GEOMETRIC CRITERIA FOR LANDWEBER EXACTNESS

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Abstract. The purpose of this paper is to give a new presentation of some of the main results concerning Landweber exactness in the context of the homotopy theory of stacks.

We present two new criteria for Landweber exactness over a flat Hopf algebroid. The first criterion is used to classify stacks arising from Landweber exact maps of rings. Using as extra input only Lazard's theorem and Cartier's classification of p-typical formal group laws, this result is then applied to deduce many of the main results concerning Landweber exactness in stable homotopy theory and to compute the Bousfield classes of certain BP-algebra spectra. The second criterion can be regarded as a generalization of the Landweber exact functor theorem and we use it to give a proof of the original theorem.

1. Introduction

Let E be a generalized homology theory. A map of graded rings $E_* \to R$ is called *Landweber exact* over E if $E_*(-) \otimes_{E_*} R$ is again a generalized homology theory. This concept was first introduced by Landweber in [La2] and plays an important role in algebraic topology (see [Mil] for an extended discussion). For example, it was by proving Landweber exactness of certain ring maps that elliptic cohomology theories were first constructed [LRS], [F]. Recently, Hovey and Strickland have made important advances [HS] toward classifying homology theories arising in this way.

When (E_*, E_*E) is a flat Hopf algebroid, the concept of Landweber exactness can be abstracted to an algebraic property. Given a flat Hopf algebroid (A, Γ) we say that a map of rings $A \to B$ is *Landweber exact* over (A, Γ) if the functor $- \otimes_A B$ from (A, Γ)-comodules to B-modules is exact. It is easy to check that Landweber exactness over (E_*, E_*E) implies Landweber exactness over E as defined before.

One of the aims of this paper is to explain the extremely simple geometric nature of many of the major results concerning Landweber exactness. We will do this by succinctly deriving them using homotopy theory of presheaves of groupoids [H1] and elementary algebraic geometry.

Recall that there is a local model structure on presheaves of groupoids on the site $\mathcal{A}\mathit{ff}_{flat}$ of affine schemes in the flat topology [H1]. We denote it by $P(\mathcal{A}\mathit{ff}_{flat}, \mathcal{G}rpd)_L$. In this model structure, stacks are the fibrant objects. Given a Hopf algebroid (A, Γ), we will write $\mathcal{M}_{A,\Gamma}$ for a fibrant replacement of the presheaf of groupoids represented by $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$.

Much of the formalism for dealing with stacks in algebraic geometry can be phrased in homotopy invariant terms within $P(\mathcal{A}\mathit{ff}_{flat}, \mathcal{G}rpd)_L$. A key reason for this is that the 2-category pullback of stacks is a model for the homotopy pullback in $P(\mathcal{A}\mathit{ff}_{flat}, \mathcal{G}rpd)_L$ [H3]. Consequently, properties of morphisms of stacks (such as being representable, open, an inclusion, etc.) are most naturally defined as homotopy invariant properties of morphisms of presheaves (see Definition 2.4). This allows us to work directly with representable presheaves of groupoids. For instance, when dealing with the stack of formal groups $\mathcal{M}_{FG}$ it is sufficient to consider the (representable) presheaf of formal group laws on $\mathcal{A}\mathrm{ff}_{flat}$.

Using properties of morphisms allows for a conceptual description of the results in this paper and understanding of their proofs. However, we have deliberately avoided using results about properties of morphisms in the proofs which we present in order to reduce the prerequisites.

It is an observation of Laures (see Remark 3.8) that a map of rings $f\colon A \to B$ is Landweber exact over (A,Γ) iff the map of rings $A \to \Gamma \otimes_A B$ is flat. In the language of stacks, this condition translates into flatness of the map of stacks $\operatorname{Spec} B \to \mathcal{M}_{A,\Gamma}$. This map factors naturally as a composition

$$\operatorname{Spec} B \to \mathcal{M}_{B,\Gamma_B} \to \mathcal{M}_{A,\Gamma}$$

where $\Gamma_B = B \otimes_A \Gamma \otimes_A B$ and (B,Γ_B) is the Hopf algebroid induced from (A,Γ) by the map $A \to B$. Taking the homotopy pullback in $P(\mathcal{A}\mathrm{ff}_{flat}, \mathcal{G}rpd)_L$ along the natural map $\operatorname{Spec} A \to \mathcal{M}_{A,\Gamma}$ we obtain a diagram of homotopy pullback squares

$$\begin{tikzcd}
\operatorname{Spec}(\Gamma \otimes_A B) \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
F \arrow[r] \arrow[d] & \mathcal{M}_{B,\Gamma_B} \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \mathcal{M}_{A,\Gamma}.
\end{tikzcd}$$

For categorical reasons, F is the sheaf theoretic image of $\operatorname{Spec}(\Gamma \otimes_A B) \to \operatorname{Spec} A$. It turns out that the map $\operatorname{Spec}(\Gamma \otimes_A B) \to \operatorname{Spec} A$ is flat iff both maps in this factorization are flat. The first factor is flat iff (B,Γ_B) is a flat Hopf algebroid while flatness of the second factor means that it is the inclusion of an intersection of open subschemes. The latter statement is a consequence of a basic property of flat maps of schemes, namely, that their sheaf theoretic image is an intersection of open subschemes.

The map $\operatorname{Spec}(\Gamma \otimes_A B) \cong \operatorname{Spec}\Gamma \times_{\operatorname{Spec} A} \operatorname{Spec} B \to \operatorname{Spec} A$ is the composite of $\mathrm{id} \times \operatorname{Spec} f$ with the action of $\operatorname{Spec}\Gamma$ on $\operatorname{Spec} A$ and so it is equivariant with respect to the action of $\operatorname{Spec}\Gamma$. It follows that its image F is an invariant subsheaf.

We prove that if $A \to B$ is Landweber exact then F is actually an *intersection of open invariant subschemes* or, in the language of stacks, $\mathcal{M}_{B,\Gamma_B}$ *is an intersection of open substacks of* $\mathcal{M}_{A,\Gamma}$. The upshot of this is the following characterization of Landweber exactness over a flat Hopf algebroid.

Theorem 1.1 (Theorem 3.9)**.** *Let (A,Γ) be a flat Hopf algebroid, $A \to B$ be a map of rings, and $\Gamma_B = B \otimes_A \Gamma \otimes_A B$. Then B is Landweber exact over (A,Γ) if and only if (B,Γ_B) is a flat Hopf algebroid and the orbit of* $\operatorname{Spec} B$ *in* $\operatorname{Spec} A$*, meaning the image of* $\operatorname{Spec}\Gamma \otimes_A B \to \operatorname{Spec} A$*, is an intersection of open invariant subschemes of* $\operatorname{Spec} A$.

The importance of Hopf algebroids in Algebraic Topology stems from the fact that the main computational tool for stable homotopy, namely the Adams Spectral Sequence, takes as input Ext over the categories of comodules. The category of comodules over (A,Γ) is equivalent to the category of quasi-coherent sheaves on $\mathcal{M}_{A,\Gamma}$ and only depends on the weak homotopy type of the presheaf $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ (see [H2, Proposition 5.15] or [Ho]). The following result classifies the homotopy types of stacks arising from a Landweber exact map of rings.

Theorem 1.2 (Theorem 3.13)**.** *Let (A,Γ) be a flat Hopf algebroid. There is a bijective correspondence between quasi-compact intersections of open invariant subschemes of* $\operatorname{Spec} A$ *and homotopy types of stacks $\mathcal{M}_{B,\Gamma_B}$ over $\mathcal{M}_{A,\Gamma}$ arising from a Landweber exact maps of rings $A \to B$.*

An equivalent statement is that the stacks $\mathcal{M}_{B,\Gamma_B}$ arising from Landweber exact maps $A \to B$ are the quasi-compact intersections of open substacks of $\mathcal{M}_{A,\Gamma}$.

In the case when $\mathcal{M}_{A,\Gamma}$ is the stack of formal groups the previous result was suggested by work of Hovey and Strickland [HS] and proved by Naumann [N] (see also [Go] for a thorough treatment of the algebraic geometry of the stack of formal groups). Our result can be seen as a generalization of this to arbitrary flat Hopf algebroids. However one of our main objectives is to explain the essential simplicity of this result. We use little more than the definition of stack in [H1], the definition of scheme and basic properties of flatness.

As we have mentioned above, the category of comodules over (A,Γ) depends only on the homotopy type of $\mathcal{M}_{A,\Gamma}$. A consequence of Theorem 1.2 is that the category of comodules over (B,Γ_B) is a function of the orbit of $\operatorname{Spec} B$ in $\operatorname{Spec} A$.

Corollary 1.3 (Corollary 3.14)**.** *Let (A,Γ) be a flat Hopf algebroid and $A \to B$, $A \to B'$ be Landweber exact. If the smallest intersections of open invariants containing the images of* $\operatorname{Spec} B$ *and* $\operatorname{Spec} B'$ *in* $\operatorname{Spec} A$ *agree then the categories of comodules over (B,Γ_B) and $(B',\Gamma_{B'})$ are equivalent.*

Theorem 1.2 also implies that the category of comodules over (B,Γ_B) is a localization of the category of (A,Γ)-comodules [HS, Theorem A]. This is analogous to the fact that when $U \subset X$ is an intersection of open subschemes, the category of quasi-coherent sheaves on U is a localization of the category of quasi-coherent sheaves on X, and follows from the description of quasi-coherent sheaves on a stack via descent in [H2]. This result is described in Section 3.3.

The effective application of the results above requires an understanding of the invariant radical ideals of the Hopf algebroid (A,Γ). An example which is very important in Algebraic Topology is the Hopf algebroid (V, VT) classifying p-typical formal group laws. There, an important theorem of Landweber [La2] (which we derive from Theorems of Lazard and Cartier in the Appendix) gives a classification of the invariant radical ideals: they are the ideals $I_n = (p, v_1, \dots v_{n-1}) \subset \mathbb{Z}_{(p)}[v_1, v_2, \dots] \cong V$ with $0 \le n \le \infty$ (setting $v_0 = p$). The corresponding invariant closed subschemes are thus codimension n affine spaces. It follows that the quasi-compact intersections of invariant opens are in fact the open subschemes

$$(\operatorname{Spec} V/I_n)^c = \bigcup_{i=0}^{n-1} \operatorname{Spec} V[v_i^{-1}],$$

with $n < \infty$. These are unions of complements of hyperplanes defined by the equations $v_i = 0$.

We would like to illustrate how easy it is to apply the result above in this important case. The algebras $B = V[v_n^{-1}]$ and $B' = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ are Landweber exact over (V, VT) (this is immediate for B and for B' follows from Landweber's exact functor Theorem, see Theorem 5.13). The description of the open invariant subsets above makes it clear that the smallest intersection of invariant opens containing either of the subschemes $\operatorname{Spec} B$ or $\operatorname{Spec} B'$ of $\operatorname{Spec} V$ is

$$(\operatorname{Spec} V/I_{n+1})^c = \bigcup_{i=0}^{n} \operatorname{Spec} V[v_i^{-1}].$$

In this case, the equivalence of categories in Corollary 1.3

$$(V[v_n^{-1}], VT_{V[v_n^{-1}]})\text{−}comod \to (\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}], VT_{\mathbb{Z}_{(p)}[v_1,\dots,v_{n-1},v_n^{\pm 1}]})\text{−}comod$$

is given by tensoring over $V[v_n^{-1}]$ with $\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$. The ring V is the coefficient ring of the Brown-Peterson spectrum BP and $V[v_n^{-1}]$ and $\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ are the coefficients of the BP-algebra spectra $BP[v_n^{-1}]$ and $E(n)$, respectively. It follows from Proposition 2.3 that tensor product induces an equivalence of categories from graded comodules over $(BP[v_n^{-1}]_*, BP[v_n^{-1}]_* BP[v_n^{-1}])$ to graded comodules over $(E(n)_*, E(n)_* E(n))$. Since $E(n)$ is Landweber exact over BP and hence also over $BP[v_n^{-1}]$ we see that $E(n)_* X \cong E(n)_* \otimes_{BP[v_n^{-1}]_*} BP[v_n^{-1}]_* X$ and so it follows that the Bousfield classes of these spectra are equal, that is,

$$\langle BP[v_n^{-1}] \rangle = \langle E(n) \rangle.$$

In fact, given any Landweber exact map $V \to B$, the orbit of $\operatorname{Spec} B$ must be either $\operatorname{Spec} V$ itself or $(\operatorname{Spec} V/I_n)^c$ for some $n < \infty$, and so the category of comodules over (B, VT_B) must be equivalent to either comodules over (V, VT) or to comodules over $(V[v_n^{-1}], VT_{V[v_n^{-1}]})$ (Corollary 4.7). This classification of the categories of comodules arising from Landweber exact cohomology theories is originally due to Hovey and Strickland [HS, Theorem C]. It implies, in particular, that the Bousfield classes of Landweber exact cohomology theories over BP are exactly the classes $\langle E(n) \rangle$ and $\langle BP \rangle$.

