form (nondegenerate) on the constraint surface
  - and that the Dirac bracket restricted to the surface agrees with the the Poisson bracket defined with the induced symplectic form ([Henneaux],p.57):

Proof: Take the coordinates $y^M$ in $M$ such that we extend the coordinates on $\Sigma$ to $M$ and take the constraint-functions $\chi_a$ as orthogonal coordinates

$$y^M = (\sigma^{\mathcal{M}}, \chi_a) \tag{2.153}$$

in such a way that

$$\{\chi_a, \sigma^{\mathcal{M}}\} \approx 0 \quad (0 \text{ on } \Sigma = \{\chi_a = 0\}) \tag{2.154}$$

If a first naive choice $\tilde{\sigma}^{\mathcal{M}}$ leads to $\{\tilde{\sigma}^{\mathcal{M}}, \chi_b\} \not\approx 0$ then redefine

$$\sigma^{\mathcal{M}} = \tilde{\sigma}^{\mathcal{M}} - \{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca} \chi_a \tag{2.155}$$

which leads to the desired relation:

$$\begin{aligned} \{\sigma^{\mathcal{M}}, \chi_b\} &= \{\tilde{\sigma}^{\mathcal{M}} - \{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca} \chi_a, \chi_b\} && (2.156) \\ &= \underbrace{\{\tilde{\sigma}^{\mathcal{M}}, \chi_b\} - \{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca} C_{ab}}_{=0} \underbrace{- \{\{\tilde{\sigma}^{\mathcal{M}}, \chi_c\} P^{ca}, \chi_b\} \chi_a}_{\approx 0} && (2.157) \end{aligned}$$

The remaining Poisson brackets are

$$\begin{aligned} \{\chi_a, \chi_b\} &= C_{ab} \quad rank\, 2N \leq D \text{ on } \Sigma && (2.158) \\ \{\sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}\} &= (\omega^{-1})^{\mathcal{M}\mathcal{N}}\ rank\, 2(D - N) && (2.159) \end{aligned}$$

The fact that we were able to implement $\{\chi_a, \sigma^{\mathcal{M}}\} \approx 0$ together with $\{y^M, y^N\}$ and $\{C_a, C_b\}$ having full rank, implies that also $(\omega^{-1})^{\mathcal{M}\mathcal{N}}$ has full rank on $\Sigma$ as indicated above. This proves the first claim.
Our choice of coordinates immediately provide the explicit embedding function of $\Sigma$ into $M$:

$$X^M: \quad \sigma^{\mathcal{M}} \mapsto X^M(\sigma) = (\sigma^{\mathcal{M}}, 0) \tag{2.160}$$

The pullback of $\omega$ onto the constraint surface then reads

$$(X^*\omega)_{\mathcal{M}\mathcal{N}} = \partial_{\mathcal{M}} X^M \omega_{MN} \partial_{\mathcal{N}} X^N = \omega_{\mathcal{M}\mathcal{N}} \tag{2.161}$$

Now build the Poisson bracket that is built using the inverse $(\omega^{-1})^{\mathcal{M}\mathcal{N}}$ of the induced 2-form $\omega_{\mathcal{M}\mathcal{N}}$ and rewrite it in terms of the Poisson bracket in the ambient phase space

$$\begin{aligned} \{X^*F, X^*G\}_D &\stackrel{?}{=} \partial_{\mathcal{M}} F\, (\omega^{-1})^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} G\big|_\Sigma = && (2.162) \\ &= \partial_M F\, (\omega^{-1})^{MN} \partial_N G\big|_\Sigma - \partial^a F\, C_{ab} \partial^b G\big|_\Sigma = && (2.163) \\ &= \{F, G\}|_\Sigma - \{F, \chi_c\} (C^{-1})^{ca} C_{ab} (C^{-1})^{bd} \{\chi_d, G\}\big|_\Sigma = && (2.164) \\ &= X^* \{F, G\} - X^*\{F, \chi_c\} (C^{-1})^{cd} X^* \{\chi_d, G\} \qquad \surd && (2.165) \end{aligned}$$

This indeed coincides with the Dirac bracket as we defined it.