This picture also lends to easy proofs of many classical relations among Bousfield classes of complex oriented cohomology theories, such as Ravenel's result [Ra2] that

$$\langle E(n) \rangle = \vee_{i \le n} \langle K(i) \rangle.$$

It is also easy to show that for X a finite spectrum, $K(n)_* X \neq 0$ implies $E(n)_* X \neq 0$. All of this is described in Section 4. There we will also prove a result of this flavor which seems new.

Proposition 1.4 (Corollary 4.11). *Let E be a BP-algebra spectrum. If the image of $\operatorname{Spec} E_*$ in $\operatorname{Spec} V$ is contained in some invariant open set $(\operatorname{Spec} V/I_{n+1})^c$, (with $n < \infty$) then $\langle E \rangle = \vee_{i \in I} \langle K(i) \rangle$ for some subset $I \subset \{0, \dots n\}$.*

Finally, it is natural to ask whether our geometric point of view sheds light on the most common criterion for testing Landweber exactness, namely Landweber's exact functor Theorem [La] (stated below as Theorem 5.13). The answer is yes and hinges on analyzing another basic property of flatness, namely, that it is an open condition. In analogy with the proof of Theorem 1.1, we prove that the subscheme over which an equivariant map is flat is not just an intersection of open subschemes which is invariant, but is in fact, an intersection of *open invariant subschemes.* From this we deduce the following result:

Theorem 1.5 (Theorem 5.1). *Let (A, Γ) be a flat Hopf algebroid with the property that every invariant radical ideal is an intersection of invariant prime ideals.*

A map $A \to B$ is Landweber exact over (A, Γ) if and only if for each invariant prime ideal $p \in \operatorname{Spec} A$ the induced map of rings

$$A_p \to A_p \otimes_A \Gamma \otimes_A B$$

is flat.

If all the rings are Noetherian we may refine the above Theorem to the following result:

Theorem 1.6 (Theorem 5.8)**.** *Let (A, Γ) be a flat Hopf algebroid. Suppose A is finite dimensional, A, Γ and B are Noetherian, and every invariant radical ideal of A is an intersection of invariant prime ideals.*

A map $A \to B$ is Landweber exact over (A, Γ) if and only if for every invariant prime ideal $p \in \operatorname{Spec} A$,

$$\operatorname{Tor}_1^A(A/p, B) = 0.$$

Finally we use Theorem 1.5 to give a new proof of Landweber's exact functor theorem which appears as Theorem 5.13.

2. Background and notation

In this section we fix notation and conventions for the rest of the paper and review the results from algebraic geometry and the homotopy theory of stacks that we will need.

2.1. **Hopf algebroids.** A (graded) Hopf algebroid (A, Γ) is a cogroupoid object in the category of (graded) commutative rings. A good general reference for Hopf algebroids is [Ra, Appendix 1].

A Hopf algebroid (A, Γ) is *flat* if the map $\eta_L \colon A \to \Gamma$ classifying the domain is flat (or equivalently if the map η_R classifying the range is flat).

If (A, Γ) is a Hopf algebroid and $A \to B$ is a map of rings, we write $B \otimes_A \Gamma$ for the tensor product when Γ is given the A-module structure via η_L, and $\Gamma \otimes_A B$ when Γ has the A-module structure determined by η_R. The composite

$$A \xrightarrow{\eta_L} \Gamma \to \Gamma \otimes_A B$$

will appear often and we will also denote it by η_L or even leave it unlabeled. Observe that there is a natural Hopf algebroid structure on $(B, B \otimes_A \Gamma \otimes_A B)$. We will write Γ_B for the ring $B \otimes_A \Gamma \otimes_A B$.

An ideal $I \subset A$ is said to be *invariant* if $\eta_R(I) \subset I\Gamma$ [Ra, Definition A1.1.21]. By symmetry, this is equivalent to the condition that $\eta_L(I) = I \subset \eta_R(I)\Gamma$, or $\eta_R(I)\Gamma = \eta_L(I)\Gamma$. There are canonical isomorphims of Γ-modules $\eta_L(I)\Gamma \cong I \otimes_A \Gamma$ and $\eta_R(I)\Gamma \cong \Gamma \otimes_A I$, so I is invariant if and only if $I \otimes_A \Gamma$ and $\Gamma \otimes_A I$ are isomorphic as Γ-modules over Γ.

An (A, Γ)-comodule M is an A-module equipped with a coassociative and unital map of left A-modules $M \to \Gamma \otimes_A M$ (see [Ra, Appendix A1] for more details).

Definition 2.1. *Let (A, Γ) be a flat Hopf algebroid. A map $A \to B$ of rings is* Landweber exact *over (A, Γ) if the functor $- \otimes_A B$ from (A, Γ)-comodules to B-modules is exact.*

The Hopf algebroids which arise in Algebraic Topology are graded. The previous definition generalizes in the obvious way to graded Hopf algebroids and maps of graded rings. Notice though, that a map of graded rings is Landweber exact in this graded sense if and only if it is after forgetting the grading. There is also a simple relation between the categories of graded and ungraded comodules which is described in the following straightforward Proposition.

Proposition 2.2. *Let (A, Γ) be a graded Hopf algebroid.*

(a) Let $A' = A[u^{\pm 1}]$ with u a homogeneous variable. Then the functor $- \otimes_A A'$ induces an equivalence of categories

$$\textit{Graded } (A,\Gamma) - \textit{comod} \to \textit{Graded } (A', \Gamma_{A'}) - \textit{comod}.$$

(b) Suppose A contains a unit u of degree $n > 0$, and A and Γ are concentrated in degrees which are multiples of n. Then there is an equivalence of categories

$$\textit{Graded } (A,\Gamma) - \textit{comod} \to ((A_0, \Gamma_0) - \textit{comod})^n$$

which assigns to a comodule M the finite sequence of comodules $(M_0, \ldots, M_{n-1})$.

(c) The categories of graded (A, Γ)-comodules and ungraded $(A, \Gamma[u^{\pm 1}])$-comodules are equivalent.

The following corollary allows us to apply the main results of the paper also in the graded case.

Corollary 2.3. *Let (A, Γ) be a graded Hopf algebroid and $A \to B$ be a map of graded rings.*

(a) $A \to B$ is Landweber exact in the graded sense if and only if it is Landweber exact in the ungraded sense.

(b) $A \to B$ induces an equivalence

$$\textit{Graded } (A,\Gamma) - \textit{comod} \to \textit{Graded } (B, \Gamma_B) - \textit{comod}$$

iff it induces an equivalence

$$(A,\Gamma) - \textit{comod} \to (B, \Gamma_B) - \textit{comod}.$$

2.2. **Homotopy theory of presheaves of groupoids.** For us a stack is a presheaf of groupoids F on a small site $\mathcal{C}$ which satisfies descent, or "the homotopy sheaf condition". This means that for all covers $\{U_i \to X\}$ in $\mathcal{C}$ the canonical map

$$F(X) \xrightarrow{\sim} \operatorname{holim} \left(\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \cdots \right)$$

is an equivalence of groupoids. The category of presheaves of groupoids on a small site $\mathcal{C}$ admits a model category structure denoted $P(\mathcal{C}, \mathcal{G}rpd)_L$ in which the fibrant objects are the stacks. This model structure is the localization of the projective model structure on the diagram category $P(\mathcal{C}, \mathcal{G}rpd)$ with respect to the collection of maps $\{U_\bullet \to X\}$ where $\{U_i \to X\}$ is a cover in $\mathcal{C}$ and $U_\bullet$ is the nerve of this cover taken in $P(\mathcal{C}, \mathcal{G}rpd)$ via the Yoneda embedding (see [H1]).

The site $\mathcal{A}\mathit{ff}_{flat}$ consists of affine schemes with covers finite collections of flat morphisms which are jointly surjective. This is sometimes called the fpqc topology. For (A, Γ) a Hopf algebroid, $\mathcal{M}_{A,\Gamma}$ denotes a fibrant replacement (or stackification) of the presheaf of groupoids represented by $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ in $P(\mathcal{A}\mathit{ff}_{flat}, \mathcal{G}rpd)_L$.

Given $\mathcal{M} \in P(\mathcal{A}\mathit{ff}_{flat}, \mathcal{G}rpd)_L$ the category $\mathcal{A}\mathit{ff}_{flat}/\mathcal{M}$ has objects maps $\operatorname{Spec} R \to \mathcal{M}$ and morphisms triangles with a commuting homotopy. By quasi-coherent sheaf on $\mathcal{M}$ we mean a contravariant functor F from $\mathcal{A}\mathit{ff}_{flat}/\mathcal{M}$ to abelian groups which assigns to each object $\operatorname{Spec} R \to \mathcal{M}$ an R-module and such that for each morphism the map $\operatorname{Spec} R \to \operatorname{Spec} S$ induces an isomorphism

$$F(\operatorname{Spec} S \to \mathcal{M}) \otimes_S R \xrightarrow{\cong} F(\operatorname{Spec} R \to \mathcal{M}).$$

In [H2] we prove that a weak equivalence $\mathcal{M} \xrightarrow{\sim} \mathcal{M}' \in P(\mathcal{A}\mathrm{ff}_{flat}, \mathcal{G}rpd)_L$ induces via the restriction functor an equivalence of categories

$$\mathrm{QC}(\mathcal{M}') \xrightarrow{\sim} \mathrm{QC}(\mathcal{M})$$

of quasi-coherent sheaves (see [H2, Theorem 4.7], and [H2, Corollary 4.10]). We also prove [H2, Proposition 5.9] the following *descent statement*: given an I-diagram $\mathcal{M}_I$ in $P(\mathcal{C}, \mathcal{G}rpd)$ there is an equivalence of categories

$$\mathrm{QC}(\operatorname{hocolim} \mathcal{M}_i) \to \operatorname{holim} \mathrm{QC}(\mathcal{M}_i)$$

(where the homotopy limit is taken in $\mathcal{C}at$ with the categorical model structure, see [R]). A simple application [H2, Proposition 5.15] of this descent statement implies that the category of comodules over a Hopf algebroid (A, Γ) is equivalent to the category of quasi-coherent sheaves on the presheaf of groupoids $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ and hence is also equivalent to the category quasi-coherent sheaves on its stackification $\mathcal{M}_{A,\Gamma}$.

In the context of the homotopy theory of stacks we can extend properties of morphisms between schemes to properties of morphisms between presheaves of groupoids in the following way.

A property P of morphisms between schemes is said to be *local on the target* if $X \to Y$ has property P whenever there is an open cover $\{U_\alpha \to Y\}$ such that each of the maps $X \times_Y U_\alpha \to U_\alpha$ has property P.

Definition 2.4. *Let P be a property of morphisms of schemes which is local on the target and stable under pullback. We say that $f \colon \mathcal{M} \to \mathcal{N} \in P(\mathcal{A}\mathrm{ff}_{flat}, \mathcal{G}rpd)_L$* **satisfies property** *P if for all maps $X \to \mathcal{N}$ with $X \in \mathcal{A}\mathrm{ff}_{flat}$, the presheaf $X \times^h_{\mathcal{N}} \mathcal{M}$ is weakly equivalent to a scheme X' and the induced map $X' \to X$ has property P.*

In [H4] we prove the following result.

Proposition 2.5. *Let P be a property of morphisms of schemes which is local on the target and stable under pullback, and let $f \colon \mathcal{M} \to \mathcal{N}$ be a map in $P(\mathcal{A}\mathrm{ff}_{flat}, \mathcal{G}rpd)_L$.*

1. *Satisfying property P is a homotopy invariant of f.*
2. *If (A, Γ) is a flat Hopf algebroid and there is a weak equivalence*

$$(\operatorname{Spec} A, \operatorname{Spec} \Gamma) \xrightarrow{\sim} \mathcal{N}$$

then f has property P iff the homotopy pullback over $\operatorname{Spec} A \to \mathcal{N}$ has property P.

We will not use this result in the proofs but we include it here as it allows for a more elegant framing of the results in this paper.

2.3. **Algebraic Geometry.** Recall that a map of commutative rings $R \to S$ is *flat* if $- \otimes_R S$ is an exact functor on the category of R-modules. A map of schemes $X \xrightarrow{f} Y$ is flat if there exist affine open covers U_α of X and V_β of Y such that $f_{|U_\alpha} \colon U_\alpha \to V_\beta$ are induced by flat maps of rings. It is easy to check that $X \xrightarrow{f} Y$ is flat iff, for each $p \in X$, the induced map of local rings $\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is flat.

A map $X \xrightarrow{f} Y$ is *faithfully flat* if it is flat and the underlying map of spaces is surjective.

If $X \xrightarrow{f} Y$ is flat, the point set theoretic image of f is *closed under generalization* (see [Mi, Corollary I.2.8] for this and more about flat morphisms). This means that if $y \in Y$ is in the image of f and $y' \in Y$ is such that $y \in \overline{y'}$ then y' is also in the image of f.

A subspace $S \subset Y$ is closed under generalization exactly when $x \notin S$ implies $\overline{x} \cap S = \emptyset$. It follows that a subspace $S \subset Y$ is closed under generalization if and only if it is an intersection of open subspaces. Explicitly,

$$S = \cap_{x \notin S} Y \setminus \overline{x}.$$

Another important property of flatness is that it is an *open condition* in the following sense. Given $X \xrightarrow{f} Y$, let $\mathrm{fl}(f)$, the *flat locus of* f, be the collection of points $p \in Y$ so that the induced map $\operatorname{Spec} \mathcal{O}_{Y,p} \times_Y X \to \operatorname{Spec} \mathcal{O}_{Y,p}$ is flat. It is easy to see that the flat locus is closed under generalization, since if q generalizes p, there is a factorization of the natural inclusions $\operatorname{Spec} \mathcal{O}_{Y,q} \to \operatorname{Spec} \mathcal{O}_{Y,p} \to Y$. Analogously, if $\mathcal{F}$ is a quasi-coherent sheaf on Y, the flat locus $\mathrm{fl}(\mathcal{F})$ is the collection of points $p \in Y$ so that $\mathcal{F}_p$ is a flat $\mathcal{O}_{Y,p}$-module. The same argument shows that for a quasi-coherent sheaf $\mathcal{F}$ the flat locus $\mathrm{fl}(\mathcal{F})$ is closed under generalization.

Let X be a scheme. An open subset A of the topological space underlying X determines a scheme U together with a map $U \to X$ with the following universal property: any map $Y \to X$ of schemes which factors through A as a map of topological spaces, factors uniquely through $U \to X$. In this case U is called an *open subscheme* of X.

We will need the following result about inverse limits of schemes and we include the proof for the convenience of the reader. See [EGA, IV,Section 8] for much more general results of this sort.

Proposition 2.6. *Let X be a scheme and B be a subset of its underlying topological space which is closed under generalization. There is a scheme V whose underlying space is B and a map $V \to X$ with the property that any map of schemes $Y \to X$ with point set image contained in B factors uniquely through V.*

Proof. We can write B as the intersection of the collection of all open subsets A_α of X containing B. Each A_α inherits a scheme structure from X which we denote by U_α. The inverse limit of the U_α exists as a locally ringed space: its underlying space is $\cap_\alpha U_\alpha$ and its structure sheaf is

$$\mathcal{O}(U \cap \lim_\alpha U_\alpha) = \operatorname*{colim}_\alpha \mathcal{O}_X(U \cap U_\alpha).$$

If $U = \operatorname{Spec} R$ is an open affine of X then $\mathcal{O}_X(U \cap U_\alpha) = R[a_\alpha^{-1}]$ for some $a_\alpha \in R$, and so the above colimit is just the localization of R with respect to all the elements a_α.

A locally ringed space is a scheme if it is locally isomorphic to the spectrum of a ring. Given an affine open $U = \operatorname{Spec} R \subset X$, the restriction of the structure sheaf of $\lim_\alpha U_\alpha$ to $U \cap (\cap_\alpha U_\alpha)$ is the affine scheme $\operatorname{Spec} R[\{a_\alpha^{-1}\}]$. It follows that the ringed space $\lim_\alpha U_\alpha$ is a scheme which we call V. Since the category of schemes is a full subcategory of locally ringed spaces, V is also the inverse limit of the U_α in the category of schemes and therefore satisfies the required universal property. $\square$

We will say that a scheme V as in the previous statement is an *intersection of open subschemes* of X.

A scheme X determines a sheaf on $\mathcal{A}\mathrm{ff}_{flat}$. By the *image* of a map of schemes $f\colon X \to Y$ we mean the image of the induced map of sheaves on $\mathcal{A}\mathrm{ff}_{flat}$.

Lemma 2.7. *If f is a flat morphism of schemes, its image is represented by an intersection of open subschemes of Y whose underlying space is the point set theoretic image of f.*

Proof. Since the point set theoretic image of f is an intersection of opens, f factors uniquely as

$$X \twoheadrightarrow V \rightarrowtail Y$$

with V an intersection of open subschemes of Y. The first map is a point set theoretic surjection and is also flat since the induced map on local rings of points is flat, i.e., it is faithfully flat. Hence the map of sheaves determined by $X \twoheadrightarrow V$ is surjective and we conclude that V represents the image of f. □

Definition 2.8. *Let R be a ring, $U \subset \operatorname{Spec} R$ an open subset and I a radical ideal with* $\operatorname{Spec} R/I = U^c$. *An R-module M is* supported on U *if* $\operatorname{Tor}^R_*(R/I, M) = 0$.

We note that if R/I is a perfect complex (i.e. a small object in the derived category $D(R)$), i denotes the inclusion of the complement U into $\operatorname{Spec} R$, and $M \neq 0$ is supported on U, then the quasi-coherent sheaf i^*M must be nontrivial [DG]. Note that this is very easy to check directly if I is a principal ideal.

3. Geometric Characterization of Landweber exactness

In this section we prove our geometric criterion for Landweber exactness Theorem 3.9 and Theorem 3.13 classifying the stacks over $\mathcal{M}_{A,\Gamma}$ which arise from Landweber exact maps of rings $A \to B$. We begin with some general results about homotopy pullbacks in the model category $P(\mathcal{C}, \mathcal{G}rpd)_L$ which apply to any site $\mathcal{C}$. In the following subsection we apply these results to the site $\mathcal{A}\mathrm{ff}_{flat}$. In the last subsection we apply Theorem 3.13 and descent statements from [H2] to obtain Hovey and Strickland's characterization of the categories of (B, Γ_B)-comodules for $A \to B$ Landweber exact.

3.1. Homotopy pullbacks in $P(\mathcal{C}, \mathcal{G}rpd)_L$.

In this section we recall the formula for the homotopy pullback in $P(\mathcal{C}, \mathcal{G}rpd)_L$ and prove results pertaining to this pullback when taken along the map including the discrete presheaf of objects into a presheaf of groupoids.

Lemma 3.1 (Lemma 2.2 [H3]). *Let*

$$\mathcal{M}_1 \xrightarrow{i} \mathcal{N} \xleftarrow{j} \mathcal{M}_2$$

be a diagram in $P(\mathcal{C}, \mathcal{G}rpd)$. The levelwise homotopy fiber product $\mathcal{M}_1 \times^h_{\mathcal{N}} \mathcal{M}_2$, which is the presheaf of groupoids with

1. *objects triples (a, b, ϕ) with $a \in \mathcal{M}_1(X), b \in \mathcal{M}_2(X)$ and an isomorphism $\phi : i(a) \xrightarrow{\sim} j(b)$, and*
2. *morphisms from (a, b, ϕ) to (a', b', ϕ') are pairs (α, β) where $\alpha : a \cong a'$ and $\beta : b \cong b'$, such that $\phi' \circ i(\alpha) = j(\beta) \circ \phi$.*

is also a model for the homotopy pullback in the local model structure $P(\mathcal{C}, \mathcal{G}rpd)_L$.

From now on, when we speak about the homotopy fiber product in $P(\mathcal{C}, \mathcal{G}rpd)_L$ we mean the functorial model described above. Note that the homotopy fiber product agrees with the usual fiber product for presheaves of sets.

Given a groupoid object (X_o, X_m) and a map $X_o \to Y$, $Y \times_{X_o} X_m$ denotes the fiber product over the domain map $d : X_m \to X_o$ while $X_m \times_{X_o} Y$ denotes the fiber product over the range map $r : X_m \to X_o$.

Definition 3.2. *Let (P_o, P_m) be a presheaf of groupoids and $Q \xrightarrow{f} P_o$ a sub-presheaf of sets. We say that f is an* **invariant sub-presheaf** *if the pre-image of Q under the domain and range maps $P_m \rightrightarrows P_o$ agree, that is, if $Q \times_{P_o} P_m = P_m \times_{P_o} Q$ as sub-presheaves of P_m.*

Notice that if (X_o, X_m) is a groupoid object in $\mathcal{C}$ and $Y \rightarrowtail X_o \in \mathcal{C}$ is a monomorphism then Y is invariant as a subpresheaf of X_o if $Y \times_{X_o} X_m \cong X_m \times_{X_o} Y \in \mathcal{C}$, since the Yoneda embedding commutes with limits.

Note also that if $Q \to P_0$ is an invariant sub-presheaf, then $(P, P \times_{P_o} P_m)$ has the natural structure of a presheaf of groupoids which comes with a natural inclusion

$$(Q, Q \times_{P_o} P_m) \to (P_o, P_m)$$

which is objectwise full and faithful.

Proposition 3.3. *Let (X_o, X_m) be a presheaf of groupoids and $(X_o, X_m) \xrightarrow{\sim} \mathcal{M} \in P(\mathcal{C}, \mathcal{G}rpd)_L$ a weak equivalence. The homotopy pullback assignment from*

$$\{\text{ homotopy types of maps } \mathcal{M}' \to \mathcal{M}\}$$

to

$$\{\text{ homotopy types of maps } \mathcal{N} \to X_o\}$$

is a monomorphism. Moreover, given $\mathcal{M}' \to \mathcal{M}$, there is a simplicial diagram in $P(\mathcal{C}, \mathcal{G}rpd)_L$

$$\mathcal{M}' \times^h_{\mathcal{M}} X_m \times_{X_o} X_m \Rrightarrow \mathcal{M}' \times^h_{\mathcal{M}} X_m \rightrightarrows \mathcal{M}' \times^h_{\mathcal{M}} X_o$$

and a weak equivalence from its geometric realization to $\mathcal{M}'$.

Proof. It suffices to prove the second statement. There is a canonical homotopy between the two maps $d_0, d_1 : X_m \rightrightarrows X_o \to \mathcal{M}$ which gives a canonical isomorphism between the homotopy pullbacks along each of these maps

$$X_m \times^{h,d_0}_{\mathcal{M}} \mathcal{M}' \xrightarrow{\sim} X_m \times^{h,d_1}_{\mathcal{M}} \mathcal{M}' \tag{3.4}$$

over X_m. There is also a canonical isomorphism $X_m \times^h_m \mathcal{M}' \cong X_m \times_{X_o} (X_o \times^h_{\mathcal{M}} \mathcal{M}')$ coming from the symmetric monoidal structure. The canonical isomorphism (3.4) satisfies the cocycle condition in the sense that pulling it back along the three maps $X_m \times_{X_o} X_m \to X_m$ one obtains a commutative triangle of canonical isomorphisms between the three different pullbacks of the form $(X_m \times_{X_o} X_m) \times^h_{\mathcal{M}} \mathcal{M}'$. Fixing particular maps $X_m \times_{X_o} X_m \to \mathcal{M}$ and $X_m \to \mathcal{M}$ we can use these canonical isomorphisms to construct a simplicial diagram

$$X_m \times_{X_o} X_m \times^h_{\mathcal{M}} \mathcal{M}' \Rrightarrow X_m \times^h_{\mathcal{M}} \mathcal{M}' \rightrightarrows X_o \times^h_{\mathcal{M}} \mathcal{M}'$$

over

$$X_m \times_{X_o} X_m \Rrightarrow X_m \rightrightarrows X_o.$$

By assumption, there is a weak equivalence

$$|X_m \times_{X_o} X_m \Longrightarrow X_m \Longrightarrow X_o| \longrightarrow \mathcal{M}.$$

and by construction we have a homotopy pullback square

$$\begin{tikzcd} {|X_m \times_{X_o} X_m \times^h_{\mathcal{M}} \mathcal{M}'} \arrow[r, Rightarrow] & X_m \times^h_{\mathcal{M}} \mathcal{M}' \arrow[r, Rightarrow] \arrow[d] & X_0 \times^h_{\mathcal{M}} \mathcal{M}'| \arrow[r] & \mathcal{M}' \arrow[d] \\ {|X_m \times_{X_o} X_m} \arrow[r, Rightarrow] & X_m \arrow[r, Rightarrow] & X_o| \arrow[r, "\sim"] & \mathcal{M}. \end{tikzcd}$$

It follows that the top map is also a weak equivalence. □

Remark 3.5. Essentially the same proof shows more generally that certain types of morphisms (open, closed, affine, flat, etc.) over $\mathcal{M}$ correspond to those types of invariant morphisms over X_o, see Proposition 2.5.

Lemma 3.6. *Let (X_o, X_m) be a presheaf of groupoids and $Y \to X_o$ a map of presheaves of sets on $\mathcal{C}$. There is an induced map of presheaves of groupoids*

$$(Y, Y \times_{X_o} X_m \times_{X_o} Y) \to (X_o, X_m) \tag{3.7}$$

and the homotopy pullback

$$F = X_o \times^h_{(X_o,X_m)} (Y, Y \times_{X_o} X_m \times_{X_o} Y)$$

is weakly equivalent to the invariant sub-presheaf Q of X_o determined by the image of $X_m \times_{X_o} Y$ in X_o under the domain map. If X_o and X_m are sheaves then F is also weakly equivalent to the subsheaf of X_o which is the sheafification of Q.