- Second class can always be seen as gauge fixed first class and vice versa ([Henneaux,p.31, p.46)
- regularity condition for constraint (example of 1st-class=2nd class):

$$\begin{aligned} \{\chi_a, \chi_b\} &= C_{ab} \quad \text{(assume 2nd class)} && (2.166) \\ \{\chi_a\chi_b, \chi_c\chi_d\} &= C_{ac}\chi_b\chi_d + C_{ad}\chi_b\chi_c + C_{bc}\chi_a\chi_d + C_{bd}\chi_a\chi_c \quad \text{(looks 1st class)} && (2.167) \end{aligned}$$

  Condition: Constraints $\chi_a$ should define local coordinates around the constraint manifold.

- Use of Dirac-bracket: Let us demonstrate that the Dirac-bracket allows to calculate with $H$ instead of $H_{tot}$. To this end, let us again split the phase space coordinates $y^M = (q^m, p_m)$ into $\Sigma$-coordinates $\sigma^{\mathcal{M}}$ (coordinates of the constraint surface) and the constraint functions $\chi_a$ themselves as orthogonal coordinates. As shown above, the Poisson-brackets can be chosen to be

$$\begin{aligned} \{\chi_a, \sigma^{\mathcal{M}}\} &\approx 0 \quad (0 \text{ on } \Sigma = \{\chi_a = 0\}) && (2.168) \\ \{\chi_a, \chi_b\} &= C_{ab} \quad rank\, 2N \le D \text{ on } \Sigma && (2.169) \\ \{\sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}\} &= (\omega^{-1})^{\mathcal{M}\mathcal{N}}\, rank\, 2(D-N) && (2.170) \end{aligned}$$

  The time evolution of a general phase space function generated by the total Hamiltonian reads

$$\begin{aligned} \dot{F}(q,p) &= \{H_{tot}, F(\sigma,\chi)\} = && (2.171) \\ &\approx \{H, F\} + u^a\{\chi_a, F(\sigma,\chi)\} && (2.172) \end{aligned}$$

  Now we can use the previous result $u^a \approx -\{H, \chi_B\}(C^{-1})^{Ba}$ to obtain

$$\dot{F}(q,p) \approx \{H,F\} - \{H,\chi_B\}(C^{-1})^{Ba}\{\chi_a, F(\sigma,\chi)\} \qquad (2.173)$$

  Using $0 \approx -\{H, \chi_B\}(C^{-1})^{B\alpha}$, we can extend the sum over $a$ (primary constraints) to a sum over $A$ (all constraints) and obtain precisely the Dirac bracket

$$\dot{F}(q,p) \approx \{H,F\}_D \qquad (2.174)$$

  Now $H(q,p)$ was a priori only one naive extension off the constraint surface and therefore quite ambiguous. But any other Hamiltonian, differing just by a linear combination of the constraint functions would work as well, in particular the total Hamiltonian: $\{H,F\}_D = \{H_{tot},F\}_D = \{H_{tot},F\} \neq \{H,F\}$.

## Main 2nd-class example: Dirac bracket directly from first order action

See also [Henneaux, p.59]

### Dirac bracket for a general first order action

Consider now a first order Lagrangian of the form

$$S[y^I] = \int \mathbf{d}t \left(\dot{y}^I A_I(y) + B(y)\right) \qquad (2.175)$$

(We call the variables $y^I$ instead of $q^i$ , because a typical case is $y^I = (q^i, p_i)$.)
Its momenta are all constrained

$$\begin{aligned} \pi_I &:= \dot{\partial}_I L(y,\dot{y}) = A_I(y) && (2.176) \\ \Rightarrow \quad \Phi_I &= \pi_I - A_I(y) && (2.177) \\ C_{IJ} &:= \{\Phi_I, \Phi_J\}_{PB} = \{\pi_I - A_I(y), \pi_J - A_J(y)\}_{PB} = && (2.178) \\ &= \partial_I A_J(y) - \partial_J A_I(y) && (2.179) \end{aligned}$$

If we define

$$\begin{aligned} \boldsymbol{A} &:= A_I \mathbf{d}y^I && (2.180) \\ C &:= \frac{1}{2} C_{IJ} \mathbf{d}x^I \mathbf{d}x^J \end{aligned}$$

then $A$ is a **symplectic (pre)potential** [6] in the sense

$$\mathbf{d}A \quad = \quad \frac{1}{2}\left(\partial_I A_J - (-)^{IJ}\partial_J A_I\right)\mathbf{d}x^I\mathbf{d}x^J = C \tag{2.181}$$