Proof. The objects of $F(Z)$ consist of maps

$$Z \xrightarrow{\phi} X_m \times_{X_o} Y.$$

Given an isomorphism $\phi_1 \to \phi_2$ in $F(Z)$, the images of ϕ_1 and ϕ_2 under the domain map $d\colon X_m \times_{X_0} Y \to X_o$ must agree. The isomorphism is then a lift in the diagram

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, shift right, "\phi_1"'] \arrow[d, shift left, "\phi_2"] & Y \times_{X_o} X_m \times_{X_o} Y \arrow[d, shift left] \arrow[d, shift right] \\ X_m \times_{X_o} Y \arrow[r] \arrow[d, "d"] & Y \\ X_o & \end{tikzcd}$$

Since $d\phi_1 = d\phi_2$, the lift is unique and is simply the composite

$$Z \to Y \times_{X_o} X_m \times_{X_o} X_m \times_{X_o} Y \overset{1\times c\times 1}{\longrightarrow} Y \times_{X_o} X_m \times_{X_o} Y$$

where c denotes composition of arrows. The same argument shows that if $\phi, \phi' \in F(Z)$ satisfy $d \circ \phi = d \circ \phi'$ then they are uniquely isomorphic in $F(Z)$. This shows that F is levelwise weakly equivalent to the invariant sub-presheaf of X_o given by the image of the domain map $X_m \times_{X_o} Y \to X_o$. Since sheafification is a weak equivalence on $P(\mathcal{C}, \mathcal{G}rpd)_L$ if X_o is a sheaf then F is weakly equivalent to the invariant subsheaf Q of X_o which is the sheafification of the image presheaf. □

Observe that in the situation of the previous Lemma, the sub(pre)sheaf $Q \subset X_o$ is the smallest invariant sub(pre)sheaf of X_o containing the image of Y.

3.2. **Characterization of Landweber exactness.** The site we are most interested in is affine schemes with the flat topology, denoted $\mathcal{A}\mathrm{ff}_{flat}$. Groupoid objects in $\mathcal{A}\mathrm{ff}_{flat}$ are given by pairs $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ and correspond precisely to cogroupoid objects (A, Γ) in commutative rings, that is, Hopf algebroids.

We will apply Lemma 3.6 in the special case where the presheaf of groupoids is represented by the pair $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ and we are given a map of representable sheaves $\operatorname{Spec} B \to \operatorname{Spec} A$. Recall that Γ_B denotes $B \otimes_A \Gamma \otimes_A B$ and that (B, Γ_B) inherits the structure of a Hopf algebroid.

We begin with the following observation.

Lemma 3.8. *[G. Laures] Let (A, Γ) be a flat Hopf algebroid. A map of rings $A \to B$ is Landweber exact over (A, Γ) if and only if $A \xrightarrow{\eta_L} \Gamma \otimes_A B$ is flat.*

Proof. First observe that this condition is necessary since any monomorphism of A modules $N \to N'$ induces a monomorphism of extended comodules $\Gamma \otimes_A N \to \Gamma \otimes_A N'$. To see that this condition is sufficient observe that for a comodule M the coaction map $M \to \Gamma \otimes_A M$ includes M as a retract and so given a monomorphism of comodules $M \to M'$, $B \otimes_A M \to B \otimes_A M'$ is a retract of $B \otimes_A \Gamma \otimes_A M \to B \otimes_A \Gamma \otimes_A M'$. □

Observe also that we have homotopy pullback squares

$$\begin{tikzcd}
\operatorname{Spec} \Gamma \otimes_A B \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] & \\
\operatorname{Spec} \Gamma \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] & \\
\operatorname{Spec} A \arrow[r] & (\operatorname{Spec} A, \operatorname{Spec} \Gamma) \arrow[r, "\simeq"] & \mathcal{M}_{(A,\Gamma)}.
\end{tikzcd}$$

It follows that, $A \to B$ is Landweber exact if and only if the morphism $\operatorname{Spec} B \to \mathcal{M}_{(A,\Gamma)}$ is flat (see Definition 2.4 and Proposition 2.5).

Theorem 3.9. *Let (A, Γ) be a flat Hopf algebroid, $A \to B$ be a map of rings. Then B is Landweber exact over (A, Γ) if and only if (B, Γ_B) is a flat Hopf algebroid and the orbit of $\operatorname{Spec} B$ in $\operatorname{Spec} A$, meaning the image of $\operatorname{Spec} \Gamma \otimes_A B \to \operatorname{Spec} A$, is an intersection of open invariant subschemes of $\operatorname{Spec} A$.*

Proof. Suppose $A \to B$ is Landweber exact over (A, Γ). By Lemma 3.3 the image as a sheaf of $\operatorname{Spec} \Gamma \otimes_A B \to \operatorname{Spec} A$ is the homotopy pullback

$$Q \simeq (\operatorname{Spec} B, \operatorname{Spec} \Gamma_B) \times^h_{(\operatorname{Spec} A, \operatorname{Spec} \Gamma)} \operatorname{Spec} A \simeq \mathcal{M}_{B,\Gamma_B} \times^h_{\mathcal{M}_{A,\Gamma}} \operatorname{Spec} A$$

which is an invariant subsheaf of $\operatorname{Spec} A$.

By Lemma 3.8, $\Gamma \otimes_A B$ is flat as a left A module, and so it follows from Lemma 2.7 that Q is an intersection of open subschemes of $\operatorname{Spec} A$.

Let S be the intersection of all the open invariant subschemes which contain Q. Let m be a prime which is not in Q. Then $\operatorname{Spec} A/m \to \operatorname{Spec} A$ is a closed subscheme which does not intersect Q. Since Q is invariant, the intersection of Q with the orbit of $\operatorname{Spec} A/m$ is also trivial. In other words,

$$(\Gamma \otimes_A A/m) \otimes_A (\Gamma \otimes_A B) = 0$$

where the A-algebra structure on both of the factors is given by η_L.

Factoring $\operatorname{Spec} \Gamma \otimes_A A/m \to \operatorname{Spec} A$ as an epimorphism followed by a monomorphism

$$A \to A/I \to \Gamma \otimes_A A/m, \tag{3.10}$$

$\operatorname{Spec} A/I$ is the closure of the set theoretic image of $\operatorname{Spec} \Gamma \otimes_A A/m$ in $\operatorname{Spec} A$, i.e. the closure of the orbit of $\operatorname{Spec} A/m$. We claim in addition that $\operatorname{Spec} A/I$ is a closed invariant subscheme. Tensoring the composite (3.10) with Γ on the left and on the right we obtain a commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, two heads] \arrow[d, "="'] & A/I \otimes_A \Gamma \arrow[r, hook, "- \otimes_A \Gamma"] \arrow[d, dashed, "\cong"] & \Gamma \otimes_A A/m \otimes_A \Gamma \arrow[d, "\cong"] \\
\Gamma \arrow[r, two heads] & \Gamma \otimes_A A/I \arrow[r, hook, "\Gamma \otimes_A -"] & \Gamma \otimes_A \Gamma \otimes_A A/m.
\end{tikzcd}$$

Here the right vertical isomorphism is induced by the extended comodule structure on $\Gamma \otimes_A A/m$, this sends 1 to 1 so the outer square commutes. Since Γ is flat over A (via η_L or η_R), each of the rows is a factorization of the composite as a surjection followed by an inclusion, so there is an induced isomorphism in the middle as indicated.

Since $\Gamma \otimes_A B$ is flat as a left A-module it follows that $A/I \otimes_A \Gamma \otimes_A B = 0$. So given $m \notin Q$ we have produced a closed invariant subscheme $\operatorname{Spec} A/I$ containing m such that $\operatorname{Spec} A/I \cap Q = \emptyset$. It follows that the intersection of $\operatorname{Spec} A/m$ with S is also trivial and so $P = S$.

Finally, as $A \to B$ is Landweber exact over (A, Γ), $A \to \Gamma \otimes_A B$ is flat and so tensoring with B we see that $B \to B \otimes_A \Gamma \otimes_A B = \Gamma_B$ is also flat. Hence (B, Γ_B) is a flat Hopf algebroid.

Conversely, assume (B, Γ_B) is a flat Hopf algebroid and $\mathcal{M}_{B,\Gamma_B} \times^h_{\mathcal{M}_{A,\Gamma}} \operatorname{Spec} A$ is an intersection of open subschemes Z. Consider the following diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d, "\pi"] \arrow[dr] & \\
Z \arrow[r] \arrow[d, hook] & \mathcal{M}_{B,\Gamma_B} \arrow[d, "\psi"] & \operatorname{Spec} A \arrow[dl] \\
\operatorname{Spec} A \arrow[r] & \mathcal{M}_{A,\Gamma} &
\end{tikzcd}$$

where the squares are homotopy pullback squares. Using the characterization of weak equivalences in $P(\mathcal{C}, \mathcal{G}rpd)_L$ as maps satisfying the local lifting conditions (see [H1, Definition 5.6]) we can find a flat cover $Z' \to Z$ and a lift up to homotopy, that is, we have a homotopy commutative diagram

$$\begin{tikzcd}
Z' \arrow[r, dashed] \arrow[d] & (\operatorname{Spec} B, \operatorname{Spec} \Gamma_B) \arrow[d, "\sim"] \\
Z \arrow[r] & \mathcal{M}_{B,\Gamma_B}.
\end{tikzcd}$$

It follows that the homotopy pullback of Z' along π is the fiber product $Z' \times_{\operatorname{Spec} B} \operatorname{Spec} \Gamma_B$, which is flat over Z'. By faithfully flat descent (see [Mi, Proposition 2.23]) W is also a scheme faithfully flat over Z.

By hypothesis $Z \to \operatorname{Spec} A$ is flat so it follows that the composition $W \to \operatorname{Spec} A$ is also flat. Now W is also the pullback

$$(\operatorname{Spec} A \times^h_{\mathcal{M}_{A,\Gamma}} \operatorname{Spec} A) \times^h_{\operatorname{Spec} A} \operatorname{Spec} B \simeq \operatorname{Spec} \Gamma \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec} \Gamma \otimes_A B.$$

Hence $A \to \Gamma \otimes_A B$ is flat. □

Remark 3.11. Proposition 2.5 tells us that the map $\pi\colon \operatorname{Spec} B \to \mathcal{M}_{B,\Gamma_B}$ is flat and so $W \to Z$ is also flat. The argument used above to prove directly that $W \to Z$ is flat actually proves that π is flat and is similar to the proof of Proposition 2.5.

Example 3.12. Let k be a field and consider the action of k^* on $\mathbb{A}^1_k$. The corresponding Hopf algebroid is $(\operatorname{Spec} k[x], \operatorname{Spec} k[x, y^{\pm 1}])$. The maps $k[x] \to k[x]/(x-a)$ are Landweber exact for $a \neq 0$ since the orbit of the point $(x-a)$ in this case is the open invariant subscheme $\mathbb{A}^1_k - 0$. In contrast $k[x] \to k[x]/(x)$ is not Landweber exact as its orbit is the invariant closed subscheme $\operatorname{Spec} k[x]/(x)$ which is not open in $\operatorname{Spec} k[x]$.

We can rephrase the previous Theorem as saying that B is Landweber exact over (A, Γ) if and only if (B, Γ_B) is a flat Hopf algebroid and $\mathcal{M}_{B,\Gamma_B}$ is weakly equivalent to $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma)$ where U is an intersection of open invariant subschemes of $\operatorname{Spec} A$. The subscheme U will necessarily be quasi-compact and we have the following bijective correspondence.

Theorem 3.13. *Let (A, Γ) be a flat Hopf algebroid. There is a bijective correspondence between equivalence classes of stacks $\mathcal{M}_{B,\Gamma_B}$ coming from Landweber exact maps $A \to B$ and quasi-compact intersections of open invariant subschemes of* $\operatorname{Spec} A$.

The bijection assigns to a stack $\mathcal{M}_{B,\Gamma_B}$ the subscheme U of $\operatorname{Spec} A$ *which is the orbit of* $\operatorname{Spec} B$ *in* $\operatorname{Spec} A$. *Conversely, it assigns to a quasi-compact intersection of open invariant subschemes U the stack associated to the groupoid $(U, U \times_{\operatorname{Spec} A}$* $\operatorname{Spec} \Gamma)$*.*

Proof. From Proposition 3.3 we know that pullback induces an injection from the homotopy types of stacks $\mathcal{M}_{B,\Gamma_B}$ to the isomorphism types of subsheaves of $\operatorname{Spec} A$ (since weak equivalences between discrete sheaves are isomorphisms). By Theorem 3.9 if $A \to B$ is Landweber exact then these subsheaves are intersections of open invariant subschemes of $\operatorname{Spec} A$. As they are images of affine schemes they are also quasi-compact.

It suffices to show that given any quasi-compact U which is an intersection of open invariant subschemes of $\operatorname{Spec} A$ there is a map $A \to B$ which is Landweber exact and so that the image of $\operatorname{Spec} \Gamma \otimes_A B \to \operatorname{Spec} A$ is U. Since U is quasi-compact it admits a finite affine open Zariski cover $\{U_i = \operatorname{Spec} R_i \to Z\}$ and so $\coprod_i U_i \cong \operatorname{Spec}(\prod_i R_i) \to Z$ is surjective and flat. Let $B = \prod_i R_i$. By construction $A \to B$ is flat and therefore it is Landweber exact and (B, Γ_B) is a flat Hopf algebroid. Moreover, (B, Γ_B) and $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma)$ both pull back to U over $\operatorname{Spec} A$ and hence are weakly equivalent by Proposition 3.3. □

Let $A = k[x_1, x_2, \ldots]$ be a polynomial ring in infinitely many variables over a field k. The open subscheme of $\operatorname{Spec} A = \mathbb{A}^\infty_k$ which is the complement of the origin is not quasi-compact. This example will come up when we apply Theorem 3.13 to formal group laws in Section 4.2.

3.3. **Categories of comodules.** Recall that the category of (A, Γ)-comodules is equivalent to the category of quasi-coherent sheaves on $\mathcal{M}_{A,\Gamma}$. The category of quasi-coherent sheaves is a homotopy invariant, in the sense that a weak equivalence $\mathcal{M} \to \mathcal{M}'$ in $P(\mathcal{A}\mathrm{ff}_{flat}, \mathcal{G}rpd)_L$ induces an equivalence of categories of quasi-coherent sheaves [H2, Proposition 5.15]. For $A \to B$ Landweber exact, Theorem 3.9 tells us that there is an intersection of open invariant subschemes U in $\operatorname{Spec} A$ so that $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma) \xrightarrow{\sim} \mathcal{M}_{B,\Gamma_B}$. It follows that the category of (B, Γ_B) comodules is equivalent to the category of quasi-coherent sheaves on $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma)$.

Corollary 3.14. *If $A \to B$ and $A \to B'$ are Landweber exact over (A, Γ) and the smallest intersection of open invariants containing the images of $\operatorname{Spec} B$ and $\operatorname{Spec} B'$ agree then the categories of comodules over (B, Γ_B) and $(B', \Gamma_{B'})$ are equivalent.*

By [H2, Proposition 5.9] the category of quasi-coherent sheaves on $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma)$ is the homotopy inverse limit of the cosimplicial diagram of categories of quasi-coherent sheaves

$$\mathrm{QC}(U) \rightrightarrows \mathrm{QC}(U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma) \Rrightarrow \mathrm{QC}(U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma) \cdots$$

where the arrows are pullbacks along the structure maps of the groupoid object $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma)$. The objects of this category are quasi-coherent sheaves on U together with an isomorphism between the two pullbacks to $U \times_{\operatorname{Spec} A} \operatorname{Spec} \Gamma$ satisfying the cocycle condition. The morphisms are the morphisms of quasi-coherent sheaves satisfying the obvious compatibility conditions.

We now present a proof of [HS, Theorem A] based on this descent description of quasi-coherent sheaves and Theorem 3.13.

Theorem 3.15 (Hovey-Strickland)**.** *Suppose (A, Γ) is a flat Hopf algebroid and $A \to B$ is Landweber exact. Then the category of Γ_B comodules is equivalent to the localization of the category of Γ-comodules with respect to the hereditary torsion theory $\mathcal{T} = \{M \mid B \otimes_A M = 0\}$.*

We will need the following Lemma which is easy to check.

Lemma 3.16. *Suppose we have a map between cosimplicial abelian categories*

$$\begin{array}{ccccc} \mathcal{C}_0 & \rightrightarrows & \mathcal{C}_1 & \Rrightarrow & \mathcal{C}_2 \dots \\ \downarrow{\scriptstyle L_0} & & \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_2} \\ \mathcal{D}_0 & \rightrightarrows & \mathcal{D}_1 & \Rrightarrow & \mathcal{D}_2 \dots \end{array}$$

so that the functors L_i admit right adjoints R_i which also form a map of cosimplicial categories. Let $\mathcal{C} = \operatorname{holim} \mathcal{C}_i$, $\mathcal{D} = \operatorname{holim} \mathcal{D}_i$.

(i) The functors $L = \operatorname{holim} L_i$ and $R = \operatorname{holim} R_i$ form an adjoint pair $L \colon \mathcal{C} \leftrightarrow \mathcal{D} \colon R$.
(ii) Furthermore, if for each i, the counit $L_i R_i \to \mathrm{id}_{\mathcal{D}_i}$ is a natural isomorphism the same is true of the counit $LR \to \mathrm{id}_{\mathcal{D}}$.

Proof of Theorem 3.15. We can apply the previous Lemma to

- $\mathcal{C}$ the category of quasi-coherent sheaves on $\mathcal{M}_{(A,\Gamma)}$,
- $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \dots$ the categories of quasi-coherent sheaves on $\operatorname{Spec} A, \operatorname{Spec} \Gamma, \operatorname{Spec} \Gamma \otimes_A \Gamma, \dots$ respectively,

- $\mathcal{D}$ the category of quasi-coherent sheaves on $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec}\Gamma)$, and
- $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \ldots$ the categories of quasi-coherent sheaves on $U, U \times_{\operatorname{Spec} A} \operatorname{Spec}\Gamma, U \times_{\operatorname{Spec} A} \operatorname{Spec}\Gamma \times_{\operatorname{Spec} A} \operatorname{Spec}\Gamma, \ldots$ respectively.

The functors $L_i \colon \mathcal{C}_i \to \mathcal{D}_i$ are given by pullback and R_i are their right adjoints, the pushforward functors (these are well defined because $U \subset \operatorname{Spec} A$ is quasi-compact and separated, see [Ha, Introduction of II.5 and II.5.8(c)]). It follows easily from the definition of the functors that the counit $L_i R_i \to \mathrm{id}_{\mathcal{D}_i}$ is an isomorphism. Applying Proposition 3.16 (ii) we see that $LR \to \mathrm{id}_{\mathcal{D}}$ is also a natural isomorphism.

By [HS, Proposition 1.4], given an adjoint pair $L \colon \mathcal{C} \leftrightarrow \mathcal{D} \colon R$ between abelian categories where the counit is an isomorphism, the image of the left adjoint is the localization of $\mathcal{C}$ at the hereditary torsion theory of objects of in $\mathcal{C}$ that map to zero under L. In our case this implies that quasi-coherent sheaves on $(U, U \times_{\operatorname{Spec} A} \operatorname{Spec}\Gamma)$, or comodules over (B, Γ_B), is the localization of quasi-coherent sheaves on $\mathcal{M}_{A,\Gamma}$, or comodules over (A, Γ), at the hereditary torsion theory of comodules M which are supported on the complement of U. Since $\operatorname{Spec}\Gamma \otimes_A B$ is faithfully flat over U, M is supported on the complement of U if and only if $M \otimes_A \Gamma \otimes_A \otimes B = 0$. As M is a comodule $M \otimes_A \Gamma \otimes_A \otimes B \simeq \Gamma \otimes_A \otimes M \otimes_A B$ and so the previous condition is equivalent to $M \otimes_A B = 0$. □

4. Applications to Stable Homotopy Theory

In this section we apply Theorem 3.13 to Hopf algebroids which arise from ring spectra. In the case of the Hopf algebroid corepresenting 1-dimensional commutative formal group laws, we obtain new proofs of the classification of stacks arising from Landweber exact complex oriented cohomology theories [N] and hence of the corresponding categories of comodules [HS, Theorem C, Theorem 8.2].

This leads to simple proofs of some classical results about Bousfield classes of complex oriented cohomology theories due to [JW], [Yos], and [Ra2] and some results which seem to be new such as the classification of Bousfield classes of certain BP-algebra spectra in Corollary 4.11.

In the second half of this section we work at a fixed prime and prove results for BP-algebras. The global analogs for MU-algebras can easily be derived from them.

4.1. **General Observations.** In this subsection, E will denote a fixed ring spectrum with (E_*, E_*E) a flat Hopf algebroid in graded commutative rings. Recall that a map of ring spectra $E \to F$ is Landweber exact if the natural transformation $E_*(-) \otimes_{E_*} F_* \to F_*(-)$ is a natural isomorphism.

The following proposition follows from the results of the last section.

Proposition 4.1. *Let $E \to F$ and $E \to F'$ be maps of ring spectra.*

1. *F is Landweber exact over E if $(F_*, F_* \otimes_{E_*} E_*E \otimes_{E_*} F_*)$ is a flat Hopf algebroid and the orbit of $\operatorname{Spec} F_*$ in $\operatorname{Spec} E_*$ is an intersection of open subschemes U.*
2. *Under this hypothesis, the category of (F_*, F_*F)-comodules is a localization of the category of (E_*, E_*E)-comodules and is equivalent to the category of quasi-coherent sheaves on the substack $\mathcal{U}$ of $\mathcal{M}_{E_*,E_*E}$ determined by U.*
3. *If the above holds for both F and F' and the orbits of $\operatorname{Spec} F_*$ and $\operatorname{Spec} F'_*$ in $\operatorname{Spec} E_*$ agree then*
$$(F_*, F_*F) - comod \simeq (F'_*, F'_*F') - comod$$

and the Bousfield classes of F and F' are equal.

Proof. The first two statements follow immediately from the results of the last section. For the third item, notice that the pullback functor $F_* \otimes_{E_*} (-) : (E_*, E_*E) - comod \to (F_*, F_*F) - comod$ factors through the equivalence of categories $\mathrm{QC}(\mathfrak{U}) \to (F_*, F_*F) - comod$ and the same holds for F'. It follows that given an (E_*, E_*E)-comodule M, $F_* \otimes_{E_*} M = 0$ if and only if the pullback of M to $\mathrm{QC}(\mathfrak{U})$ is 0, if and only if $F'_* \otimes_{E_*} M = 0$ (see [H2, Section 5] and [H4] for more details on the pullback functor for quasi-coherent sheaves). □

Proposition 4.2. *Let $E \to F$ be a map of ring spectra and $a \in E_*$ a primitive element which is a nonzero divisor. Suppose furthermore that in the cofiber sequence*

$$E \xrightarrow{a} E \to E/a \tag{4.3}$$

the map $E \to E/a$ is a map of ring spectra.

1. *The open invariant subschemes of $\operatorname{Spec} E/a_*$ with respect to the flat Hopf algebroid $(E/a_*, E/a_*E/a)$ correspond to the open invariant subschemes of $\operatorname{Spec} E_*$ containing the complement of $\operatorname{Spec} E_*/a$.*
2. *If F_* is Landweber exact over E_* and the orbit of $\operatorname{Spec} F_*$ in $\operatorname{Spec} E_*$ covers the complement of $\operatorname{Spec} E_*/a$, then $\langle E\rangle = \langle E/a\rangle \vee \langle F\rangle$.*

Proof. If I is an invariant ideal, pullback induces a bijection between invariant opens of $\operatorname{Spec} A$ containing the complement of $\operatorname{Spec} A/I$ and invariant opens of $\operatorname{Spec} A/I$ with respect to $(A/I, \Gamma_{A/I})$. In particular, the open invariants of $\operatorname{Spec} E/a_*$ with respect to $(E/a_*, E/a_* \otimes_{E_*} E_*E \otimes_{E_*} E/a_*)$ are exactly the open invariants of $\operatorname{Spec} E_*$ containing the complement of $\operatorname{Spec} E/a_*$. Since a is not a zero divisor, the long exact sequences in homotopy for the cofiber sequence (4.3) smashed with E and E/a show that $E/a_*E/a$ is free of rank 2 over $E/a_* \otimes_{E_*} E_*E \otimes_{E_*} E/a_*$. It follows that the natural map induces a bijection between open subschemes of $\operatorname{Spec} E/a_*E/a$ and $\operatorname{Spec} E/a_* \otimes_{E_*} E_*E \otimes_{E_*} E/a_*$. Hence open invariants of $\operatorname{Spec} E/a_*$ with respect to $(E/a_*, E/a_*E/a)$ are the same as the open invariants with respect to $(E/a_*, E/a_* \otimes_{E_*} E_*E \otimes_{E_*} E/a_*)$ and so the above classification applies to the open invariants of $\operatorname{Spec} E/a_*$ with respect to $(E/a_*, E/a_*E/a)$ also.

For the second statement observe that if $E/a_*X = 0$ the long exact sequence in homotopy for the cofiber sequence (4.3) smashed with X implies that a acts as an isomorphism on E_*X and so E_*X is supported on the complement of $\operatorname{Spec} E_*/a$. If furthermore $F_*X = E_*X \otimes_{E_*} F_* = 0$, then E_*X is 0 when restricted to the open subscheme which is the complement of $\operatorname{Spec} E_*/a$ and so E_*X must be trivial. □

Recall that the Adams conditions [A, III.13.3] ensure there is a universal coefficient spectral sequence for computing F_*X from E_*X when F is an E-module.