The Dirac bracket belonging to a first order action of the upper type implements a symplectic structure within the coordinate-space (so actually the coordinate space is already a phase space):

$$\{F(y),G(y)\}_D \quad := \quad \underbrace{\{F,G\}_{PB}}_{0} - \{F,\Phi_I\}_{PB}\,C^{IJ}\,\{\Phi_J,G\}_{PB} = \tag{2.182}$$

$$= \quad -\Big(\underbrace{\partial^K\Phi_I}_{\delta_I^K}\partial_K F - \underbrace{\partial^K F}_{0}\partial_K\Phi_I\Big)C^{IJ}\Big(\underbrace{\partial^K G}_{0}\partial_K\Phi_J - \underbrace{\partial^K\Phi_J}_{\delta_J^K}\partial_K G\Big) = \tag{2.183}$$

$$= \quad \partial_I F C^{IJ}\partial_J G \tag{2.184}$$

**Poisson bracket <-> Dirac bracket of first order Lagrangian**

If the first order action is just the first order formalism of a second order action, then the Dirac-bracket on this "extended phase space" coincides with the Poisson bracket of the original phase space:

$$\left(\tilde{S}[q^i,p_i]\equiv\right) \quad S[y^I] \quad = \quad \int \dot{q}^i p_i - H(q,p) \tag{2.185}$$

$$\Rightarrow \quad A_I \quad = \quad (A_i, A^{\underline{i}}) = (p_i, 0), \quad \partial_I = (\partial_i, \partial^{\underline{i}}) \tag{2.186}$$

$$C_{IJ} \quad = \quad \partial_I A_J(y) - \partial_J A_I(y) = \tag{2.187}$$

$$= \quad \begin{pmatrix} 0 & -\partial^{\underline{j}}p_i \\ \partial^{\underline{i}}p_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\delta_i{}^{\underline{j}} \\ \delta^{\underline{i}}{}_j & 0 \end{pmatrix} \tag{2.188}$$

$C_{IJ}$ thus coincides with the symplectic two form $\omega_{IJ}$ of the Poisson bracket of the original action!

**Calculation of the symplectic 2-form via variation of the action (Witten's method)**

Consider now the variation of a first-order action

$$\left(\tilde{S}[q^i,p_i]\equiv\right) \quad S[y^I] \quad = \quad \int \mathbf{d}t\,\left(\dot{y}^J A_J(y) + B(y)\right) \tag{2.189}$$

$$\delta S[y^I] \quad = \quad \int \mathbf{d}t\,\left(\delta\dot{y}^J A_J + \dot{y}^J\delta y^I\partial_I A_J + \delta y^J\partial_J B\right) = \tag{2.190}$$

$$\overset{p.I.}{=} \quad \int \mathbf{d}t\Big(\dot{y}^J\delta y^I\underbrace{(\partial_I A_J - \partial_J A_I)}_{C_{IJ}} + \delta y^J\partial_J B\Big) \tag{2.191}$$

One can thus read off the symplectic two form of the variation of the first order action. In the point particle case there seems no advantage over just reading off $A_J$ and then calculating its exterior derivative. However, in field theory we enter the $\infty$-dimensional case with continous index $\vec{\sigma}$ over which is integrated. The method of partial integration then becomes quite convenient. In particular in the 2-dimensional $WZW$-sigma model which has the special property that part of it can only be locally written via a 2-dimensional integral, for a global description instead requires a 3-dimensional one. This mixture of dimensions makes it quite challenging to read of $A_J(y)$, while the above method to determine $C_{IJ}$ still works. (see Wittens "nonabelian bosonization"-paper).

[6] The symplectic prepotential of the canonical phase space symplectic 2-form $\omega = \mathbf{d}q^i\mathbf{d}p_i$ is $\boldsymbol{A} := -\mathbf{d}q^i p_i$ for which clearly

$$\omega = \mathbf{d}\boldsymbol{A} \qquad \diamond$$