Proposition 4.4. *Assume that E satisfies the Adams conditions. Let $E \to E'$ and $E \to F$ be maps of ring spectra such that F_* is Landweber exact over (E_*, E_*E) and the image of $\operatorname{Spec} E'_*$ is contained in the orbit U of $\operatorname{Spec} F_*$. Then*

$$\langle F\rangle \geq \langle E'\rangle = \langle E' \wedge F\rangle.$$

Proof. Suppose $F_*X = 0$ then $(E \wedge F)_*X = E_*X \otimes_{E_*} (E_*E \otimes_{E_*} F_*) = 0$ and so E_*X pulls back to 0 on the subscheme U. Since the map $\operatorname{Spec} E'_* \to \operatorname{Spec} E_*$ factors through U, we have $\operatorname{Tor}_k^{E_*}(E_*X, E'_*) = 0$. It follows from the universal coefficient spectral sequence that $E'_*X = 0$.

To see the equality, suppose $(E' \wedge F)_*X = F_* \otimes_{E_*} E_*E \otimes_{E_*} E'_*X = 0$. Since $\operatorname{Spec} E'_* \to \operatorname{Spec} E_*$ factors through U, we then have that $E'_* \otimes_{E_*} E'_*X = 0$. As E'_*X is a retract of this, it is also 0. $\square$

4.2. **The stack $\mathcal{M}_{FG,p}$.** Recall that the functor assigning to a commutative ring the groupoid of 1-dimensional formal group laws and strict isomorphisms is corepresented by the Hopf algebroid (L, LB), where L is the Lazard ring. Quillen showed that the (ungraded) Hopf algebroid (MU_*, MU_*MU) is isomorphic to (L, LB) (see [A, II.8a]).

Cartier proved that, over $\mathbb{Z}_{(p)}$, any formal group law is canonically isomorphic to a p-typical formal group law. This result yields an idempotent self map of L whose image is denoted V. The Hopf algebroid corepresenting the full subgroupoid of p-typical formal group laws over $\mathbb{Z}_{(p)}$ and strict isomorphisms between them is denoted (V, VT). As rings, $V = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ and $VT = V[t_1, t_2, \ldots]$ are infinite polynomial algebras. For an exposition of these results see [Ra, Appendix A2.1.25-27]).

Using Cartier's idempotent, Quillen gave a construction of the Brown-Peterson spectrum BP and proved that the ungraded Hopf algebroid (BP_*, BP_*BP) is isomorphic to (V, VT). We will now show that the map $L \to V$ is Landweber exact over (L, LB) which gives another construction of BP.

Corollary 4.5. [La2] *The map $L \to V$ classifying the universal p-typical formal group law is Landweber exact over (L, LB). This map of rings is realized by the map of ring spectra $MU \to BP$ which is Landweber exact.*

Proof. Since VT is a polynomial algebra over V, (V, VT) is a flat Hopf algebroid. Cartier's Theorem [Ra, A2.1.18] says that over $\mathbb{Z}_{(p)}$ the inclusion of p-typical formal group laws into all formal group laws is an equivalence of groupoids. It follows that the orbit of $\operatorname{Spec} V$ in $\operatorname{Spec} L$ is the open invariant subscheme $\operatorname{Spec} L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. By Theorem 3.9, $L \to V$ is Landweber exact over (L, LB).

Given the appropriate grading on V the map $L \cong MU_* \to V$ is a map of graded rings and so it is also Landweber exact as a map of graded rings (Corollary 2.3). It follows that there is a complex oriented cohomology theory with coefficients V and co-operations VT. Quillen's construction of BP shows that the formal group law on BP_* is the universal p-typical formal group law (see [A, II.15.2], compare with [Ra, A2.1.18-24]). It follows that the cohomology theory obtained from MU and the map $L \to V$ is equivalent to BP and the induced map $MU_*X \otimes_{MU_*} BP_* \to BP_*X$ is an isomorphism for all spaces X. $\square$

Recall that fibrant replacement in $P(\mathcal{C}, \mathcal{G}rpd)_L$ associates to a presheaf of groupoids a locally equivalent stack. We take as our definition that the *stack of formal groups* $\mathcal{M}_{FG}$ is the stack associated to the presheaf on $\mathcal{A}\mathit{ff}_{flat}$ of formal group laws and strict isomorphisms between them. It follows from Cartier's Theorem that the stack associated to $(\operatorname{Spec} V, \operatorname{Spec} VT)$ is the stack of formal groups on affine schemes over $\operatorname{Spec} \mathbb{Z}_{(p)}$, which we denote by $\mathcal{M}_{FG,p}$. Alternatively, $\mathcal{M}_{FG,p} \simeq \mathcal{M}_{FG} \times^h_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)}$.

Landweber classified the invariant radical ideals of V; they are the ideals $I_n = (p, v_1, \ldots, v_{n-1})$ (see the Appendix for a proof of this). This is also a classification of the invariant reduced closed subschemes of $\operatorname{Spec} V$ (see Definition 3.2).

Let $C_n = \operatorname{Spec} V/I_n$. Geometrically C_n is a codimension $n-1$ hyperplane in $\operatorname{Spec} V/p \cong \mathbb{A}^\infty_{\mathbb{F}_p}$. Since C_n is an invariant subscheme,

$$(C_n, C_n \times_{\operatorname{Spec} V} \operatorname{Spec} VT)$$

is a groupoid object in $\mathcal{A}\mathit{ff}$ and we will call its stackification $\mathcal{C}_n$. Similarly, let

$$U_n = \operatorname{Spec} V - C_n = \cup_{i=0}^{i=n-1} \operatorname{Spec} V[v_i^{-1}].$$

Since U_n is an invariant subscheme of $\operatorname{Spec} V$ the pair

$$(U_n, U_n \times_{\operatorname{Spec} V} \operatorname{Spec} VT)$$

forms a groupoid object in schemes and we will call its stackification $\mathcal{U}_n$.

Remark 4.6. Using the language of Definition 2.4, $\mathcal{C}_n$ is a closed substack of $\mathcal{M}_{FG,p}$ and $\mathcal{U}_n$ is an open substack of $\mathcal{M}_{FG,p}$. Landweber's classification of the invariant prime ideals gives a classification of the closed and open substacks of $\mathcal{M}_{FG,p}$: they are the $\mathcal{C}_n$ and the $\mathcal{U}_n$ respectively.

Since any collection of open invariants has a smallest member, an intersection of open invariants is again an open invariant. Since U_∞ is not quasi-compact, it follows that the non-trivial quasi-compact intersections of open invariants are the subschemes U_n with $n < \infty$. Theorem 3.13 then implies that the non-trivial substacks associated to $(\operatorname{Spec} E_*, \operatorname{Spec} E_*E)$ with $BP_* \to E_*$ Landweber exact over (BP_*, BP_*BP) correspond to the natural numbers $\mathbb{N}$.

Corollary 4.7. *Let E be BP-algebra spectrum such that E_* is Landweber exact over (BP_*, BP_*BP). The smallest open invariant subscheme of* $\operatorname{Spec} V$ *containing the image of* $\operatorname{Spec} E_*$ *is either the entire space* $\operatorname{Spec} V$ *or U_n for some $n < \infty$. In the second case*

(1) $(\operatorname{Spec} E_*, \operatorname{Spec} E_*E) \xrightarrow{\sim} (U_n, U_n \times_{\operatorname{Spec} V} \operatorname{Spec} VT)$.
(2) In particular, $(\operatorname{Spec} V[v_{n-1}^{-1}], \operatorname{Spec} VT_{V[v_{n-1}^{-1}]}) \xrightarrow{\sim} (U_n, U_n \times_{\operatorname{Spec} V} \operatorname{Spec} VT)$.
(3) The category of (E_*, E_*E)-comodules is equivalent to the category of $(V[v_{n-1}^{-1}], VT_{V[v_{n-1}^{-1}]})$-comodules.
(4) The Bousfield class of E is equal to that of $BP[v_n^{-1}]$.

Remark 4.8. A somewhat stronger statement holds in the case of BP, in that, the classification given in the last Corollary also applies to Landweber exact maps of ring spectra $BP \to E$. In other words, Landweber exactness of the map of ring spectra $BP \to E$ implies Landweber exactness of $BP_* \to E_*$ over (BP_*, BP_*BP). The proof of this strengthening follows from the Landweber exact functor Theorem (Theorem 5.13). If $BP \to E$ is Landweber exact the cofiber sequences (4.10) imply that $\operatorname{Tor}_k^{BP_*}(BP_*/I_n, E_*) = 0$ for all n and $k > 0$.

Recall the Johnson-Wilson spectra $E(n)$ whose coefficients are $\mathbb{Z}_{(p)}[v_1, \ldots, v_n^{\pm 1}]$ as a V-module. That the Johnson-Wilson spectra are Landweber exact over BP follows from the Landweber exact functor theorem [La], (proved below as Theorem 5.13). Since U_{n+1} is the smallest open invariant subscheme containing $\operatorname{Spec} E(n)_*$, $(\operatorname{Spec} E(n)_*, \operatorname{Spec} E(n)_*E(n))$ is weakly equivalent to $\mathcal{U}_{n+1}$. It follows that for every Landweber exact cohomology theory E over BP either $(\operatorname{Spec} E_*, \operatorname{Spec} E_*E)$ is equivalent to $(\operatorname{Spec} BP_*, \operatorname{Spec} BP_*BP)$ or it is equivalent to $(\operatorname{Spec} E(n)_*, \operatorname{Spec} E(n)_*E(n))$ for some n.

Example 4.9. Since $E(n)$ is Landweber exact, $E(n)[v_k^{-1}]$ is also Landweber exact. The smallest open invariant subset which contains the image of $\operatorname{Spec} E(n)[v_k^{-1}]_*$ in $\operatorname{Spec} V$ is U_{k+1}. It follows that

$$(\operatorname{Spec} E(n)[v_k^{-1}]_*, \operatorname{Spec} E(n)[v_k^{-1}]_* E(n)[v_k^{-1}]) \xrightarrow{\sim} \mathfrak{U}_{k+1}.$$

Since $(\operatorname{Spec} E(k)_*, \operatorname{Spec} E(k)_* E(k)) \xrightarrow{\sim} \mathfrak{U}_{k+1}$ also, the Bousfield classes of $E(n)[v_k^{-1}]$ and $E(k)$ are equal.

Recall that there are BP-algebra spectra $P(n), B(n)$, and $K(n)$ with coefficients $P(n)_* = BP_*/I_n$, $B(n)_* = P(n)_*[v_n^{-1}]$, and $K(n)_* = E(n)_*/I_n$ (see [Ra2, Section 2]). Moreover there are cofibration sequences

$$P(n) \xrightarrow{v_n} P(n) \to P(n+1). \tag{4.10}$$

We assume for simplicity that $p > 2$ so that $(P(n)_*, P(n)_*P(n))$ is a Hopf algebroid. An inductive application of Proposition 4.2 shows that the map $(P(n)_*, VT \otimes_V P(n)_*) \to (P(n)_*, P(n)_*P(n))$ induces a bijection on invariant open subsets of $\operatorname{Spec} P(n)_* = C_n$.

The analog of Corollary 4.7 for $P(n)$ implies that if $P(n) \to E$ is Landweber exact then

$$\langle E \rangle = \langle P(n)[v_m^{-1}] \rangle, \quad \text{for some } m \geq n.$$

As the orbit of $\operatorname{Spec} K(n)_*$ in $\operatorname{Spec} P(n)_*$ is an open invariant, $K(n)$ is Landweber exact over $P(n)$ (first proved in [Yos]) and that $\langle B(n) \rangle = \langle K(n) \rangle$ (first proved in [JW]).

Using Proposition 4.2(2) inductively we have the equality

$$\langle E(n) \rangle = \vee_{i \leq n} \langle K(i) \rangle$$

first proved in [Ra2].

Applying Proposition 4.4 to the spectrum BP we see that:

Corollary 4.11. *If E is a BP-algebra spectrum and the image of $\operatorname{Spec} E_*$ in $\operatorname{Spec} BP_*$ is contained in some invariant open U_{n+1} (with $n < \infty$) then $\langle E \rangle = \vee_{i \in I} \langle K(i) \rangle$ for some subset $I \subset \{0, \dots n\}$.*

From our perspective it is also easy to prove the classical theorem of Ravenel [Ra2, Theorem 2.11]: If X is a finite spectrum and $K(n+1)_*X = 0$ then $K(n)_*X = 0$. To see this, note that if X is a finite spectrum then BP_*X is a finitely presented module and hence is supported on a closed subscheme $\operatorname{Spec} BP_*/I$ with I finitely generated. Since BP_*X is a comodule, its support is an invariant subscheme and so must be one of the closed invariant subschemes $\operatorname{Spec} BP*/I_n$ for some $n < \infty$. The result now follows from looking at the localizations of the cofiber sequence (4.10) with respect to v_{n+1}.

5. The Landweber exact functor theorem

In this section we present another criterion for Landweber exactness which is inspired by Landweber's exact functor theorem [La]. We follow up in Section 5.2 with a proof of Landweber's theorem based on this criterion.

5.1. The generalized Landweber exact functor theorem.

The Landweber exact functor theorem gives a criterion for when a map $V \xrightarrow{\phi} R$ is Landweber exact over (V, VT). The criterion is that $\{\phi(v_0), \phi(v_1), \dots\}$ form an almost regular sequence[1] in R; that is, multiplication by $\phi(v_n)$ is an injection on $R/(\phi(v_0), \dots \phi(v_{n-1}))$. We will call this the *Landweber condition.*

It is easy to see that the Landweber condition is the same as the requirement that $\mathrm{Tor}^i_V(V/I_n, R)$ vanish for $i > 0$ and even that $\mathrm{Tor}^V_1(V/I_n, R) = 0$. Since VT is faithfully flat over V this is equivalent to $\mathrm{Tor}^V_1(V/I_n, VT \otimes_V R)$ vanishing, where $VT \otimes_V R$ is taken as a V-module via the action on the right. Since I_n is an invariant ideal this is the same as the requirement that $\mathrm{Tor}^V_1(V/I_n, VT \otimes_V R)$ vanish for $i > 0$, where the $VT \otimes_V R$ is taken as a V-module via the action on the left.

On the other hand, we know from Lemma 3.8 that Landweber exactness is equivalent to the requirement that $VT \otimes_V R$ be a flat left V-module. Thus Landweber's Theorem reduces to the statement that if $\mathrm{Tor}^V_1(V/I_n, VT \otimes_V R) = 0$ then $VT \otimes_V R$ is flat as a left V-module.

The analog of the Landweber condition for a flat Hopf algebroid (A, Γ) and a ring homomorphism $A \to B$ is that

- for all invariant primes $p \subset A$, $\mathrm{Tor}^A_1(A/p, B) = 0$ (or equivalently $\mathrm{Tor}^A_1(A/p, \Gamma \otimes_A B) = 0$).

We will prove below in Theorem 5.8 that this is indeed a criterion for Landweber exactness, under certain additional hypothesis on A, Γ, and B.

What is true much more generally (Theorem 5.1 below) and gives rise to this criterion is the fact that $\Gamma \otimes_A B$ is flat as an A-module if and only if it is flat at the invariant primes. One can think of this as an equivariant version of the local nature of flatness: a map of rings $f\colon R \to S$ is flat iff for each prime ideal $p \subset R$ the induced map of rings $R_p \to R_p \otimes_R S$ is flat.

Theorem 5.1. *Let (A, Γ) be a flat Hopf algebroid with the property that every invariant radical ideal is an intersection of invariant prime ideals.*

A map of rings $A \xrightarrow{f} B$ is Landweber exact over (A, Γ) if and only if for each invariant prime ideal $p \in \operatorname{Spec} A$

$$A_p \to A_p \otimes_A \Gamma \otimes_A B$$

is flat.

Since Landweber exactness is equivalent to $A \xrightarrow{\eta_L} \Gamma \otimes_A B$ being flat (see Lemma 3.8), this follows from the following result taking M to be the extended comodule $\Gamma \otimes_A B$.

Theorem 5.2. *Let (A, Γ) be a flat Hopf algebroid with the property that every invariant radical ideal is an intersection of invariant prime ideals.*

An (A, Γ)-comodule M is flat as an A-module if and only if $A_p \otimes_A M$ is a flat A_p-module for each invariant prime ideal $p \in \operatorname{Spec} A$.

Proof. The condition is obviously necessary.

Let $Q = \{p \in \operatorname{Spec} A \colon A_p \otimes_A M \text{ is flat}\}$. We will show that Q is an intersection of open invariant subschemes of $\operatorname{Spec} A$. Assuming this, the complement Q^c is the

[1] An almost regular sequence is a sequence of elements satisfying the requirements of a regular sequence except the one about forming a proper ideal (see [Ei, Exercise 6.7])

union of closed invariant subspaces and, by hypothesis, this is a union of irreducible closed invariant subspaces. Concretely, $Q^c = \cup_i \operatorname{Spec} A/p_i$ for some set of invariant prime ideals p_i and for each such p_i, $A_{p_i} \otimes_A M$ is not flat. This shows that it suffices to check flatness at the invariant primes.

We will now prove the claim that Q is an intersection of open invariant subschemes of $\operatorname{Spec} A$. Observe that if $R \xrightarrow{f} S$ is faithfully flat, $p \in \operatorname{Spec} R$ and $q \in \operatorname{Spec} S$ is such that $(\operatorname{Spec} f)(q) = p$ then $R_p \to S_q$ is also faithfully flat (since $R \to S_q$ is flat, the image of $\operatorname{Spec} S_q \to \operatorname{Spec} R$ is closed under generalization and contains p so contains all of $\operatorname{Spec} R_p$). In this case, if M is an R-module, $R_p \otimes_R M$ is a flat R_p module if and only if $S_q \otimes_R M$ is a flat S_q module.

Let $p \in Q^c$. Since Q is an intersection of opens $\operatorname{Spec} A/p \subset Q^c$. Applying the previous statement to $f = \eta_R$ and the A-module M, we see that, for all $q \in \operatorname{Spec}(\Gamma \otimes_A A/p) \subset \operatorname{Spec} \Gamma$, $\Gamma_q \otimes_A M$ is not flat. Since M is a comodule, the two Γ-modules $\Gamma \otimes_A M$ and $\Gamma^L \otimes_A M$ are isomorphic, where Γ^L denotes Γ with the A-module structure given by η_L. It follows that $\Gamma_q^L \otimes_A M$ is also not flat.

Let p' be the image of q under $\operatorname{Spec} \eta_L$. Taking f as above to be η_L we see that $A_{p'} \otimes_A M$ is also not flat. This means that for $p \in Q^c$, the image of

$$\operatorname{Spec}(\Gamma \otimes_A A/p) \xrightarrow{\operatorname{Spec} \eta_L} \operatorname{Spec} A$$

also lies in Q^c; in other words, the orbit of $\overline{p}$ is contained in Q^c.

To complete the proof we show that the closure of the orbit of $\overline{p}$ is also contained in Q^c. Consider the factorization of the map induced by η_L

$$A \to A/I \to \Gamma \otimes_A A/p$$

into a surjection followed by an injection so that $\operatorname{Spec} A/I$ is the closure of the orbit of $\operatorname{Spec} A/p$. $\operatorname{Spec} A/I$ is also an invariant subscheme of $\operatorname{Spec} A$ (see the argument following (3.10)). For $q \in Q$, the primes of the ring $A_q \otimes_A \Gamma \otimes_A A/p$ are the primes of $\Gamma \otimes_A A/p$ whose image via η_L is contained in q. As Q is closed under generalization, a prime contained in q is also a member of Q. On the other hand, η_L applied to a prime in $\Gamma \otimes_A A/p$ is in Q^c. Hence $A_q \otimes_A \Gamma \otimes_A A/p = 0$. Now since A_q is flat over A, tensoring the above factorization with A_q we see that $A_q \otimes_A A/I = 0$, that is, $(\operatorname{Spec} A/I) \cap Q = \emptyset$. □

Example 5.3. (i) Landweber's classification of the invariant radical ideals of (V, VT) (see Theorem A.1) shows that (V, VT) satisfies the hypothesis of Theorem 5.1.

(ii) The Hopf algebroids which arise from the action ϕ of a connected affine algebraic group G on an affine scheme $\operatorname{Spec} A$ satisfy the hypothesis of Theorem 5.1. To see this, let I be an invariant radical ideal of A and let p be a minimal prime of I. Geometrically, $\operatorname{Spec} A/p$ is an irreducible component of $\operatorname{Spec} A/I$. Since G is irreducible, the image of $G \times \operatorname{Spec} A/p \xrightarrow{\phi} \operatorname{Spec} A/I$ is an irreducible subspace. It also contains $\operatorname{Spec} A/p$ and therefore must be equal to $\operatorname{Spec} A/p$. This means that p is itself invariant, and since I is the intersection of its minimal primes the statement follows.

(iii) Let k be a field of characteristic not equal to 2. The Hopf algebroid corresponding to the sign action of $\mathbb{Z}/2$ on $\mathbb{A}_k^1$ does not satisfy the hypothesis of Theorem 5.1 as the ideal $(x^2 - a^2) \subset k[x]$, with $a \in k^*$, is invariant and reduced but the only primes containing it are $(x - a)$ and $(x + a)$ neither of which is invariant.

Remark 5.4. Using Definition 2.4 we may rephrase the hypothesis on (A, Γ) in Theorem 5.1 as the requirement that every closed reduced substack of $\mathcal{M}_{A,\Gamma}$ be a union of irreducible closed substacks. This shows that satisfaction of this hypothesis is invariant under weak equivalence in $P(\mathcal{A}\mathrm{ff}_{flat}, \mathcal{G}rpd)_L$. One can then show that if (A, Γ) satisfies this hypothesis and $A \to B$ is Landweber exact then (B, Γ_B) also satisfies the hypothesis.

The following special case of Theorem 5.1 will be important to us later.

Corollary 5.5. *Let A be a domain and (A, Γ) be a flat Hopf algebroid such that 0 is the only invariant radical ideal. Then every map of rings $A \to B$ is Landweber exact over (A, Γ).*

Remark 5.6. The previous Corollary can also be deduced directly from Corollary 3.14. Under the hypothesis, the only open invariant is $\operatorname{Spec} A$ and so the inclusion of A in its fraction field $A \to K(A)$ induces an equivalence between the categories of (A, Γ) and $(K(A), \Gamma_{K(A)})$-comodules. It follows that $A \to B$ is Landweber exact iff $- \otimes_A B$ is exact on the category of $(K(A), \Gamma_{K(A)})$-comodules. This is always the case since $K(A)$ is a field.

Example 5.7. $(V/I_n[v_n^{-1}], V/I_n[v_n^{-1}] \otimes_V VT)$ satisfies the hypothesis of the previous Corollary so, given an arbitrary map of rings $V \to B$, the induced map $V/I_n[v_n^{-1}] \to V/I_n[v_n^{-1}] \otimes_V B$ is Landweber exact.

Under more restrictive hypothesis on (A, Γ) we can use the local criterion for flatness (see [Ei, Theorem 6.8]) to strengthen Theorem 5.1 as follows.

Theorem 5.8. *Let (A, Γ) be a flat Hopf algebroid with A of finite Krull dimension, and with the property that every invariant radical ideal is an intersection of invariant prime ideals.*

(1) A map of rings $A \xrightarrow{f} B$ is Landweber exact over (A, Γ) if and only if for each prime $q \in \operatorname{Spec} \Gamma \otimes_A B$ whose image in $\operatorname{Spec} A$ is an invariant prime ideal p, the map
$$A_p \to (\Gamma \otimes_A B)_q$$
is flat.

(2) If, in addition, A, Γ, and B are Noetherian, then $A \to B$ is Landweber exact if and only if $\operatorname{Tor}_1^A(A/p, B) = 0$ for every invariant prime ideal $p \subset A$.

Proof. Let $Q = \{p \in \operatorname{Spec} A \colon A_p \to A_p \otimes_A \Gamma \otimes_A B \text{ is flat}\}$ and suppose Q^c is not empty. By Theorem 5.1, $Q^c = \cup \operatorname{Spec} A/p_i$ for some collection of invariant primes p_i. It follows that if $p' \in Q^c$ then for some i, $p_i \subset p'$ and
$$A_{p_i} \to A_{p_i} \otimes_A \Gamma \otimes_A B$$
is not flat. Therefore there is some $p'' \subset p_i$ and $q' \in \operatorname{Spec} A_{p_i} \otimes_A \Gamma \otimes_A B$ lying over it so that the map
$$A_{p''} \to (\Gamma \otimes_A B)_{q'}$$
is not flat (see for example [Ma, Theorem 7.1]) and in this case $p'' \in Q^c$. Iterating this argument one can form a descending chain of prime ideals in A (and in Q^c). By hypothesis A is finite dimensional and so this chain stabilizes, which means that there is an invariant prime p and $q \in \operatorname{Spec} \Gamma \otimes_A B$ lying over it so that the map
$$A_p \to (\Gamma \otimes_A B)_q$$

is not flat.

To prove the second statement we apply the local criterion for flatness [Ei, Theorem 6.8], which says that $(\Gamma \otimes_A B)_q$ is flat over A_p if and only if

$$\mathrm{Tor}_1^{A_p}(A_p/p, (\Gamma \otimes_A B)_q) = 0.$$

This group is a localization of $\mathrm{Tor}_1^A(A/p, \Gamma \otimes_A B)$ and so Landweber exactness follows from the vanishing of the groups $\mathrm{Tor}_1^A(A/p, \Gamma \otimes_A B)$. Since p is invariant and $A \to \Gamma$ is faithfully flat $\mathrm{Tor}_1^A(A/p, \Gamma \otimes_A B) = 0$ if and only if $\mathrm{Tor}_1^A(A/p, B) = 0$, which proves the theorem. □

Example 5.9. Let (A, Γ) be the Hopf algebroid corresponding to the action of $GL_2(k)$ on $\mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$ with k a field. The only invariant primes are (0) and (x, y) so it is immediate from Theorem 5.8 that $k[x, y]/y$ is not Landweber exact even though the orbit of $\operatorname{Spec} k[x]$ in $\mathbb{A}_k^2$ is everything. On the other hand an inclusion of a line $\mathbb{A}_k^1 \hookrightarrow \mathbb{A}_k^2$ which does not go through the origin will be Landweber exact.

Example 5.10. There is a Hopf algebroid given by the pair of rings

$$V_n = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$$
$$VT_n = \mathbb{Z}_{(p)}[v_1, \ldots, v_n][t_1, \ldots t_n]$$

where the Hopf algebroid structure is just the restriction of the one on (V, VT). This corepresents p^n-buds of p-typical formal group laws. It's easy to see that the classification of invariant radical ideals in (V, VT) implies that the invariant radical ideals in (V_n, VT_n) are simply the $I_k \cap V_n$ for $k \leq n$. Applying Theorem 5.8, we see that $\phi \colon V_n \to R$ is Landweber exact over (V_n, VT_n) if and only if $\{\phi(v_0), \phi(v_1), \ldots \phi(v_n)\}$ is an almost regular sequence in R. For example $V_n \to E(k)_*$ is Landweber exact over (V_n, VT_n).

5.2. **The classical Landweber exact functor theorem.** In the previous section we proved a general criterion for Landweber exactness (Theorem 5.1). Under the hypothesis that A, Γ, and B are Noetherian, and A is finite dimensional, we then showed Landweber exactness is equivalent to $\mathrm{Tor}_1^A(A/p, B)$ vanishing for all invariant primes $p \subset A$.

In this section we will prove that this last criterion holds when (A, Γ) is the Hopf algebroid (V, VT). This is the Landweber exact functor theorem [La].

The approach we take here bears similarities to the one outlined by Hopkins [Hop, Go], in particular Proposition 5.12. However, our proof depends essentially on Corollary 5.5 which is not a part of his proof (and Proposition 5.11 makes it unnecessary for us to put a bound on the height).

Let $V_n = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$. Then the canonical maps $V_n \to V$ are flat and $V = \operatorname{colim} V_n$. We start by noting that to prove flatness over V it suffices to prove flatness over each V_n.

Proposition 5.11. *Let $A_n \to A_{n+1}$ be a sequence of maps of rings, $A = \operatorname{colim} A_n$, and suppose $A_n \to A$ are flat. Then an A-module M is flat iff it is flat as a module over each A_n.*

Proof. M is flat as an A-module iff $\mathrm{Tor}_1^A(A/q, M) = 0$ for every finitely generated ideal $q \in A$. Since q is finitely generated, there exists an n and an ideal q' in A_n such that $q = q'A$.

Since A is flat over A_n we have

$$\mathrm{Tor}_1^A(A/q, M) = \mathrm{Tor}_1^A(A_n/q' \otimes_{A_n} A, M) = \mathrm{Tor}_1^{A_n}(A_n/q', M).$$

$\square$

The Landweber exact functor theorem will now follow from the following simple algebraic result.

Proposition 5.12. *Let R be a Noetherian ring, $x \in R$ be a non-zero divisor, and M be an R-module. Then M is flat if the following conditions are satisfied:*

(i) *M/x is flat over R/x,*
(ii) $\mathrm{Tor}_1^R(R/x, M) = 0$,
(iii) *$M[x^{-1}]$ is flat over $R[x^{-1}]$.*

Proof. We will show that for any R-module N, $\mathrm{Tor}_1^R(N, M) = 0$.

Condition (i) implies that $\mathrm{Tor}_i^R(N, M/x) = \mathrm{Tor}_i^R(N, R/x) \otimes_{R/x} M/x$.

Condition (iii) implies that $\mathrm{Tor}_1^R(N, M) \otimes_R R[x^{-1}] = \mathrm{Tor}_1^R(N, M[x^{-1}]) = 0$.

By condition (ii), $0 \to M \xrightarrow{x} M \to M/x \to 0$ is an exact sequence and therefore we have a long exact sequence

$$\mathrm{Tor}_2^R(N, M/x) \to \mathrm{Tor}_1^R(N, M) \xrightarrow{x} \mathrm{Tor}_1^R(N, M) \to \mathrm{Tor}_1^R(N, M/x)$$

where the Tor_2 term vanishes because $\mathrm{Tor}_i^R(N, R/x)$ vanishes unless $i = 0$ or 1. Thus multiplication by x is an injection on $\mathrm{Tor}_1^R(N, M)$ and since $\mathrm{Tor}_1^R(N, M)[x^{-1}] = 0$, it follows that $\mathrm{Tor}^1_R(N, M) = 0$. $\square$

Theorem 5.13 (Landweber)**.** *A map $V \xrightarrow{\phi} B$ is Landweber exact over (V, VT) iff*

$$\phi(p), \phi(v_1), \ldots, \phi(v_n), \ldots$$

is an almost regular sequence in B or equivalently, if

$$\mathrm{Tor}_i^V(V/I_n, B) = 0 \textit{ for all } i > 0, n \geq 0.$$

Proof. Since I_n is invariant and VT is faithfully flat over V, the homological condition in the statement is equivalent to $\mathrm{Tor}_i^V(V/I_n, VT \otimes_V B) = 0$ for all $i > 0$ and $n \geq 0$. Since Landweber exactness is equivalent to $VT \otimes_V B$ being flat as a V-module, the condition is clearly necessary.

Now suppose the condition holds. To prove that B is Landweber exact it suffices, by Proposition 5.11, to show that $VT \otimes_V B$ is flat over V_n for each n. We will show that $V_n/I_k \otimes_{V_n} VT \otimes_V B$ is flat over V_n/I_k by downward induction on k (where we set $I_0 = 0$).

We start the induction with $k = n$. Since V_n/I_n is a PID, flatness of $V_n/I_n \otimes_{V_n} VT \otimes_V B \cong V/I_n \otimes_V VT \otimes_V B$ is the condition that v_n is a nonzero divisor, and this is true by hypothesis.

For the inductive step we apply Proposition 5.12 to $v_k \in V_n/I_k$. The inductive hypothesis is condition (i) in Proposition 5.12. Condition (ii) is that there is no v_k torsion in $V_n/I_k \otimes_{V_n} VT \otimes_V B$, which again holds by hypothesis. Condition (iii) is that $V_n/I_k[v_k^{-1}] \otimes_{V_n} VT \otimes_V B$ is flat over $V_n/I_k[v_k^{-1}]$. The Hopf algebroid $(V/I_k[v_k^{-1}], V/I_k[v_k^{-1}] \otimes_V VT)$ has no nontrivial invariant radical ideals and so by Corollary 5.5 all $V/I_k[v_k^{-1}]$ algebras are Landweber exact over it (see Example 5.7). Hence $V/I_k[v_k^{-1}] \otimes_V VT \otimes_V B \cong V/I_k[v_k^{-1}] \otimes_V VT \otimes_V (V/I_k[v_k^{-1}] \otimes_V B)$ is flat over $V/I_k[v_k^{-1}]$ and so also over $V_n/I_k[v_k^{-1}]$. $\square$

The same proof gives the following result.

Theorem 5.14. *Given a map $V/I_n \xrightarrow{\phi} B$ the following conditions are equivalent:*
(1) ϕ is Landweber exact over $(V/I_n, V/I_n \otimes_V VT)$
(2) ϕ is Landweber exact over $(P(n)_*, P(n)_*P(n))$
(3) The sequence
$$\phi(v_n), \phi(v_{n+1}), \dots, \phi(v_m), \dots$$
is an almost regular sequence in B,
(4) $\mathrm{Tor}_i^V(V/I_k, B) = 0$ for all $k > n, i \geq 0$.

Proof. The equivalence of the statements (1) and (2) follows from Proposition 4.2 (see (4.10)). The equivalence of statements (1), (3), and (4) is exactly as before in the proof of Theorem 5.13. □

Recall that (L, LB) is the Hopf algebroid corepresenting formal group laws and strict isomorphisms (see [Ra, Appendix A2]). There is an isomorphism of the Lazard ring L with the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ (see [Ra, A2.1.10]) and, in terms of this choice of generators, the invariant radical ideals can be written as
$$I_{p,n} = (p, x_{p-1}, x_{p^2-1}, \dots x_{p^n-1}),$$
(see [La3] and Remark A.2). The argument used in the proof of Theorem 5.13 yields the following integral version of Landweber's Theorem.

Theorem 5.15 (Landweber)**.** *A map $L \xrightarrow{\phi} R$ is Landweber exact over (L, LB) iff for all prime numbers p the sequences*
$$\phi(p), \phi(x_{p-1}), \phi(x_{p^2-1}), \dots \phi(x_{p^n-1}), \dots$$
are almost regular sequences in R or equivalently, if
$$\mathrm{Tor}_i^L(L/I_{p,n}, R) = 0 \text{ for all primes } p \text{ and all } i > 0, n \geq 0.$$

Appendix A. Classification of invariant radical ideals of (V, VT).

The main algebraic input needed to apply Theorem 3.13 is a description of the invariant closed subspaces of $\mathrm{Spec}\, A$, or equivalently the invariant radical ideals of (A, Γ).

In this section we deduce the classification of invariant radical ideals of (V, VT) from the uniqueness of height n formal group laws. This uniqueness Theorem due to Lazard (see [Ra, Theorem A2.2.11]) says that if K is separably closed and $\phi_1, \phi_2 : V \to K$ represent formal group laws with $\phi_i(v_n)$ a unit and $\phi_i(v_j) = 0$ for $i < n$, then ϕ_1 and ϕ_2 are isomorphic.

Theorem A.1. [La]. *The proper invariant radical ideals of (V, VT) are exactly the ideals $I_n = (p, v_1, \dots v_{n-1})$, for $0 \leq n \leq \infty$.*

Proof. Let $I \subset V$ be a proper nonzero invariant radical ideal and $C = \mathrm{Spec}\, V/I$ the corresponding proper invariant closed subscheme of $\mathrm{Spec}\, V$. We begin by showing that the prime number $p \in I$ and so V/I is an $\mathbb{F}_p$-algebra.

Since I is invariant $\mathrm{Spec}\, p^{-1}V/I$ is also invariant. As all formal group laws over $\mathbb{Q}$ are isomorphic (to the additive one) there are no nontrivial invariant subschemes of $\mathrm{Spec}\, p^{-1}V$. Therefore $p^{-1}V/I$ is either 0 or $p^{-1}V$; in other words I either generates the zero ideal or the unit ideal in $p^{-1}V$. The first of these possibilities can not occur as $I \neq 0$, and the second means that p is in I.

If all $v_n \in I$ then $I = I_\infty$ and we are done. Otherwise let n be the largest integer such that $I_n \subset I$. Let m be a maximal ideal containing I and not containing v_n, i.e. a closed point of $C \cap \operatorname{Spec} V[v_n^{-1}]$. Let K be the separable closure of V/m and let x denote the point of C corresponding to the composite $V/I \to V/m \to K$.

Given a map $V \xrightarrow{\phi} \mathbb{F}_p$ such that $\phi(I_n) = 0$ and $\phi(v_n) \neq 0$ we can compose it with the inclusion $\mathbb{F}_p \to K$ to obtain a map $V \xrightarrow{\phi'} K$. The uniqueness of the height n formal group law over a separably closed field implies the formal group laws associated to x and ϕ' are isomorphic. As C is invariant and $x \in C$, ϕ must factor through V/I and so the kernel of ϕ contains I.

We have proved that I is contained in the intersection of the maximal ideals $\{m \mid I_n \subset m, v_n \notin m, V/m \cong \mathbb{F}_p\}$. This intersection corresponds to the Zariski closure of the collection of maximal ideals and so is the ideal I_n. By hypothesis $I_n \subset I$, hence $I = I_n$. □

Remark A.2. Recall that (L, LB) is the Hopf algebroid corepresenting formal group laws and strict isomorphisms and the Lazard ring L is isomorphic to a polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ (see [Ra, Appendix A2]).

It is not difficult to see that the ideals

$$I_{p,n} = (p, x_{p-1}, x_{p^2-1}, \dots x_{p^n-1})$$

are invariant using, for instance, [Ra, Theorem A2.1.10].

Cartier's Theorem tells us that localizing at the prime p, V is a retract of $L_{(p)}$. Moreover, the inclusion induces a weak equivalence $(\operatorname{Spec} V, \operatorname{Spec} VT) \xrightarrow{\sim} (\operatorname{Spec} L_{(p)}, \operatorname{Spec} LB_{(p)})$. Pulling back along such a weak equivalence induces a bijection on invariant closed subsets and so the invariant radical ideals of $(L_{(p)}, LB_{(p)})$ must be precisely the ideals $I_{p,n}$. From this it is easy to check that the $I_{p,n}$ are also the only invariant radical ideals of (L, LB) as originally proved in [La3].

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